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Groups, Geometry, and Dynamics

On growth of random groups of intermediate growth

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Dedicated to Pierre de la Harpe
on the occasion of his 70th birthday

Abstract. We study the growth of typical groups from the family of p-groups of intermediate growth constructed by the second author. We find that, in the sense of category, a generic group exhibits oscillating growth with no universal upper bound. At the same time, from a measure-theoretic point of view (i.e., almost surely relative to an appropriately chosen probability measure), the growth function is bounded by e^{n^α} for some $\alpha < 1$.

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1. Introduction

There are few approaches to randomness in group theory. The most known are associated with the names of Gromov and Olshanskii. For the account of these approaches and further literature; see [34]. As far as the authors are concerned, these approaches deal with the models when in certain classes of finitely presented groups one locates "generic" finite presentations with prescribed properties. These models are variation of the *density model* and randomness in them appears in the form of *frequency* or *density*, which correspond to the classic "naive" approach to probability in mathematics.

The modern Kolmogorov's approach to probability assumes existence of a space supplied with a sigma-algebra of measurable sets and a probability measure on it.

There can be different constructions of spaces of groups and one of them was suggested in [16], where Gromov's idea from [27] of convergence of marked metric spaces was transformed into the notion of the compact totally disconnected topological metrizable space $\mathcal{M}_k$ of marked k-generated groups, $k \geq 2$. Later it was

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discovered that this topology is related to the Chabauty topology on the space of normal subgroups of the free group of rank k; see [12]. Observe that in general the Chabauty topology is defined in the space of closed subgroups of a locally compact group (there is analogous notion in differential geometry [8]), and that in the case of a discrete group G it is nothing but the topology induced on the set of subgroups of G by the Tychonoff topology of the space $\{0, 1\}^G$.

The main result of [16] is the construction of the first examples of groups of intermediate growth, thus answering a question of Milnor [31]. In fact, an uncountable family of 3-generated groups was introduced and studied in [16], and among other results it was shown there that the set of rates of growth of finitely generated groups has the cardinality of the continuum, and that there are pairs of groups with incomparable growth (the growth rates of two groups are different but neither grows faster than the other; in fact, the space of rates of growth of finitely generated groups contains an anti-chain of the cardinality of the continuum). The possibility of such phenomenon is based on the fact that there are groups with *oscillating growth*, i.e., groups whose growth on different parts of the range of the argument of growth function (which is a set of natural numbers) behaves alternatively in two fashions: in the intermediate (between polynomial and exponential) way and exponential way. In this paper we will use one particular form of oscillating property which will be defined below. Recent publications [5], [4] [10], and [28] added a lot of new information about oscillating phenomenon and possible rates of growth of finitely generated groups. The survey [25] summarizes some of these achievements.

The construction in [16] deals with torsion 2-groups of intermediate growth. It also provides interesting examples of self-similar groups and first examples of just-infinite branch groups; see [26]. Later a similar construction of p-groups of intermediate growth was produced for arbitrary prime p as well as the first example of a torsion free group of intermediate growth [17]. Observe that, as indicated in [16] and [17], the torsion 2-group constructed in [1] and torsion p-groups constructed in [36] also happen to have intermediate growth and for some periodic sequences (like $\omega = (012)^\infty$ for $p = 2$) have similar features of groups G_ω discussed in this paper.

The reason for introducing the space $\mathcal{M}_k$ of marked groups in [16] was to show that this space in the cases $k = 2, 3$ (and hence for all k) contains a closed subset of groups homeomorphic to a Cantor set consisting primarily of torsion groups of intermediate growth. Later other interesting families of groups constituting a Cantor set of groups and satisfying various properties were produced and used for answering different questions; see [12] and [33]. The topology on the space $\mathcal{M}_k$ was used in [16] and [18] not only for study of growth but also for investigations of algebraic properties of the involved groups. For instance, among many ways of showing that the involved groups are not finitely presented, there is one which makes use of this topology (the topic of finite presentability of groups in the context of growth, amenability and topology is discussed in detail in [9]). At present there is a big account of results related to the space of marked groups and various algebraic, geometric and asymptotic properties of groups including such properties as (T)-property of Kazhdan, local embeddability into finite or amenable groups (so-called LEF and LEA properties), being sofic and various other properties (see [11] for a comprehensive source of these).

For each $k \geq 2$ there is a natural embedding of $\mathcal{M}_k$ into $\mathcal{M}_{k+1}$ and one can consider the inductive limit $\mathcal{M} = \lim_k \mathcal{M}_k$ which is a locally compact totally disconnected space. As was observed by Champetier in [12], the group of Nielsen transformations over infinite generating set acts naturally on this space with orbits consisting of isomorphic groups. Any Baire measure on $\mathcal{M}$, i.e., a measure defined on the sigma algebra generated by compact G_δ sets (countable intersections of open sets) with finite values on compact sets, that is invariant (or at least quasi-invariant) with respect to this action would be a good choice for the model of random finitely generated group (this approach based on discussions of the second author with E. Ghys is presented in [23]). Unfortunately, at the moment no such measures were produced. This is also related to the question of existence of "good" measures invariant (or quasi-invariant) under the action of the automorphism group of a free group F_k of rank $k \geq 2$, with support in the set of normal subgroups of F_k.

Fortunately, another approach can be used. It is based on the following idea. Assume we have a compact $X \subset \mathcal{M}_k$ of groups and a continuous map $\tau\colon X \to X$. Then by the Bogolyubov–Krylov Theorem there is at least one τ-invariant probability measure μ on X. Suppose also that we have a certain group property $\mathcal{P}$ (or a family of properties), and that the subset $X_{\mathcal{P}} \subset X$ of groups satisfying this property is measurable τ-invariant (i.e., $\tau^{-1}(X_{\mathcal{P}}) = X_{\mathcal{P}}$). Then one may be interested in the measure $\mu(X_{\mathcal{P}})$ which is 0 or 1, in the case of ergodic measure (i.e., when the only invariant measurable subsets up to sets of μ measure 0 are empty set and the whole set X). Observe that by (another) Bogolyubov–Krylov Theorem, ergodic measures always exist in the situation of a continuous map on a metrizable compact space and are just extreme points of the simplex of invariant measures. The described model allows to speak about typical properties of a random group from the family (X, μ).

The alternative approach when the measure μ is not specified is the study of the typical properties of groups in compact X from topological (or categorical) point of view i.e., in the sense of Baire category. Under this approach a group property $\mathcal{P}$ is typical if the subset $X_{\mathcal{P}}$ is co-meager i.e., its complement $X \setminus X_{\mathcal{P}}$ is meager (a countable union of nowhere dense subsets of X). It happens quite often that what is typical in the measure sense is not typical in the sense of category and this paper gives one more example of this sort.

2. Statement of main results

In Ergodic Theory (and more generally in Probability Theory), one of the most important models is the model of a shift in a space of sequences. Given a finite alphabet $Y = \{s_1, \dots, s_k\}$, one considers a space $\Omega = Y^{\mathbb{N}}$ of infinite sequences $\omega = (\omega_n)_{n=1}^{\infty}, \omega_n \in Y$ endowed with the Tychonoff product topology. A natural transformation in such space is a shift $\tau: \Omega \to \Omega$, $(\tau(\omega))_n = \omega_{n+1}$. There is a lot of invariant measures for the dynamical system (Ω, τ) and in fact the simplex $M_\tau(\Omega)$ of invariant measures is Poulsen simplex (i.e., ergodic measures are dense in weak-$*$ topology).

Let p be prime and consider the set $\{0, 1, \dots, p\}$ as an alphabet with the corresponding set $\Omega_p = \{0, 1, \dots, p\}^{\mathbb{N}}$ of infinite sequences endowed with the shift $\tau: \Omega_p \to \Omega_p$. Let $\Omega_{p,0}$ denote the subset of sequences which are eventually constant and $\Omega_{p,\infty}$ the set of sequences in which all symbols $\{0, 1, \dots, p\}$ appear infinitely often. Note that $\Omega_{p,\infty}$ and $\Omega_{p,0}$ are τ invariant.

In [16] and [17] for each $\omega \in \Omega_p$ a group G_ω with a set $S_\omega = \{a, b_\omega, c_\omega\}$ of three generators acting on the interval $[0, 1]$ by Lebesgue measure preserving transformations was constructed. One of the specific features of this construction is that if two sequences $\omega, \eta \in \Omega_p$, which are not eventually constant have the same prefix of length n, then the corresponding groups G_ω, G_η have isomorphic Cayley graphs in the neighborhood of the identity element of radius 2^{n-1}. Replacing the groups $G_\omega, \omega \in \Omega_{p,0}$ with appropriate limits (again denoted by G_ω), that is, taking the closure of the set $\{(G_\omega, S_\omega) \mid \omega \in \Omega_p \setminus \Omega_{p,0})\}$ in $\mathcal{M}_3$, one obtains a compact subset $\mathcal{G}_p = \{(G_\omega, S_\omega) \mid \omega \in \Omega_p\}$ of $\mathcal{M}_3$ which is homeomorphic to Ω_p (via the correspondence $\omega \mapsto (G_\omega, S_\omega)$) and hence homeomorphic to a Cantor set. In what follows we will continue to keep the notation $G_\omega,\ \omega \in \Omega_p$ to denote these groups after this modification. Also, quite often we will identify $\mathcal{G}_p$ with Ω_p. In the case $p = 2$ the new limit groups $G_\omega,\ \omega \in \Omega_{2,0}$ are known to be virtually metabelian groups of exponential growth while there is no analogous result for the case $p > 2$. This is the underlying reason that Theorem 3 below is stated only for $p = 2$.

Another important feature of the construction is that for all but countably many $\omega \in \Omega_2$ and for all $\omega \in \Omega_p$ the groups G_ω and $(G_{\tau(\omega)})^p$ (direct product of p copies of G_ω) are abstractly commensurable (i.e., contain isomorphic subgroups of finite index). Thus the shift τ preserves many of group properties on the set of full measure when μ is a τ invariant measure supported on $\Omega_{p,\infty}$, for instance, the property to be a torsion group. While for some properties of the groups G_ω it is quite easy to decide whether it is typical or not, there are some properties for which such a question is more difficult to answer. Among them is the property to have a growth function bounded from above (or below) by a specific function.

Given functions $f_1, f_2: \mathbb{N} \to \mathbb{N}$, we write $f_1 \preceq f_2$ if f_1 grows no faster than f_2 and $f_1 \sim f_2$ if $f_1 \preceq f_2$ and $f_2 \preceq f_1$. $f_1 \prec f_2$ means that $f_1 \preceq f_2$ but $f_1 \nsim f_2$ (precise definitions are given in Section 3). For any $\omega \in \Omega_p$ let $\gamma_\omega(n)$ denote the growth function of the group G_ω. It was shown in [16] and [17] that if every symbol of the alphabet $\{0, 1, \dots, p\}$ appears in any sufficiently large sub word of a sequence ω then $\gamma_\omega(n)$ grows slower than e^{n^α} with constant $\alpha < 1$. At the same time, in the case $p = 2$ for any function $f(n) \prec e^n$ there is a sequence ω such that $\gamma_\omega(n)$ grows not slower than $f(n)$.

The upper and lower bounds by functions of the type e^{n^α} with constant $0 < \alpha < 1$ are of special importance in the study of growth of finitely generated groups and there is a number of interesting results and conjectures associated with them. One of the main conjectures says that if the growth of a group G is slower than $e^{\sqrt{n}}$ then it is actually polynomial ([20], [25], and [24]). The results of [5] and [4] provide great progress on the study of intermediate growth. It was shown that groups of the form $A \wr_X G_\omega$, with A being a finite group and suitable X, are the first examples of groups with growth functions exactly equivalent to functions of the form e^{n^α}, $0 < \alpha < 1$. In contrast, the precise computation of growth rate of the groups G_ω is still open.

We are ready to formulate our results.

Theorem 1. *Suppose μ is a Borel probability measure on Ω_p that is invariant and ergodic relative to the shift transformation $\tau\colon \Omega_p \to \Omega_p$.*

a) *If the measure μ is supported on $\Omega_{p,\infty}$, then, there exists $\alpha = \alpha(\mu, p) < 1$ such that $\gamma_\omega(n) \preceq e^{n^\alpha}$ for μ-almost all $\omega \in \Omega_p$.*

b) *In the case μ is the uniform Bernoulli measure on Ω_2, one can take $\alpha = 0.999$.*

Note that the upper bound e^{n^α} in the theorem is universal only as a rate. Namely, the inequality $\gamma_\omega(n) \leq e^{n^\alpha}$ holds for some $\alpha < 1$ and $n \geq n_0$ where n_0 depends on ω.

If $T\colon \mathcal{G}_p \to \mathcal{G}_p$ is the map induced by the shift τ, our result can be interpreted as follows. For any "reasonable" T-invariant measure μ on $\mathcal{G}_p \subset \mathcal{M}_3$, a typical group in $\mathcal{G}_p$ has growth bounded by e^{n^α}, where $\alpha = \alpha(\mu, p) < 1$.

The bound for α given in part (b) of the Theorem 1 is far from to be optimal, but getting an essentially better bound would require more work. In any case it can not be below $1/2$ as for all groups G_ω of intermediate growth the corresponding growth function is bounded from below by $e^{n^{1/2}}$; see [16], [17], and [19]. The gap conjecture, discussed in [24] and proven in certain cases, gives more information about what one can expect concerning possible optimal values of α.

In fact, there is nothing special about the space $\mathcal{M}_3$ and the following holds.

Theorem 1′. *For any $k \geq 2$ and prime p, $\mathcal{M}_k$ contains a compact subset $\mathcal{K}_k = \{(M_\omega, L_\omega) \mid \omega \in \Omega_p\}$ homeomorphic to Ω_p (via the map $\omega \mapsto (M_\omega, L_\omega)$) such that if μ is an invariant and ergodic measure supported on $\Omega_{p,\infty}$ there exists $\alpha = \alpha(\mu, p) < 1$ such that $\gamma_{M_\omega}(n) \preceq e^{n^\alpha}$ for μ-almost all $\omega \in \Omega_p$.*

For $k \geq 3$, the group M_ω is the same as G_ω, with an appropriate generating set L_ω of size k. For $k = 2$, M_ω is a 2-generated group constructed from G_ω as a subgroup of $G_\omega \wr \mathbb{Z}_4$. When $p = 2$ and $\omega = (012)^\infty$ it is isomorphic to the 2-group of Aleshin from [1].

The proof of Theorem 1 is based on the next result, which has its own interest. It improves Theorem 3 in [17] (a generalization of Theorem 3.2 in [16]), which states that if the sequence ω is regularly packed by symbols 0, 1, 2, namely, if there exists $k = k(\omega)$ such that each subsequence of length k of ω contains all symbols $\{0, 1, \dots, p\}$, then there is $\alpha < 1$ such that $\gamma_\omega(n) \preceq e^{n^\alpha}$.

To every infinite word $\omega = l_1 l_2 \dots$ in $\Omega_{p,\infty}$ we associate an increasing sequence of integers $t_i = t_i(\omega)$, $i = 0, 1, 2, \dots$. Namely, t_i is the smallest integer such that the finite word $l_1 l_2 \dots l_{t_i}$ can be split into i subwords each containing all letters $\{0, 1, \dots, p\}$. For any $C \geq p + 1$ let $\Omega_{p,C}$ denote the set of all infinite words $\omega \in \Omega_{p,\infty}$ such that $t_n(\omega) \leq Cn$ for sufficiently large n. Given $\varepsilon > 0$, let $\Omega_{p,C,\varepsilon}$ denote the set of all $\omega \in \Omega_{p,C}$ such that $t_{n+1}(\omega) - t_n(\omega) \leq \varepsilon t_n(\omega)$ for sufficiently large n.

Theorem 2. *Given $C \geq p + 1$, there exist $\varepsilon > 0$ and $0 < \alpha < 1$ such that $\gamma_\omega(n) \preceq e^{n^\alpha}$ for any $\omega \in \Omega_{p,C,\varepsilon}$.*

Our next result deals exclusively with the case $p = 2$ (the reason was explained earlier). Given two functions $\gamma_1, \gamma_2 \colon \mathbb{N} \to \mathbb{N}$ such that $\gamma_1(n) \prec \gamma_2(n) \prec e^n$, let us say that a group G has *oscillating growth of type* (γ_1, γ_2) if $\gamma_1 \not\preceq \gamma_G$ and $\gamma_G \not\preceq \gamma_2$. The existence of groups with oscillating growth follows from the results of [16]. Theorem 3 shows that the oscillating growth of [10] and [28] are in fact topologically generic in $\mathcal{G}_2$.

Let $\theta_0 = \log(2)/\log(2/x_0)$, where x_0 is the real root of the polynomial $x^3 + x^2 + x - 2$. We have $\theta_0 < 0.767429$.

Theorem 3. a) *For any $\theta > \theta_0$ and any function f satisfying $e^{n^\theta} \prec f(n) \prec e^n$, there exists a dense G_δ subset $\mathcal{Z} \subset \mathcal{G}_2$ such that any group in $\mathcal{Z}$ has oscillating growth of type (e^{n^θ}, f).*

b) *In particular, there exists a dense G_δ subset of $\mathcal{G}_2$ which consists of groups with oscillating growth of type $(e^{n^\theta}, e^{n^\beta})$ for every θ and β, $\theta_0 < \theta < \beta < 1$.*

c) *Given any $\varepsilon > 0$ and function f satisfying $\exp(\frac{n}{\log^{1-\varepsilon} n}) \prec f(n) \prec e^n$, there is a dense G_δ subset $\mathcal{E} \subset \{(G_\omega, S_\omega) \mid \omega \in \{0,1\}^{\mathbb{N}}\}$ such that any group in $\mathcal{E}$ has oscillating growth of type $(\exp(\frac{n}{\log^{1-\varepsilon} n}), f)$.*

Again, all these results generalize to arbitrary $k \geq 2$. In particular, the following theorem holds.

Theorem 3′. *For each $k \geq 2$, $\theta > \theta_0$ and function f satisfying $e^{n^\theta} \prec f(n) \prec e^n$, $\mathcal{M}_k$ contains a compact subset $\mathcal{C}_k$ homeomorphic to Ω_2, such that there is a dense G_δ subset $\mathcal{C}'_k \subset \mathcal{C}_k$ which consists of groups with oscillating growth of type (e^{n^θ}, f).*

The reason why oscillating groups are typical in the categorical sense is the existence of a countable dense subset in $\mathscr{G}_2$ consisting of (virtually metabelian) groups of exponential growth and a dense subset of groups with the growth equivalent to the growth of the *first Grigorchuk group* $G_{(012)^\infty}$, which is bounded by $e^{n^{\theta_0}}$ due to a result of Bartholdi [2] (see also [32]). Note also that this is the smallest upper bound known for any group of intermediate growth. It is used to prove part a). To prove part c), we use instead a result of Erschler [14] stating that the growth of the group G_ω for $\omega = (01)^\infty \in \Omega_2$ is slower than $\exp(\frac{n}{\log^{1-\varepsilon} n})$ for all $\varepsilon > 0$.

Note that the categorical approach for study of amenability of groups from the family $\mathscr{G}_2$ was suggested by Stepin in [35] where the fact that this family contains a dense set of virtually metabelian (and hence amenable) groups was used to show that amenability is a typical property of this family. In fact, all groups in $\mathscr{G}_2$ are amenable as was shown in [16], but Stepin's paper provided for the first time a categorical approach to study typical groups in compact subsets of the space of marked groups.

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3. Preliminaries

Definition of the groups . The original definition of groups in [16] and [17] is in terms of measure preserving transformations of the unit interval. We will give here the alternative definition in terms of automorphisms of rooted trees. For the sake of notation we will focus on the case $p = 2$ and the construction in the case $p \geq 3$ is analogous. For more detailed account of this construction; see [21] and [23].

Let us recall some notation: Ω_2 denotes the set all infinite sequences over the alphabet $\{0, 1, 2\}$. We identify Ω_2 with the product $\{0, 1, 2\}^{\mathbb{N}}$ and endow it with the product topology. Let $\Omega_{2,0}$ be the set of eventually constant sequences and $\Omega_{2,\infty}$ be the set of sequences in which each letter 0,1,2 appears infinitely often. Our notation here is different from [16] and [17]. Let $\tau \colon \Omega_2 \to \Omega_2$ denote the shift transformation, that is if $\omega = l_1 l_2 \dots$ then $\tau(\omega) = l_2 l_3 \dots$. Note that both $\Omega_{2,0}$ and $\Omega_{2,\infty}$ are τ invariant.

For each $\omega \in \Omega_2$ we will define a subgroup G_ω of $\mathrm{Aut}(\mathcal{T}_2)$, where the latter denotes the automorphism group of the binary rooted tree $\mathcal{T}_2$ whose vertices are identified with the set of finite sequences $\{0, 1\}^*$. Each group G_ω is the subgroup generated by the four automorphisms denoted by $a, b_\omega, c_\omega, d_\omega$ whose action onto the tree is as follows.

For $v \in \{0, 1\}^*$

$$a(0v) = 1v \quad \text{and} \quad a(1v) = 0v$$

and

$$b_\omega(0v) = 0\beta(\omega_1)(v), \quad c_\omega(0v) = 0\zeta(\omega_1)(v), \quad d_\omega(0v) = 0\delta(\omega_1)(v),$$

$$b_\omega(1v) = 1b_{\tau(\omega)}(v), \quad c_\omega(1v) = 1c_{\tau\omega}(v), \quad d_\omega(1v) = 1d_{\tau\omega}(v),$$

where

$$\beta(0) = a, \quad \beta(1) = a, \quad \beta(2) = e,$$
$$\zeta(0) = a, \quad \zeta(1) = e, \quad \zeta(2) = a,$$
$$\delta(0) = e, \quad \delta(1) = a, \quad \delta(2) = a,$$

and e denotes the identity. From the definition, the following relations are immediate:

$$a^2 = b_\omega^2 = c_\omega^2 = d_\omega^2 = b_\omega c_\omega d_\omega = e. \tag{1}$$

Observe that the group G_ω is in fact 3-generated as one of generators $b_\omega, c_\omega, d_\omega$ can be deleted from the generating set. We will use the notation $A_\omega = \{a, b_\omega, c_\omega, d_\omega\}$ for this generating set while S_ω will denote the reduced generating set $\{a, b_\omega, c_\omega\}$ (as in Section 2). Algebraically, the action defines an embedding into the semi-direct product

$$\begin{array}{rcll} \varphi_\omega\colon G_\omega & \longrightarrow & S_2 \ltimes (G_{\tau(\omega)} \times G_{\tau(\omega)}) & \\ a & \longmapsto & (01) \quad (e \quad , \quad e) & \\ b_\omega & \longmapsto & \quad (\beta(\omega_1) \ , \ b_{\tau(\omega)}), & \\ c_\omega & \longmapsto & \quad (\zeta(\omega_1) \ , \ c_{\tau(\omega)}), & \\ d_\omega & \longmapsto & \quad (\delta(\omega_1) \ , \ d_{\tau(\omega)}), & \end{array}$$

where S_2 is the symmetric group of order 2 and (01) denotes its non-identity element.

Given $g \in G_\omega$ and $x \in \{0, 1\}$, let us denote the x coordinate of $\varphi_\omega(g)$ by g_x (or by $g|_x$ to avoid possible confusion) so that $\varphi_\omega(g) = \sigma_g(g_0, g_1)$. Let us also extend this to all $\{0, 1\}^*$ by

$$g_{xv} = (g_x)_v$$

where $x \in \{0, 1\}$ and $v \in \{0, 1\}^*$. For $g \in G_\omega$ and $v \in \{0, 1\}^*$ the automorphism g_v will be called *the section of g at vertex v*. Note that if v has length n and $g \in G_\omega$, then g_v is an element of $G_{\tau^n(\omega)}$. Given $g, h \in G_\omega$ and $v \in \{0, 1\}^*$, we have

$$(gh)_v = g_{h(v)} h_v \tag{2}$$

Topology on the space of marked groups. A *marked k-generated group* is a pair (G, S), where G is a group and $S = \{s_1, \ldots, s_k\}$ is an ordered set of (not necessarily distinct) generators of G. The *canonical map* between two marked k-generated groups (G, S) and (H, T) is the map that sends s_i to t_i for $i = 1, 2, \ldots, k$. Let $\mathcal{M}_k$ denote the *space of marked k-generated groups* consisting of marked k-generated groups, where two marked groups are identified whenever the canonical map between them extends to an isomorphism of the groups.

There is a natural metric on $\mathcal{M}_k$: two marked groups (G,S), (H,T) are of distance $\frac{1}{2^m}$, where m is the largest natural number such that the canonical map between (G,S) and (H,T) extends to an isomorphism (of labeled graphs) from the ball of radius m (around the identity) in the Cayley graph of (G,S) onto the ball of radius m in the Cayley graph of (H,T). This makes M_k into a compact, totally disconnected topological space.

Alternatively, let a group F_k be free over the ordered basis $X = \{x_1,\dots,x_k\}$ and let $\mathcal{N}(F_k)$ denote the set of normal subgroups of F_k. $\mathcal{N}(F_k)$ has a natural topology inherited from the space $\{0,1\}^{F_k}$ of all subsets of F_k. $\mathcal{M}_k$ can be identified with $\mathcal{N}(F_k)$ in the following way. Each $(G,S) \in \mathcal{M}_k$ is identified with the kernel of the canonical map between (F_k,X) and (G,S). Conversely, each $N \lhd F_k$ is identified with $(F_k/N, \{\bar{x}_1,\dots,\bar{x}_k\})$, where $\{\bar{x}_1,\dots,\bar{x}_k\}$ is the image of the basis of F_k in F_k/N. A system of basic open sets are sets of the form $\mathcal{O}_{A,B} = \{N \lhd F_k \mid A \subset N, B \cap N = \varnothing\}$, where A and B are finite subsets of F_k. Or the topology can be defined by the metric $d(N_1,N_2) = 2^{-m}$, where $m = \max\{n \mid B_{F_k}(n) \cap N_1 = B_{F_k} \cap N_2\}$. It is easy to see that the topology defined in this way agree with the definition given in the previous paragraph (see [13] for a survey of alternative definitions).

Let $A_\omega = \{a, b_\omega, c_\omega, d_\omega\}$ so that $\mathcal{F}_2 = \{(G_\omega, A_\omega) \mid \omega \in \Omega_2\}$ is a subset of $\mathcal{M}_4$. $\mathcal{F}_2$ is not closed in $\mathcal{M}_4$ (see [16]). Given $\omega \in \Omega_2$, let $\{\omega^{(n)}\} \subset \Omega_2 \setminus \Omega_{2,0}$ be a sequence converging to ω. It was shown in [16] that the sequence $\{(G_{\omega^{(n)}}, A_{\omega^{(n)}})\}$ converges in $\mathcal{M}_4$ to a marked group $(\widetilde{G}_\omega, \widetilde{A}_\omega)$ that depends only on ω. Moreover, $(\widetilde{G}_\omega, \widetilde{A}_\omega) = (G_\omega, A_\omega)$ if and only if $\omega \in \Omega_2 \setminus \Omega_{2,0}$. By construction, the group $\widetilde{G}_\omega$ acts naturally on the binary rooted tree for any $\omega \in \Omega_2$. However the action is not faithful when $\omega \in \Omega_{2,0}$. The modified family $\{(\widetilde{G}_\omega, \widetilde{A}_\omega) \mid \omega \in \Omega_2\}$ is a compact subset of $\mathcal{M}_4$ homeomorphic to Ω_2 via the map $\widetilde{G}_\omega \mapsto \omega$.

Observe that a similar procedure can be applied to the family $\mathcal{G}_2 = \{(G_\omega, S_\omega) \mid \omega \in \Omega_2\}$ to obtain a closed subset $\{(\widetilde{G}_\omega, \widetilde{S}_\omega) \mid \omega \in \Omega_2\}$ in $\mathcal{M}_3$. In what follows we will mostly be concerned with the modified groups. Therefore, we use notation $\mathcal{F}_2$ and $\mathcal{G}_2$ for the modified families and also drop all tildes.

Growth functions of groups. Given a group G and a finite generating set S of G, the growth function of G with respect to S is defined as $\gamma_G^S(n) = |B(n)|$ where $B(n)$ is the ball of radius n around the identity in the Cayley graph of G with respect to the generating set S.

Given two increasing functions $f, g\colon \mathbb{N} \to \mathbb{N}$, we write $f \preceq g$ if there exists a constant $C > 0$ such that $f(n) \le g(Cn)$ for all $n \in \mathbb{N}$. Also, let $f \sim g$ mean that $f \preceq g$ and $g \preceq f$ with the convention that $f \prec g$ means $f \preceq g$ but $f \nsim g$. It can be easily observed that $\sim$ is an equivalence relation and the growth functions of a group with respect to different generating sets are $\sim$ equivalent. Therefore one can speak of the growth of a group meaning the $\sim$ equivalence class of its growth functions. Note that if two groups $(G, S), (H, T) \in \mathcal{M}_k$ are of distance 2^{-m}, then $\gamma_G^S(n) = \gamma_H^T(n)$ for $n \leq m$.

If G is an *infinite* group and H a subgroup of finite index, then the growth functions of G and H are $\sim$ equivalent by Proposition 3.1 in [16] (note that this is not true if G is a finite group). Therefore if two finitely generated infinite groups G_1 and G_2 are commensurable (i.e., have finite index subgroups H_1 and H_2 which are isomorphic) then their growth functions are $\sim$ equivalent.

There are three types of growth for groups. If $\gamma_G \preceq n^d$ for some $d \geq 0$ then G is said to be of polynomial growth, if $\gamma_G \sim e^n$ then it is said to have exponential growth. If neither of this happens then the group is said to have *intermediate growth*. Also the condition $\gamma_G \prec e^n$ means that G has *subexponential* growth.

Definition. Let G be a finitely generated group with growth function γ_G corresponding to some generating set. Let γ_1, γ_2 be two functions such that $\gamma_1(n) \prec \gamma_2(n) \prec e^n$. G is said to have *oscillating growth of type* (γ_1, γ_2) if $\gamma_1 \npreceq \gamma_G$ and $\gamma_G \npreceq \gamma_2$ (i.e., neither $\gamma_1 \preceq \gamma_G$ nor $\gamma_G \preceq \gamma_2$).

Equivalently, the group G has oscillating growth of type (γ_1, γ_2) if for some (and hence for all) generating set S the following condition is satisfied: for every $C \in \mathbb{N}$ there exists $m = m(C)$ such that $\gamma_G^S(Cm) < \gamma_1(m)$ and for every $D \in \mathbb{N}$ there exists $k = k(D)$ such that $\gamma_2(Dk) < \gamma_G^S(k)$.

Regarding the growth of the groups $\mathcal{G}_p$ the following are known (recall that $\gamma_\omega(n)$ denotes the growth function of G_ω and when $\omega \in \Omega_{p,0}$, G_ω denotes the limit group obtained by the procedure described above).

Theorem 4. (1) *If* $\omega \in \Omega_2 \setminus \Omega_{2,0}$ *or* $\omega \in \Omega_{p,\infty}$ *if* $p \geq 3$*, then* G_ω *is of intermediate growth.*

(2) *If* $\omega \in \Omega_{2,0}$ *then* G_ω *is of exponential growth.*

(3) *For every* $\omega \in \Omega_2$ *or* $\omega \in \Omega_{p,\infty}$, $p \geq 3$*, we have* $e^{\sqrt{n}} \preceq \gamma_\omega(n)$.

(4) *If there exists a number* r *such that every subword of* ω *of length* r *contains all the symbols* $\{0, 1, \dots, p\}$ *then* $\gamma_\omega(n) \preceq e^{n^\alpha}$ *for some* $0 < \alpha < 1$ *depending only on* r.

(5) *There is a subset* $\Lambda \subset \Omega_2$ *of the cardinality of continuum such that the functions* $\{\gamma_\omega(n) \mid \omega \in \Lambda\}$ *are incomparable with respect to* $\preceq$.

(6) *For any function* $f(n)$ *such that* $f(n) \prec e^n$*, there exists* $\omega \in \Omega_2 \setminus \Omega_{2,0}$ *for which* $\gamma_\omega(n) \npreceq f(n)$.

(7) *If* $\omega = (012)^\infty \in \Omega_2$ *is the periodic sequence with period* 012 *then* $e^{n^{\alpha_0}} \preceq \gamma_\omega(n) \preceq e^{n^{\theta_0}}$*, where* $\alpha_0 = 0.5157, \theta_0 = \log(2)/\log(2/x_0)$ *and* x_0 *is the real root of the polynomial* $x^3 + x^2 + x - 2$ *(*$\theta_0 \approx 0.7674$*).*

(8) *If* $\omega = (01)^\infty \in \Omega_2$ *is periodic with period* 01 *then* $\exp(\frac{n}{\log^{2+\varepsilon} n}) \preceq \gamma_\omega(n) \preceq \exp(\frac{n}{\log^{1-\varepsilon} n})$ *for any* $\varepsilon > 0$.

Proof. (1) See Theorem 3.1 in [16] and [17].

(2) See Lemma 6.1 in [16].

(3) See Theorem 3.2 in [16] and Theorem 4.4 in [17] where the lower bound $e^{\sqrt{n}}$ is proven for a certain subset of Ω_2 and for $\Omega_{p,\infty},\ p \geq 3$. As all groups mentioned are residually finite p-groups for some prime p and are not virtually nilpotent (which can be shown in various ways, for example using the fact that the groups are periodic), the lower bound $e^{\sqrt{n}}$ follows from a general result of [19].

(4) See [16] and [7] and [32] for explicit upper bounds depending on r.

(5) See Theorem 7.2 in [16].

(6) See Theorem 7.1 in [16].

(7) See [3] for the lower bound which improved upon [29]. See [2] for the upper bound.

(8) See [14]. □

4. Proof of Theorem 1

This section is devoted to the proof of Theorems 1 and 2. We prove these theorems in the case $p = 2$. The proof in the case $p \geq 3$ is completely analogous. To simplify notation, we set $\Omega = \Omega_2$ and $\Omega_\infty = \Omega_{2,\infty}$ for the rest of this section.

For any element g of a group G_ω, $\omega \in \Omega$, we denote by $|g|$ its length relative to the generating set $A_\omega = \{a, b_\omega, c_\omega, d_\omega\}$. If $|g| = n$, then g can be expanded into a product $s_1 s_2 \dots s_n$, called a geodesic representation, where each $s_i \in A_\omega$. For every generator $s \in A_\omega$ we denote by $|g|_s$ the number of times this generator occurs in the sequence $s_1, s_2, \dots, s_n$. Note that the element g may admit several geodesic representations and $|g|_s$ may depend on a representation (for example, $b_\omega a d_\omega a b_\omega = c_\omega a d_\omega a c_\omega$ for any ω starting with 0). Lemmas 1 and 4 below hold for any possible value of the corresponding number $|g|_s$.

Lemma 1. $(|g| - 1)/2 \leq |g|_a \leq (|g| + 1)/2$ *for all* $g \in G_\omega$.

Proof. It follows from relations (1) that any geodesic representation of an element $g \in G_\omega$ is of the form

$$g = (s_1) a s_2 a \dots a (s_k),$$

where each $s_i \in \{b_\omega, c_\omega, d_\omega\}$ and parentheses indicate optional factors. The lemma follows. □

Lemma 2. *For any word $w \in \{0,1\}^*$ of length q we have $|g_w| \le 2^{-q}|g| + 1 - 2^{-q}$.*

Proof. First consider the case when w is 0 or 1. Let $g = s_1 s_2 \dots s_n$ be a geodesic representation, where each $s_i \in A_\omega$. It follows by induction from equation (2) that $g_w = s_1|_{w_1} s_2|_{w_2} \dots s_n|_{w_n}$, where $w_n = w$ and $w_i = (s_{i+1} \dots s_n)(w)$ for $1 \le i \le n-1$. Note that each section $s_i|_{w_i}$ is a generator of the group $G_{\tau(\omega)}$ or e. Moreover, $s_i|_{w_i} = e$ if $s_i = a$. Therefore $|g_w| \le |g| - |g|_a$. By Lemma 1, $|g|_a \ge (|g|-1)/2$. Hence $|g_w| \le (|g|+1)/2$. Equivalently, $|g_w|-1 \le 2^{-1}(|g|-1)$.

Now it follows by induction on $|w|$ that $|g_w| - 1 \le 2^{-q}(|g|-1)$ for any word w of length q.

□

For any element $g \in G_\omega$ and any integer $q \ge 0$, let

$$L_q(g) = \sum_{|w|=q} |g_w|$$

Lemma 3. *$L_q(gh) \le L_q(g) + L_q(h)$ for all $g, h \in G_\omega$.*

Proof. Since $(gh)_w = g_{h(w)} h_w$ for any word $w \in \{0,1\}^*$, it follows that $|(gh)_w| \le |g_{h(w)}| + |h_w|$. Summing this inequality over all words w of length q and using the fact that h acts bijectively on such words, we obtain $L_q(gh) \le L_q(g) + L_q(h)$.

□

Lemma 4. *$L_q(g) \le |g| + 1 - |g|_{h_q}$ for any $q \ge 1$, where $h_q = b_\omega$, c_ω, or d_ω if the q-th letter of ω is 2, 1, or 0, respectively.*

Proof. Let $n = |g|$. Consider an arbitrary geodesic representation $g = s_1 s_2 \dots s_n$, where each $s_i \in A_\omega$. It follows by induction from Lemma 3 that $L_q(g) \le L_q(s_1) + L_q(s_2) + \dots + L_q(s_n)$. Fix an arbitrary word $w \in \{0,1\}^*$ of length q. Clearly, $a|_w = 1$. Further, $h_q|_w = 1$ unless $w = 1\dots1$ (in which case $h_q|_w \in A_{\tau^q\omega} \setminus \{a\}$). If s is any of the other two generators in A_ω, then $s|_w = 1$ unless $w = 1\dots1$ (in which case $s|_w \in A_{\tau^q\omega} \setminus \{a\}$) or $w = 1\dots10$ (in which case $s|_w = a$). Therefore $L_q(a) = 0$, $L_q(h_q) = 1$, and $L_q(s) = 2$ if $s \in A_\omega$ is neither a nor h_q. It follows that $L_q(g) \le 2(|g| - |g|_a) - |g|_{h_q}$. By Lemma 1, $|g|_a \ge (|g|-1)/2$. Hence $2(|g| - |g|_a) \le |g| + 1$.

□

Lemma 5. *Suppose that the beginning of length q of the sequence ω contains each of the letters 0, 1, and 2. Then*

$$L_q(g) \le \frac{5}{6}|g| + \frac{7}{6} + 2^{q-1}$$

for all $g \in G_\omega$.

Proof. We have $|g| = |g|_a + |g|_{b_\omega} + |g|_{c_\omega} + |g|_{d_\omega}$ whenever all numbers in the right-hand side are computed for the same geodesic representation of g. By Lemma 1, $|g|_a \le (|g|+1)/2$. It follows that $|g|_s \ge (|g|-1)/6$ for some $s \in \{b_\omega, c_\omega, d_\omega\}$. Lemma 4 implies that $L_{q_0}(g) \le |g|+1-|g|_s \le \frac{5}{6}|g|+\frac{7}{6}$ for some $1 \le q_0 \le q$. In the case $q_0 = q$, we are done. Otherwise we notice that $L_q(g) = \sum_{|w|=q_0} L_{q-q_0}(g_w)$. By Lemma 4, $L_{q-q_0}(g_w) \le |g_w|+1$ for any word w. Therefore $L_q(g) \le L_{q_0}(g) + 2^{q_0} \le \frac{5}{6}|g| + \frac{7}{6} + 2^{q-1}$.

□

Note that the growth function of a group is *sub-multiplicative*, that is, $\gamma(n+m) \le \gamma(n)\gamma(m)$ for every $n, m \in \mathbb{N}$. It is convenient to extend the argument of a growth function to non-integer values. Given increasing $f\colon \mathbb{N} \to \mathbb{N}$, define $\tilde{f}\colon \mathbb{R}^+ \to \mathbb{N}$ by $\tilde{f}(x) = f(\lceil x \rceil)$ for all x where $\lceil x \rceil$ is the least natural number bigger than or equal to x. Observe that $f(x+\kappa) \le \tilde{f}(x)$ whenever $\kappa < 1$. If $f\colon \mathbb{N} \to \mathbb{N}$ is sub-multiplicative then it is easy to see that $\tilde{f}(x+y) \le \tilde{f}(x)\tilde{f}(y)$ for any $x, y > 0$.

For the remainder of this section let $\rho = \frac{131}{132}$.

Lemma 6. *Suppose that the beginning of length q of the sequence ω features each of the letters* 0, 1, *and* 2. *Then*

$$\tilde{\gamma}_\omega(x) \le 2^{2^{q+1}} \Big(\tilde{\gamma}_{\tau^q\omega}\Big(\frac{x}{11\cdot 2^q}\Big)\Big)^{\rho(11\cdot 2^q)}$$

for any $x > 0$.

Proof. Let $n = \lceil x \rceil$ and consider an arbitrary element $g \in G_\omega$ of length at most n. By Lemma 2, we have $|g_w| \le 2^{-q}n + 1 - 2^{-q}$ for any word $w \in \{0,1\}^*$ of length q. We denote by W the set of all words w of length q such that $|g_w| > \frac{12}{11}\cdot 2^{-q}(\frac{5}{6}n + \frac{7}{6} + 2^{q-1})$. In view of Lemma 5, the cardinality of W satisfies $|W| < \frac{11}{12}\cdot 2^q$.

The element g is uniquely determined by its sections on words of length q and its restriction to the qth level of the binary rooted tree. The number of possible choices for the restriction is at most 2^{2^q}. The number of possible choices for the set W is also at most 2^{2^q}. Once the set W is specified, the number of possible choices for a particular section g_w is at most $\gamma_{\tau^q\omega}(\frac{12}{11}\cdot 2^{-q}(\frac{5}{6}n + \frac{7}{6} + 2^{q-1}))$ if $w \notin W$ and at most $\gamma_{\tau^q\omega}(2^{-q}n + 1 - 2^{-q})$ otherwise. Since $n < x+1$, we have $2^{-q}n+1-2^{-q} < 2^{-q}x+1$ so that $\gamma_{\tau^q\omega}(2^{-q}n+1-2^{-q}) \le \tilde{\gamma}_{\tau^q\omega}(x/2^q)$. Besides, $\frac{12}{11}\cdot 2^{-q}(\frac{5}{6}n + \frac{7}{6} + 2^{q-1}) < \frac{10}{11}2^{-q}x + 1$ so that $\gamma_{\tau^q\omega}(\frac{12}{11}\cdot 2^{-q}(\frac{5}{6}n + \frac{7}{6} + 2^{q-1})) \le \tilde{\gamma}_{\tau^q\omega}(\frac{10}{11}x/2^q)$. Finally, for a fixed set W the number of possible choices for all sections of g is

$$\begin{aligned}\tilde{\gamma}_{\tau^q\omega}\Big(\frac{10x}{11\cdot 2^q}\Big)^{2^q-|W|}\tilde{\gamma}_{\tau^q\omega}\Big(\frac{x}{2^q}\Big)^{|W|} &\le \tilde{\gamma}_{\tau^q\omega}\Big(\frac{x}{11\cdot 2^q}\Big)^{10(2^q-|W|)+11|W|}\\ &\le \tilde{\gamma}_{\tau^q\omega}\Big(\frac{x}{11\cdot 2^q}\Big)^{\frac{131}{132}\cdot 11\cdot 2^q}.\end{aligned}$$

Consequently,

$$\tilde{\gamma}_\omega(x) \le 2^{2^{q+1}} \tilde{\gamma}_{\tau^q(\omega)}\Big(\frac{x}{11 \cdot 2^q}\Big)^{\rho(11\cdot 2^q)}. \qquad \square$$

As in Section 2, to every infinite word $\omega = l_1 l_2 \dots$ in Ω_∞ we associate an increasing sequence of integers $t_i = t_i(\omega)$, $i = 0, 1, 2, \dots$. The sequence is defined inductively. First we let $t_0 = 0$. Then, once some t_i is defined, we let t_{i+1} to be the smallest integer such that the finite word $l_{t_i+1} l_{t_i+2} \dots l_{t_{i+1}}$ features each of the letters 0, 1, and 2. Further, let $q_i = t_i - t_{i-1}$ for $i = 1, 2, \dots$.

Lemma 7. *Let $x_m = 11^m \cdot 2^{t_m}$ for any integer $m > 0$. Then $\gamma_\omega(x_m) \le 10^{\rho^m x_m}$.*

Proof. For any integer $m > 0$ let $\alpha_m = \rho(11 \cdot 2^{q_m})$ and $\beta_m = 2^{q_m+1}$. Lemma 6 implies that

$$\tilde{\gamma}_{\tau^{t_{m-1}}(\omega)}(x) \le 2^{\beta_m}\, \tilde{\gamma}_{\tau^{t_m}(\omega)}\Big(\frac{x}{11 \cdot 2^{q_m}}\Big)^{\alpha_m}$$

for any $x > 0$. Since $q_1 + q_2 + \dots + q_m = t_m$, it follows that for any integer $m > 0$ and real $x > 0$,

$$\tilde{\gamma}_\omega(x) \le 2^{S_m}\, \tilde{\gamma}_{\tau^{t_m}(\omega)}\Big(\frac{x}{11^m \cdot 2^{t_m}}\Big)^{R_m},$$

where $R_m = \alpha_1 \dots \alpha_m$ and $S_m = \beta_1 + \alpha_1\beta_2 + \dots + \alpha_1 \dots \alpha_{m-1}\beta_m$. In particular, $\gamma_\omega(x_m) \le 2^{S_m} \gamma_{\tau^{t_m}(\omega)}(1)^{R_m} = 2^{S_m} 5^{R_m}$. Since $R_m = \rho^m x_m$, it remains to show that $S_m \le R_m$.

We have $\alpha_m = \frac{11}{2}\rho\beta_m = \frac{131}{24}\beta_m > 5\beta_m$. Note that $q_m \ge 3$ so that $\beta_m \ge 16$. Hence $\alpha_m - \beta_m > 64$. Now the inequality $S_m \le R_m$ is proved by induction on m. First of all, $S_1 = \beta_1 < \alpha_1 = R_1$. Then, assuming $S_m \le R_m$ for some $m > 0$, we get $S_{m+1} = S_m + \alpha_1 \dots \alpha_m \beta_{m+1} \le R_m + \alpha_1 \dots \alpha_m \beta_{m+1} = \alpha_1 \dots \alpha_m(1 + \beta_{m+1}) < R_{m+1}$. $\square$

Recall some notation from Section 2 (for brevity, we drop index p). For any $C \ge 0$ let Ω_C denote the set of all infinite words $\omega \in \Omega_\infty$ such that $t_n(\omega) \le Cn$ for sufficiently large n. Given $\varepsilon > 0$, let $\Omega_{C,\varepsilon}$ denote the set of all $\omega \in \Omega_C$ such that $q_{n+1} = t_{n+1}(\omega) - t_n(\omega) \le \varepsilon t_n(\omega)$ for sufficiently large n.

Now we can prove the next theorem, which is a more detailed version of Theorem 2 (in the case $p = 2$).

Theorem 5. *Let $C > 0$ and*

$$\alpha > 1 - \frac{\log(\rho^{-1})}{\log(11 \cdot 2^C)}.$$

Then there exists $\varepsilon > 0$ such that $\gamma_\omega(n) \preceq e^{n^\alpha}$ for any $\omega \in \Omega_{C,\varepsilon}$.

Proof. Let
$$\kappa = \frac{\log(\rho^{-1})}{\log(11 \cdot 2^C)}.$$
Note that $0 < \kappa < 1$. Choose $\varepsilon > 0$ small enough so that $(\varepsilon + 1)(1 - \kappa) < \alpha$. Let $\omega \in \Omega_{C,\varepsilon}$. Then there exists an integer $N > 0$ such that $t_m \le Cm$ and $q_{m+1} = t_{m+1} - t_m \le \varepsilon t_m$ for $m \ge N$. By the choice of κ we have
$$\rho^m = \Big(\frac{1}{(11 \cdot 2^C)^\kappa}\Big)^m = \frac{1}{(11^m \cdot 2^{Cm})^\kappa} \le \frac{1}{(11^m \cdot 2^{t_m})^\kappa} = x_m^{-\kappa}$$
for any $m \ge N$. Since
$$x_{m+1} = 11^{m+1} \cdot 2^{t_{m+1}} = 11 \cdot 2^{q_{m+1}} \cdot 11^m \cdot 2^{t_m} = 11 \cdot 2^{q_{m+1}} \cdot x_m,$$
we obtain
$$\begin{aligned} x_{m+1}^{1-\kappa} = (11 \cdot 2^{q_{m+1}} \cdot x_m)^{1-\kappa} &\le 11^{1-\kappa}(11^{\varepsilon m} \cdot 2^{\varepsilon t_m} \cdot x_m)^{1-\kappa} \\ &= 11^{1-\kappa} x_m^{(\varepsilon+1)(1-\kappa)} \\ &\le 11^{1-\kappa} x_m^{\alpha}. \end{aligned}$$

Consider an arbitrary integer $n \ge x_N$. We have $x_m \le n \le x_{m+1}$ for some $m \ge N$. By Lemma 7,
$$\gamma_\omega(n) \le \gamma_\omega(x_{m+1}) \le 10^{\rho^{m+1} x_{m+1}}.$$
By the above,
$$\rho^{m+1} x_{m+1} \le x_{m+1}^{1-\kappa} \le 11^{1-\kappa} x_m^\alpha \le 11^{1-\kappa} n^\alpha,$$
hence
$$\gamma_\omega(n) \le 10^{11^{1-\kappa} n^\alpha} = D^{n^\alpha},$$
where $D = 10^{11^{1-\kappa}}$. Thus $\gamma_\omega(n) \preceq D^{n^\alpha} \sim e^{n^\alpha}$. □

Suppose μ is a Borel probability measure on Ω that is invariant and ergodic relative to the shift transformation $\tau\colon \Omega \to \Omega$. Since Ω_∞ is a Borel, shift invariant set, the measure μ is either supported on Ω_∞ or else $\mu(\Omega_\infty) = 0$. Theorem 1 will be derived from Theorem 5 using the following lemma.

Lemma 8. *If the measure μ is supported on Ω_∞, then there exists $C_0 > 0$ such that $t_n(\omega)/n \to C_0$ as $n \to \infty$ for μ-almost all $\omega \in \Omega$. Consequently, $\mu(\Omega_{C,\varepsilon}) = 1$ for any $C > C_0$ and $\varepsilon > 0$. In the case μ is the uniform Bernoulli measure on Ω, we can take $C_0 < 7.3$.*

Proof. For any finite word w over the alphabet $\{0, 1, 2\}$ let $T(w)$ denote the maximal number of non overlapping sub-words of w each containing all the letters. Clearly, $T(w) \leq |w|/3$. It is easy to see that $T(w_1) + T(w_2) \leq T(w_1 w_2) \leq T(w_1) + T(w_2) + 1$ for any words w_1 and w_2. It follows by induction that $T(w_0) + T(w_1) + \cdots + T(w_k) \leq T(w_0 w_1 \dots w_k) \leq T(w_0) + T(w_1) + \cdots + T(w_k) + k$ for any words $w_0, w_1, \dots, w_k$.

For any $\omega \in \Omega$ and integer $m > 0$ let $T_m(\omega) = T(\omega_m)$, where ω_m is the beginning of length m of the sequence ω. We are going to show that for μ-almost all ω there is a limit of $T_m(\omega)/m$ as $m \to \infty$. Note that T_m/m is a bounded ($0 \leq T_m \leq m/3$) Borel function on Ω. Let

$$I_m = \int_\Omega T_m \, d\mu.$$

If $\omega \in \Omega_\infty$ then $T_m(\omega) > 0$ for m large enough. Since the measure μ is supported on the set Ω_∞, it follows that $I_m > 0$ for m large enough.

Given integers $m_1, m_2 > 0$, the beginning of length $m_1 + m_2$ of any sequence $\omega \in \Omega$ is represented as the concatenation of two words, the beginning of length m_1 of the same sequence and the beginning of length m_2 of the sequence $\tau^{m_1}(\omega)$. Therefore $T_{m_1+m_2}(\omega) \geq T_{m_1}(\omega) + T_{m_2}(\tau^{m_1}(\omega))$. Integrating this inequality over Ω and using shift-invariance of the measure μ, we obtain $I_{m_1+m_2} \geq I_{m_1} + I_{m_2}$. Now the standard argument implies that $I_m/m \to I$ as $m \to \infty$, where $I = \sup_{k \geq 1} I_k/k$. Note that $0 < I \leq 1/3$.

Let Ω_μ denote the Borel set of all sequences $\omega \in \Omega$ such that for any integer $m > 0$ we have

$$\lim_{k \to \infty} \frac{1}{k} \sum\nolimits_{i=0}^{k-1} T_m(\tau^i(\omega)) = I_m.$$

Birkhoff's ergodic Theorem implies that Ω_μ is a set of full measure: $\mu(\Omega_\mu) = 1$. Consider an arbitrary $\omega \in \Omega$ and integers $m > 0$ and $k \geq 2m$. Let $l = \lfloor k/m \rfloor$, the integer part of k/m. For any integer j, $0 \leq j < m$, we represent the beginning of length k of ω as the concatenation of $l+1$ words $w_0 w_1 \dots w_l$, where w_0 is of length j, w_l is of length $k - lm + m - j$, and the other words are of length m. By the above,

$$T_k(\omega) - l \leq \sum\nolimits_{i=0}^{l} T(w_i) \leq T_k(\omega).$$

By construction, $T(w_i) = T_m(\tau^{(i-1)m+j}(\omega))$ for $1 \leq i \leq l-1$. Besides, $l \leq k/m$ and $0 \leq T(w_0) + T(w_l) \leq (k - lm + m)/3 < 2m/3$. Therefore

$$T_k(\omega) - k/m - 2m/3 \leq \sum\nolimits_{i=1}^{l-1} T_m(\tau^{(i-1)m+j}(\omega)) \leq T_k(\omega).$$

Summing the latter inequalities over j ranging from 0 to $m-1$, we obtain

$$mT_k(\omega) - k - 2m^2/3 \leq \sum\nolimits_{i=0}^{(l-1)m-1} T_m(\tau^i(\omega)) \leq mT_k(\omega).$$

Since $0 \le \sum_{i=(l-1)m}^{k-1} T_m(\tau^i(\omega)) \le (k - lm + m)m/3 < 2m^2/3$, it follows that

$$mT_k(\omega) - k - 2m^2/3 \le \sum_{i=0}^{k-1} T_m(\tau^i(\omega)) \le mT_k(\omega) + 2m^2/3.$$

Then

$$\left|\frac{1}{k}T_k(\omega) - \frac{1}{mk}\sum_{i=0}^{k-1} T_m(\tau^i(\omega))\right| \le \frac{1}{m} + \frac{2m}{3k}.$$

At this point, let us assume that $\omega \in \Omega_\mu$. Fixing m and letting k go to infinity in the latter estimate, we obtain that all limit points of the sequence $\{T_k(\omega)/k\}_{k\ge1}$ lie in the interval $[I_m/m - 1/m, I_m/m + 1/m]$. Letting m go to infinity as well, we obtain that $T_k(\omega)/k \to I$ as $k \to \infty$.

Given $\omega \in \Omega_\infty$, there is a simple relation between sequences $\{T_m(\omega)\}_{m\ge1}$ and $\{t_n(\omega)\}_{n\ge1}$. Namely, $t_n(\omega) \le m$ if and only if $T_m(\omega) \ge n$. In particular, $T_{t_n(\omega)}(\omega) = n$. Since $T_m(\omega)/m \to I$ as $m \to \infty$ for any $\omega \in \Omega_\mu$, it easily follows that $t_n(\omega)/n \to C_0$, where $C_0 = I^{-1}$, for any ω in $\Omega_\mu \cap \Omega_\infty$, a set of full measure.

Now consider the case μ is the uniform Bernoulli measure. To estimate the limit C_0 in this case, we are going to evaluate the integral I_7. For any integer $k \ge 0$ let N_k denote the number of words w of length 7 over the alphabet $\{0, 1, 2\}$ such that $T(w) = k$. Then $I_7 = 3^{-7}\sum_{k\ge0} kN_k$. Since $N_k = 0$ for $k > 2$, we have $N_0 + N_1 + N_2 = 3^7$ and $I_7 = (N_1 + 2N_2)/3^7$. Let us compute the numbers N_0 and N_2. A word w of length 7 satisfies $T(w) = 0$ if it does not use one of the letters. The number of words missing one particular letter is 2^7. Also, there are three words 0000000, 1111111, and 2222222 that miss two letters. It follows that $N_0 = 3\cdot2^7 - 3 = 381$. To compute N_2, we represent an arbitrary word w of length 7 as w_1lw_2, where w_1 and w_2 are words of length 3 and l is a letter. There are two cases when $T(w) = 2$. In the first case, each of the words w_1 and w_2 contains all letters, then l can be arbitrary. In the second case, either w_1 or w_2 misses exactly one letter, then l must be the missing letter and the other word must contain all letters. It follows that $N_2 = (3!)^2 \cdot 3 + 2M \cdot 3!$, where M is the number of words of length 3 that miss exactly one of the letters 0, 1, and 2. It is easy to observe that $M = 3^3 - 3! - 3 = 18$, then $N_2 = 324$. Now $N_1 = 3^7 - N_0 - N_2 = 2187 - 381 - 324 = 1482$. Finally, $I_7 = (N_1 + 2N_2)/3^7 = (1482 + 2 \cdot 324)/3^7 = 710/729$. Now we can estimate the limits. As shown earlier, $I \ge I_7/7 = 710/(7 \cdot 729) > 100/729$, then $C_0 = I^{-1} < 7.3$. □

Now we are ready to complete the proof of Theorem 1.

Proof of Theorem 1. Take any $C > C_0$, where C_0 is as in Lemma 8. By Theorem 5, there exists $\varepsilon > 0$ and $0 < \alpha < 1$ such that $\gamma_\omega(n) \preceq e^{n^\alpha}$ for all $\omega \in \Omega_{C,\varepsilon}$. The set $\Omega_{C,\varepsilon}$ has full measure by Lemma 8. In the case when μ is the uniform Bernoulli measure, we can assume that $C < 7.3$ by Lemma 8. Consequently, we can choose $\alpha = 1 - \kappa$, where $\kappa = \log\frac{132}{131}/\log(11 \cdot 2^{7.3})$. One can compute that $\kappa > 0.001$. □

5. Proof of Theorem 3

Recall that we are in the case $p = 2$ so that we use the notation $\Omega = \Omega_2$ and $\Omega_0 = \Omega_{2,0}$. Also, recall θ_0 is as defined before Theorem 3. We begin with preliminary lemmas.

Lemma 9. *Let g be a function of natural argument and let $\mathcal{L}_g \subset \mathcal{M}_k$ be the subset consisting of marked groups (G, S) such that $g \not\preceq \gamma_G^S$. Then $\mathcal{L}_g$ is a G_δ subset of $\mathcal{M}_k$ (i.e., a countable intersection of open sets).*

Proof. Given $(G, S) \in \mathcal{L}_g$ and $C \in \mathbb{N}$, let $K = K((G,S),C)$ be such that $\gamma_G^S(CK) < g(K)$ (such K exists since $g \not\preceq \gamma_G^S$). Let $\mathcal{B}_{((G,S),C)}$ denote the ball of radius 2^{-CK} (in the metric defined in the space of marked groups) centered at (G, S). We claim that

$$\mathcal{L}_g = \bigcap_{C \in \mathbb{N}} \bigcup_{(G,S)\in\mathcal{L}_g} \mathcal{B}_{((G,S),C)}.$$

The inclusion $\subset$ is clear. For the other inclusion, let (H, T) be an element of the right hand side. Then for any $C \in \mathbb{N}$ there is $(G, S) \in \mathcal{L}_g$ such that $(H, T) \in \mathcal{B}_{((G,S),C)}$. Therefore for $K = K((G,S),C)$ we have $\gamma_H^T(CK) = \gamma_G^S(CK) < g(K)$ and hence $g \not\preceq \gamma_H^T$, which shows that $(H, T) \in \mathcal{L}_g$. ☐

Lemma 10. *Let f be a function of natural argument and let $\mathcal{U}_f \subset \mathcal{M}_k$ be the subset consisting of marked groups (G, S) such that $\gamma_G^S \not\preceq f$. Then $\mathcal{U}_f$ is a G_δ subset of $\mathcal{M}_k$.*

Proof. The proof is analogous to the proof of Lemma 9. ☐

Now we are going to prove each part of Theorem 3.

Proof of Theorem 3. a) Suppose we are given $\theta > \theta_0$ and a function $f(n) \prec e^n$. Let $\eta = (012)^\infty \in \Omega$ and recall that we have $\gamma_\eta(n) \preceq e^{n^{\theta_0}}$ (Theorem 4, part 7). Consider the set $\mathcal{X} = \{(G_\omega, S_\omega) \mid \tau^k(\omega) = \eta \text{ for some } k\} \subset \mathcal{G}_2$. Since $\mathcal{G}_2$ is homeomorphic to Ω via $(G_\omega, S_\omega) \mapsto \omega$, the set $\mathcal{X}$ is dense in $\mathcal{G}_2$. For any $\omega \in \Omega \setminus \Omega_0$, the groups G_ω and $G_{\tau(\omega)} \times G_{\tau(\omega)}$ are commensurable by Theorem 2.2 in [16]. Therefore we have

$$\gamma_\omega \sim \gamma_{\tau(\omega)}^2$$

and for any ω, $\tau^k(\omega) = \eta$ it follows that

$$\gamma_\omega \sim \gamma_\eta^{2^k} \preceq (e^{n^{\theta_0}})^{2^k} \sim e^{n^{\theta_0}}.$$

Let $g(n) = e^{n^\theta}$ so that $\mathcal{X} \subset \mathcal{L}_g$, where $\mathcal{L}_g$ is defined in Lemma 9.

According to Theorem 4, part 2, the set $\mathcal{Y} = \{(G_\omega, S_\omega) \mid \omega \in \Omega_0\}$, which is dense in $\mathcal{G}_2$, consists of groups of exponential growth. In particular, $\mathcal{Y} \subset \mathcal{U}_f$, where $\mathcal{U}_f \subset \mathcal{M}_3$ is defined in Lemma 10. By Lemmas 9 and 10, the sets $\mathcal{L}_g \cap \mathcal{G}_2$ and $\mathcal{U}_f \cap \mathcal{G}_2$ are dense G_δ subsets of $\mathcal{G}_2$. Since $\mathcal{G}_2$ is compact, their intersection is also a dense G_δ subset of $\mathcal{G}_2$. For any $(G, S) \in \mathcal{L}_g \cap \mathcal{U}_f \cap \mathcal{G}_2$, we have $g \not\precsim \gamma_G^S$ and $\gamma_G^S \not\precsim f$.

b) This part is a corollary of part a) with $f(n) = e^{\frac{n}{\log n}}$ as $e^{n^\beta} \prec f(n)$ for any $\beta < 1$.

c) The proof of this part is analogous to part a). Let us denote by $\Omega' = \{0,1\}^{\mathbb{N}} \subset \Omega$. We set $\zeta = (01)^\infty$, $\mathcal{X} = \{(G_\omega, S_\omega) \mid \tau^k(\omega) = \zeta \quad \text{for some } k\}$ and $\mathcal{Y} = \{(G_\omega, S_\omega) \mid \omega \in \Omega_0 \cap \Omega'\}$. According to Theorem 4, part 8, for any $\varepsilon > 0$ the function $g(n) = \exp\left(\frac{n}{\log^{1-\varepsilon} n}\right)$ grows faster than γ_ζ. Since $g \sim g^2$, it follows that $\gamma_\omega \preceq g$ whenever $(G_\omega, S_\omega) \in \mathcal{X}$. It remains to apply Lemmas 9 and 10. □

6. Proof of Theorems 1′ and 3′

As it was mentioned in the introduction there is a natural embedding $\iota_k \colon \mathcal{M}_k \to \mathcal{M}_{k+1}$ given by $\iota_k((G, A)) = (G, A')$, where $A' = \{a_1, \dots, a_k, a_{k+1}\}$ if $A = \{a_1, \dots, a_k\}$ and $a_{k+1} = 1$ in G. This induces an embedding $\iota_{k,n} \colon \mathcal{M}_k \to \mathcal{M}_{k+n}$ for all k, n and, given a subset $X \subset \mathcal{M}_k$, one can consider its homeomorphic image $\iota_{k,n}(X) \subset \mathcal{M}_{k+n}$.

There are two natural ways of replacing one generating set A of a group G by another. We can, as just suggested, add one more formal generator representing the identity (and place it for definiteness at the end), or apply to a generating set *Nielsen transformations,* which are given by (see [30]):

i) exchanging two generators,

ii) replacing a generator $a \in A$ by its inverse a^{-1},

iii) replacing $a_i \in A$ by $a_i a_j$, where $a_i \neq a_j$.

Note that these transform generating sets into generating sets. It is in general incorrect that two generating sets of size k of a group are related by a sequence of Nielsen transformations (i.e., by an automorphism of the free group F_k), but we have the following result.

Proposition 1. *Let $(G, A) \in \mathcal{M}_k$ and $(G, B) \in \mathcal{M}_n$. Let $\iota_{k,n}(G, A) = (G, A')$ and $\iota_{n,k}(G, B) = (G, B')$, so that $(G, A'), (G, B') \in \mathcal{M}_{n+k}$. Then A' can be transformed into B' by a sequence of Nielsen transformations.*

Proof. Let

$$A' = \{a_1, \dots, a_k, a_{k+1}, \dots, a_{k+n}\}$$

and

$$B' = \{b_1, \dots, b_n, b_{n+1}, \dots, b_{n+k}\},$$

where $a_{k+1} = \dots = a_{k+n} = b_{n+1}, \dots = b_{n+k} = 1$. For any $b_i,\ 1 \leq i \leq n$ there is a word $B_i \in \{a_1, \dots, a_k\}^{\pm}$ such that $b_i = B_i$. By a sequence of Nielsen transformations (acting trivially on $a_i,\ i \leq k$) we can transform A' into $A'' = \{a_1, \dots, a_k, B_1, \dots, B_n\} = \{a_1, \dots, a_k, b_1, \dots, b_n\}$. In a similar way B' can be transformed into $B'' = \{b_1, \dots, b_n, a_1, \dots, a_k\}$ by a sequence of Nielsen transformations. It is clear that B'' can be obtained from A'' by permuting the generators, which can be achieved by a sequence of Nielsen transformations. □

Taking the inductive limit $\mathcal{M} = \lim_{\to} \mathcal{M}_k$ and setting $A_\infty = \{a_1, a_2, \dots\}$, the previous proposition shows (as observed by Champetier in [12]) that the group of Nielsen transformations over an infinite alphabet (that is, the group $\mathrm{Aut}_{\mathrm{fin}}(F_\infty)$ of finitary automorphisms of a free group F_∞ of countably infinite rank) acts on $\mathcal{M}$ in such a way that if two pairs $(G, A), (G, B) \in \mathcal{M}$ represent the same group then they belong to the same orbit of the action of $\mathrm{Aut}_{\mathrm{fin}}(F_\infty)$ on $\mathcal{M}$ (and it is clear the points in the orbit all represent the same group). In [12] it was shown that this action, which is by homeomorphisms and hence is Borel, is not *tame* (in other terminology, not measurable or not smooth). As was mentioned in the introduction, the question of existence of $\mathrm{Aut}_{\mathrm{fin}}(F_\infty)$-invariant (or at least quasi-invariant) measure is important for the topic of random groups.

There are more general ways of embedding $\mathcal{M}_k$ into $\mathcal{M}_l$. Assume we have a subset $X = \{(G_i, A_i) \mid i \in I\} \subset \mathcal{M}_k$, where $A_i = \{a_1^{(i)}, \dots, a_k^{(i)}\}$. Let F_k be a free group on $\{a_1, \dots, a_k\}$ and suppose that there are words $B_j(a_1, \dots, a_k) \in F_k$ for $1 \leq j \leq m$ such that for all $i \in I$, the set

$$B_i = \{B_1(a_1^{(i)}, \dots, a_k^{(i)}), \dots, B_m(a_1^{(i)}, \dots, a_k^{(i)})\}$$

is a generating set for G_i. Let $Y = \{(G_i, B_i) \mid i \in I\} \subset \mathcal{M}_m$.

Proposition 2. *The map $\varphi\colon X \to Y$ given by $\varphi((G_i, A_i)) = (G_i, B_i)$ is a homeomorphism.*

Proof. Let $X' = \iota_{k+m}(X)$ and $Y' = \iota_{m+k}(Y)$. By the previous proposition, there is an automorphism of F_{k+m} (realized by a sequence of Nielsen transformations) which induces a homeomorphism φ of $\mathcal{M}_{k+m}$ which maps X' onto Y'. It is clear that φ is the restriction of $\iota_{m+k}^{-1} \circ \varphi \circ \iota_{k+m}$ to X. □

We are ready to prove the theorems.

Theorems 1 and 3 show that Theorems 1′ and 3′ hold for $k = 4$. Using the previous propositions, it immediately follows that they hold for values $k \geq 4$.

For $k = 3$ observe that by virtue of equations (1), for every $\omega \in \Omega$ we have $d_\omega = b_\omega c_\omega$ and hence the groups G_ω are generated by $\{a, b_\omega, c_\omega\}$. Therefore by proposition 2 we obtain the result for $k = 3$.

The case $k = 2$ is more delicate. There are several methods of embedding a group into a 2-generated group. We need an embedding that preserves the property to have intermediate growth. To accomplish this, we use the following trick. Let T be the rooted tree with branch index 4,2,2,2,… . Given $\omega \in \Omega$, let e be the automorphism of T which cyclically permutes the first level vertices and let f_ω be the automorphism given by $(b_\omega, c_\omega, a, 1)$. Set $M_\omega = \langle x, y_\omega \rangle$. This gives an embedding

$$\begin{aligned} \psi\colon M_\omega &\longrightarrow S_4 \ltimes \quad G_\omega^4, \\ x &\longmapsto \sigma \quad (1,1,1,1), \\ y_\omega &\longmapsto \quad (b_\omega, c_\omega, a, 1), \end{aligned}$$

where σ is the cyclic permutation of order 4 in S_4. Let

$$\bar{M}_\omega = \langle y_\omega, x y_\omega x^{-1}, x^2 y_\omega x^{-2}, x^3 y_\omega x^{-3} \rangle$$

and observe that $\bar{M}_\omega$ has index 4 in M_ω. The equalities

$$\begin{aligned} \psi(y_\omega) &= (b_\omega, c_\omega, a, 1) \\ \psi(x y_\omega x^{-1}) &= (c_\omega, a, 1, b_\omega) \\ \psi(x^2 y_\omega x^{-2}) &= (a, 1, b_\omega, c_\omega) \\ \psi(x^3 y_\omega x^{-3}) &= (1, b_\omega, c_\omega, a) \end{aligned}$$

show that $\bar{M}_\omega$ is a sub-direct product of G_ω^4. Hence if G_ω has intermediate growth so does M_ω and if the growth of G_ω is bounded above by a function of the form e^{n^α} then the same holds for the growth function of M_ω. Similarly, if G_ω has oscillating growth of type (e^{n^θ}, f), so does M_ω.

One can observe that the branch algorithm solving the word problem for groups G_ω (described in [16]) can be adapted to the groups M_ω: the covering group will be $\mathbb{Z}_4 * \mathbb{Z}_2$, given a word g in the normal form in $\mathbb{Z}_4 * \mathbb{Z}_2$, one first checks whether the exponent of e in g is divisible by 4 or not. If not then the element g does not belong to the first level stabilizer and hence $g \neq 1$. Otherwise one computes the sections of g and then applies the classical branch algorithm to the sections of g with oracle ω. This shows that for two sequences $\omega, \eta \in \Omega \setminus \Omega_0$ which have common prefix of length n, the Cayley graphs of the groups M_ω and M_η will have isomorphic balls of radius 2^{n-1}. Therefore we consider the subset $X = \{(M_\omega, L_\omega) \mid \omega \in \Omega \setminus \Omega_0\} \subset \mathcal{M}_2$ where $L_\omega = \{x, y_\omega\}$ and take its closure in $\mathcal{M}_2$ to obtain a Cantor set in $\mathcal{M}_2$. The new limit groups M_ω for $\omega \in \Omega_0$ will have a finite index subgroup which is a sub-direct product in the group G_ω^4, and therefore are of exponential growth. Thus the limit groups $M_\omega,\ \omega \in \Omega_0$ will have exponential growth and therefore similar arguments used to prove Theorem 3 can be applied in this case too. Also note that when $\omega \in \Omega_\infty$ then M_ω and G_ω^4 are abstractly commensurable i.e., have finite index subgroups which are isomorphic.

For $p \geq 3$ a similar construction can be done by setting $M_\omega = \langle x, y_\omega \rangle$ as the group of automorphisms of the tree with branch index $p^2, p, p, \ldots$, where x is the cyclic permutation of order p and $y_\omega = (b_\omega, c_\omega, a, 1, \ldots, 1)$. One can observe that M_ω in this case is a sub-direct product in $G_\omega^{p^2}$ and M_ω is abstractly commensurable with $G_{\tau(\omega)}^{p^2}$ when $\omega \in \Omega_{p,\infty}$. This allows to prove Theorem 1' in the case $p \geq 3$.

7. Concluding Remarks

Let $G_\omega^{\mathrm{um}},\ \omega \in \Omega_{p,0}$ denote the unmodified groups as defined in Section 3 (i.e., the groups before modifying countably many groups corresponding to eventually constant sequences). Note that for fixed prime p we have $G_{0^\infty}^{\mathrm{um}} = G_{1^\infty}^{\mathrm{um}} = \cdots = G_{p^\infty}^{\mathrm{um}}$ as subgroups of the p-ary rooted tree. The limit groups $G_\omega,\ \omega \in \Omega_{p,0}$ map onto the corresponding group G_ω^{um}. When $p = 2$ and $\omega \in \Omega_2$ is a constant sequence, G_ω^{um} is isomorphic to the infinite dihedral group (Lemma 2.1 in [16]) and hence has linear growth. This shows that G_ω^{um} has polynomial growth for $\omega \in \Omega_{2,0}$. For $p \geq 3$ and $\omega \in \Omega_p$ a constant sequence, the groups G_ω^{um} were considered in [22] and were shown to be regular branch self-similar groups. As these groups are residually finite p-groups, the main result of [19] shows that for all such groups $e^{\sqrt{n}}$ is a lower bound for their growth functions. Therefore for all primes p and $\omega \in \Omega_{p,0}$, the groups G_ω have super-polynomial growth. As mentioned in Theorem 4, for $p = 2$ the groups $G_\omega,\ \omega \in \Omega_{2,0}$, are known to have exponential growth. An extension of this fact to $p > 2$ would generalize Theorem 3 to all primes p. For $p = 3$ and $\omega \in \Omega_p$ a constant sequence, the group G_ω^{um} coincides with the Fabrykowski–Gupta group studied in [15]. In [6] it was shown that the growth of this group satisfies

$$e^{n^{\frac{\log 3}{\log 6}}} \preceq \gamma(n) \preceq e^{\frac{n(\log\log n)^2}{\log n}}.$$

A more general problem is the following. Given two increasing functions γ_1, γ_2 such that $\gamma_i(n) \sim \gamma_i(n)^p,\ i = 1, 2$ and $\gamma_1(n) \prec \gamma_2(n)$, consider the set

$$\mathcal{W}_{\gamma_1,\gamma_2} = \{\omega \in \Omega_p \mid \gamma_1(n) \preceq \gamma_\omega(n) \preceq \gamma_2(n)\}.$$

As mentioned before, for $\omega \in \Omega_p \setminus \Omega_{p,0}$ the groups G_ω and $G_{\tau(\omega)}$ are commensurable and hence $\gamma_\omega \sim \gamma_{\tau(\omega)}^p$. This shows that $\mathcal{W}_{\gamma_1,\gamma_2}$ is τ invariant, and hence for any τ invariant ergodic measure μ defined on Ω_p we have $\mu(\mathcal{W}_{\gamma_1,\gamma_2}) = 0$ or 1. A natural direction for investigation would be to determine functions γ_1, γ_2 for which the set $\mathcal{W}_{\gamma_1,\gamma_2}$ has full measure (and make γ_1, γ_2 as close to each other as possible while keeping $\mu(\mathcal{W}_{\gamma_1,\gamma_2}) = 1$). Theorem 1, part (b) together with Theorem 4, part (3) can be interpreted as $\mu(\mathcal{W}_{\gamma_1,\gamma_2}) = 1$, where $\gamma_1(n) = e^{n^{0.5}}$, $\gamma_2(n) = e^{n^{0.999}}$ and μ is the uniform Bernoulli measure on Ω_2.

The idea of statements similar to Lemmas 9 and 10, which descends to the paper of A. Stepin [35], is based on the fact that many group properties are formulated in "local terms" with respect to the topology on $\mathcal{M}_k$. This includes properties such as to be amenable, to be LEK (locally embeddable into the class K of groups), to be sofic, to be hyperfinite, etc. (see, e.g., [11]).

In all these and other cases one can state that for any k the subset $X_{\mathcal{P}} \subset \mathcal{M}_k$ of groups satisfying a local property $\mathcal{P}$ is a G_δ set in $\mathcal{M}_k$. So if a subset $Y \subset \mathcal{M}_k$ has a dense subset of groups satisfying property $\mathcal{P}$ then it contains dense G_δ subset satisfying property $\mathcal{P}$. In some cases like LEF (locally embeddable into finite groups), LEA (locally embeddable into amenable groups), sofic and hyperfinite groups, the corresponding set is a closed subset in $\mathcal{M}_k$ and there is not a big outcome of the above argument. But for properties such as to be amenable, to have particular type of growth and some other properties, the above observation gives nontrivial information about the structure of the space of groups.

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