

## Imbeddings into groups of intermediate growth

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*To Pierre de la Harpe,  
who introduced us to the beauty and diversity  
of the world of infinite groups,  
in gratitude*

**Abstract.** Every countable group that does not contain a finitely generated subgroup of exponential growth imbeds in a finitely generated group of subexponential word growth.

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### 1. Introduction

A classical result by Higman, Neumann, and Neumann [10] states that every countable group imbeds in a finitely generated group. It was then shown that many properties of the group can be inherited by the imbedding: in particular, solvability (Neumann and Neumann [12]), torsion (Phillips [14]), residual finiteness (Wilson [15]), and amenability (Olshansky and Osin [13]).

Seen the other way round, these results show that there is little restriction, apart from being countable, on the subgroups of a finitely generated group.

A finitely generated group  $G$  has *polynomial growth* if there is a polynomial function  $p(n)$  bounding from above the number of group elements that are products of at most  $n$  generators; it has *subexponential growth* if  $p(n)$  may be chosen subexponential in  $n$ , and has *intermediate growth* if  $G$  has subexponential but not polynomial growth.

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By a theorem of Gromov [9], groups of polynomial growth are virtually nilpotent, so all its subgroups are finitely generated (see e.g. Corollary 9.10 in [11]). On the other hand, there are groups of intermediate growth such as the “first Grigorchuk group” [6] with infinitely generated subgroups. We are therefore led to ask which groups may appear as subgroups of a group of subexponential growth.

**1.1. Main result.** Let us say that a group has *locally subexponential growth* if all of its finitely generated subgroups have subexponential growth. Clearly, if  $G$  has subexponential growth then all its subgroups have locally subexponential growth. Our main result shows that this is the only restriction:

**Theorem A.** *Let  $B$  be a countable group of locally subexponential growth. Then there exists a finitely generated group of subexponential growth in which  $B$  imbeds as a subgroup.*

Furthermore, this group may be assumed to have two generators, see Remark 6.5, and to contain  $B$  in its derived subgroup.

In contrast, there exist nilpotent (and even abelian) countable groups that do not imbed into finitely generated nilpotent groups. Gromov’s theorem mentioned above has the consequence that there exist countable groups of locally polynomial growth that do not imbed in groups of polynomial growth. Mann noted in Corollary 9.11 in [11] that torsion-free groups locally of polynomial growth of bounded degree are also virtually nilpotent.

It is a tantalizing open question to understand which properties are shared by groups of intermediate growth and by nilpotent and virtually nilpotent groups. It is clear that a group of intermediate growth cannot contain non-abelian free subgroups or even free subsemigroups. Groups of intermediate growth were constructed by Grigorchuk in [6] and his first example, known as the “first Grigorchuk group,” admits a pair of dilating endomorphisms with commuting images. This property can be viewed as a higher dimensional analogue of groups with dilation; and any group admitting a dilation has polynomial growth.

We may also ask which groups may appear as subgroups of a specific group of intermediate growth such as the first Grigorchuk group  $G_{012}$ . For example,  $G_{012}$  is known to contain every finite 2-group, and all its subgroups are countable, residually-2 and have locally smaller growth.

There are other restrictions, apart from these obvious ones, for a countable group to be imbedded as a subgroup of a generalised Grigorchuk group. For example, only a finite number of primes appears as exponents in a Grigorchuk group [7]; see also §3.6 of [2]. Extensions of Grigorchuk groups constructed by the authors in [3] admit a larger class of possible subgroups, but some restrictions appear nevertheless. In particular, Theorem A gives the first groups of subexponential growth containing  $\mathbb{Q}$ .

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## 2. Sketch of the proofs

The original imbedding result by Higman, Neumann, and Neumann [10] mentioned in the introduction proceeds by a sequence of “HNN extensions.” We recall the later construction by Neumann and Neumann [12], which uses wreath products rather than HNN extensions. The *unrestricted* wreath products of two groups  $H, G$  is the group  $H \wr G = H^G \rtimes G$ , the split extension of the set of maps  $G \rightarrow H$  by  $G$ , where the action of  $g \in G$  on  $f: G \rightarrow H$  is  ${}^g f(x) = f(xg)$ . The Neumann–Neumann construction proceeds in two steps.

(i) Starting with a countable group  $B$ , one imbeds it into a countable subgroup  $G$  of the unrestricted wreath product  $B \wr \mathbb{Z}$  in such a way that  $B$  is imbedded into the commutator group  $[G, G]$ . The group  $G$  is generated by  $\mathbb{Z}$  and, for all  $b \in B$ , the function  $f_b: \mathbb{Z} \rightarrow B$  defined by  $f_b(m) = b^m$ . Denoting by  $t$  the generator of  $\mathbb{Z}$ , we see that  $[t, f_b]$  is the constant function  $b$ ; so  $B$  is in fact imbedded in  $[t, G]$ .

(ii) Starting with a countable group  $G$ , one imbeds the commutator subgroup  $[G, G]$  into a two-generated subgroup  $W$  of the unrestricted wreath product of  $G \wr \mathbb{Z}$ . More generally, one constructs imbeddings into  $G \wr P$  for a finitely generated group  $P$ . Denoting a generating set of  $G$  by  $\{b_1, b_2, \dots\}$ , the group  $W$  is generated by  $P$  and  $f: P \rightarrow G$  with  $f(x_i) = b_i$  along a sparse-enough sequence  $(x_i)_{i \geq 1}$  of elements of  $P$ . In fact, since it suffices in (i) to imbed  $[t, G]$  in  $W$ , one sets  $f(1) = t$  and the exact requirement on the sequence  $(x_i)$  is:  $x_i \neq 1$  for all  $i$ ; all  $x_i$  are distinct; and  $x_i x_j \notin \{1, x_k\}$  for all  $i, j, k \in \mathbb{N}$ . One then sees that  $[f, f^{x_i^{-1}}]$  is a function supported only at 1, with value  $[t, b_i]$  there. This is the imbedding of  $[t, G]$ .

The combination of both steps imbeds  $B$  into the finitely generated group  $W$ . If  $B$  is solvable, then so are  $B \wr \mathbb{Z}$  and  $G$ ; and similarly, if  $G$  is solvable, then so are  $G \wr \mathbb{Z}$  and  $W$ .

This construction may be applied to an arbitrary countable group  $B$ , but some properties of  $B$ , such as amenability, may be lost along the way. Olshansky and Osin introduce in [13] the following slightly stronger condition on  $(x_i)_{i \geq 1}$ : by definition, a *parallelogram* in a sequence  $(x_i)_{i \geq 1}$  is a quadruple of elements  $p_1 \neq p_2 \neq p_3 \neq p_4 \neq p_1$ , each belonging to  $\{x_i\}$ , such that  $p_1 p_2^{-1} p_3 p_4^{-1} = 1$ . A sequence is *parallelogram-free* if it contains no parallelogram. They show that, if  $(x_i)$  is parallelogram-free, then the group  $W$  is obtained from  $G$  and  $P$  by extensions, subgroups, quotients and directed limits, so in particular is amenable as soon as  $G$  and  $P$  are amenable.

They also modify slightly step (i), by defining rather  $f_b(m) = b$  for  $m \geq 0$  and  $f_b(m) = 1$  for  $m < 0$ ; then  $[t, f_b]$  is the function supported at 0 with value  $b$  there, and the group  $G = \langle t, f_b : b \in B \rangle$  is also obtained from  $B$  and  $\mathbb{Z}$  by elementary operations, so is amenable as soon as  $B$  is amenable.

Note that the group  $W$  contains the standard wreath product  $B \wr \mathbb{Z}$ , so always has exponential growth.

**2.1. Imbedding in groups of subexponential growth.** Our goal is, starting from a countable group  $B$  of locally subexponential growth, to construct a finitely generated group  $W$  of subexponential growth containing  $B$ . We exhibit analogues of steps (i) and (ii) among *permutational wreath products*. Given groups  $H, G$  and an action of  $G$  on a set  $X$ , the *unrestricted permutational wreath product* is  $H \wr_X G = H^X \rtimes G$ , and the *restricted permutational wreath product* is the extension of finitely supported functions  $X \rightarrow H$  by  $G$ . We also introduce the *finite-valued permutational wreath product*  $H \wr^{f.v.}_X G$ , defined as the extension of functions  $X \rightarrow H$  with finite image by  $G$ . Clearly

$$H \wr_X G \leq H \wr^{f.v.}_X G \leq H \wr G.$$

Our previous work [3] gives a criterion, in terms of *inverted orbits*, that guarantees that the restricted permutational wreath product  $W = H \wr_X G$  has subexponential growth as soon as  $H$  and  $G$  have subexponential growth. The inverted orbit of a point  $x \in X$  under a word  $w = g_1 \dots g_n$  in  $G$  is the set  $\{xg_1 \dots g_n, xg_2 \dots g_n, \dots, xg_n, x\}$ . If its cardinality may be bounded sublinearly in  $n$ , then  $W$  has subexponential growth. We compare subgroups  $\langle G, f \rangle$  of the unrestricted wreath product with  $W$  to bound its growth.

Ad step (i), we show in Proposition 3.1 that for every group  $B$  there exists a group  $G$  that is a directed union of finite extensions of finite powers of  $B$  and such that  $[G, G]$  contains  $B$ . In particular, if  $B$  has locally subexponential growth, so does  $G$ , and if  $B$  is countable then  $G$  may be so chosen.

Ad step (ii), we consider separately the group  $P$  and the set  $X$  on which it acts. As a replacement for parallelogram-free sequences, we introduce *rectifiable* sequences, which are sequences  $(x_i)$  in  $X$  such that, for all  $i \neq j$ , there exists  $g \in P$  with  $x_i g = x_j$  and  $x_k g \neq x_\ell$  for all  $\ell \neq k \neq i$ . We show that such sequences exist in the action of the first Grigorchuk group on the orbit of a ray, and more generally for all “weakly branched” groups.

The next step in the proof is an argument controlling the growth of a subgroup of the form  $W = \langle P, f \rangle \leq G \wr_X P$ , for a function  $f : X \rightarrow G$  with sparse-enough (but infinite!) support. The rectifiability of the sequence  $(x_i)$  guarantees that functions with singleton support and arbitrary values in  $[G, G]$  belong to  $W$ . Using the sparseness of the support of  $f$ , we show that balls in  $W$  can be approximated by balls in subgroups of restricted wreath products  $\langle S \rangle \wr_X f$  for finite subsets  $S$  of  $G$ . By [3], these restricted wreath products have subexponential growth. We recall that, in general, a limit in the Cayley topology of groups of subexponential growth may

have exponential growth (see Theorem C in [5]); the Cayley topology on the space of finitely generated groups is the topology in which groups are close if their labeled Cayley graphs agree on a large ball. We control more precisely the approximation of  $W$  so that the growth estimates pass to the limit. Finally, in contrast with standard wreath products, the space  $X$  is not homogeneous, so an extra condition of stabilisation of balls around the  $x_i$  is required (even to ensure that  $W$  be amenable).

### 3. Imbedding in the derived subgroup

Let  $B$  be a group. We call a group  $G$  *hyper- $B$*  if it is a directed union of finite extensions of finite powers of  $B$ . In this section, our goal is to prove the following proposition.

**Proposition 3.1.** *Let  $B$  be a countable group. Then there exists a hyper- $B$  group  $G$  such that  $[G, G]$  contains  $B$  as a subgroup. In particular, if  $B$  has locally subexponential growth, then so does  $G$ .*

*If  $B$  is infinite, then  $G$  may furthermore be supposed to have the same cardinality as  $B$ .*

In order to prove Proposition 3.1, we first introduce the following notation. For groups  $H, U$  we denote by

$$H \wr^{f.v.} U = \{(\phi, u) \in H^U \times U : \#\phi(U) < \infty\}$$

the subgroup of the unrestricted wreath product  $H^U \rtimes U$  in which the function  $U \rightarrow H$  takes finitely many values. Note that  $H \wr^{f.v.} U$  is a subgroup, because if  $(\phi, u)^{-1}(\phi', u') = (\phi'', u^{-1}u')$  then  $\phi''(U) \subseteq \phi(U)^{-1}\phi'(U)$  is finite.

**Lemma 3.2.** *Let  $G$  be a hyper- $B$  group, and let  $H$  be a hyper- $G$  group. Then  $H$  is hyper- $B$ .*

*Proof.* Consider  $h \in H$ ; then  $h$  belongs to a finite extension of a finite power of  $G$ , which may be assumed of the form  $G \wr F$  for a finite group  $F$ . Let us write  $h = \phi f$  with  $\phi: F \rightarrow G$  and  $f \in F$ ; then  $\phi(f)$  belongs for all  $f \in F$  to a finite extension of a finite power of  $B$ , which can be assumed to be the same for all  $f$ . This extension may be assumed to be of the form  $B \wr E$  for a finite group  $E$ . It follows that  $h$  belongs to  $B \wr_{E \times F} (E \wr F)$ , a finite extension of a finite power of  $B$ ; so  $H$  is hyper- $B$ .  $\square$

**Lemma 3.3.** *If  $H$  is a hyper- $B$  group and  $U$  is locally finite, then  $H \wr^{f.v.} U$  is a hyper- $B$  group.*

*Proof.* We first show that  $H \wr^{f.v.} U$  is hyper- $H$ . By hypothesis,  $U$  is a directed union of finite subgroups  $E$ . The partitions  $\mathcal{P}_0$  of  $U$  into finitely many parts also form a

directed poset; and for every such partition  $\mathcal{P}_0$  and every finite subgroup  $E \leq U$  there exists a finite partition  $\mathcal{P}$  of  $U$  that is invariant under  $E$  and refines  $\mathcal{P}_0$ , namely the wedge (= least upper bound) of all  $E$ -images of  $\mathcal{P}_0$ .

Consider now the directed poset of pairs  $(E, \mathcal{P})$  consisting of finite subgroups  $E \leq U$  and  $E$ -invariant partitions of  $U$ . Consider the corresponding subgroups  $H^{\mathcal{P}} \rtimes E$  of  $H \wr^{f.v.} U$ . If  $(E, \mathcal{P}) \leq (E', \mathcal{P}')$  then  $H^{\mathcal{P}} \rtimes E$  is naturally contained in  $H^{\mathcal{P}'} \rtimes E'$ , so these subgroups of  $H \wr^{f.v.} U$  form a directed poset, which exhausts  $H \wr^{f.v.} U$ .

It follows that  $H \wr^{f.v.} U$  is a hyper- $H$  group, and we are done by Lemma 3.2.  $\square$

**Lemma 3.4.** *Let  $B$  be a group. Then there exists a subgroup  $C$  of  $B$ , containing  $[B, B]$ , such that  $B/C$  is torsion and  $C/[B, B]$  is free abelian.*

*Proof.*  $B/[B, B] \otimes_{\mathbb{Z}} \mathbb{Q}$  is a  $\mathbb{Q}$ -vector space, hence has a basis, call it  $X$ . It generates a free abelian group  $\mathbb{Z}X$  within  $B/[B, B]$ , whose full preimage in  $B$  we call  $C$ . Then  $B/C \otimes_{\mathbb{Z}} \mathbb{Q} = 0$  so  $B/C$  is torsion.  $\square$

We set up the following notation for the proof of Proposition 3.1. We choose a subgroup  $C \leq B$  as in Lemma 3.4 and write  $T := B/C$ . We choose a basis  $X$  of  $C/[B, B]$ , for every  $x \in X$  we choose an element  $b_x \in C$  representing it, and we define a homomorphism  $\theta_x: C \rightarrow \langle b_x \rangle \subseteq B$ , trivial on  $[B, B]$ , by  $\theta_x(b_x) = b_x^{-1}$  and  $\theta_x(b_y) = 1$  for all  $y \neq x \in X$ . In particular, we have for all  $b \in C$

$$b \cdot \prod_{x \in X} \theta_x(b) \in [B, B]$$

and the product is finite.

We write  $\pi: B \rightarrow T$  the natural projection, and define inductively a set-theoretic section  $\sigma: T \rightarrow B$  as follows. Since  $T$  is torsion, it is locally finite, hence may be written as a directed union  $T = \bigcup_{\alpha \in \mathbb{N}} T_\alpha$  of finite groups. Assume  $\sigma$  has already been defined on  $T'_\alpha := \bigcup_{\beta < \alpha} T_\beta$ . Choose a transversal  $T''_\alpha$  of  $T'_\alpha$  in  $T_\alpha$ , namely a set of coset representatives of  $T'_\alpha$  in  $T_\alpha$ , and define  $\sigma$  on  $T''_\alpha$  by choosing arbitrarily for each  $t'' \in T''_\alpha$  a  $\pi$ -preimage in  $B$ . Extend then  $\sigma$  to  $T_\alpha$  by  $\sigma(t''t') = \sigma(t'')\sigma(t')$  for  $t'' \in T''_\alpha, t' \in T'_\alpha$ .

Let  $F$  be a locally finite group of cardinality  $> \#X$ , and fix an imbedding of  $X$  in  $F \setminus \{1\}$ . As a first step, we consider the group  $G_0 = B \wr^{f.v.} (T \times F)$ , and define a map  $\Phi_0: B \rightarrow G_0$  as follows:

$$\Phi_0(b) = (\phi, \pi(b), 1) \quad \text{with } \phi(t, f) = \begin{cases} b & \text{if } f = 1, \\ \theta_f(\sigma(t)b\sigma(t\pi(b))^{-1}) & \text{if } f \in X, \\ 1 & \text{otherwise.} \end{cases} \quad (1)$$

**Lemma 3.5.** *The map  $\Phi_0$  is well-defined and is an injective homomorphism into  $G_0$ .*

*Proof.* To see that  $\Phi_0$  is well-defined, note that the argument  $\sigma(t)b\sigma(t\pi(b))^{-1}$  belongs to  $\ker(\pi) = C$ , so that  $\theta_f$  may be applied to it.

We next show that the image of  $\Phi_0$  belongs to  $G_0$ . Consider  $b \in B$ . Let  $\alpha$  be such that  $\pi(b)$  belongs to the finite group  $T_\alpha$ . Now given  $t \in T$ , let  $\omega \in \mathbb{N}$  be such that  $t \in T_\omega$ . Write  $t$  using  $T_\alpha$  and transversal elements as  $t''_\omega \dots t''_\beta u$  with  $\omega > \dots > \beta > \alpha$  and  $t''_\omega \in T''_\omega, \dots, t''_\beta \in T''_\beta, u \in T_\alpha$ . Then  $\sigma(t) = \sigma(t''_\omega) \dots \sigma(t''_\beta)\sigma(u)$  and  $\sigma(t\pi(b)) = \sigma(t''_\omega) \dots \sigma(t''_\beta)\sigma(u\pi(b))$ , so that  $\sigma(t)b\sigma(t\pi(b))^{-1}$  is conjugate to  $\sigma(u)b\sigma(u\pi(b))^{-1}$ , and therefore  $\theta_f(\sigma(t)b\sigma(t\pi(b))^{-1}) = \theta_f(\sigma(u)b\sigma(u\pi(b))^{-1})$  takes only finitely many values because  $\theta_f$  vanishes on  $[B, B]$ . Also,  $\theta_f(\sigma(t)b\sigma(t\pi(b))^{-1}) = 1$  except for finitely many values of  $f \in X$ . In summary, the function  $\phi \in B^{T \times F}$  is such that  $\phi(t, f)$  takes only finitely many values.

It is clear that  $\Phi_0$  is injective: if  $b \neq 1$  and  $\Phi_0(b) = (\phi, \pi(b), 1)$  then  $\phi(1, 1) = b \neq 1$ . It is a homomorphism because all  $\theta_f$  are homomorphisms. □

**Lemma 3.6.** *We have  $\Phi_0(C) \leq [G_0, G_0]$ .*

*Proof.* If  $b \in [B, B]$  then clearly  $\Phi_0(b) \in [G_0, G_0]$ . Since  $C$  is generated by  $[B, B] \cup \{b_x\}_{x \in X}$ , it suffices to consider  $b = b_x$ .

We define  $g \in G_0$  by

$$g = (\psi, 1, 1) \quad \text{with } \psi(t, f) = \begin{cases} b_x & \text{if } f = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then  $\Phi_0(b_x) = (\phi, 1, 1)$  with  $\phi(t, 1) = b_x$  and  $\phi(t, x) = b_x^{-1}$ , all other values being trivial, according to (1); so, as was to be shown,

$$\Phi_0(b_x) = (\phi, 1, 1) = ({}^x\psi^{-1} \cdot \psi, 1, 1) = [(1, 1, x^{-1}), g] \in [G_0, G_0]. \quad \square$$

We next define

$$G = G_0 \wr^{\text{f.v.}} (\mathbb{Q}/\mathbb{Z})$$

and a map  $\Phi: B \rightarrow G$  by

$$\Phi(b) = (\phi, 0) \text{ with } \phi(r) = \Phi_0(b) \text{ for all } r \in \mathbb{Q}/\mathbb{Z}.$$

**Lemma 3.7.** *The map  $\Phi$  is an injective homomorphism, and  $\Phi(B) \leq [G, G]$ .*

*Proof.* Clearly  $\Phi$  is an injective homomorphism, since  $\Phi_0$  is an injective homomorphism by Lemma 3.5.

We identify  $\mathbb{Q}/\mathbb{Z}$  with  $\mathbb{Q} \cap [0, 1)$ . For every  $n \in \mathbb{N}$ , consider the map  $\Psi_n: B \rightarrow G$  defined by

$$\Psi_n(b) = (\phi, 0) \quad \text{with } \phi(r) = \begin{cases} \Phi_0(b) & \text{if } r \in [0, 1/n), \\ 1 & \text{otherwise;} \end{cases}$$

so  $\Phi = \Psi_1$ . We know from Lemma 3.6 that  $\Psi_n(C)$  is contained in  $[G, G]$ .

Consider now  $b \in B$ . Since  $B/C$  is torsion, there exists  $n \in \mathbb{N}$  such that  $b^n \in C$ . We define  $g \in G$  by

$$g = (\psi, 0) \quad \text{with } \psi(r) = \Phi_0(b)^{\lfloor rn \rfloor} \quad \text{for } r \in \mathbb{Q} \cap [0, 1).$$

Let us write  $h = \Phi_0(b)$ , and consider the element  $[(1, 1/n), g] \cdot \Psi_n(b^n) = (\phi, 0)$ . If  $r \in [0, 1/n)$  then  $\phi(r) = \psi(r - 1/n)^{-1} \psi(r) h^n = h$ , while if  $r \in [1/n, 1)$  then  $\phi(r) = \psi(r - 1/n)^{-1} \psi(r) = h$ ; therefore

$$\Phi(b) = [(1, 1/n), g] \cdot \Psi_n(b^n) \in [G, G]. \quad \square$$

*Proof of Proposition 3.1.* The first assertion is simply Lemma 3.7.

Assume that  $B$  has locally subexponential growth, and consider a finite subset  $S$  of  $G$ . Then there exists a subgroup of  $G$  that contains  $S$  and is virtually a finite power of  $B$ , hence has subexponential growth. This shows that  $G$  has locally subexponential growth.

For the last assertion: if  $B$  is infinite, we wish to find a subgroup  $H$  of  $G$  with the same cardinality as  $B$ , such that  $\Phi$  maps into  $[H, H]$ . For each  $b \in B$ , choose a finite subset  $S_b$  of  $G$  such that  $\Phi(b) \in [\langle S_b \rangle, \langle S_b \rangle]$ , and a subgroup  $G_b$ , containing  $S_b$ , that is virtually a finite power of  $B$ . Consider the group  $H$  generated by the union of all the  $G_b$ . As soon as  $B$  is infinite, all  $G_b$  have the same cardinality as  $B$ , and so does  $H$ .  $\square$

#### 4. Orbits and inverted orbits

Let  $P = \langle S \rangle$  be a finitely generated group acting on the right on a set  $X$ . We consider  $X$  as the vertex set of a graph still denoted  $X$ , with for all  $x \in X, s \in S$  an edge labelled  $s$  from  $x$  to  $xs$ . We denote by  $d$  the path metric on this graph.

**Definition 4.1.** A sequence  $(x_0, x_1, \dots)$  in  $X$  is *spreading* if for all  $R$  there exists  $N$  such that if  $i, j \geq N$  and  $i \neq j$  then  $d(x_i, x_j) \geq R$ .

**Example 4.2.** If all  $x_i$  lie in order on a geodesic ray starting from  $x_0$  (for example if  $X$  itself is a ray starting from  $x_0$ ) and for all  $i$  we have  $d(x_0, x_{i+1}) \geq 2d(x_0, x_i)$ , then  $(x_i)$  is spreading.

**Lemma 4.3.** *Equivalently, a sequence  $(x_0, x_1, \dots)$  in  $X$  is spreading if and only if for all  $R$  there exists  $N$  such that if  $i \neq j$  and  $i \geq N$  then  $d(x_i, x_j) \geq R$ .*

*Proof.* Assume the converse, namely  $d(x_i, x_j) < R$  along a sequence with  $i \rightarrow \infty$  and  $j \not\rightarrow \infty$ . Then, up to passing to a subsequence,  $j$  may be assumed constant. There are then  $i, i' \rightarrow \infty$  with  $i \neq i'$  and  $d(x_i, x_{i'}) < 2R$ , so  $(x_i)$  is not spreading.  $\square$

**Definition 4.4.** A sequence  $(x_i)$  in  $X$  *locally stabilises* if for all  $R$  there exists  $N$  such that if  $i, j \geq N$  then the  $S$ -labelled radius- $R$  balls centered at  $x_i$  and  $x_j$  in  $X$  are equal.

**Definition 4.5.** A sequence of points  $(x_i)$  in  $X$  is *rectifiable* if for all  $i, j$  there exists  $g \in P$  with  $x_i g = x_j$  and  $x_k g \neq x_\ell$  for all  $k \notin \{i, \ell\}$ .

For example, if  $X = \mathbb{Z}$  and  $P = \mathbb{Z}$  acting by translations, then  $\Sigma = \{2^i : i \in \mathbb{N}\}$  is rectifiable, since  $2^j - 2^i = 2^\ell - 2^k$  only has trivial solutions  $i = k, j = \ell$  and  $i = j, k = \ell$ .

**Remark 4.6.** The sequence  $\Sigma = (x_i) \subseteq X$  is rectifiable if and only if for all  $i, j$  there exists  $g \in P$  with  $x_i g = x_j$  and  $\Sigma \cap \Sigma g \subseteq \{x_j\} \cup \text{fixed.points}(g)$ .

The following property is closer to Olshansky-Osin’s notion of parallelogram-free sequence, see Definition 2.3 in [13].

**Definition 4.7.** Fix a point  $z \in X$ . A sequence  $(g_i)$  in  $P$  is *parallelogram-free at  $z$*  if, for all  $i, j, k, \ell$  with  $i \neq j$  and  $j \neq k$  and  $k \neq \ell$  and  $\ell \neq i$  one has  $z g_i^{-1} g_j g_k^{-1} g_\ell \neq z$ .

**Lemma 4.8.** *If  $z \in X$  and  $(g_i)$  is parallelogram-free at  $z$ , then  $(z g_i^{-1})$  is a rectifiable sequence in  $X$ .*

*Proof.* Set  $x_i = z g_i^{-1}$  for all  $i \in \mathbb{N}$ . Given  $i, j \in \mathbb{N}$ , consider  $g = g_i g_j^{-1}$ , so  $x_i g = x_j$ . If furthermore we have  $x_k g = x_\ell$ , then we have  $z g_k^{-1} g_i g_j^{-1} g_\ell = z$ , so either  $k = i$ , or  $i = j$  which implies  $k = \ell$ , or  $j = \ell$  which implies  $k = i$ , or  $\ell = k$ . In all cases  $k \in \{i, \ell\}$  as was to be shown.  $\square$

It is clear that, if  $P$  is finitely generated and  $X$  is infinite, then it admits spreading and locally stabilizing sequences. Also, a subsequence of a spreading or locally stabilizing sequence is again spreading, respectively locally stabilizing. We give in the next section a general construction of rectifiable sequences, and in §4.2 a concrete example in the first Grigorchuk group.

**4.1. Separating actions.** Consider a group  $P$  acting on a set  $X$ . We recall that the *fixator* of the subset  $Y \subseteq X$  is the set  $\text{Fix}(Y) := \{g \in P : y g = y \text{ for all } y \in Y\}$ .

**Definition 4.9** (Abért [1]). The group  $P$  *separates*  $X$  if for every finite subset  $Y \subseteq X$  and every  $y_0 \notin Y$  there exists  $g \in \text{Fix}(Y)$  with  $y_0 g \neq y_0$ .

**Lemma 4.10.** *Let  $P$  be a group acting on a non-empty set  $X$  and separating it. Then there exists a rectifiable sequence  $(x_i)$  in  $X$ .*

*Proof.* We choose an arbitrary point  $z \in X$ , and construct iteratively a parallelogram-free sequence  $(g_i)$  at  $z$ ; by Lemma 4.8, this proves the lemma. Suppose that we have already constructed  $g_j$  for all  $j < i$ . For  $i \geq 0$ , we then construct  $g_i$  in the following way. We define

$$X_i^r := \{z g_{j_1} \cdots g_{j_s} : s \leq r, 0 \leq j_1, \dots, j_s < i\} \setminus \{z\} \quad \text{for } r = 1, 2, 3,$$

and choose an element  $g_i \in \text{Fix}(X_i^3)$  that moves  $z$ . In particular,  $z g_i^{-1} \notin X_i^3 \cup \{z\}$ .

Let us suppose that we have  $z g_k^{-1} g_i g_j^{-1} g_\ell = z$  with  $i \neq j \neq k \neq \ell \neq i$ , and seek a contradiction. If needed, we switch  $i \leftrightarrow j$  and  $k \leftrightarrow \ell$  and consider the equivalent equality  $z g_\ell^{-1} g_j g_i^{-1} g_k = z$ , to reduce to the case  $k < \ell$ .

We note, first,  $u := z g_k^{-1} \in X_\ell^1$ , because  $k < \ell$ . Next, we consider  $v := z g_k^{-1} g_i = u g_i$  and claim  $v \in X_\ell^2$ . By assumption  $i \neq k$  so  $v \neq z$ ; if  $i < \ell$  then  $v \in X_\ell^2$  by definition of  $X_\ell^2$ , while if  $i \geq \ell$  then  $i > k$  so  $u \in X_i^1$  and  $v = u g_i = u \in X_\ell^1 \subseteq X_\ell^2$ . Finally, we consider  $w = z g_k^{-1} g_i g_j^{-1} = v g_j^{-1}$ . By assumption  $j \neq \ell$ ; if  $j < \ell$  then  $w \in X_\ell^3 \cup \{z\}$  by definition of  $X_\ell^3$ , while if  $j > \ell$  then  $v \in X_j^2$  so  $w = v g_j^{-1} = v \in X_\ell^2 \subseteq X_\ell^3$ . We have in both cases reached the contradiction  $w = z g_\ell^{-1} \in X_\ell^3 \cup \{z\}$ .  $\square$

We quote from Proof of Corollary 1.4 in Abért [1] (see also Lemma 6.11 in [5]) that the action of weakly branch groups separates the boundary of their tree. Since the first Grigorchuk group is weakly branched (see Theorem 1 in [8] or Proposition 1.25 in [2]), it provides by Lemma 4.10 an example of a group action with rectifiable sequences. We also see it directly in the following section.

**4.2. An orbit for the first Grigorchuk group.** In this subsection, we consider the first Grigorchuk group  $G_{012} = \langle a, b, c, d \rangle$ . Recall that it acts on the set of infinite sequences  $\{\mathbf{0}, \mathbf{1}\}^\infty$  over a two-letter alphabet, which is naturally the boundary of a binary rooted tree; the action may be found in [6] and in §1.6.1 of [2]. We denote by  $X = \mathbf{1}^\infty G_{012}$  the orbit of the rightmost ray, and view it as a graph with vertex set  $X$  and for each  $x \in X$  and each generator  $s$  of  $G_{012}$  an edge labeled  $s$  from  $x$  to  $x s$ ; such graphs are called *Schreier graphs*. We construct explicitly a spreading, locally stabilizing, rectifiable sequence for the action of  $G_{012}$  on  $X$ : for all  $i \in \mathbb{N}$ , let us define

$$x_i = \mathbf{0}^i \mathbf{1}^\infty.$$

The geometric image of the Schreier graph  $X$  is that of a half-infinite line. The point  $x_i$  is at position  $2^i$  along this ray.

**Lemma 4.11.** *For all  $i, j \in \mathbb{N}$ ,*

- (1) *the marked balls of radius  $2^{\min(i,j)}$  in  $X$  around  $x_i$  and  $x_j$  coincide;*
- (2) *the distance  $d(x_i, x_j)$  is  $|2^i - 2^j|$ ;*
- (3) *there exists  $g_{i,j} \in G_{012}$  of length  $|2^i - 2^j|$  with  $x_i g_{i,j} = x_j$  and  $x_k g_{i,j} \neq x_\ell$  for all  $(k, \ell) \neq (i, j)$ .*

*Proof.* (1, 2) Consider the map  $\sigma: a \mapsto c, b \mapsto d^a, c \mapsto b^a, d \mapsto c^a$ . It defines a self-map of  $X$  by sending  $\mathbf{1}^\infty g$  to  $\mathbf{1}^\infty \sigma(g)$ . A direct calculation shows that it sends  $x \in X$  to  $\mathbf{0}x$ .

Since  $\sigma$  is 2-Lipschitz on words of even length in  $\{a, b, c, d\}$ , it maps the ball of radius  $n$  around  $x$  to the ball of radius  $2n$  around  $\mathbf{0}x$ . Its image is in fact a net in the ball of radius  $2n$ : two points at distance 1 in the ball of radius  $n$  around  $x$  will be mapped to points at distance 1 or 3 in the image, connected either by a path  $a$  or by a segment  $a - b - a, a - c - a$  or  $a - d - a$ . In particular, the  $2^n$ -neighbourhoods of the balls about the  $x_m$  coincide for all  $m \geq n$ .

(3) Note, first, that there exists  $g_{i,j}$  with  $x_i g_{i,j} = x_j$ , because the rays ending in  $\mathbf{1}^\infty$  form a single orbit. Note, also, that we have  $x_k g_{i,j} = x_\ell$  for either finitely many  $(k, \ell) \neq (i, j)$  or for all but finitely many  $(k, \ell)$ , because there is a level  $N$  at which the decomposition of  $g_{i,j}$  consists entirely of generators; if the entry at  $\mathbf{0}^N$  of  $g_{i,j}$  is trivial or ‘ $d$ ’ then all but finitely many of the  $x_k$  are fixed; while otherwise (up to increasing  $N$  by at most one) we may assume it is an ‘ $a$ ’; then  $\mathbf{0}^{N+1} g_{i,j} = \mathbf{0}^N \mathbf{1}$ , so  $x_k \neq x_\ell$  for all  $k > N + 1$ .

We use the following property of the Grigorchuk group: for every finite sequence  $u \in \{\mathbf{0}, \mathbf{1}\}^*$  there exists an element  $h_u \in G_{012}$  whose fixed points are precisely those sequences in  $\{\mathbf{0}, \mathbf{1}\}^\infty$  that do not start with  $u$ ; see Proposition 1.25 in [2].

If the entry at  $\mathbf{0}^N$  of  $g_{i,j}$  is trivial, then we multiply  $g_{i,j}$  with  $h_{\mathbf{0}^M}$  for some  $M > \max(N, i)$ , so as to fall back to the second case.

Then, for each pair  $(k, \ell) \neq (i, j)$  with  $x_k g_{i,j} = x_\ell$ , we multiply  $g_{i,j}$  with  $h_{\mathbf{0}^\ell \mathbf{1}}$ , so as to destroy the relation  $x_k g_{i,j} = x_\ell$ .

The resulting element  $g_{i,j}$  satisfies the required conditions. □

### 5. Subexponential growth of wreath products

In this section, we show how some permutational wreath products have subexponential growth.

**Definition 5.1.** The group  $P$  acting on  $X$  has the *subexponential wreathing property* if for any finitely generated group of subexponential growth  $H$  the restricted wreath product  $H \wr_X P$  has subexponential growth.

**Lemma 5.2.** *Let  $f$  be a positive sublinear function, namely  $f(n)/n \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $f$  is bounded from above by a concave sublinear function.*

*Proof.* For every  $\theta \in (0, 1)$ , let  $n_\theta$  be such that  $f(n) - \theta n$  is maximal. Given  $n \in \mathbb{R}$ , let  $\zeta < \theta$  be such that  $n \in [n_\theta, n_\zeta]$  with maximal  $\zeta$  and minimal  $\theta$ , and define  $\bar{f}(n)$  on  $[n_\theta, n_\zeta]$  by linear interpolation between  $(n_\theta, f(n_\theta))$  and  $(n_\zeta, f(n_\zeta))$ . Clearly  $\bar{f} \geq f$ , and  $\bar{f}(n)/n$  is decreasing and coincides infinitely often with  $f(n)/n$ , so it converges to 0. □

**Lemma 5.3.** *Let the Schreier graph of  $X$  have linear growth, and assume that  $P$  has sublinear inverted orbit growth on  $X$ . Assume also that  $P$  has subexponential growth. Then  $P$  has the subexponential wreathing property.*

*Proof.* We essentially follow Lemma 5.1 in [3].

Fix some  $x_0 \in X$  and let  $\rho(n)$  be the growth of inverted orbits starting from  $x_0$ . By assumption,  $\rho(n)/n \rightarrow 0$ , and there is a constant  $C$  such that the ball of radius  $n$  around  $x_0$  has cardinality  $\leq Cn$ .

Let  $H$  be a group of subexponential growth, and choose a finite generating set for  $H$ . By Lemma 5.2, there exists a log-concave subexponential function  $\bar{v}_H$  bounding the growth function  $v_H(n)$  of  $H$ .

We view  $H \wr_X P$  as generated by the generating set of  $P$  and the imbedding of the generating set of  $H$  as functions supported at  $\{x_0\}$ .

Consider an element  $(c, g) \in H \wr_X P$  of norm at most  $R$ . The function  $c: X \rightarrow H$  has support of cardinality  $k \leq \rho(R)$ , and this support is contained in the ball of radius  $R$  around  $x_0$ . Since the ball of radius  $R$  has cardinality at most  $CR$ , the number of possible choices for this support is at most  $\binom{CR}{\rho(R)}$ . Let  $\{z_1, \dots, z_k\}$  denote the support of  $c$ . The values of  $c$  belong to  $H$ , and their total norm is  $\leq R$ , so the number of choices for  $c$  is at most  $v_H(n_1) \cdots v_H(n_k)$  subject to the constraint  $n_1 + \dots + n_k \leq R$ . Since  $v_H(n_i) \leq \bar{v}_H(n_i)$  and  $\bar{v}_H(n_i)$  is log-concave, the number of choices for  $v_H$  is at most  $\bar{v}_H(R/k)^k$ . On the other hand, the number of choices for  $g$  is at most  $v_P(R)$ . All in all, the cardinality of the ball of radius  $R$  in  $H \wr_X P$  is bounded from above as

$$v_{H \wr_X P}(R) \leq v_P(R) \binom{CR}{\rho(R)} \bar{v}_H\left(\frac{R}{\rho(R)}\right)^{\rho(R)}.$$

Since it is a product of subexponential functions, it is itself subexponential. □

We now quote Proposition 4.4 in [3]: the inverted orbit growth of the first Grigorchuk group  $G_{012}$  on  $X = \mathbf{1}^\infty G_{012}$  is sublinear (actually of the form  $n^\alpha$  for some  $\alpha < 1$ ); therefore, by Lemma 5.3, the action of  $G_{012}$  on  $X$  has the subexponential wreathing property. (It follows from [4] that all Grigorchuk groups  $G_\omega$  also have the subexponential wreathing property, as soon as  $\omega \in \{0, 1, 2\}^\infty$  contains infinitely many copies of each symbol.)

### 6. The construction of $W$

Using the results of the previous section, we select a finitely generated group  $P$  acting on a set  $X$ , and a rectifiable, spreading, locally stabilizing sequence  $(x_i)$  of elements of  $X$ .

Let  $(b_1, b_2, \dots)$  be a sequence in  $B$ . We will specify later a rapidly increasing sequence  $0 \leq n(1) < n(2) < \dots$ ; assuming this sequence given, we define  $f: X \rightarrow B$  by

$$f(x_{n(1)}) = b_1, \quad f(x_{n(2)}) = b_2, \quad \dots, \quad f(x) = 1, \quad \text{for other } x.$$

We then consider the subgroup  $W = \langle P, f \rangle$  of the unrestricted wreath product  $B^X \rtimes P$ .

**Lemma 6.1.** *Denote by  $B_0$  the subgroup of  $B$  generated by  $\{b_1, b_2, \dots\}$ . If the sequence  $(x_i)$  is rectifiable, then  $[W, W]$  contains  $[B_0, B_0]$  as a subgroup.*

*Proof.* Without loss of generality and to lighten notation, we rename  $B_0$  into  $B$ . We also denote by  $\iota: B \rightarrow B^X \rtimes P$  the imbedding of  $B$  mapping the element  $b \in B$  to the function  $X \rightarrow B$  with value  $b$  at  $x_0$  and 1 elsewhere. We shall show that  $[W, W]$  contains  $\iota([B, B])$ . For this, denote by  $H$  the subgroup  $\iota([B, B]) \cap [W, W]$ .

We first consider an elementary commutator  $g = [b_i, b_j]$ . Let  $g_i, g_j \in P$  respectively map  $x_i, x_j$  to  $x_0$ , and be such that  $g_i g_j^{-1}$  maps no  $x_k$  to  $x_\ell$  with  $k \neq \ell$ , except for  $x_i g_i g_j^{-1} = x_j$ . Consider  $[f^{g_i}, f^{g_j}] \in [W, W]$ ; it belongs to  $B^X$ , and has value  $[b_i, b_j]$  at  $x_0$  and is trivial elsewhere, so equals  $\iota(g)$  and therefore  $\iota(g) \in H$ .

We next show that  $H$  is normal in  $B^X$ . For this, consider  $h \in H$ . It suffices to show that  $h^{(b_i)}$  belongs to  $H$  for all  $i$ . Now  $h^{(b_i)} = h^{f^{g_i}}$  belongs to  $H$ , and we are done. □

**Proposition 6.2.** *Let  $P$  be a finitely generated group acting on  $X$ . Let the sequence  $(x_i)$  in  $X$  be spreading and locally stabilizing. Let a sequence of elements  $(b_i)$  be given in the group  $B$ , all of the same order  $\in \mathbb{N} \cup \{\infty\}$ .*

*Then for every increasing sequence  $(m(i))$  there is a choice of increasing sequence  $(n(i))$  with the following property.*

*For all  $i \in \mathbb{N}$ , let  $f_i$  be the finitely supported function  $X \rightarrow B$  with  $f_i(x_{n(j)}) = b_j$  for all  $j \leq i$ , all other values being trivial, and denote by  $W_i$  the group  $\langle P, f_i \rangle$ . Let also  $f: X \rightarrow B$  be defined by  $f(x_{n(j)}) = b_j$  for all  $j \in \mathbb{N}$ , all other values being trivial, and write  $W = \langle P, f \rangle$ .*

*Then the ball of radius  $m(i)$  in  $W$  coincides with the ball of radius  $m(i)$  in  $W_i$ , via the identification  $f \leftrightarrow f_i$ .*

*Furthermore, the term  $n(i)$  depends only on the previous terms  $n(1), \dots, n(i-1)$ , on the initial terms  $m(1), \dots, m(i-1)$ , and on the ball of radius  $m(i)$  in the subgroup  $\langle b_1, \dots, b_{i-1} \rangle$  of  $B$ .*

*Proof.* Choose  $n(i)$  such that  $d(x_j, x_k) \geq m(i)$  for all  $j \neq k$  with  $k \geq n(i)$ , and such that the balls of radius  $m(i)$  around  $x_{n(i)}$  and  $x_j$  coincide for all  $j > n(i)$ .

Consider then an element  $h \in W$  in the ball of radius  $m(i)$ , and write it in the form  $h = (c, g)$  with  $c: X \rightarrow B$  and  $g \in P$ . The function  $c$  is a product of conjugates of  $f$  by words of length  $< m(i)$ . Its support is therefore contained in the union of balls of radius  $m(i) - 1$  around the  $x_j$ , with  $j$  either  $\geq n(i)$  or of the form  $n(k)$  for  $k < i$ . In particular, the entries of  $c$  are in  $\langle b_1, \dots, b_{i-1} \rangle \cup \bigcup_{j \geq i} \langle b_j \rangle$ . For  $j > n(i)$ , the restriction of  $c$  to the ball around  $x_j$  is determined by the restriction of  $c$  to the ball around  $x_{n(i)}$ , via the identification  $b_i \mapsto b_j$ , because the neighbourhoods in  $X$  coincide and all cyclic groups  $\langle b_j \rangle$  are isomorphic.

It follows that the element  $h \in W$  is uniquely determined by the corresponding element in  $W_i$ .  $\square$

**Corollary 6.3.** *Let  $P$  be a group acting on  $X$  with the subexponential wreathing property, and let  $(x_i)$  be a spreading and locally stabilizing sequence in  $X$ . Let  $B$  be a group and let  $(b_i)$  be a sequence in  $B$ .*

*If  $B$  has locally subexponential growth, then there exists a sequence  $(n(i))$  such that the group  $W$  has subexponential growth.*

*Proof.* Let  $Z = \langle z \rangle$  be a cyclic group whose order (possibly  $\infty$ ) is divisible by the order of the  $b_i$ 's. We replace  $B$  by  $B \times Z$  and each  $b_i$  by  $b_i z$ , so as to guarantee that all generators in  $B$  have the same order.

Let  $\epsilon_i$  be a decreasing sequence tending to 1. We construct a sequence  $m(i)$  inductively, and obtain the sequence  $n(i)$  by Proposition 6.2, making always sure that  $m(i)$  depends only on  $m(j), n(j)$  for  $j < i$ .

Denote by  $v_i$  the growth function of the group  $W_i$  introduced in Proposition 6.2. Since the group  $W_i$  is contained in  $B \wr_X P$  and  $P$  has the subexponential wreathing property, it has subexponential growth. Therefore, there exists  $m(i)$  be such that

$$v_i(m(i)) \leq \epsilon_i^{m(i)}.$$

By Proposition 6.2, the terms  $n(i+1), n(i+2), \dots$  can be chosen in such a manner that the balls of radius  $m(i)$  coincide in  $W$  and  $W_i$ .

Denote now by  $w$  the growth function of  $W$ . We then have  $w(m(i)) \leq \epsilon_i^{m(i)}$ . Therefore,

$$w(R) \leq \epsilon_i^{R+m(i)} \text{ for all } R > m(i),$$

so  $\limsup_{R \rightarrow \infty} \sqrt[R]{w(R)} \leq \epsilon_i$ . Since this holds for all  $i$ , the growth of  $W$  is subexponential.  $\square$

*Proof of Theorem A.* By Proposition 3.1, the countable, locally subexponentially growing group  $B$  imbeds in  $[G, G]$  for a countable, locally subexponentially growing group  $G$ . Let  $(b_1, b_2, \dots)$  be a generating set for  $G$ . By Lemma 6.1, the derived subgroup  $[G, G]$  imbeds in  $[W, W]$ , and by Corollary 6.3, the finitely generated group  $W$  has subexponential growth.  $\square$

**Remark 6.4.** If the sequence  $(x_i)$  is only spreading, or only stabilizing, then it may happen that  $W$  have exponential growth, even if the sequence  $(n(i))$  grows arbitrarily fast.

*Proof.* We first consider an example where the sequence  $(x_i)$  is spreading but not stabilizing. Consider  $P = G_{012}$  acting on  $X = \mathbf{1}^\infty P$ , and let  $Q$  denote the stabilizer of  $\mathbf{1}^\infty$  so that  $X = Q \backslash P$ . Since the action is faithful, we have  $\bigcap_{g \in P} Q^g = 1$ , and in fact  $\bigcap_{g \in T} Q^g = 1$  for a sequence  $T$  in  $P$  such that  $(\mathbf{1}^\infty t : t \in T)$  is spreading. Take  $B = \langle z \rangle \cong \mathbb{Z}$  and define  $f : X \rightarrow B$  by  $f(\mathbf{1}^\infty t) = z$  for all  $t \in T$ , all other values being 1. Then  $\langle P, f \rangle \cong \mathbb{Z} \wr P$  has exponential growth.

We next consider an example where the sequence  $(x_i)$  is stabilizing but not spreading. Again, consider  $P = G_{012}$  acting on  $X$ , and consider a spreading, stabilizing sequence  $(x_{2i})$  in  $X$ . Set  $x_{2i+1} = x_{2i}a$ . Consider  $B = G_{012}$ , and note that, since  $P$  does not satisfy any law, there are sequences  $(g_0, h_0), (g_1, h_1), \dots$  of pairs of elements of  $P$  such that the groups  $\langle g_i, h_i \rangle$  converge to a free group of rank 2 in the Cayley topology. Set then  $f(x_{2i}) = g_i$  and  $f(x_{2i+1}) = h_i$ , and note that  $\langle P, f \rangle$  contains the free group  $\langle f, f^a \rangle$ .  $\square$

**Remark 6.5.** If  $B$  has locally subexponential growth, then it may be imbedded in a 2-generator group of subexponential growth.

*Proof.* We make the following general claim about finitely generated groups: if  $W$  is finitely generated, then there exists a 2-generated hyper- $W$  group in which  $[W, W]$  imbeds. Since the imbedding given by Theorem A is actually into  $[W, W]$ , this is sufficient to prove the remark.

Let us now turn to the claim, and consider a group  $W$  generated by a set  $S = \{s_1, \dots, s_n\}$ . Let  $C = \langle t \mid t^{2^n} \rangle$  be a cyclic group, and consider the subgroup

$$\bar{W} = \langle x, t \rangle \leq W \wr C,$$

with  $x : C \rightarrow W$  defined by  $x(t^{2^{i-1}}) = s_i$  for all  $i \in \{1, \dots, n\}$ , all other values being trivial. The imbedding of  $[W, W]$  into  $\bar{W}$  is as functions  $x : C \rightarrow W$  whose support is contained in  $\{t\}$ . Indeed, given  $w \in [W, W]$ , write it as a balanced word (i.e. with exponent sum zero in each variable)  $w$  over  $S$ , and replace each  $s_i$  by  $\bar{s}_i := x^{t^{1-2^{i-1}}}$ , yielding  $\bar{w} \in [\bar{W}, \bar{W}]$ . The functions  $\bar{s}_i : C \rightarrow W$  all have disjoint supports, except at  $t$  where their respective values are  $s_i$ . Therefore,  $\bar{w}$  is supported only at  $t$  and has value  $w$  there.  $\square$

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