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Groups, Geometry, and Dynamics

Geometry of locally compact groups of polynomial growth and shape of large balls

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Dedicated to Pierre de la Harpe on the occasion of his 70^{th} birthday

Abstract. We generalize Pansu's thesis [27] about asymptotic cones of finitely generated nilpotent groups to arbitrary locally compact groups G of polynomial growth. We show that any such G is weakly commensurable to some simply connected solvable Lie group S, the Lie shadow of G and that balls in any reasonable left invariant metric on G admit a well-defined asymptotic shape. By-products include a formula for the asymptotics of the volume of large balls and an application to ergodic theory, namely that the Ergodic Theorem holds for all ball averages. Along the way we also answer negatively a question of Burago and Margulis [7] on asymptotic word metrics and give a geometric proof of some results of Stoll [33] of the rationality of growth series of Heisenberg groups.

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Contents

1. Introduction

1.1. Groups with polynomial growth. Let G be a locally compact group with left Haar measure vol_G. We will assume that G is generated by a compact symmetric subset Ω. Classically, G is said to have *polynomial growth* if there exist $C > 0$ and $k > 0$ such that for any integer $n \geq 1$

$$\mathrm{vol}_G(\Omega^n) \leq C \cdot n^k,$$

where $\Omega^n = \Omega \ldots \Omega$ is the n-fold product set. Another choice for Ω would only change the constant C, but not the polynomial nature of the bound. One of the consequences of the analysis carried out in this paper is the following theorem.

Theorem 1.1 (Volume asymptotics). *Let G be a locally compact group with polynomial growth and Ω a compact symmetric generating subset of G. Then there exists $c(\Omega) > 0$ and an integer $d(G) \geq 0$ depending on G only such that the following holds:*

$$\lim_{n \to +\infty} \frac{\mathrm{vol}_G(\Omega^n)}{n^{d(G)}} = c(\Omega)$$

This extends the main result of Pansu [27]. The integer $d(G)$ coincides with the exponent of growth of a naturally associated graded nilpotent Lie group, the asymptotic cone of G, and is given by the Bass–Guivarc'h formula (4) below. The constant $c(\Omega)$ will be interpreted as the volume of the unit ball of a sub-Riemannian Finsler metric on this nilpotent Lie group. Theorem 1.1 is a by-product of our study of the asymptotic behavior of *periodic pseudodistances* on G, that is pseudodistances that are invariant under a co-compact subgroup of G and satisfy a weak kind of the existence of geodesics axiom (see Definition 4.1).

Our first task is to get a better understanding of the structure of locally compact groups of polynomial growth. Guivarc'h [21] proved that locally compact groups of polynomial growth are amenable and unimodular and that every compactly generated[1] closed subgroup also has polynomial growth.

Guivarc'h [21] and Jenkins [15] also characterized connected Lie groups with polynomial growth: a connected Lie group S has polynomial growth if and only if it is of type (R), that is if for all $x \in \mathrm{Lie}(S)$, $\mathrm{ad}(x)$ has only purely imaginary eigenvalues. Such groups are solvable-by-compact and every connected nilpotent Lie group is of type (R).

It is much more difficult to characterize discrete groups with polynomial growth, and this was done in a celebrated paper of Gromov [17], proving that they are virtually nilpotent. Losert [24] generalized Gromov's method of proof and showed that it applied with little modification to arbitrary locally compact groups with polynomial growth. In particular he showed that they contain a normal compact subgroup modulo

[1] In fact it follows from the Gromov–Losert structure theory that every closed subgroup is compactly generated.

which the quotient is a (not necessarily connected) Lie group. We will prove the following refinement.

Theorem 1.2 (Lie shadow). *Let G be a locally compact group of polynomial growth. Then there exists a connected and simply connected solvable Lie group S of type (R), which is weakly commensurable to G. We call such a Lie group a Lie shadow of G.*

Two locally compact groups are said to be *weakly commensurable* if, up to moding out by a compact kernel, they have a common closed co-compact subgroup. More precisely, we will show that, for some normal compact subgroup K, G/K has a co-compact subgroup H/K which can be embedded as a closed and co-compact subgroup of a connected and simply connected solvable Lie group S of type (R).

We must be aware that being weakly commensurable is not an equivalence relation among locally compact groups (unlike among finitely generated groups). Additionally, the Lie shadow S is not unique up to isomorphism (e.g. $\mathbb{Z}^3$ is a co-compact lattice in both $\mathbb{R}^3$ and the universal cover of the group of motions of the plane).

We cannot replace the word solvable by the word nilpotent in the above theorem. We refer the reader to Example 7.9 for an example of a connected solvable Lie group of type (R) without compact normal subgroups, which admits no co-compact nilpotent subgroup. In fact this is typical for Lie groups of type (R). So in the general locally compact case (or just the Lie case) groups of polynomial growth can be genuinely not nilpotent, unlike what happens in the discrete case. There are important differences between the discrete case and the general case. For example, we will show that no rate of convergence can be expected in Theorem 1.1 when G is solvable not nilpotent, while some polynomial rate always holds in the nilpotent discrete case [9].

Theorem 1.2 will enable us to reduce most geometric questions about locally compact groups of polynomial growth, and in particular the proof of Theorem 1.1, to the connected Lie group case. Observe also that Theorem 1.2 subsumes Gromov's theorem on polynomial growth, because it is not hard to see that a co-compact lattice in a solvable Lie group of polynomial growth must be virtually nilpotent (see Remark 7.8). Of course in the proof we make use of Gromov's theorem, in its generalized form for locally compact groups due to Losert. The rest of the proof combines ideas of Y. Guivarc'h and D. Mostow and a crucial embedding theorem of H. C. Wang. It is given in §7.1 and is largely independent of the rest of the paper.

1.2. Asymptotic shapes. The main part of the paper is devoted to the asymptotic behavior of *periodic pseudodistances* (also called *periodic metric*) on G. We refer the reader to Definition 4.1 for the precise definition of this term, suffices it to say now that it is a class of pseudodistances which contains both left-invariant word metrics on G and geodesic metrics on G that are left-invariant under a co-compact subgroup of G.

Theorem 1.2 enables us to assume that G is a co-compact subgroup of a simply connected solvable Lie group S, and rather than looking at pseudodistances on G, we will look at pseudodistances on S that are left-invariant under a co-compact subgroup H. More precisely a direct consequence of Theorem 1.2 is the following proposition.

Proposition 1.3. *Let G be a locally compact group with polynomial growth and ρ a periodic metric on G. Then (G,ρ) is $(1,C)$-quasi-isometric to (S,ρ_S) for some finite $C>0$, where S is a connected and simply connected solvable Lie group of type (R) and ρ_S some periodic metric on S.*

Recall that two metric spaces (X,d_X) and (Y,d_Y) are called $(1,C)$-quasi-isometric if there exists a map $\varphi: X \to Y$ such that any $y \in Y$ is at distance at most C from some element in the image of φ and if $|d_Y(\varphi(x),\varphi(x')) - d_X(x,x')| \leq C$ for all $x,x' \in X$.

In the case when S is $\mathbb{R}^d$ and H is $\mathbb{Z}^d$, it is a simple exercise to show that any periodic pseudodistance is asymptotic to a norm on $\mathbb{R}^d$, i.e. $\rho(e,x)/\|x\| \to 1$ as $x \to \infty$, where $\|x\| = \lim \frac{1}{n}\rho(e,nx)$ is a well defined norm on $\mathbb{R}^d$. Burago in [6] showed a much finer result, namely that if ρ is coarsely geodesic, then $\rho(e,x) - \|x\|$ is bounded when x ranges over $\mathbb{R}^d$. When S is a nilpotent Lie group and H a lattice in S, then Pansu proved in his thesis [27], that a similar result holds, namely that $\rho(e,x)/|x| \to 1$ for some (unique only after a choice of a one-parameter group of dilations) homogeneous quasi-norm $|x|$ on the nilpotent Lie group. However, we show in Section 8, that it is not true in general that $\rho(e,x) - |x|$ stays bounded, even for finitely generated nilpotent groups, thus answering a question of Burago (see also Gromov [20]). Our main purpose here will be to extend Pansu's result to solvable Lie groups of polynomial growth.

As was first noticed by Guivarc'h in his thesis [21], when dealing with geometric properties of solvable Lie groups, it is useful to consider the so-called nilshadow of the group, a construction first introduced by Auslander and Green in [2]. According to this construction, it is possible to modify the Lie product on S in a natural way, by so to speak removing the semisimple part of the action on the nilradical, in order to turn S into a nilpotent Lie group, its nilshadow S_N. The two Lie groups have the same underlying manifold, which is diffeomorphic to $\mathbb{R}^n$, only a different Lie product. They also share the same Haar measure. This "semisimple part" is a commutative relatively compact subgroup $T(S)$ of automorphisms of S, image of S under a homomorphism $T: S \to \mathrm{Aut}(S)$. The new product $g * h$ is defined as follows by twisting the old one $g \cdot h$ by means of $T(S)$,

$$g * h := g \cdot T(g^{-1})h. \tag{1}$$

The two groups S and S_N are easily seen to be quasi-isometric, and this is why any locally compact group of polynomial growth G is quasi-isometric to some nilpotent Lie group. In particular, their asymptotic cones are bi-Lipschitz. The asymptotic cone of a nilpotent Lie group is a certain associated graded nilpotent Lie group endowed with a left invariant geodesic distance (or Carnot group). The graded group associated to S_N will be called the *graded nilshadow* of S. Section 3 will be devoted to the construction and basic properties of the nilshadow and its graded group.

In this paper, we are dealing with a finer relation than quasi-isometry. We will be interested in when do two left invariant (or periodic) distances are asymptotic[2] (in the sense that $\frac{d_1(e,g)}{d_2(e,g)} \to 1$ when $g \to \infty$). In particular, for every locally compact group G with polynomial growth, we will identify its asymptotic cone up to isometry and not only up to quasi-isometry or bi-Lipschitz equivalence (see Corollary 1.9 below). One of our main results is the following theorem.

Theorem 1.4 (Main theorem). *Let S be a simply connected solvable Lie group with polynomial growth. Let $\rho(x, y)$ be a periodic pseudodistance on S which is invariant under a co-compact subgroup H of S (see Definition 4.1). On the manifold S, one can put a new Lie group structure, which turns S into a stratified nilpotent Lie group, the graded nilshadow of S, and a subFinsler metric $d_\infty(x, y)$ on S which is left-invariant for this new group structure such that*

$$\frac{\rho(e,g)}{d_\infty(e,g)} \longrightarrow 1$$

as $g \to \infty$ in S. Moreover every automorphism in $T(H)$ is an isometry of d_∞.

The reader who wishes to see a simple illustration of this theorem can go directly to subsection 8.1, where we have treated in detail a specific example of periodic metric on the universal cover of the groups of motions of the plane.

The new stratified nilpotent Lie group structure on S given by the graded nilshadow comes with a one-parameter family of so-called *homogeneous dilations* $\{\delta_t\}_{t>0}$. It also comes with an extra group of automorphisms, namely the image of H under the homomorphism T. This yields automorphisms of S for both the original group structure on S and the new graded nilshadow group structure. Moreover the dilations $\{\delta_t\}_{t>0}$ are automorphisms of the graded nilshadow and they commute with $T(H)$.

A subFinsler metric is a geodesic distance which is defined exactly as subRiemannian (or Carnot–Caratheodory) metrics on Carnot groups are defined (see e.g. [25]), except that the norm used to compute the length of horizontal paths is not necessarily a Euclidean norm. We refer the reader to Section 2.1 for a precise definition.

In Theorem 1.4 the subFinsler metric d_∞ is left invariant for the new Lie structure on S and it is also invariant under all automorphisms in $T(H)$ (these form a relatively compact commutative group of automorphisms). Moreover it satisfies the following pleasing scaling law:

$$d_\infty(\delta_t(x), \delta_t(y)) = t d_\infty(x, y), \quad t > 0.$$

[2] Yet a finer equivalence relation is $(1, C)$-quasi-isometry, i.e. being at bounded distance in Gromov–Hausdorff metric; classifying periodic metrics up to this kind of equivalence is much harder.

The proof of Theorem 1.4 splits in two important steps. The first is a reduction to the nilpotent case and is performed in Section 5. Using a double averaging of the pseudodistance ρ over both $K := \overline{T(H)}$ and S/H, we construct an associated pseudodistance, which is periodic for the nilshadow structure on S (i.e. left-invariant by a co-compact subgroup for this structure), and we prove that it is asymptotic to the original ρ. This reduces the problem to nilpotent Lie groups. The key to this reduction is the following crucial observation: that unipotent automorphisms of S induce only a sublinear distortion, forcing the metric ρ to be asymptotically invariant under $T(H)$. The second step of the proof assumes that S is nilpotent. This part is dealt with in Section 6 and is essentially a reformulation of the arguments used by Pansu in [27].

Incidently, we stress the fact that the generality in which Section 6 is treated (i.e. for general coarsely geodesic, and even asymptotically geodesic periodic metrics) is necessary to prove even the most basic case (i.e. word metrics) of Theorem 1.4 for non-nilpotent solvable groups. So even if we were only interested in the asymptotics of left invariant word metrics on a solvable Lie group of polynomial growth S, we would still need to understand the asymptotics of arbitrary coarsely geodesic left invariant distances (and not only word metrics!) on nilpotent Lie groups. This is because the new pseudodistance obtained by averaging, see (30), is no longer a word metric.

The subFinsler metric $d_\infty(e,x)$ in the above theorem is induced by a certain $T(H)$-invariant norm on the first stratum m_1 of the graded nilshadow (which is a $T(H)$-invariant complementary subspace of the commutator subalgebra of the nilshadow). This norm can be described rather explicitly as follows.

Recall that we have[3] a canonical map $\pi_1\colon S \to m_1$, which is a group homomorphism for both the nilshadow and graded nilshadow structures. Then

$$\{v \in m_1, \|v\|_\infty \leq 1\} = \bigcap_{F\subset S} \overline{\text{CvxHull}}\Big\{\frac{\pi_1(h)}{\rho(e,h)}, h \in H\backslash F\Big\},$$

where the right hand side is the intersection over all compact subsets F of S of the closed convex hull of the points $\pi_1(h)/\rho(e,h)$ for $h \in H\backslash F$.

Figure 1.2 gives an illustration of the limit shape corresponding to the word metric on the 3-dimensional discrete Heisenberg group with standard generators. We explain in the Appendix how one can compute explicitly the geodesics of the limit metric and the limit shape in this example.

[3] The subspace m_1 can be identified with the abelianized nilshadow (or abelianized graded nilshadow) by first identifying the nilshadow with its Lie algebra via the exponential map and then projecting modulo the commutator subalgebra. The map does not depend on the choice involved in the construction of the nilshadow. See also Remark 3.7.

When S itself is nilpotent to begin with and ρ is (in restriction to H) the word metric associated to a symmetric compact generating set Ω of H (namely $\rho_\Omega(e,h) := \inf\{n \in \mathbb{N}; h \in \Omega^n\}$), the above norm takes the simple form

$$\{v \in m_1, \|v\|_\infty \leq 1\} = \mathrm{CvxHull}\{\pi_1(\omega), \omega \in \Omega\}. \tag{2}$$

For instance, in the special case when H is a torsion-free finitely generated nilpotent group with generating set Ω and S is its Malcev closure, the unit ball $\{v \in m_1, \|v\|_\infty \leq 1\}$ is a polyhedron in m_1. This was Pansu's description in [27].

However when S is not nilpotent, and is equipped with a word metric ρ_Ω on a co-compact subgroup, then the determination of the limit shape, i.e. the determination of the limit norm $\|\cdot\|_\infty$ on the abelianized nilshadow, is much more difficult. Clearly $\|\cdot\|_\infty$ is K-invariant and it is a simple observation that the unit ball for $\|\cdot\|_\infty$ is always contained in the convex hull of the K-orbit of $\pi_1(\Omega)$. Nevertheless the unit ball is typically smaller than that (unless Ω was K-invariant to begin with).

In general it would be interesting to determine whether there exists a simple description of the limit shape of an arbitrary word metric on a solvable Lie group with polynomial growth. We refer the reader to Section 8 and §8.2 for an example of a class of word metrics on the universal cover of the group of motions of the plane, for which we were able to compute the limit shape.

Another by-product of Theorem 1.4 is the following result.

Corollary 1.5 (Asymptotic shape). *Let S be a simply connected solvable Lie group with polynomial growth and H a co-compact subgroup. Let ρ be an H-periodic pseudodistance on S. Then in the Hausdorff metric,*

$$\lim_{t\to+\infty} \delta_{\frac{1}{t}}(B_\rho(t)) = \mathcal{C},$$

where $\mathcal{C}$ is a $T(H)$-invariant compact neighborhood of the identity in S, $B_\rho(t)$ is the ρ-ball of radius t in S and $\{\delta_t\}_{t>0}$ is a one-parameter group of dilations on S (equipped with the graded nilshadow structure). Moreover,

$$\mathcal{C} = \{g \in S, d_\infty(e,g) \leq 1\}$$

is the unit ball of the limit subFinsler metric from Theorem 1.4.

Proof. By Theorem 1.4, for every $\varepsilon > 0$ we have $B_{d_\infty}(t-\varepsilon t) \subset B_\rho(t) \subset B_{d_\infty}(t+\varepsilon t)$ if t is large enough. Since $\delta_{\frac{1}{t}}(B_{d_\infty}(t)) = \mathcal{C}$, for all $t > 0$, we are done. □

Combining this with Theorem 1.2, we also get the following corollary, of which Theorem 1.1 is only a special case with ρ the word metric associated to the generating set Ω.

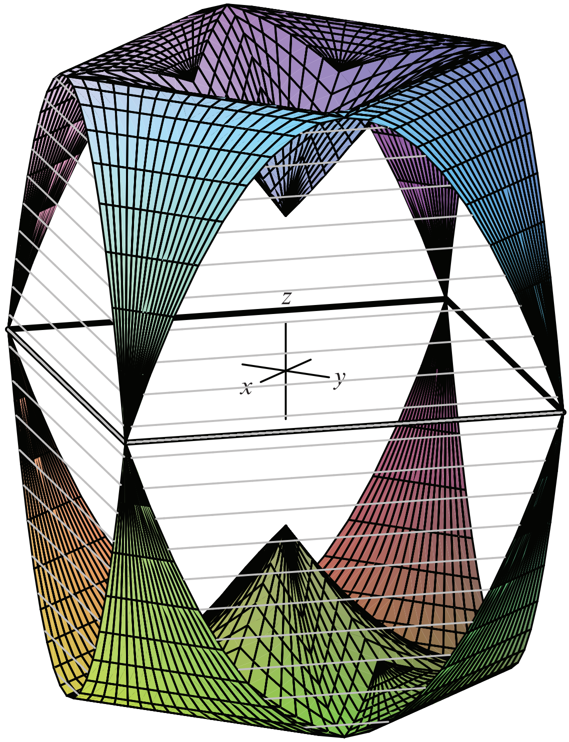

Figure 1. The asymptotic shape of large balls in the Cayley graph of the Heisenberg group $H(\mathbb{Z}) = \langle x, y \mid [x,[x,y]] = [y,[x,y]] = 1\rangle$ viewed in exponential coordinates.

Corollary 1.6 (Volume asymptotics). *Suppose that G is a locally compact group with polynomial growth and ρ is a periodic pseudodistance on G. Let $B_\rho(t)$ be the ρ-ball of radius t in G, i.e. $B_\rho(t) = \{x \in G, \rho(e,x) \leq t\}$, then there exists a constant $c(\rho) > 0$ such that the limit*

$$\lim_{t\to+\infty} \frac{\mathrm{vol}_G(B_\rho(t))}{t^{d(G)}} = c(\rho) \tag{3}$$

exists.

Here $d(G)$ is the integer $d(S_N)$, the so-called homogeneous dimension of the nilshadow S_N of a Lie shadow S of G (obtained by Theorem 1.2), and is given by the Bass–Guivarc'h formula

$$d(S_N) = \sum_{k\geq 0} \dim(C^k(S_N)), \tag{4}$$

where $\{C^k(S_N)\}_k$ is the descending central series of S_N.

The limit $c(\rho)$ is equal to the volume $\mathrm{vol}_S(\mathcal{C})$ of the limit shape $\mathcal{C}$ from Corollary 1.5 once we make the right choice of Haar measure on a Lie shadow S of G. Let us explain this choice. Recall that according to Theorem 1.2, G/K admits a co-compact subgroup H/K which embeds co-compactly in S. Starting with a Haar measure vol_G on G, we get a Haar measure on G/K after fixing the Haar measure of K to be of total mass 1, and we may then choose a Haar measure on H/K so that the compact quotient G/H has volume 1. Finally we choose the Haar measure on S so that the other compact quotient $S/(H/K)$ has volume 1. This gives the desired Haar measure vol_S such that $c(\rho) = \mathrm{vol}_S(\mathcal{C})$.

Note that Haar measure on S is also invariant under the group of automorphisms $T(S)$ and is thus left invariant for the nilshadow structure on S. It is also left invariant for the graded nilshadow structure. In both exponential coordinates of the first kind (on S_N) and of the second kind (as in Lemma 3.10), Haar measure is just Lebesgue measure.

In the case of the discrete Heisenberg group of dimension 3 equipped with the word metric given by the standard generators, it is possible to compute the constant $c(\rho)$ and the volume of the limit shape as shown in Figure 1.2. In this case the volume is $\frac{31}{72}$ (see the Appendix). The 5-dimensional Heisenberg group can also be worked out and the volume of its limit shape (associated to the word metric given by standard generators) is equal to $\frac{2009}{21870} + \frac{\log 2}{32805}$. The fact that this number is transcendental implies that the growth series of this group, i.e. the formal power series $\sum_{n \geq 0} |B_\rho(n)| z^n$ is not algebraic in the sense that it is not a solution of a polynomial equation with rational functions in $\mathbb{C}(z)$ as coefficients (see [33], Proposition 3.3.). This was observed by Stoll in [33] by more direct combinatorial means. Stoll also shows there the interesting fact that the growth series can be rational for some other choices of generating sets in the 5-dimensional Heisenberg group. So rationality of the growth series depends on the generating set.

Another interesting feature is asymptotic invariance.

Corollary 1.7 (Asymptotic invariance). *Let S be a simply connected solvable Lie group with polynomial growth and ρ a periodic pseudodistance on S. Let $*$ be the new Lie product on S given by the nilshadow group structure* (*or the graded nilshadow group structure*)*. Then $\rho(e, g * x)/\rho(e, x) \to 1$ as $x \to \infty$ for every $g \in S$.*

This follows immediately from Theorem 1.4, when $*$ is the graded nilshadow product, and from Theorem 6.2 below in the case $*$ is the nilshadow group structure.

It is worth observing that we may not in general replace $*$ by the ordinary product on S. Indeed, let for instance $S = \mathbb{R} \ltimes \mathbb{R}^2$ be the universal cover of the group of motions of the Euclidean plane, then S, like its nilshadow $\mathbb{R}^3$, admits a lattice $\Gamma \simeq \mathbb{Z}^3$. The quotient S/Γ is diffeomorphic to the 3-torus $\mathbb{R}^3/\mathbb{Z}^3$ and it is easy to find Riemannian metrics on this torus so that their lift to $\mathbb{R}^3$ is not invariant under rotation around the z-axis. Hence this metric, viewed on the Lie group S will not be asymptotically invariant under left translation by elements of S. Nevertheless, if the metric is left-invariant and not just periodic, then we have the following corollary of the proof of Theorem 1.4.

Corollary 1.8 (Left-invariant pseudodistances are asymptotic to subFinsler metrics). *Let S be a simply connected solvable Lie group of polynomial growth and ρ be a periodic pseudodistance on S which is invariant under all left-translations by elements of S (e.g. a left-invariant coarsely geodesic metric on S). Then there is a left-invariant subFinsler metric d on S which is asymptotic to ρ in the sense that $\frac{\rho(e,g)}{d(e,g)} \to 1$ as $g \to \infty$.*

We already mentioned above that determining the exact limit shape of a word metric on S is a difficult task. Consequently so is the task of telling when two distinct word metrics are asymptotic. The above statement says that in any case every word metric on S is asymptotic to some left-invariant subFinsler metric. So the set of possible limit shapes is no richer for word metrics than for left-invariant subFinsler metrics.

We note that in the case of nilpotent Lie groups (where K is trivial), Theorem 1.4 shows that every periodic metric is asymptotic to a left-invariant metric. It is still an open problem to determine whether every coarsely geodesic periodic metric is at a bounded distance from a left-invariant metric (this is Burago's theorem in $\mathbb{R}^n$, more about it below).

Theorems 1.2 and 1.4 allow us to describe the asymptotic cone of (G, ρ) for any periodic pseudodistance ρ on any locally compact group with polynomial growth.

Corollary 1.9 (Asymptotic cone). *Let G be a locally compact group with polynomial growth and ρ a periodic pseudodistance on G. Then the sequence of pointed metric spaces $\{(G, \frac{1}{n}\rho, e)\}_{n\geq 1}$ converges in the Gromov–Hausdorff topology. The limit is the metric space (N, d_∞, e), where N is a graded simply connected nilpotent Lie group and d_∞ a left invariant subFinsler metric on N. Moreover the Lie group N is (up to isomorphism) independent of ρ. The space (N, d_∞) is isometric to "the asymptotic cone" associated to (G, ρ). This asymptotic cone is independent of the choice of ultrafilter used to define it.*

This corollary is a generalization of Pansu's theorem (eq. (10) in [27]). We refer the reader to the book [18] for the definitions of the asymptotic cone and the Gromov–Hausdorff convergence. We discuss in Section 8 the speed of convergence (in the Gromov–Hausdorff metric) in this theorem and its corollaries about volume growth. In particular there is a major difference between the discrete nilpotent case and the solvable non nilpotent case. In the former, one can find a polynomial rate of convergence [9], while in the latter no such rate exist in general (see Theorem 8.1).

1.3. Folner sets and ergodic theory. A consequence of Corollary 1.6 is that sequences of balls with radius going to infinity are Folner sequences; namely, we have the following corollary.

Corollary 1.10. *Let G be a locally compact group with polynomial growth and ρ a periodic pseudodistance on G. Let $B_\rho(t)$ be the ρ-ball of radius t in G. Then $\{B_\rho(t)\}_{t>0}$ form a Folner family of subsets of G namely, for any compact set F in G, we have (Δ denotes the symmetric difference)*

$$\lim_{t\to+\infty} \frac{\mathrm{vol}_G(FB_\rho(t)\Delta B_\rho(t))}{\mathrm{vol}_G(B_\rho(t))} = 0. \tag{5}$$

Proof. Indeed $FB_\rho(t)\Delta B_\rho(t) \subset B_\rho(t+c)\backslash B_\rho(t)$ for some $c >$ depending on F. Hence (5) follows from (3). □

This settles the so-called "localization problem" of Greenleaf for locally compact groups of polynomial growth (see [16]), i.e. determining whether the powers of a compact generating set $\{\Omega^n\}_n$ form a Folner sequence. At the same time it implies that the ergodic theorem for G-actions holds along any sequence of balls with radius going to infinity.

Theorem 1.11 (Ergodic theorem). *Let be given a locally compact group G with polynomial growth together with a measurable G-space X endowed with a G-invariant ergodic probability measure m. Let ρ be a periodic pseudodistance on G and $B_\rho(t)$ the ρ-ball of radius t in G. Then for any p, $1 \le p < \infty$, and any function $f \in \mathbb{L}^p(X,m)$ we have*

$$\lim_{t\to+\infty} \frac{1}{\mathrm{vol}_G(B_\rho(t))} \int_{B_\rho(t)} f(gx)dg = \int_X f dm$$

for m-almost every $x \in X$ and also in $\mathbb{L}^p(X,m)$.

In fact, Corollary 1.10 above, was the "missing block" in the proof of the ergodic theorem on groups of polynomial growth. So far and to my knowledge, Corollary 1.10 and Theorem 1.11 were known only along some subsequence of balls $\{B_\rho(t_n)\}_n$ chosen so that (5) holds (see for instance [10] or [34]). This issue was drawn to my attention by A. Nevo and was my initial motivation for the present work. We refer the reader to the A. Nevo's survey paper [26], Section 5.

It later turned out that the mere fact that balls are Folner in a given polynomial growth locally compact group can also be derived from the fact these groups are doubling metric spaces (which is an easier result than the precise asymptotics $\mathrm{vol}(\Omega^n) \sim c_\Omega n^{d(G)}$ proved in this paper and only requires lower and upper bounds of the form $c_1 n^{d(G)} \le \mathrm{vol}(\Omega^n) \le c_2 n^{d(G)}$). This was observed by R. Tessera [35] who rediscovered a cute argument of Colding and Minicozzi [11], Lemma 3.3, showing that the volume of spheres $\Omega^{n+1} \setminus \Omega^n$ is at most some $O(n^{-\delta})$ times the volume of the ball Ω^n, where $\delta > 0$ is a positive constant depending only on the doubling constant the word metric induced by Ω in G.

In [9], we give a better upper bound (which depends only on the nilpotency class and not on the doubling constant) for the volume of spheres in the case of finitely generated nilpotent groups. This is done by showing the following error term in the asymptotics of the volume of balls: we have $\mathrm{vol}(\Omega^n) = c_\Omega n^{d(G)} + O(n^{d(G)-\alpha_r})$, where $\alpha_r > 0$ depends only on the nilpotency class r of G. We refer the reader to Section 8 and to the preprint [9] for more information on this.

Finally we note that, while an error term for the volume of balls yields immediately an upper bound on the volume of spheres, the converse is not true. An example is given in §8.1 of a Lie group of polynomial growth for which the error term in the asymptotics for the volume of balls tends to zero arbitrarily slowly. However the above Colding–Minicozzi–Tessera upper bound on the volume of spheres holds very generally for any locally compact group with polynomial growth.

1.4. A conjecture of Burago and Margulis. In [7], D. Burago and G. Margulis conjectured that any two word metrics on a finitely generated group which are asymptotic (in the sense that $\frac{\rho_1(e,\gamma)}{\rho_2(e,\gamma)}$ tends to 1 at infinity) must be at a bounded distance from one another (in the sense that $|\rho_1(e,\gamma) - \rho_2(e,\gamma)| = O(1)$). This holds for abelian groups. An analogous result was proved by Abels and Margulis for word metrics on reductive groups [1]. S. Krat [23] established this property for word metrics on the Heisenberg group $H_3(\mathbb{Z})$. However using Theorem 1.4 (which in this particular case of finitely generated nilpotent groups is just Pansu's theorem [27]) we will show in Section 8.3, that there are counter-examples and exhibit two word metrics on $H_3(\mathbb{Z}) \times \mathbb{Z}$ which are asymptotic and yet are not at a bounded distance. For more on this counter-example, and how to adequately modify the conjecture of Burago and Margulis, we refer the interested reader to [9].

1.5. Organization of the paper. Sections 2–4 are devoted to preliminaries. In Section 2 we present the basic nilpotent theory as can be found in Guivarc'h's thesis [21]. In particular, a full proof of the Bass–Guivarc'h formula is given. In Section 3, we recall the construction of the nilshadow of a solvable Lie group. In Section 4 we set up the axioms and basic properties of the (pseudo)distance functions that are studied in this paper.

Sections 5–7 contain the core of the proof of the main theorems. In Section 5, we assume that G is a simply connected solvable Lie group and reduce the problem to the nilpotent case. In Section 6, we assume that G is a simply connected nilpotent Lie group and prove Theorem 1.4 in this case following the strategy used by Pansu in [27]. In Section 7, we prove Theorem 1.2 for general locally compact groups and reduce the proof of the results of the introduction to the Lie case.

In the last section we make further comments about the speed of convergence. In particular we give examples answering negatively the aforementioned question of Burago and Margulis.

The appendix is devoted to the discrete Heisenberg groups of dimension 3 and 5. We compute their limit balls, explain Figure 1.2, and recover the main result of Stoll [33].

The reader who is mainly interested in the nilpotent group case can read directly Section 6 while keeping an eye on Sections 2 and 4 for background notations and elementary facts.

Finally, let us mention that the results and methods of this paper were largely inspired by the works of Y. Guivarc'h [21] and P. Pansu [27].

Nota Bene. A version of this article circulated since 2007. The present version contains essentially the same material; only the exposition has been improved and several somewhat sketchy arguments have been replaced by full fledged proofs (in particular in Sections 3 and 7). This delay is due to the fact that the author was planning for a long time to improve Section 6 and show an error term in the volume asymptotics of balls in nilpotent groups. E. Le Donne and the author recently managed to achieve this and it has now become an independent joint paper [9].

2. Quasi-norms and the geometry of nilpotent Lie groups

In this section, we review the necessary background material on nilpotent Lie groups. In §2.4, we give some crucial properties of homogeneous quasi norms and reproduce some lemmata originally due to Y. Guivarc'h which will be used in the sequel. Meanwhile, we prove the Bass–Guivarc'h formula for the degree of polynomial growth of nilpotent Lie groups, following Guivarc'h's original argument.

2.1. Carnot–Caratheodory metrics. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let m_1 be a vector subspace of $\mathfrak{g}$. We denote by $\|\cdot\|$ a norm on m_1.

We now recall the definition of a left-invariant *Carnot–Carathéodory metric* also called *subFinsler metric* on G. Let $x, y \in G$. We consider all possible piecewise smooth paths $\xi\colon [0,1] \to G$ going from $\xi(0) = x$ to $\xi(1) = y$. Let $\xi'(u)$ be the tangent vector which is pulled back to the identity by a left translation, i.e.

$$\frac{d\xi}{du} = \xi(u) \cdot \xi'(u), \tag{6}$$

where $\xi'(u) \in \mathfrak{g}$ and the notation $\xi(u) \cdot \xi'(u)$ means the image of $\xi'(u)$ under the differential at the identity of the left translation by the group element $\xi(u)$. We say that the path ξ is *horizontal* if the vector $\xi'(u)$ belongs to m_1 for all $u \in [0,1]$. We denote by $\mathcal{H}$ the set of piecewise smooth horizontal paths. The Carnot–Carathéodory metric associated to the norm $\|\cdot\|$ is defined by

$$d(x,y)=\inf\left\{\int_0^1\|\xi'(u)\|du,\xi\in\mathscr{H},\ \xi(0)=x,\xi(1)=y\right\},$$

where the infimum is taken over all piecewise smooth paths $\xi\colon[0,1]\to N$ with $\xi(0)=x,\xi(1)=y$ that are horizontal in the sense that $\xi'(u)\in m_1$ for all u. If $\|\cdot\|$ is a Euclidean norm, the metric $d(x,y)$ is also called *subRiemannian*. In this paper however the norm $\|\cdot\|$ will typically not be Euclidean (it can be polyhedral like in the case of word metrics on finitely generated nilpotent groups) and $d(x,y)$ will only be *subFinsler*. If $m_1=\mathfrak{g}$, and $\|\cdot\|$ is a Euclidean (resp. arbitrary) norm on $\mathfrak{g}$, then d is simply the usual left-invariant Riemannian (resp. Finsler) metric associated to $\|\cdot\|$.

Chow's theorem (see, e.g. [19] or [25]) tells us that $d(x,y)$ is finite for all x and y in G if and only if the vector subspace m_1, together with all brackets of elements of m_1, generates the full Lie algebra $\mathfrak{g}$. If this condition is satisfied, then d is a distance on G which induces the original topology of G.

In this paper, we will only be concerned with Carnot–Caratheodory metrics on a simply connected nilpotent Lie group N. In the sequel, whenever we speak of a Carnot–Carathéodory metric on N, we mean one that is associated to a norm $\|\cdot\|$ on a subspace m_1 such that $\mathfrak{n}=m_1\oplus[\mathfrak{n},\mathfrak{n}]$ where $\mathfrak{n}=\mathrm{Lie}(N)$. It is easy to check that any such m_1 generates the Lie algebra $\mathfrak{n}$.

Remark 2.1. Let us observe here that for such a metric d on N, for all $x\in N$,

$$\|\pi_1(x)\|\leq d(e,x),$$

where π_1 is the linear projection map from $\mathfrak{n}$ (identified with N via exp) to m_1 with kernel $[\mathfrak{n},\mathfrak{n}]$. Indeed, π_1 gives rise to a homomorphism from N to the vector space m_1. And if $\xi(u)$ is a horizontal path from e to x, then applying π_1 to (6) we get $\frac{d}{du}\pi_1(\xi(u))=\xi'(u)$, hence $\pi_1(x)=\int_0^1\xi'(u)du$. Hence $\|\pi_1(x)\|\leq d(e,x)$. Moreover if $x\in m_1$, there is equality in the above upper bound, namely $||x||=d(e,x)$, because then $\{tx\}_{t\in[0,1]}$ is a horizontal path of length $\|x\|$ connecting the identity to x, hence a geodesic. Consequently we see that the image of the unit ball centered at the identity in N under the projection π_1 coincides with the unit ball for $\|\cdot\|$ in m_1 and that

$$\{v\in m_1,\|v\|\leq 1\}=\mathrm{CvxHull}\left\{\frac{\pi_1(x)}{d(e,x)},x\in N\setminus\{e\}\right\}.$$

2.2. Dilations on a nilpotent Lie group and the associated graded group. We now focus on the case of simply connected nilpotent Lie groups. Let N be such a group with Lie algebra $\mathfrak{n}$ and nilpotency class r. For background about analysis on such groups, we refer the reader to the book [12]. The exponential map is a diffeomorphism between $\mathfrak{n}$ and N. Most of the time, if $x\in\mathfrak{n}$, we will abuse notation and denote the group element $\exp(x)$ simply by x. We denote by $\{C^p(\mathfrak{n})\}_p$ the central descending series for $\mathfrak{n}$, i.e. $C^{p+1}(\mathfrak{n}) = [\mathfrak{n}, C^p(\mathfrak{n})]$ with $C^0(\mathfrak{n}) = \mathfrak{n}$ and $C^r(\mathfrak{n}) = \{0\}$.

Let $(m_p)_{p\geq 1}$ be a collection of vector subspaces of $\mathfrak{n}$ such that for each $p \geq 1$,

$$C^{p-1}(\mathfrak{n}) = C^p(\mathfrak{n}) \oplus m_p. \tag{7}$$

Then $\mathfrak{n} = \bigoplus_{p\geq 1} m_p$ and in this decomposition, any element x in $\mathfrak{n}$ (or N by abuse of notation) will be written in the form

$$x = \sum_{p\geq 1} \pi_p(x)$$

where $\pi_p(x)$ is the linear projection onto m_p.

To such a decomposition is associated a one-parameter group of dilations $(\delta_t)_{t>0}$. These are the linear endomorphisms of $\mathfrak{n}$ defined by

$$\delta_t(x) = t^p x$$

for any $x \in m_p$ and for every p. Conversely, the one-parameter group $(\delta_t)_{t\geq 0}$ determines the $(m_p)_{p\geq 1}$'s since they appear as eigenspaces of each δ_t, $t \neq 1$. The dilations δ_t do not preserve *a priori* the Lie bracket on $\mathfrak{n}$. This is the case if and only if

$$[m_p, m_q] \subseteq m_{p+q} \tag{8}$$

for every p and q (where $[m_p, m_q]$ is the subspace spanned by all commutators of elements of m_p with elements of m_q). If (8) holds, we say that the $(m_p)_{p\geq 1}$ form a *stratification* of the Lie algebra $\mathfrak{n}$, and that $\mathfrak{n}$ is a *stratified* (or homogeneous) Lie algebra. It is an exercise to check that (8) is equivalent to require $[m_1, m_p] = m_{p+1}$ for all p.

If (8) does not hold, we can however consider a new Lie algebra structure on the real vector space $\mathfrak{n}$ by defining the new Lie bracket as $[x, y]_\infty = \pi_{p+q}([x, y])$ if $x \in m_p$ and $y \in m_q$. This new Lie algebra $\mathfrak{n}_\infty$ is stratified and has the same underlying vector space as $\mathfrak{n}$. We denote by N_∞ the associated simply connected Lie group. Moreover the $(\delta_t)_{t>0}$ form a one-parameter group of automorphisms of $\mathfrak{n}_\infty$. In fact the original Lie bracket $[x, y]$ on $\mathfrak{n}$ can be deformed continuously to $[x, y]_\infty$ through a continuous family of Lie algebra structures by setting

$$[x, y]_t = \delta_{\frac{1}{t}}([\delta_t x, \delta_t y]) \tag{9}$$

and letting $t \to +\infty$. Note that conversely, if the δ_t's are automorphisms of $\mathfrak{n}$, then $[x, y] = \pi_{p+q}([x, y])$ for all $x \in m_p$ and $y \in m_q$, and $\mathfrak{n} = \mathfrak{n}_\infty$.

The graded Lie algebra associated to $\mathfrak{n}$ is by definition

$$\mathrm{gr}(\mathfrak{n}) = \bigoplus_{p\geq 0} C^p(\mathfrak{n})/C^{p+1}(\mathfrak{n})$$

endowed with the Lie bracket induced from that of $\mathfrak{n}$. The quotient map

$$m_p \longrightarrow C^p(\mathfrak{n})/C^{p+1}(\mathfrak{n})$$

gives rise to a linear isomorphism between $\mathfrak{n}$ and $\mathrm{gr}(\mathfrak{n})$, which is a Lie algebra isomorphism between the new Lie algebra structure $\mathfrak{n}_\infty$ and $\mathrm{gr}(\mathfrak{n})$. Hence stratified Lie algebra structures induced by a choice of supplementary subspaces $(m_p)_{p\geq 1}$ as in (7) are all isomorphic to $\mathrm{gr}(\mathfrak{n})$.

On N_∞ the left-invariant subFinsler metrics d_∞ associated to a choice of norm on m_1 are of special interest. The one-parameter group of dilations $\{\delta_t\}_t$ is an automorphism of N_∞ and that

$$d_\infty(\delta_t x, \delta_t y) = t d_\infty(x, y) \tag{10}$$

for any $x, y \in N_\infty$. The metric space (N_∞, d_∞) is called a *Carnot group*.

If on the other hand the simply connected nilpotent Lie group N is not stratified, then the group of dilations $(\delta_t)_t$ associated to a choice of supplementary vector subspaces m_i's as in (7) will not consist of automorphisms of N and the relation (10) will not hold.

Note also that if we are given two different choices of supplementary subspaces m_i's and m_i''s as in (7), then the left-invariant Carnot–Caratheodory metrics on the corresponding stratified Lie groups are isometric if and only if $(m_1, \|\cdot\|)$ and $(m_1', \|\cdot\|')$ are isometric (a linear isomorphism from m_1 to m_1' that sends $\|\cdot\|$ to $\|\cdot\|'$ extends to an isometry of the two Carnot groups).

2.3. The Campbell–Hausdorff formula. The exponential map $\exp\colon \mathfrak{n} \to N$ is a diffeomorphism. In the sequel, we will often abuse notation and identify N and $\mathfrak{n}$ without further notice. In particular, for two elements x and y of $\mathfrak{n}$ (or N equivalently) xy will denote their product in N, while $x + y$ denotes the sum in $\mathfrak{n}$. Let $(\delta_t)_t$ be a one-parameter group of dilations associated to a choice of supplementary subspaces m_i's as in (7). We denote the corresponding stratified Lie algebra by $\mathfrak{n}_\infty$ as above and the Lie group by N_∞. The product on N_∞ is denoted by $x * y$. On N_∞ the dilations $(\delta_t)_t$ are automorphisms.

The Campbell–Hausdorff formula (see [12]) allows to give a more precise form of the product in N. Let $(e_i)_{1\leq i\leq d}$ be a basis of $\mathfrak{n}$ adapted to the decomposition into m_i's, that is $m_i = \mathrm{span}\{e_j, e_j \in m_i\}$. Let $x = x_1e_1 + \cdots + x_d e_d$ the corresponding decomposition of an element $x \in \mathfrak{n}$. Then define the degree $d_i = \deg(e_i)$ to be the largest j such that $e_i \in C^{j-1}(\mathfrak{n})$. If $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ is a multi-index, then let $d_\alpha = \deg(e_1)\alpha_1 + \cdots + \deg(e_d)\alpha_d$.

The Campbell–Hausdorff formula yields

$$(xy)_i = x_i + y_i + \sum C_{\alpha,\beta} x^\alpha y^\beta, \tag{11}$$

where $C_{\alpha,\beta}$ are real constants and the sum is over all multi-indices α and β such that $d_\alpha + d_\beta \le \deg(e_i)$, $d_\alpha \ge 1$ and $d_\beta \ge 1$.

From (9), it is easy to give the form of the associated stratified Lie group law

$$(x * y)_i = x_i + y_i + \sum C_{\alpha,\beta} x^\alpha y^\beta, \tag{12}$$

where the sum is restricted to those α's and β's such that $d_\alpha + d_\beta = \deg(e_i)$, $d_\alpha \ge 1$ and $d_\beta \ge 1$.

2.4. Homogeneous quasi-norms and Guivarc'h's theorem on polynomial growth. Let $\mathfrak{n}$ be a finite dimensional real nilpotent Lie algebra and consider a decomposition

$$\mathfrak{n} = m_1 \oplus \cdots \oplus m_r$$

by supplementary vector subspaces as in (7). Let $(\delta_t)_{t>0}$ be the one parameter group of dilations associated to this decomposition, that is $\delta_t(x) = t^i x$ if $x \in m_i$. We now introduce the following definition.

Definition 2.2 (Homogeneous quasi-norm). A continuous function $|\cdot|\colon \mathfrak{n} \to \mathbb{R}_+$ is called a homogeneous quasi-norm associated to the dilations $(\delta_t)_t$, if it satisfies the properties

(i) $|x| = 0 \iff x = 0$.

(ii) $|\delta_t(x)| = t|x|$ for all $t > 0$.

Example 2.3. (1) Quasi-norms of supremum type, i.e. $|x| = \max_p \|\pi_p(x)\|_p^{1/p}$ where $\|\cdot\|_p$ are ordinary norms on the vector space m_p and π_p is the projection on m_p as above.

(2) $|x| = d_\infty(e,x)$, where d_∞ is a Carnot–Carathéodory metric on a stratified nilpotent Lie group (as the relation (10) shows).

Clearly, a quasi-norm is determined by its sphere of radius 1 and two quasi-norms (which are homogeneous with respect to the same group of dilations) are always equivalent in the sense that

$$\frac{1}{c}\,|\cdot|_1 \le |\cdot|_2 \le c\,|\cdot|_1 \tag{13}$$

for some constant $c > 0$ (indeed, by continuity, $|\cdot|_2$ admits a maximum on the "sphere" $\{|x|_1 = 1\}$). If the two quasi-norms are homogeneous with respect to two distinct semi-groups of dilations, then the inequalities (13) continue to hold outside a neighborhood of 0, but may fail near 0.

Homogeneous quasi-norms satisfy the following properties.

Proposition 2.4. *Let $|\cdot|$ be a homogeneous quasi-norm on $\mathfrak{n}$, then there are constants $C, C_1, C_2 > 0$ such that*

(*a*) $|x_i| \leq C \cdot |x|^{\deg(e_i)}$ *if* $x = x_1e_1 + \cdots + x_ne_n$ *in an adapted basis* $(e_i)_i$*;*

(*b*) (b) $|x^{-1}| \leq C \cdot |x|$*;*

(*c*) (c) $|x + y| \leq C \cdot (|x| + |y|)$*;*

(*d*) (d) $|xy| \leq C_1(|x| + |y|) + C_2$.

Properties (a), (b) and (c) are straightforward because $|x| = \max_p \|\pi_p(x)\|_p^{1/p}$ is a homogeneous quasi-norm and because of (13). Property (d) justifies the term "quasi-norm" and follows from Lemma 2.5 below. It can be a problem that the constant C_1 in (d) may not be equal to 1. In fact, this is why we use the word quasi-norm instead of just norm, because we do not require the triangle inequality axiom to hold. However the following lemma of Guivarc'h is often a good enough remedy to this situation. Let $\|\cdot\|_p$ be an arbitrary norm on the vector space m_p.

Lemma 2.5 (Guivarc'h [21], Lemme II.1). *Let $\varepsilon > 0$. Up to rescaling each $\|\cdot\|_p$ into a proportional norm $\lambda_p\|\cdot\|_p$ ($\lambda_p > 0$) if necessary, the quasi-norm $|x| = \max_p \|\pi_p(x)\|_p^{1/p}$ satisfies*

$$|xy| \leq |x| + |y| + \varepsilon \tag{14}$$

for all $x, y \in N$. If N is stratified with respect to $(\delta_t)_t$ we can take $\varepsilon = 0$.

This lemma is crucial also for computing the coarse asymptotics of volume growth. For the reader's convenience, we reproduce here Guivarc'h's argument, which is based on the Campbell–Hausdorff formula (11).

Proof. We fix $\lambda_1 = 1$ and we are going to give a condition on the λ_i's so that (14) holds. The λ_i's will be taken to be smaller and smaller as i increases. We set $|x| = \max_p \|\pi_p(x)\|_p^{1/p}$ and let $|x|_\lambda = \max_p \|\lambda_p\pi_p(x)\|_p^{1/p}$ for any r-tuple of λ_i's. We want that for any index $p \leq r$,

$$\lambda_p\|\pi_p(xy)\|_p \leq (|x|_\lambda + |y|_\lambda + \varepsilon)^p. \tag{15}$$

By (11) we have $\pi_p(xy) = \pi_p(x) + \pi_p(y) + P_p(x, y)$ where P_p is a polynomial map into m_p depending only on the $\pi_i(x)$ and $\pi_i(y)$ with $i \leq p-1$ such that

$$\|P_p(x, y)\|_p \leq C_p \cdot \sum_{l,m\geq 1, l+m\leq p} M_{p-1}(x)^l M_{p-1}(y)^m,$$

where $M_k(x) := \max_{i\leq k} \|\pi_i(x)\|_i^{1/i}$ and $C_p > 0$ is a constant depending on P_p and on the norms $\|\cdot\|_i$'s. Since $\varepsilon > 0$, when expanding the right hand side of (15) all terms of the form $|x|_\lambda^l |y|_\lambda^m$ with $l + m \leq p$ appear with some positive coefficient, say $\varepsilon_{l,m}$. The terms $|x|_\lambda^p$ and $|y|_\lambda^p$ appear with coefficient 1 and cause no trouble since we always have $\lambda_p\|\pi_p(x)\|_p \leq |x|_\lambda^p$ and $\lambda_p\|\pi_p(y)\|_p \leq |y|_\lambda^p$. Therefore, for (15) to hold, it is sufficient that

$$\lambda_p C_p M_{p-1}(x)^l M_{p-1}(y)^m \leq \varepsilon_{l,m}|x|_\lambda^l |y|_\lambda^m$$

for all remaining l and m. However, clearly $M_k(x) \leq \Lambda_k \cdot |x|_\lambda$ where $\Lambda_k := \max_{i\leq k}\{1/\lambda_i^{1/i}\} \geq 1$. Hence a sufficient condition for (15) to hold is

$$\lambda_p \leq \frac{\bar{\varepsilon}}{C_p \Lambda_{p-1}^p},$$

where $\bar{\varepsilon} = \min \varepsilon_{l,m}$. Since Λ_{p-1} depends only on the first $p-1$ values of the λ_i's, it is obvious that such a set of conditions can be fulfilled by a suitable r-tuple λ. □

Remark 2.6. The constant C_2 in Property (d) above can be taken to be 0 when N is stratified with respect to the m_i's (i.e. the δ_t's are automorphisms), as is easily seen after changing x and y into their image under δ_t. And conversely, if $C_2 = 0$ for some δ_t-homogeneous quasi-norm on N, then N admits a stratification. Indeed, from (11) and (12), we see that if the δ_t's are not automorphisms, then one can find $x, y \in N$ such that, when t is small enough, $|\delta_t(xy) - \delta_t(x)\delta_t(y)| \geq ct^{(r-1)/r}$ for some $c > 0$. However, combining Properties (c) and Property (d) with $C_2 = 0$ above we must have $|\delta_t(xy) - \delta_t(x)\delta_t(y)| = O(t)$ near $t = 0$. A contradiction.

Guivarc'h's lemma enables us to show the following theorem.

Theorem 2.7 (Guivarc'h ibid.). *Let Ω be a compact neighborhood of the identity in a simply connected nilpotent Lie group N and $\rho_\Omega(x, y) = \inf\{n \geq 1, x^{-1}y \in \Omega^n\}$. Then for any homogeneous quasi-norm $|\cdot|$ on N, there is a constant $C > 0$ such that*

$$\frac{1}{C}|x| \leq \rho_\Omega(e, x) \leq C|x| + C. \tag{16}$$

Proof. Since any two homogeneous quasi-norms (with respect to the same one-parameter group of dilations) are equivalent, it is enough to do the proof for one of them, so we consider the quasi-norm obtained in Lemma 2.5 with the extra property (14). The lower bound in (16) is a direct consequence of (14) and one can take there C to be $\max\{|x|, x \in \Omega\} + \varepsilon$. For the upper bound, it suffices to show that there is $C \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $|x| \leq n$ then $x \in \Omega^{Cn}$. To achieve this, we proceed by induction of the nilpotency length of N. The result is clear when N is abelian. Otherwise, by induction we obtain $C_0 \in \mathbb{N}$ such that $x = \omega_1 \cdot \cdots \cdot \omega_{C_0 n} \cdot z$ where $\omega_i \in \Omega$ and $z \in C^{r-1}(N)$ whenever $|x| \leq n$. Hence $|z| \leq |x| + C_0 n \cdot \max|\omega_i^{-1}| + \varepsilon C_0 \cdot n \leq C_1 n$ for some other constant $C_1 \in \mathbb{N}$. So we have reduced the problem to $x = z \in m_r = C^{r-1}(N)$ which is central in N. We have $z = z_1^{n^r}$ where $|z_1| = |z|/n \leq C_1$. Since Ω is a neighborhood of the identity in N, the set $\mathcal{U}$ of all products of at most $\dim(m_r)$ simple commutators of length r of elements in Ω is a neighborhood of the identity in $C^{r-1}(N)$ (see e.g. [19], p. 113). It follows that there is a constant $C_2 \in \mathbb{N}$ such that z_1 is in $\mathcal{U}^{C_2}$, hence the product of at most $C_2 \dim(m_r)$ simple commutators. Then we are done because z itself will be equal to the same product of commutators where each letter $x_i \in \Omega$ is replaced by x_i^n. This last fact follows from the following lemma.

Lemma 2.8. *Let G be a nilpotent group of nilpotency class r and $n_1, \ldots, n_r$ be positive integers. Then for any $x_1, \ldots, x_r \in G$*

$$[x_1^{n_1}, [x_2^{n_2}, [\ldots, x_r^{n_r}] \ldots] = [x_1, [x_2, [\ldots, x_r] \ldots]^{n_1 \cdot \cdots \cdot n_r}.$$

To prove the lemma it suffices to use induction and the following obvious fact: if $[x, y]$ commutes to x and y then $[x^n, y] = [x, y]^n$. □

Finally, we obtain the following corollary.

Corollary 2.9. *Let Ω be a compact neighborhood of the identity in N. Then there are positive constants C_1 and C_2 such that for all $n \in \mathbb{N}$,*

$$C_1 n^d \leq \mathrm{vol}_N(\Omega^n) \leq C_2 n^d,$$

where d is given by the Bass–Guivarc'h formula

$$d = \sum_{i \geq 1} i \cdot \dim m_i. \tag{17}$$

Proof. By Theorem 2.7, it is enough to estimate the volume of the quasi-norm balls. By homogeneity of the quasi-norm, we have

$$\mathrm{vol}_N\{x, |x| \leq t\} = t^d \mathrm{vol}_N\{x, |x| \leq 1\}.$$ □

Remark 2.10. The use of Malcev's embedding theorem allows, as Guivarc'h observed, to deduce immediately that the analogous result holds for virtually nilpotent finitely generated groups. This fact that was also proven independently by H. Bass [3] by a direct combinatorial argument. See also Tits' appendix to Gromov's paper [17]. In fact Guivarc'h's Theorem 2.7 seems to have been rediscovered several times in the past 40 years, including by Pansu in his thesis [27], the latest example of that being [22].

3. The nilshadow

The goal of this section is to introduce the nilshadow of a simply connected solvable Lie group G. We will assume that G has polynomial growth, although this last assumption is not necessary for almost everything we do in this section. The only statement which will be used afterwards in the paper (in Section 5) is Lemma 3.12 below. The reader familiar with the nilshadow can jump directly to the statement of this lemma and skip the forthcoming discussion.

3.1. Construction of the nilshadow. The nilshadow of G is a simply connected nilpotent Lie group G_N, which is associated to G in a natural way. This notion was first introduced by Auslander and Green in [2] in their study of flows on solvmanifolds. They defined it as the unipotent radical of a *semi-simple splitting* of G. However, we are going to follow a different approach for its construction by working first at the Lie algebra level. We refer the reader to the book [13] where this approach is taken up.

Let $\mathfrak{g}$ be a solvable real Lie algebra and $\mathfrak{n}$ the nilradical of $\mathfrak{g}$. We have $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{n}$. If $x \in \mathfrak{g}$, we write $\mathrm{ad}(x) = \mathrm{ad}_s(x) + \mathrm{ad}_n(x)$ the Jordan decomposition of $\mathrm{ad}(x)$ in $\mathrm{GL}(\mathfrak{g})$. Since $\mathrm{ad}(x) \in \mathrm{Der}(\mathfrak{g})$, the space of derivations of $\mathfrak{g}$, and $\mathrm{Der}(\mathfrak{g})$ is the Lie algebra of the *algebraic* group $\mathrm{Aut}(\mathfrak{g})$, the Jordan components $\mathrm{ad}_s(x)$ and $\mathrm{ad}_n(x)$ also belong to $\mathrm{Der}(\mathfrak{g})$. Moreover, for each $x \in \mathfrak{g}$, $\mathrm{ad}_s(x)$ sends $\mathfrak{g}$ into $\mathfrak{n}$ (because so does $\mathrm{ad}(x)$ and $\mathrm{ad}_s(x)$ is a polynomial in $\mathrm{ad}(x)$). Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, namely a nilpotent self-normalizing subalgebra. Recall that the image of a Cartan subalgebra by a surjective Lie algebra homomorphism is again a Cartan subalgebra. Now since $\mathfrak{g}/\mathfrak{n}$ is abelian, it follows that $\mathfrak{h}$ maps onto $\mathfrak{g}/\mathfrak{n}$, i.e. $\mathfrak{h} + \mathfrak{n} = \mathfrak{g}$. Moreover $\mathrm{ad}_s(x)_{|\mathfrak{h}} = 0$ if $x \in \mathfrak{h}$, because $\mathfrak{h}$ is nilpotent.

Now pick any real vector subspace $\mathfrak{v}$ of $\mathfrak{h}$ in direct sum with $\mathfrak{n}$. Then the following two conditions hold:

(i) $\mathfrak{v} \oplus \mathfrak{n} = \mathfrak{g}$;

(ii) $\mathrm{ad}_s(x)(y) = 0$, for all $x, y \in \mathfrak{v}$.

From (i) and (ii), it follows easily that $\mathrm{ad}_s(x)$ commutes with $\mathrm{ad}(y)$, $\mathrm{ad}_s(y)$ and $\mathrm{ad}_n(y)$, for all x, y in $\mathfrak{v}$. We have the following lemma.

Lemma 3.1. *The map* $\mathfrak{v} \to \mathrm{Der}(\mathfrak{g})$ *defined by* $x \mapsto \mathrm{ad}_s(x)$ *is a Lie algebra homomorphism.*

Proof. First let us check that this map is linear. Let $x, y \in \mathfrak{v}$. By the above $\mathrm{ad}_s(y)$ and $\mathrm{ad}_s(x)$ commute with each other (hence their sum is semi-simple) and commute with $\mathrm{ad}_n(x) + \mathrm{ad}_n(y)$. From the uniqueness of the Jordan decomposition it remains to check that $\mathrm{ad}_n(x) + \mathrm{ad}_n(y)$ is nilpotent if x, y in $\mathfrak{v}$. To see this, apply the following obvious remark twice to $a = \mathrm{ad}_n(x)$ and $V = \mathrm{ad}(\mathfrak{n})$ first and then to $a = \mathrm{ad}_n(y)$ and $V = \mathrm{span}\{\mathrm{ad}_n(x), \mathrm{ad}((\mathrm{ad}(y))^n x), n \geq 1\}$: *Let* V *be a nilpotent subspace of* $\mathrm{GL}(\mathfrak{g})$ *and* $a \in \mathrm{GL}(\mathfrak{g})$ *nilpotent, i.e.* $V^n = 0$ *and* $a^m = 0$ *for some* $n, m \in \mathbb{N}$ *and assume* $[a, V] \subset V$. *Then* $(a + V)^{nm} = 0$.

The fact that this map is a Lie algebra homomorphism follows easily from the fact that all $\mathrm{ad}_s(x)$, $x \in \mathfrak{v}$ commute with one another and with $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$. □

We define a new Lie bracket on $\mathfrak{g}$ by setting

$$[x, y]_N = [x, y] - \mathrm{ad}_s(x_v)(y) + \mathrm{ad}_s(y_v)(x), \tag{18}$$

where x_v is the linear projection of x on $\mathfrak{v}$ according to the direct sum $\mathfrak{v} \oplus \mathfrak{n} = \mathfrak{g}$. The Jacobi identity is checked by a straightforward computation where the following fact is needed: $\mathrm{ad}_s\,(\mathrm{ad}_s(x)(y)) = 0$ for all $x, y \in \mathfrak{g}$. This holds because, as we just saw, $\mathrm{ad}_s(x)(\mathfrak{g}) \subset \mathfrak{n}$ for all $x \in \mathfrak{g}$, and $\mathrm{ad}_s(a) = 0$ if $a \in \mathfrak{n}$.

Definition 3.2. Let $\mathfrak{g}_N$ be the vector space $\mathfrak{g}$ endowed with the new Lie algebra structure $[\cdot, \cdot]_N$ given by (18). The *nilshadow* G_N of G is defined to be the simply connected Lie group with Lie algebra $\mathfrak{g}_N$.

It is easy to check that $\mathfrak{g}_N$ *is a nilpotent Lie algebra*. To see this, note first that $[\mathfrak{g}_N, \mathfrak{g}_N]_N \subset \mathfrak{n}$, and if $x \in \mathfrak{g}_N$ and $y \in \mathfrak{n}$ then $[x, y]_N = (\mathrm{ad}_n(x_v) + \mathrm{ad}(x_n))(y)$. However, $\mathrm{ad}_n(x_v) + \mathrm{ad}(x_n)$ is a nilpotent endomorphism of $\mathfrak{n}$ as follows from the same remark used in the proof of Lemma 3.1. Hence $\mathfrak{g}_N$ is a nilpotent.

The nilshadow Lie product on G_N will be denoted by $*$ in order to distinguish it from the original Lie product on G. In the sequel, we will often identify G (resp. G_N) with its Lie algebra $\mathfrak{g}$ (resp. $\mathfrak{g}_N$) via their respective exponential map. Since the underlying space of $\mathfrak{g}_N$ was $\mathfrak{g}$ itself, this gives an identification (although not a group isomorphism) between G and G_N. Then the nilshadow Lie product can be expressed in terms of the original product as

$$g * h = g \cdot (T(g^{-1})h).$$

Here T is the Lie group homomorphism $G \to \mathrm{Aut}(G)$ induced by the above choice of supplementary subspace $\mathfrak{v}$ as follows:

$$T(e^a)(e^b) = \exp(e^{\mathrm{ad}_s(a_v)} b), \quad a, b \in \mathfrak{g}. \tag{19}$$

In other words, T is the unique Lie group homomorphism whose differential at the identity is the Lie algebra homomorphism $d_e T\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ given by $d_e T(a)(b) = \mathrm{ad}_s(a_v)b$, that is the composition of the map $\mathfrak{v} \to \mathrm{Der}(\mathfrak{g})$ from Lemma 3.1 with the linear projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{n} \simeq \mathfrak{v}$.

It is easy to check that this definition of the new product is compatible with the definition of the new Lie bracket.

It can also be checked that two choices of supplementary spaces $\mathfrak{v}$ as above yield isomorphic Lie structures (see [13], Chapter III). Hence by abuse of language, we speak of *the* nilshadow of $\mathfrak{g}$, when we mean the Lie structure on G induced by a choice of $\mathfrak{v}$ as above.

The following example shows several of the features of a typical solvable Lie group of polynomial growth.

Example 3.3 (Nilshadow of a semi-direct product). Let $G = \mathbb{R} \ltimes_\varphi \mathbb{R}^n$ where $\varphi_t \in \mathrm{GL}_n(\mathbb{R})$ is some one parameter subgroup given by $\varphi_t = \exp(tA) = k_t u_t$ where A is some matrix in $M_n(\mathbb{R})$ and $A = A_s + A_u$ is its Jordan decomposition, giving rise to $k_t = \exp(tA_s)$ and $u_t = \exp(tA_u)$. The group G is diffeomorphic to $\mathbb{R}^{n+1}$, hence simply connected. If all eigenvalues of A_s are purely imaginary, then G has polynomial growth. However G is not nilpotent unless $A_s = 0$. So let us assume that neither A_s nor A_u is zero. Then the nilshadow G_N is the semi-direct product $\mathbb{R} \ltimes_u \mathbb{R}^n$ where u_t is the unipotent part of φ_t.

It is easy to compute the homogeneous dimension of G (or G_N) in terms of the dimension of the Jordan blocs of A_u. If n_k is the number of Jordan blocks of A_u of size k, then

$$d(G) = 1 + \sum_{k \geq 1} \frac{k(k+1)}{2} n_k.$$

3.2. Basic properties of the nilshadow. We now list in the form of a few lemmata some basic properties of the nilshadow.

Lemma 3.4. *The image of $T: G \to \mathrm{Aut}(G)$ is abelian and relatively compact. Moreover $T(T(g)h) = T(h)$ for any $g, h \in G$.*

Proof. Since G has polynomial growth it is of type (R) by Guivarc'h's theorem. Hence all $\mathrm{ad}_s(x)$ have purely imaginary eigenvalues. It follows that K is compact. Since T factors through the nilradical, its image is abelian. The last equality follows from (19) and the fact that, for all $x, y \in \mathfrak{g}$, $\mathrm{ad}_s(\mathrm{ad}_s(x)(y)) = 0$. □

Lemma 3.5. *$T(G)$ also belongs to $\mathrm{Aut}(G_N)$ and T is a group homomorphism*

$$G_N \longrightarrow \mathrm{Aut}(G_N).$$

Proof. The first assertion follows from (19) and the fact that $d_e T$ is a derivation of $\mathfrak{g}_N$ as one can check from (18) and the fact that, for all $x, y \in \mathfrak{g}$, $\mathrm{ad}_s(\mathrm{ad}_s(x)(y)) = 0$. The second assertion then follows from Lemma 3.4. □

We denote by K the closure of $T(G)$ in $\mathrm{Aut}(G) = \mathrm{Aut}(\mathfrak{g})$.

Lemma 3.6 (K-action on $\mathfrak{g}_N$). *K preserves $\mathfrak{v}$ and acts trivially on it. It also preserves the ideals $\mathfrak{n}$ and the central descending series $\{C^i(\mathfrak{g}_N)\}_i$ of $\mathfrak{g}_N$.*

Proof. It suffices to check that $\mathrm{ad}_s(\mathfrak{v})$ preserves $\mathfrak{n}$ and $C^i(\mathfrak{g}_N)$. It preserves $\mathfrak{n}$ because $\mathrm{ad}(x)$ preserves $\mathfrak{n}$ for all $x \in \mathfrak{g}$. It preserves $C^i(\mathfrak{g}_N)$ because it acts as a derivation of $\mathfrak{g}_N$ as we have already checked in the proof of Lemma 3.5. □

Remark 3.7 (Well-definiteness of π_1). It is also easy to check from the definition of the nilshadow bracket that the commutator subalgebra $[\mathfrak{g}_N, \mathfrak{g}_N]$ and in fact each term of the central descending series $C^i(\mathfrak{g}_N)$ is an ideal in $\mathfrak{g}$ and *does not depend* on the choice of supplementary subspace $\mathfrak{v}$ used to defined the nilshadow bracket. In particular the projection map $\pi_1 : \mathfrak{g}_N \to \mathfrak{g}_N/[\mathfrak{g}_N, \mathfrak{g}_N]$ is a well defined linear map on $\mathfrak{g} = \mathfrak{g}_N$ (i.e. independently of the choice involved in the construction of the nilshadow Lie bracket).

Lemma 3.8 (Exponential map). *The respective exponential maps* $\exp : \mathfrak{g} \to G$ *and* $\exp_N : \mathfrak{g}_N \to G_N$ *coincide on* $\mathfrak{n}$ *and on* $\mathfrak{v}$.

Proof. Since the two Lie products coincide on $N = \exp(\mathfrak{n})$, so do their exponential map. For the second assertion, note that $T(e^{-tv})v = v$ for every $v \in \mathfrak{v}$ because $\mathrm{ad}_s(x)(y) = 0$ for all $x, y \in v$. It follows that $\{e^{tv}\}_t$ is a one-parameter subgroup for both Lie structures, hence it is equal to $\{\exp_N(tv)\}_t$. □

Remark 3.9 (Surjectivity of the exponential map). The exponential map is not always a diffeomorphism, as the example of the universal cover $\widetilde{E}$ of the group E of motions of the plane shows (indeed any 1-parameter subgroup of E is either a translation subgroup or a rotation subgroup, but the rotation subgroup is compact hence a torus, so its lift will contain the (discrete) center of $\widetilde{E}$, hence will miss every lift of a non trivial translation). In fact, it is easy to see that if $\mathfrak{g}$ is the Lie algebra of a solvable (non-nilpotent) Lie group of polynomial growth, then $\mathfrak{g}$ maps surjectively on the Lie algebra of E. Hence, *for a simply connected solvable and non-nilpotent Lie group of polynomial growth, the exponential map is never onto*. Nevertheless its image is easily seen to be dense.

However, exponential coordinates of the second kind behave nicely. Note that $[\mathfrak{g}_N, \mathfrak{g}_N] \subset \mathfrak{n}$.

Lemma 3.10 (Exponential coordinates of the second kind). *Let* $\{C^i(\mathfrak{g}_N)\}_{i\geq 0}$ *be the central descending series of* $\mathfrak{g}_N$ *(with* $C^1(\mathfrak{g}_N) = [\mathfrak{g}_N, \mathfrak{g}_N]$*) and pick linear subspaces* m_i *in* $\mathfrak{g}_N$ *such that* $C^i(\mathfrak{g}_N) = m_i \oplus C^{i-1}(\mathfrak{g}_N)$ *for* $i \geq 2$. *Let* ℓ *be a supplementary subspace of* $C^1(\mathfrak{g}_N)$ *in* $\mathfrak{n}$. *Define exponential coordinates of the second kind by setting*

$$\begin{aligned} m_r \oplus \cdots \oplus m_2 \oplus \ell \oplus \mathfrak{v} &\longrightarrow G, \\ (\xi_r, \ldots, \xi_1, v) &\longmapsto \exp_N(\xi_r) * \cdots * \exp_N(\xi_1) * \exp_N(v). \end{aligned}$$

This map is a diffeomorphism. Moreover $\exp_N(\xi_r) * \cdots * \exp_N(\xi_1) * \exp_N(v) = e^{\xi_r} \cdot \cdots \cdot e^{\xi_1} \cdot e^v$ *for all choices of* $v \in \mathfrak{v}$ *and* $\xi_i \in m_i$.

Proof. By Lemma 3.8 the exponential maps of G and G_N coincide on $\mathfrak{n}$ and on $\mathfrak{v}$. Moreover $g * h = g \cdot h$ whenever g belongs to the nilradical $\exp(\mathfrak{n})$ of G. Hence

$$\begin{aligned}\exp_N(\xi_r) * \cdots * \exp_N(\xi_1) * \exp_N(v) &= \exp_N(\xi_r) \cdot \cdots \cdot \exp_N(\xi_1) \cdot \exp_N(v) \\ &= e^{\xi_r} \cdot \cdots \cdot e^{\xi_1} \cdot e^v.\end{aligned}$$

The restriction of the map to $\mathfrak{n}$ is a diffeomorphism onto $\exp(\mathfrak{n})$, because this map and its inverse are explicit polynomial maps (the ξ_i's are coordinates of the second kind, see the book [12]). Now the map $\mathfrak{n} \oplus \mathfrak{v} \to G$ sending (n, v) to $e^n \cdot e^v$ is a diffeomorphism, because G is simply connected and hence the quotient group $G/\exp(\mathfrak{n})$ isomorphic to a vector space and hence to $\exp(\mathfrak{v})$. □

Lemma 3.11 ("Bi-invariant" Riemannian metric). *There exists a Riemannian metric on G which is left invariant under both Lie structures.*

Proof. Indeed it suffices to pick a scalar product on $\mathfrak{g}$ which is invariant under the compact subgroup $K = \overline{T(G)} \subset \mathrm{Aut}(\mathfrak{g})$. □

We identify $K = \overline{\{T(g), g \in G\}}$ with its image in $\mathrm{Aut}(\mathfrak{g})$ under the canonical isomorphism between $\mathrm{Aut}(G)$ and $\mathrm{Aut}(\mathfrak{g})$. Recall that, according to Lemma 3.6, the central descending series of $\mathfrak{g}_N$ is invariant under $\mathrm{ad}_s(x)$ for all $x \in \mathfrak{v}$ and consists of ideals of $\mathfrak{g}$. The same holds for $\mathfrak{n}$. It follows that these linear subspaces also invariant under K. However since K is compact, its action on $\mathfrak{g}$ is completely reducible. Therefore we have proved the following lemma.

Lemma 3.12 (K-invariant stratification of the nilshadow). *Let $\mathfrak{g}$ be the Lie algebra of a simply connected Lie group G with polynomial growth. Let $\mathfrak{g}_N$ be the nilshadow Lie algebra obtained from a splitting $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{v}$ as above (i.e. $\mathfrak{n}$ is the nilradical and $\mathfrak{v}$ satisfies $\mathrm{ad}_s(x)(y) = 0$ for every $x, y \in \mathfrak{v}$). Let $K := \overline{\{T(g), g \in G\}} \subset \mathrm{Aut}(G)$, where T is defined by (19). Then there is a choice of linear subspaces m_i's and ℓ such that*

$$\mathfrak{g}_N = m_r \oplus \cdots \oplus m_2 \oplus \ell \oplus \mathfrak{v}, \tag{20}$$

where each term is K-invariant, $m_1 := \ell \oplus \mathfrak{v}$ and the central descending series of $\mathfrak{g}_N$ satisfies $C^i(\mathfrak{g}_N) = m_i \oplus C^{i-1}(\mathfrak{g}_N)$. Moreover the action on K can be read off on the exponential coordinates of second kind in this decomposition; namely,

$$\begin{aligned}k(e^{\xi_r} \cdot \cdots \cdot e^{\xi_0}) &= k(e^{\xi_r}) \cdot \cdots \cdot k(e^{\xi_0}) \\ &= e^{k(\xi_r)} \cdot \cdots \cdot e^{k(\xi_0)} \\ &= \exp_N(k(\xi_r)) * \cdots * \exp_N(k(\xi_0)).\end{aligned}$$

4. Periodic metrics

In this section, unless otherwise stated, G will denote an arbitrary locally compact group.

4.1. Definitions. By a *pseudodistance* (or metric) on a topological space X, we mean a function $\rho: X \times X \to \mathbb{R}_+$ satisfying $\rho(x,y) = \rho(y,x)$ and $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for any triplet of points of X. However $\rho(x,y)$ may be equal to 0 even if $x \neq y$.

We will require our pseudodistances to be *locally bounded*, meaning that the image under ρ of any compact subset of $G \times G$ is a bounded subset of $\mathbb{R}_+$. To avoid irrelevant cases (for instance $\rho \equiv 0$) we will also assume that ρ is *proper*, i.e. the map $y \mapsto \rho(e,y)$ is a proper map, namely the preimage of a bounded set is bounded (we do not ask that the map be continuous). When ρ is locally bounded then it is proper if and only if $y \mapsto \rho(x,y)$ is proper for any $x \in G$.

A pseudodistance ρ on G is said to be *asymptotically geodesic* if for every $\varepsilon > 0$ there exists $s > 0$ such that for any $x, y \in G$ one can find a sequence of points $x_1 = x, x_2, \ldots, x_n = y$ in G such that

$$\sum_{i=1}^{n-1} \rho(x_i, x_{i+1}) \leq (1+\varepsilon)\rho(x,y) \tag{21}$$

and $\rho(x_i, x_{i+1}) \leq s$ for all $i = 1, \ldots, n-1$.

We will consider exclusively pseudodistances on a group G that are *invariant* under left translations by all elements of a fixed closed and co-compact subgroup H of G, meaning that for all $x, y \in G$ and all $h \in H$, $\rho(hx, hy) = \rho(x,y)$.

Combining all previous axioms, we set the following definition.

Definition 4.1. Let G be a locally compact group. A pseudodistance ρ on G will be said to be a **periodic metric** (or H-periodic metric) if it satisfies the following properties:

(i) ρ is invariant under left translations by a closed co-compact subgroup H;

(ii) ρ is locally bounded and proper;

(iii) ρ is asymptotically geodesic.

Remark 4.2. The assumption that ρ is symmetric, i.e. $\rho(x,y) = \rho(y,x)$ is here only for the sake of simplicity, and most of what is proven in this paper can be done without this hypothesis.

4.2. Basic properties. Let ρ be a periodic metric on G and H some co-compact subgroup of G. The following properties are straightforward.

(1) ρ is at a bounded distance from its restriction to H. This means that if F is a bounded fundamental domain for H in G and for an arbitrary $x \in G$, if h_x denotes the element of H such that $x \in h_x F$, then $|\rho(x, y) - \rho(h_x, h_y)| \leq C$ for some constant $C > 0$.

(2) For all $t > 0$ there exists a compact subset K_t of G such that, for all $x, y \in G$, we have $\rho(x, y) \leq t \implies x^{-1} y \in K_t$. Conversely, if K is a compact subset of G, then there exists $t(K) > 0$ such that $x^{-1} y \in K \implies \rho(x, y) \leq t(K)$.

(3) If $\rho(x, y) \geq s$, the x_i's in (21) can be chosen in such a way that $s \leq \rho(x_i, x_{i+1}) \leq 2s$ (one can take a suitable subset of the original x_i's).

(4) The restriction of ρ to $H \times H$ is a periodic pseudodistance on H. This means that the x_i's in (21) can be chosen in H.

(5) Conversely, given a periodic pseudodistance ρ_H on H, it is possible to extend it to a periodic pseudodistance on G by setting $\rho(x, y) = \rho_H(h_x, h_y)$ where $x = h_x F$ for some bounded fundamental domain F for H in G.

4.3. Examples. Let us give a few examples of periodic pseudodistances.

(1) Let Γ be a finitely generated torsion free nilpotent group which is embedded as a co-compact discrete subgroup of a simply connected nilpotent Lie group N. Given a finite symmetric generating set S of Γ, we can consider the corresponding word metric d_S on Γ which gives rise to a periodic metric on N given by $\rho(x, y) = d_S(\gamma_x, \gamma_y)$ where $x \in \gamma_x F$ and $y \in \gamma_y F$ if F is some fixed fundamental domain for Γ in N.

(2) Another example, given in [27], is as follows. Let N/Γ be a nilmanifold with universal cover N and fundamental group Γ. Let g be a Riemannian metric on N/Γ. It can be lifted to the universal cover and thus gives rise to a Riemannian metric $\tilde{g}$ on N. This metric is Γ-invariant, proper and locally bounded. Since Γ is co-compact in N, it is easy to check that it is also asymptotically geodesic hence periodic.

(3) Any word metric on G. That is, if Ω is a compact symmetric generating subset of G, let $\Delta_\Omega(x) = \inf\{n \geq 1, x \in \Omega^n\}$. Then define $\rho(x, y) = \Delta_\Omega(x^{-1} y)$. Clearly ρ is a pseudodistance (although not a distance) and it is G-invariant on the left, it is also proper, locally bounded and asymptotically geodesic, hence periodic.

(4) If G is a connected Lie group, any left invariant Riemannian metric on G. Here again $H = G$ and we obtain a periodic distance. Similarly, any left invariant Carnot–Carathéodory metric on G will do.

Remark 4.3 (Berestovski's theorem). According to a result of Berestovski [5] every left-invariant geodesic distance on a connected Lie group is a subFinsler metric as defined in §2.1.

4.4. Coarse equivalence between invariant pseudodistances. The next proposition is basic.

Proposition 4.4. *Let ρ_1 and ρ_2 be two periodic pseudodistances on G. Then there is a constant $C > 0$ such that for all $x, y \in G$*

$$\frac{1}{C}\rho_2(x,y) - C \leq \rho_1(x,y) \leq C\rho_2(x,y) + C. \tag{22}$$

Proof. Clearly it suffices to prove the upper bound. Let $s > 0$ be the number corresponding to the choice $\varepsilon = 1$ in (21) for ρ_2. From 4.2(2), there exists a compact subset K_s in G such that $\rho_2(x,y) \leq 2s \implies x^{-1}y \in K_{2s}$, and there is a constant $t = t(K_{2s}) > 0$ such that $x^{-1}y \in K_{2s} \implies \rho_1(x,y) \leq t$. Let $C = \max\{2t/s, t\}$, and let $x, y \in G$. If $\rho_2(x,y) \leq s$ then $\rho_1(x,y) \leq t$ so the right hand side of (22) holds. If $\rho_2(x,y) \geq s$ then, from (21) and 4.2(3), we get a sequence of x_i's in G from x to y such that $s \leq \rho_2(x_i, x_{i+1}) \leq 2s$ and $\sum_1^N \rho_2(x_i, x_{i+1}) \leq 2\rho_2(x,y)$. It follows that $\rho_1(x_i, x_{i+1}) \leq t$ for all i. Hence $\rho_1(x,y) \leq \sum \rho_1(x_i, x_{i+1}) \leq Nt \leq \frac{2}{s}t\rho_2(x,y)$ and the right hand side of (22) holds. □

In the particular case when $G = N$ is a simply connected nilpotent Lie group, the distance to the origin $x \mapsto \rho(e,x)$ is also coarsely equivalent to any homogeneous quasi-norm on N. We have,

Proposition 4.5. *Suppose N is a simply connected nilpotent Lie group. Let ρ_1 be a periodic pseudodistance on N and $|\cdot|$ be a homogeneous quasi-norm, then there exists $C > 0$ such that for all $x \in N$*

$$\frac{1}{C}|x^{-1}y| - C \leq \rho_1(x,y) \leq C|x^{-1}y| + C. \tag{23}$$

Moreover, if ρ_2 is a periodic pseudodistance on the stratified nilpotent group N_∞ associated to N, then again, there is a constant $C > 0$ such that

$$\frac{1}{C}\rho_2(e,x) - C \leq \rho_1(e,x) \leq C\rho_2(e,x) + C. \tag{24}$$

The proposition follows at once from Guivarc'h's theorem (see Corollary 2.7 above), the equivalence of homogeneous quasi-norms, and the fact that left-invariant Carnot–Caratheodory metrics on N_∞ are homogeneous quasi norms. However, since the group structures on N and N_∞ differ, (24) cannot in general be replaced by the stronger relation (22) as simple examples show.

The next proposition is of fundamental importance for the study of metrics on Lie groups of polynomial growth.

Proposition 4.6. *Let G be a simply connected solvable Lie group of polynomial growth and G_N its nilshadow. Let ρ and ρ_N be arbitrary periodic pseudodistances on G and G_N respectively. Then there is a constant $C > 0$ such that for all $x, y \in G$*

$$\frac{1}{C}\rho_N(x, y) - C \leq \rho(x, y) \leq C\rho_N(x, y) + C. \tag{25}$$

Proof. According to Proposition 4.4, it is enough to show (25) for *some* choice of periodic metrics on G and G_N. But in Lemma 3.11 we constructed a Riemannian metric on G which is left invariant for both G and G_N. We are done. □

4.5. Right invariance under a compact subgroup. Here we verify that, given a compact subgroup of G, any periodic metric is at bounded distance from another periodic metric which is invariant on the right by this compact subgroup. Let K be a compact subgroup of G and ρ a periodic pseudodistance on G. We average ρ with the help of the normalized Haar measure on K to get

$$\rho^K(x, y) = \int_{K\times K} \rho(xk_1, yk_2)dk_1dk_2. \tag{26}$$

Then the following holds.

Lemma 4.7. *There is a constant $C_0 > 0$ depending only on ρ and K such that for all $k_1, k_2 \in K$ and all $x, y \in G$*

$$|\rho(xk_1, yk_2) - \rho(x, y)| \leq C_0. \tag{27}$$

Proof. From 4.2(2), there exists $t = t(K) > 0$ such that, for all $x \in G$, $\rho(x, xk) \leq t$. Applying the triangle inequality, we are done. □

Hence we obtain the following proposition.

Proposition 4.8. *The pseudodistance ρ^K is periodic and lies at a bounded distance from ρ. In particular, as x tends to infinity in G the following limit holds*

$$\lim_{x\to\infty} \frac{\rho^K(e, x)}{\rho(e, x)} = 1. \tag{28}$$

Proof. From Lemma 4.7 and 4.2(3), it is easy to check that ρ^K must be asymptotically geodesic, and periodic. Integrating (27) we get that ρ^K is at a bounded distance from ρ and (28) is obvious. □

If K is normal in G, we thus obtain a periodic metric ρ^K on G/K such that $\rho^K(p(x), p(y))$ is at a bounded distance from $\rho(x, y)$, where p is the quotient map $G \to G/K$.

5. Reduction to the nilpotent case

In this section, G denotes a *simply connected* solvable Lie group of polynomial growth. We are going to reduce the proof of the theorems of the Introduction to the case of a nilpotent G. This is performed by showing that any *periodic* pseudodistance ρ on G is asymptotic to some associated *periodic* pseudodistance ρ_N on the nilshadow G_N. We state this in Proposition 5.1 below.

The key step in the proof is Proposition 5.2 below, which shows the asymptotic invariance of ρ under the "semisimple part" of G. The crucial fact there is that the displacement of a distant point under a fixed unipotent automorphism is negligible compared to the distance from the identity (see Lemmata 5.4 and 5.5), so that the action of the semisimple part of large elements can be simply approximated by their action by left translation.

5.1. Asymptotic invariance under a compact group of automorphisms of G. The main result of this section is the following. Let G be a connected and simply connected solvable Lie group with polynomial growth and G_N its nilshadow (see Section 3).

Proposition 5.1. *Let H be a closed co-compact subgroup of G and ρ an H-periodic pseudodistance* (*see Definition 4.1*) *on G. There exist a closed subset H_K containing H which is a co-compact subgroup for both G and G_N, and an H_K-periodic* (*for both Lie structures*) *pseudodistance ρ_K such that*

$$\lim_{x\to\infty}\frac{\rho_K(e,x)}{\rho(e,x)}=1. \tag{29}$$

The closed subgroup H_K will be taken to be the closure of the group generated by all elements of the form $k(h)$, where h belongs to H and k belongs to the closure K in the group $\mathrm{Aut}(G)$ of the image of H under the homomorphism $T\colon G\to \mathrm{Aut}(G)$ introduced in Section 3. It is easy to check from the definition of the nilshadow product (1) that this is indeed a subgroup in both G and its nilshadow G_N.

The new pseudodistance ρ_K is defined as follows, using a double averaging procedure.

$$\rho_K(x,y):=\int_{H\backslash H_K}\int_K \rho(gk(x),gk(y))\,dk\,d\mu(g). \tag{30}$$

Here the measure μ is the normalized Haar measure on the coset space $H\backslash H_K$ and dk is the normalized Haar measure on the compact group K. Recall that all closed subgroups of S are unimodular (since they have polynomial growth by [21], Lemme I.3). Hence the existence of invariant measures on the coset spaces.

An essential part of the proof of Proposition 5.1 is enclosed in the following statement.

Proposition 5.2. *Let ρ be a periodic pseudodistance on G which is invariant under a co-compact subgroup H. Then ρ is asymptotically invariant under the action of $K = \overline{\{T(h), h \in H\}} \subseteq \mathrm{Aut}(G)$. Namely, (uniformly) for all $k \in K$,*

$$\lim_{x\to\infty} \frac{\rho(e,k(x))}{\rho(e,x)} = 1. \tag{31}$$

The proof of Proposition 5.2 splits into two steps. First we show that it is enough to prove (31) for a dense subset of k's. This is a consequence of the following continuity statement.

Lemma 5.3. *Let $\varepsilon > 0$, then there is a neighborhood U of the identity in K such that, for all $k \in U$,*

$$\overline{\lim}_{x\to\infty} \frac{\rho(x,k(x))}{\rho(e,x)} < \varepsilon.$$

Then we show that the action of $T(g)$ can be approximated by the conjugation by g, essentially because the unipotent part of this conjugation does not move x very much when x is far. This is the content of the following lemma.

Lemma 5.4. *Let ρ be a periodic pseudodistance on G which is invariant under a co-compact subgroup H. Then for any $\varepsilon > 0$, and any compact subset F in H there is $s_0 > 0$ such that*

$$|\rho(e, T(h)x) - \rho(e, hx)| \leq \varepsilon\rho(e,x)$$

for any $h \in F$ and as soon as $\rho(e,x) > s_0$.

Proof of Proposition 5.2 modulo Lemmata 5.3 and 5.4. As ρ is assumed to be H-invariant, for every $h \in H$, we have $\rho(e,h^{-1}x)/\rho(e,x) \to 1$. The proof of the proposition then follows immediately from the combination of the last two lemmata. □

5.2. Proof of Lemmata (5.3) and (5.4). We choose K-invariant subspaces m_i's and ℓ of the nilshadow $\mathfrak{g}_N$ of $\mathfrak{g}$ as in Lemma 3.12 from Section 3. In particular

$$\mathfrak{g}_N = m_r \oplus \cdots \oplus m_2 \oplus \ell \oplus \mathfrak{v},$$

where each term is K-invariant, $\mathfrak{n} = [\mathfrak{g}_N, \mathfrak{g}_N] \oplus \mathfrak{l}$ and $C^i(\mathfrak{g}_N) = m_i \oplus C^{i-1}(\mathfrak{g}_N)$. Moreover $\delta_t(x) = t^i x$ if $x \in m_i$ (here $m_1 = \ell \oplus \mathfrak{v}$).

We also set $v(x) = \max_i \|\xi_i\|_i^{1/d_i}$ if $x = \exp_N(\xi_r) * \cdots * \exp_N(\xi_0)$ and $d_i = i$ if $i > 0$ and $d_0 = 1$. And we let $|x| := \max_i \|x_i\|^{1/d_i}$ if $x = x_r + \cdots + x_1 + x_0$ in the above direct sum decomposition.

Note that $|\cdot|$ is a δ_t-homogeneous quasi-norm. Moreover, it is straightforward to verify (using the Campbell–Hausdorff formula (12) and Proposition 2.4) that $v(x) \leq C|x| + C$ for some constant $C > 0$. In particular $\xi_i/|x|^{d_i}$ remains bounded as $|x|$ becomes large.

Proof of Lemma 5.3. Combining Propositions 4.5 and 4.6, there is a constant $C > 0$ such that for all $x, y \in G$, $\rho(x, y) \leq C|x^{*-1} * y| + C$. Therefore we have reduced to prove the statement for $|\cdot|$ instead of ρ, namely it is enough to show that $|x^{*-1} * k(x)|$ becomes negligible compared to $|x|$ as $|x|$ goes to infinity and k tends to 1.

It follows from the Campbell–Baker–Hausdorff formula (11) and (12) that, if $x, y \in G_N$ and $|x|, |y|$ are $O(t)$, then $|\delta_{\frac{1}{t}}(x * y) - \delta_{\frac{1}{t}}(x) * \delta_{\frac{1}{t}}(y)| = O(t^{-1/r})$, and similarly $|\delta_{\frac{1}{t}}(x_1 * \cdots * x_m) - \delta_{\frac{1}{t}}(x_1) * \cdots * \delta_{\frac{1}{t}}(x_m)| = O_m(t^{-1/r})$, for m elements x_i with $|x_i| = O(t)$. Hence when writing $x = \exp_N(\xi_r) * \cdots * \exp_N(\xi_0)$, and setting $t = |x|$, we thus obtain that the following quantity

$$\left|\delta_{\frac{1}{t}}(x^{*-1} * k(x)) - \prod_{0 \leq i \leq r}^{*} \exp_N(-t^{-d_i}\xi_i) * \prod_{0 \leq i \leq r}^{*} \exp_N(t^{-d_{r-i}} k(\xi_{r-i}))\right|$$

is a $O(t^{-1/r})$. Indeed recall from Lemma 3.12 that $k(x) = \exp_N(k(\xi_r)) * \cdots * \exp_N(k(\xi_0))$. As x gets larger, each $t^{-d_i}\xi_i$ remains in a compact subset of m_i. Therefore, as k tends to the identity in K, each $t^{-d_i}k(\xi_i)$ becomes uniformly close to $t^{-d_i}\xi_i$ independently of the choice of $x \in G_N$ as long as $t = |x|$ is large. The result follows. □

Proof of Lemma 5.4. Recall that $hx = h * T(h)x$ for all $x, h \in G$ (see (1). By the triangle inequality it is enough to bound $\rho(y, h * y)$, where $y = T(h)x$. From Propositions 4.5 and 4.6, ρ is comparable (up to multiplicative and additive constants to the homogeneous quasi-norm $|\cdot|$. Hence the Lemma follows from the following lemma.

Lemma 5.5. *Let N be a simply connected nilpotent Lie group and let $|\cdot|$ be a homogeneous quasi norm on N associated to some 1-parameter group of dilations $(\delta_t)_t$. For any $\varepsilon > 0$ and any compact subset F of N, there is a constant $s_2 > 0$ such that*

$$|x^{-1}gx| \leq \varepsilon|x|$$

for all $g \in F$ and as soon as $|x| > s_2$.

Proof. Recall, as in the proof of the last lemma, that for any $c_1 > 0$ there is a $c_2 > 0$ such that if $t > 1$ and $x, y \in N$ are such that $|x|, |y| \leq c_1 t$, then

$$|\delta_{\frac{1}{t}}(xy) - \delta_{\frac{1}{t}}(x) * \delta_{\frac{1}{t}}(y)| \leq c_2 t^{-1/r}.$$

In particular, if we set $t = |x|$, then

$$|\delta_{\frac{1}{t}}(x^{-1}gx) - \delta_{\frac{1}{t}}(x)^{-1} * \delta_{\frac{1}{t}}(g) * \delta_{\frac{1}{t}}(x)| \leq c_2 t^{-1/r}.$$

On the other hand, as g remains in the compact set F, $\delta_{\frac{1}{t}}(g)$ tends uniformly to the identity when $t = |x|$ goes to infinity, and $\delta_{\frac{1}{t}}(x)$ remains in a compact set. By continuity, we see that $\delta_{\frac{1}{t}}(x)^{-1} * \delta_{\frac{1}{t}}(g) * \delta_{\frac{1}{t}}(x)$ becomes arbitrarily small as t increases. We are done. □

5.3. Proof of Proposition 5.1. First we prove the following continuity statement.

Lemma 5.6. *Let ρ be a periodic pseudodistance on G and $\varepsilon > 0$. Then there exists a neighborhood of the identity U in G and $s_3 > 0$ such that*

$$1 - \varepsilon \leq \frac{\rho(e, gx)}{\rho(e, x)} \leq 1 + \varepsilon$$

as soon $g \in U$ and $\rho(e, x) > s_3$.

Proof. Let ρ_N be a left invariant Riemannian metric on the nilshadow G_N.

$$|\rho(e, x) - \rho(e, gx)| \leq \rho(x, gx) \leq \rho(x, g * x) + \rho(g * x, gx)$$

However $\rho(a, b) \leq C\rho_N(a, b) + C$ for some $C > 0$ by Proposition 4.6. Moreover by (1) we have $gx = g * T(g)x$. Hence

$$|\rho(e, x) - \rho(e, gx)| \leq C\rho_N(x, g * x) + C\rho_N(x, T(g)x) + 2C.$$

To complete the proof, we apply Lemmata 5.5 and 5.3 to the right hand side above. □

Proof of Proposition 5.1. Let L be the set of all $g \in G$ such that $\rho(e, gx)/\rho(e, x)$ tends to 1 as x tends to infinity in G. Clearly L is a subgroup of G. Lemma 5.6 shows that L is closed. The H-invariance of ρ insures that L contains H. Moreover, Proposition 5.2 implies that L is invariant under K. Consequently L contains H_K, the closed subgroup generated by all $k(h)$, $k \in K$, $h \in H$. This, together with Proposition 5.2, grants pointwise convergence of the integrand in (29). Convergence of the integral follows by applying Lebesgue's dominated convergence theorem.

The fact that ρ_K is invariant under left multiplication by H and invariant under precomposition by automorphisms from K insures that ρ_K is invariant under $*$-left multiplication by any element $h \in H$, where $*$ is the multiplication in the nilshadow G_N. Moreover we check that $T(g) \in K$ if $g \in H_K$, hence H_K is a *subgroup* of G_N. It is clearly co-compact in G_N too (if F is compact and $HF = G$ then $H * F_K = G$ where F_K is the union of all $k(F)$, $k \in K$).

Clearly ρ_K is proper and locally bounded, so in order to finish the proof, we need only to check that ρ_K is asymptotically geodesic. By H-invariance of ρ_K and since H is co-compact in G, it is enough to exhibit a pseudogeodesic between e and a point $x \in H$. Let $x = z_1 \cdot \cdots \cdot z_n$ with $z_i \in H$ and $\sum \rho(e, z_i) \leq (1 + \varepsilon) \cdot \rho(e, x)$. Fix a compact fundamental domain F for H in H_K so that integration in (29) over $H \backslash H_K$ is replaced by integration over F. Then for some constant $C_F > 0$ we have $|\rho(g, gz) - \rho(e, gz)| \leq C_F$ for $g \in F$ and $z \in H$. Moreover, it follows from Proposition 5.2, Lemma 5.6 and the fact that $H_K \subset L$, that

$$\rho(e, gk(z)) \leq (1 + \varepsilon) \cdot \rho(e, z) \tag{32}$$

for all $g \in F$, $k \in K$ and as soon as $z \in G$ is large enough. Fix s large enough so that $C_F \leq \varepsilon s$ and so that (32) holds when $\rho(e,z) \geq s$. As already observed in the discussion following Definition 4.1, property 4.2(3), we may take the z_i's so that $\frac{s}{2} \leq \rho(e,z_i) \leq s$. Then $nC_F \leq ns\varepsilon \leq 3\varepsilon\rho(e,x)$. Finally we get for $\varepsilon < 1$ and x large enough

$$\begin{aligned}\sum \rho_K(e,z_i) &\leq C_F n + (1+\varepsilon)^2 \rho(e,x)\\ &\leq C_F n + (1+\varepsilon)^3 \rho_K(e,x)\\ &\leq (1+10\varepsilon)\cdot \rho_K(e,x)\end{aligned}$$

where we have used the convergence $\rho_K/\rho \to 1$ that we just proved. □

6. The nilpotent case

In this section, we prove Theorem 1.4 and its corollaries stated in the Introduction for a simply connected nilpotent Lie group. We essentially follow Pansu's argument from [27], although our approach differs somewhat in its presentation. Throughout the section, the nilpotent Lie group will be denoted by N, and its Lie algebra by $\mathfrak{n}$.

Let m_1 be any vector subspace of $\mathfrak{n}$ such that $\mathfrak{n} = m_1 \oplus [\mathfrak{n},\mathfrak{n}]$. Let π_1 the associated linear projection of $\mathfrak{n}$ onto m_1. Let H be a closed co-compact subgroup of N. To every H-periodic pseudodistance ρ on N we associate a norm $\|\cdot\|_0$ on m_1 which is the norm whose unit ball is defined to be the closed convex hull of all elements $\pi_1(h)/\rho(e,h)$ for all $h \in H\backslash\{e\}$. In other words,

$$E := \{x \in m_1, \|x\|_0 \leq 1\} = \overline{\mathrm{CvxHull}}\Big\{\frac{\pi_1(h)}{\rho(e,h)}, h \in H\backslash\{e\}\Big\}. \tag{33}$$

The set E is clearly a convex subset of m_1 which is symmetric around 0 (since ρ is symmetric). To check that E is indeed the unit ball of a norm on m_1 it remains to see that E is bounded and that 0 lies in its interior. The first fact follows immediately from (23) and Example 2.3. If 0 does not lie in the interior of E, then E must be contained in a proper subspace of m_1, contradicting the fact that H is co-compact in N.

Taking large powers h^n, we see that we can replace the set $H \setminus \{e\}$ in the above definition by any neighborhood of infinity in H. Similarly, it is easy to see that the following holds.

Proposition 6.1. *For $s > 0$ let E_s be the closed convex hull of all $\pi_1(x)/\rho(e,x)$ with $x \in N$ and $\rho(e,x) > s$. Then $E = \bigcap_{s>0} E_s$.*

Proof. Since ρ is H-periodic, we have $\rho(e, h^n) \leq n\rho(e, h)$ for all $n \in \mathbb{N}$ and $h \in H$. This shows $E \subset \bigcap_{s>0} E_s$. The opposite inclusion follows easily from the fact that ρ is at a bounded distance from its restriction to H, i.e. from 4.2(1). □

We now choose a set of supplementary subspaces (m_i) starting with m_1 as in §2.2. This defines a new Lie product $*$ on N so that $N_\infty = (N, *)$ is stratified. We can then consider the $*$-left invariant Carnot–Carathéodory metric associated to the norm $\|\cdot\|_0$ as defined in §2.1 on the stratified nilpotent Lie group N_∞. In this section, we will prove Theorem 1.4 for nilpotent groups in the following form.

Theorem 6.2. *Let ρ be a periodic pseudodistance on N and d_∞ the Carnot–Carathéodory metric defined above, then as x tends to infinity in N*

$$\lim \frac{\rho(e, x)}{d_\infty(e, x)} = 1. \tag{34}$$

Note that d_∞ is left-invariant for the N_∞ Lie product, but not the original Lie product on N.

Before going further, let us draw some simple consequences.

(1) In Theorem 6.2 we may replace $d_\infty(e, x)$ by $d(e, x)$, where d is the left invariant Carnot–Caratheodory metric on N (rather than N_∞) defined by the norm $\|\cdot\|_0$ (as opposed to d_∞ which is $*$-left invariant). Hence ρ, d and d_∞ are asymptotic. This follows from the combination of Theorem 6.2 and Remark 2.1.

(2) Observe that the choice of m_1 was arbitrary. Hence two Carnot–Carathéodory metrics corresponding to two different choices of a supplementary subspace m_1 with the same induced norm on $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$, are asymptotically equivalent (i.e. their ratio tends to 1), and in fact isometric; see Remark 2.1. Conversely, if two Carnot–Carathéodory metrics are associated to the same supplementary subspace m_1 and are asymptotically equivalent, they must be equal. This shows that the set of all possible norms on the quotient vector space $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ is in bijection with the set of all classes of asymptotic equivalence of Carnot–Carathéodory metrics on N_∞.

(3) As another consequence we see that if a locally bounded proper and asymptotically geodesic left-invariant pseudodistance on N is also homogeneous with respect to the 1-parameter group $(\delta_t)_t$ (i.e. $\rho(e, \delta_t x) = t\rho(e, x)$) then it has to be of the form $\rho(x, y) = d_\infty(e, x^{-1}y)$ where d_∞ is a Carnot–Carathéodory metric on N_∞.

6.1. Volume asymptotics. Theorem 6.2 also yields a formula for the asymptotic volume of ρ-balls of large radius. Let us fix a Haar measure on N (for example Lebesgue measure on $\mathfrak{n}$ gives rise to a Haar measure on N under exp). Since d_∞ is homogeneous, it is straightforward to compute the volume of a d_∞-ball:

$$\mathrm{vol}(\{x \in N, d_\infty(e, x) \leq t\}) = t^{d(N)}\mathrm{vol}(\{x \in N, d_\infty(e, x) \leq 1\})$$

where $d(N) = \sum_{i\geq 1} \dim(C^i(\mathfrak{n}))$ is the *homogeneous dimension* of N. For a pseudodistance ρ as in the statement of Theorem 6.2, we define the *asymptotic volume of* ρ to be the volume of the unit ball for the associated Carnot–Carathéodory metric d_∞,

$$\mathrm{AsVol}(\rho) = \mathrm{vol}(\{x \in N, d_\infty(e, x) \leq 1\}).$$

Then we obtain as an immediate corollary of Theorem 6.2.

Corollary 6.3. *Let* ρ *be periodic pseudodistance on* N. *Then*

$$\lim_{t\to+\infty} \frac{1}{t^{d(N)}} \mathrm{vol}(\{x \in N, \rho(e, x) \leq t\}) = \mathrm{AsVol}(\rho) > 0.$$

Finally, if Γ is an arbitrary finitely generated nilpotent group, we need to take care of the torsion elements. They form a normal finite subgroup T and applying Theorem 6.2 to Γ/T, we obtain the following corollary.

Corollary 6.4. *Let* S *be a finite symmetric generating set of* Γ *and* S^n *the ball of radius* n *is the word metric* ρ_S *associated to* S, *then*

$$\lim_{n\to+\infty} \frac{1}{n^{d(N)}} \#S^n = \#T \cdot \frac{\mathrm{AsVol}(\rho_{\overline{S}})}{\mathrm{vol}(N/\overline{\Gamma})} > 0,$$

where N *is the Malcev closure of* $\overline{\Gamma} = \Gamma/T$, *the torsion free quotient of* Γ, *and* $d_{\overline{S}}$ *is the word pseudodistance associated to* $\overline{S}$, *the projection of* S *in* $\overline{\Gamma}$.

Moreover, it is possible to be a bit more precise about $\mathrm{AsVol}(\rho_{\overline{S}})$. In fact, the norm $\|\cdot\|_0$ on m_1 used to define the limit Carnot–Carathéodory distance d_∞ associated to $\rho_{\overline{S}}$ is a simple polyhedral norm defined by

$$\{\|x\|_0 \leq 1\} = \mathrm{CvxHull}(\pi_1(\bar{s}), s \in S).$$

More generally the following holds. Let H be any closed, co-compact subgroup of N. Choose a Haar measure on H so that $\mathrm{vol}_N(N/H) = 1$. Theorem 6.2 yields the following statement.

Corollary 6.5. *Let* Ω *be a compact symmetric (i.e.* $\Omega = \Omega^{-1}$*) neighborhood of the identity, which generates* H. *Let* $\|\cdot\|_0$ *be the norm on* m_1 *whose unit ball is* $\overline{\mathrm{CvxHull}\{\pi_1(\Omega)\}}$ *and let* d_∞ *be the corresponding Carnot–Carathéodory metric on* N_∞. *Then we have the following limit in the Hausdorff topology*

$$\lim_{n\to+\infty} \delta_{\frac{1}{n}}(\Omega^n) = \{g \in N, d_\infty(e, g) \leq 1\}$$

and

$$\lim_{n\to+\infty} \frac{\mathrm{vol}_H(\Omega^n)}{n^{d(N)}} = \mathrm{vol}_N(\{g \in N, d_\infty(e, g) \leq 1\}).$$

6.2. Outline of the proof. We first devise some standard lemmata about piecewise approximations of horizontal paths (Lemmata 6.6, 6.7, and 6.10). Then it is shown (Lemma 6.11) that the original product on N and the product in the associated graded Lie group are asymptotic to each other, namely, if $(\delta_t)_t$ is a 1-parameter group of dilations of N, then after renormalization by $\delta_{\frac{1}{t}}$, the product of $O(t)$ elements lying in some bounded subset of N, is very close to the renormalized product of the same elements in the graded Lie group N_∞. This is why all complications due to the fact that N may not be *a priori* graded and the δ_t's may not be automorphisms disappear when looking at the large scale geometry of the group. Finally, we observe (Lemma 6.13), as follows from the very definition of the unit ball E for the limit norm $\|\cdot\|_0$, that any vector in the boundary of E, can be approximated, after renormalizing by $\delta_{\frac{1}{s}}$ by some element $x \in N$ lying in a fixed annulus $s(1-\varepsilon) \leq \rho(e,x) \leq s(1+\varepsilon)$. This enables us to assert that any ρ-quasi geodesic gives rise, after renormalization, to a d_∞-geodesic (this gives the lower bound in Theorem 6.2). And vice-versa, that any d_∞-geodesic can be approximated uniformly by some renormalized ρ-quasi geodesic (this gives the upper bound in Theorem 6.2).

6.3. Preliminary lemmata

Lemma 6.6. *Let G be a Lie group and let $\|\cdot\|_e$ be a Euclidean norm on the Lie algebra of G and $d_e(\cdot,\cdot)$ the associated left invariant Riemannian metric on G. Let K be a compact subset of G. Then there is a constant $C_0 = C_0(d_e, K) > 0$ such that whenever $d_e(e,u) \leq 1$ and $x, y \in K$*

$$|d_e(xu, yu) - d_e(x, y)| \leq C_0 d_e(x, y) d_e(e, u).$$

Proof. The proof reduces to the case when u and $x^{-1}y$ are in a small neighborhood of e. Then the inequality boils down to the following $\|[X,Y]\|_e \leq c\|X\|_e\|Y\|_e$ for some $c > 0$ and every X, Y in $\mathrm{Lie}(G)$. □

Lemma 6.7. *Let G be a Lie group, let $\|\cdot\|$ be some norm on the Lie algebra of G and let $d_e(\cdot,\cdot)$ be a left invariant Riemannian metric on G. Then for every $L > 0$ there is a constant $C = C(d_e, \|\cdot\|, L) > 0$ with the following property. Assume $\xi_1, \xi_2 \colon [0,1] \to G$ are two piecewise smooth paths in the Lie group G with $\xi_1(0) = \xi_2(0) = e$. Let $\xi_i' \in \mathrm{Lie}(G)$ be the tangent vector pulled back at the identity by a left translation of G. Assume that $\sup_{t\in[0,1]} \|\xi_1'(t)\| \leq L$, and that $\int_0^1 \|\xi_1'(t) - \xi_2'(t)\| dt \leq \varepsilon$. Then*

$$d_e(\xi_1(1), \xi_2(1)) \leq C\varepsilon.$$

Proof. The function $f(t) = d_e(\xi_1(t), \xi_2(t))$ is piecewise smooth. For small dt we may write, using Lemma 6.6,

$$\begin{aligned} f(t+dt) - f(t) &\leq d_e(\xi_1(t)\xi_1'(t)dt, \xi_1(t)\xi_2'(t)dt) \\ &\quad + d_e(\xi_1(t)\xi_2'(t)dt, \xi_2(t)\xi_2'(t)dt) - f(t) + o(dt) \\ &\leq \|\xi_1'(t) - \xi_2'(t)\|_e dt + C_0 f(t)\|\xi_2'(t)dt\|_e + o(dt) \\ &\leq \varepsilon(t)dt + C_0 L f(t)dt + o(dt), \end{aligned}$$

where $\varepsilon(t) = \|\xi_1'(t) - \xi_2'(t)\|_e$. In other words,

$$f'(t) \leq \varepsilon(t) + C_0 L f(t)$$

Since $f(0) = 0$, Gronwall's lemma implies that

$$f(1) \leq e^{C_0 L} \int_0^1 \varepsilon(s) e^{-C_0 L s} ds \leq C\varepsilon. \qquad \square$$

From now on, we will take G to be the stratified nilpotent Lie group N_∞, and $d_e(\cdot,\cdot)$ will denote a left invariant Riemannian metric on N_∞ while $d_\infty(\cdot,\cdot)$ is a left invariant Carnot–Caratheodory Finsler metric on N_∞ associated to some norm $\|\cdot\|$ on m_1.

Remark 6.8. There is $c_0 > 0$ such that $c_0^{-1} d_e(e,x) \leq d_\infty(e,x) \leq c_0 d_e(e,x)^{\frac{1}{r}}$ in a neighborhood of e. Hence in the situation of the lemma we get $d_\infty(\xi_1(1), \xi_2(1)) \leq C_1 \varepsilon^{\frac{1}{r}}$ for some other constant $C_1 = C_1(L, d_\infty, d_e)$.

Lemma 6.9. *Let $N \in \mathbb{N}$ and $d_N(x,y)$ be the function in N_∞ defined by*

$$d_N(x,y) = \inf\left\{\int_0^1 \|\xi'(u)\|du, \xi \in \mathcal{H}_{\mathrm{PL}(N)},\ \xi(0) = x, \xi(1) = y\right\},$$

where $\mathcal{H}_{\mathrm{PL}(N)}$ is the set of horizontal paths ξ which are piecewise linear with at most N possible values for ξ'. Then we have $d_N \to d_\infty$ uniformly on compact subsets of N_∞.

Proof. Note that it follows from Chow's theorem (see e.g. [25] or [19]) that there exists $K_0 \in \mathbb{N}$ such that $A := \sup_{d_\infty(e,x)=1} d_{K_0}(e,x) < \infty$. Moreover, since piecewise linear paths are dense in L^1, it follows for example from Lemma 6.7 that for each fixed x, $d_n(e,x) \to d_\infty(e,x)$. We need to show that $d_N(e,x) \to d_\infty(e,x)$ uniformly in x satisfying $d_\infty(e,x) = 1$. By contradiction, suppose there is a sequence $(x_n)_n$ such that $d_\infty(e,x_n) = 1$ and $d_n(e,x_n) \geq 1 + \varepsilon_0$ for some $\varepsilon_0 > 0$. We may assume that $(x_n)_n$ converges to say x. Let $y_n = x^{-1} * x_n$ and $t_n = d_\infty(e,y_n)$. Then $d_{K_0}(e,y_n) = t_n d_{K_0}(e, \delta_{\frac{1}{t_n}}(y_n)) \leq At_n$. Thus $d_n(e,x_n) \leq d_n(e,x) + d_n(e,y_n) \leq d_n(e,x) + At_n$ as soon as $n \geq K_0$. As n tends to ∞, we get a contradiction. $\square$

This lemma prompts the following notation. For $\varepsilon > 0$, we let $N_\varepsilon \in \mathbb{N}$ be the first integer such that $1 \leq d_{N_\varepsilon}(e, x) \leq 1 + \varepsilon$ for all x with $d_\infty(e, x) = 1$. Then we have the following lemma.

Lemma 6.10. *For every $x \in N_\infty$ with $d_\infty(e, x) = 1$, and all $\varepsilon > 0$ there exists a path $\xi\colon [0,1] \to N_\infty$ in $\mathcal{H}_{\mathrm{PL}(N_\varepsilon)}$ with unit speed (i.e. $\|\xi'\| = 1$) such that $\xi(0) = e$ and $d_\infty(x, \xi(1)) \leq C_2\varepsilon$ and ξ' has at most one discontinuity on any subinterval of $[0,1]$ of length $\varepsilon^r / N_\varepsilon$.*

Proof. We know that there is a path in $\mathcal{H}_{\mathrm{PL}(N_\varepsilon)}$ connecting e to x with length $\ell \leq 1+\varepsilon$. Reparametrizing the path so that it has unit speed, we get a path $\xi_0\colon [0,\ell] \to N_\infty$ in $\mathcal{H}_{\mathrm{PL}(N_\varepsilon)}$ with $d_\infty(x, \xi_0(1)) = d_\infty(\xi_0(\ell), \xi_0(1)) \leq \varepsilon$. The derivative ξ_0' is constant on at most N_ε different intervals say $[u_i, u_{i+1})$. Let us remove all such intervals of length $\leq \varepsilon^r / N_\varepsilon$ by merging them to an adjacent interval and let us change the value of ξ_0' on these intervals to the value on the adjacent interval (it doesn't matter if we choose the interval on the left or on the right). We obtain a new path $\xi\colon [0,1] \to N_\infty$ in $\mathcal{H}_{\mathrm{PL}(N_\varepsilon)}$ with unit speed and such that ξ' has at most one discontinuity on any subinterval of $[0,1]$ of length $\varepsilon^r / N_\varepsilon$. Moreover $\int_0^1 \|\xi'(t) - \xi_0'(t)\| dt \leq \varepsilon^r$. By Lemma 6.7 and Remark 6.3, we have $d_\infty(\xi(1), \xi_0(1)) \leq C_1\varepsilon$, hence

$$d_\infty(\xi(1), x) \leq d_\infty(x, \xi_0(1)) + d_\infty(\xi_0(1), \xi(1)) \leq (C_1 + 1)\varepsilon. \qquad \square$$

Lemma 6.11 (Piecewise horizontal approximation of paths). *Let $x * y$ denote the product inside the stratified Lie group N_∞ and $x \cdot y$ the ordinary product in N. Let $n \in \mathbb{N}$ and $t \geq n$. Then for any compact subset K of N, and any $x_1, \ldots, x_n$ elements of K, we have*

$$d_e(\delta_{\frac{1}{t}}(x_1 \cdot \cdots \cdot x_n), \delta_{\frac{1}{t}}(x_1 * \cdots * x_n)) \leq c_1 \frac{1}{t}$$

and

$$d_e(\delta_{\frac{1}{t}}(x_1 * \cdots * x_n), \delta_{\frac{1}{t}}(\pi_1(x_1) * \cdots * \pi_1(x_n))) \leq c_2 \frac{1}{t},$$

where c_1, c_2 depend on K and d_e only.

Proof. Let $\|\cdot\|$ be a norm on the Lie algebra of N. For $k = 1, \ldots, n$ let $z_k = x_1 \cdot \cdots \cdot x_{k-1}$ and $y_k = x_{k+1} * \cdots * x_n$. Since all x_i's belong to K, it follows from (24) that as soon as $t \geq n$, all $\delta_{\frac{1}{t}}(z_k)$ and $\delta_{\frac{1}{t}}(y_k)$ for $k = 1, \ldots, n$ remain in a bounded set depending only on K. Comparing (12) and (11), we see that whenever $y = O(1)$ and $\delta_{\frac{1}{t}}(x) = O(1)$, we have

$$\|\delta_{\frac{1}{t}}(xy) - \delta_{\frac{1}{t}}(x * y)\| = O\Big(\frac{1}{t^2}\Big). \tag{35}$$

On the other hand, from (12) it is easy to verify that right $*$-multiplication by a bounded element is Lipschitz for $\|\cdot\|$ and the Lipschitz constant is locally bounded.

It follows that there is a constant $C_1 > 0$ (depending only on K and $\|\cdot\|$) such that for all $k \leq n$

$$\|\delta_{\frac{1}{t}}((z_k \cdot x_k) * y_k) - \delta_{\frac{1}{t}}(z_k * x_k * y_k)\| \leq C_1 \|\delta_{\frac{1}{t}}(z_k \cdot x_k) - \delta_{\frac{1}{t}}(z_k * x_k)\|.$$

Applying n times the relation (35) with $x = x_1 \cdot \dots \cdot x_{k-1}$ and $y = x_k$, we finally obtain

$$\|\delta_{\frac{1}{t}}(x_1 \cdot \dots \cdot x_n) - \delta_{\frac{1}{t}}(x_1 * \dots * x_n)\| = O\Big(\frac{n}{t^2}\Big) = O\Big(\frac{1}{t}\Big),$$

where $O()$ depends only on K. On the other hand, using (11), it is another simple verification to check that if x, y lie in a bounded set, then $\frac{1}{c_2} d_e(x, y) \leq \|x - y\| \leq c_2 d_e(x, y)$ for some constant $c_2 > 0$. The first inequality follows.

For the second inequality, we apply Lemma 6.7 to the paths ξ_1 and ξ_2 starting at e and with derivative equal on $[\frac{k}{n}, \frac{k+1}{n})$ to $n\delta_{\frac{1}{t}}(x_k)$ for ξ_1 and to $n\frac{\pi_1(x_k)}{t}$ for ξ_2. We get

$$d_e(\delta_{\frac{1}{t}}(x_1 * \dots * x_n), \delta_{\frac{1}{t}}(\pi_1(x_1) * \dots * \pi_1(x_n))) = O\Big(\frac{1}{t}\Big). \qquad \square$$

Remark 6.12. From Remark 6.3 we see that if we replace d_e by d_∞ in the above lemma, we get the same result with $\frac{1}{t}$ replaced by $t^{-\frac{1}{r}}$.

Lemma 6.13 (Approximation in the abelianized group). *Recall that $\|\cdot\|_0$ is the norm on m_1 defined in (33). For any $\varepsilon > 0$, there exists $s_0 > 0$ such that for every $s > s_0$ and every $v \in m_1$ such that $\|v\|_0 = 1$, there exists $h \in H$ such that*

$$(1 - \varepsilon)s \leq \rho(e, h) \leq (1 + \varepsilon)s$$

and

$$\left\| \frac{\pi_1(h)}{\rho(e, h)} - v \right\|_0 \leq \varepsilon.$$

Proof. Let $\varepsilon > 0$ be fixed. Considering a finite ε-net in E, we see that there exists a finite symmetric subset $\{g_1, \dots, g_p\}$ of $H \backslash \{e\}$ such that, if we consider the closed convex hull of $\mathfrak{F} = \{f_i = \pi_1(g_i)/\rho(e, g_i) | i = 1, \dots, p\}$ and $\|\cdot\|_\varepsilon$ the associated norm on m_1, then $\|\cdot\|_0 \leq \|\cdot\|_\varepsilon \leq (1 + 2\varepsilon)\|\cdot\|_0$. Up to shrinking $\mathfrak{F}$ if necessary, we may assume that $\|f_i\|_\varepsilon = 1$ for all i's. We may also assume that the f_i's generate m_1 as a vector space. The sphere $\{x, \|x\|_\varepsilon = 1\}$ is a symmetric polyhedron in m_1 and to each of its facets corresponds $d = \dim(m_1)$ vertices lying in $\mathfrak{F}$ and forming a vector basis of m_1. Let $f_1, \dots, f_d$, say, be such vertices for a given facet. If $x \in m_1$ is of the form $x = \sum_{i=1}^d \lambda_i f_i$ with $\lambda_i \geq 0$ for $1 \leq i \leq d$ then we see that $\|x\|_\varepsilon = \sum_{i=1}^d \lambda_i$, because the convex hull of $f_1, \dots, f_d$ is precisely that facet, hence lies on the sphere $\{x, \|x\|_\varepsilon = 1\}$.

Now let $v \in m_1$, $\|v\|_0 = 1$, and let $s > 0$. The half line tv, $t > 0$, hits the sphere $\{x, \|x\|_\varepsilon = 1\}$ in one point. This point belongs to some facet and there are d linearly independent elements of $\mathfrak{F}$, say $f_1, \ldots, f_d$, the vertices of that facet, such that this point belongs to the convex hull of $f_1, \ldots, f_d$. The point sv then lies in the convex cone generated by $\pi_1(g_1), \ldots, \pi_1(g_d)$. Moreover, there is a constant $C_\varepsilon > 0$ ($C_\varepsilon \le \frac{d}{2} \max_{1 \le i \le p} \rho(e, g_i)$) such that

$$\left\| sv - \sum_{i=1}^{d} n_i \pi_1(g_i) \right\|_\varepsilon \le C_\varepsilon$$

for some non-negative integers $n_1, \ldots, n_d$ depending on $s > 0$. Hence

$$\begin{aligned}\frac{1}{s}\sum_{i=1}^{d} n_i \rho(e, g_i) = \frac{1}{s}\left\|\sum_{i=1}^{d} n_i \pi_1(g_i)\right\|_\varepsilon &\le \frac{1}{s}(\|sv\|_\varepsilon + C_\varepsilon)\\ &\le 1 + 2\varepsilon + \frac{C_\varepsilon}{s} \le 1 + 3\varepsilon,\end{aligned}$$

where the last inequality holds as soon as $s > C_\varepsilon/\varepsilon$.

Now let $h = g_1^{n_1} \cdot \cdots \cdot g_d^{n_d} \in H$. We have $\pi_1(h) = \sum_{i=1}^{d} n_i \pi_1(g_i)$ and

$$\rho(e, h) \ge \|\pi_1(h)\|_0 \ge s - C_\varepsilon \ge s(1 - \varepsilon).$$

Moreover

$$\rho(e, h) \le \sum_{i=1}^{d} n_i \rho(e, g_i) \le s(1 + 3\varepsilon).$$

Changing ε into say $\frac{\varepsilon}{5}$ and for say $\varepsilon < \frac{1}{2}$, we get the desired result with $s_0(\varepsilon) = \frac{d}{\varepsilon} \max_{1 \le i \le p} \rho(e, g_i)$. □

6.4. Proof of Theorem 6.2. We need to show that as $x \to \infty$ in N

$$1 \le \underline{\lim} \frac{\rho(e, x)}{d_\infty(e, x)} \le \overline{\lim} \frac{\rho(e, x)}{d_\infty(e, x)} \le 1.$$

First note that it is enough to prove the bounds for $x \in H$. This follows from 4.2(1).

Let us begin with the lower bound. We fix $\varepsilon > 0$ and $s = s(\varepsilon)$ as in the definition of an asymptotically geodesic metric; see (21). We know by 4.2(3) and 4.2(4) that as soon as $\rho(e, x) \ge s$ we may find $x_1, \ldots, x_n$ in H with $s \le \rho(e, x_i) \le 2s$ such that $x = \prod x_i$ and $\sum \rho(e, x_i) \le (1 + \varepsilon)\rho(e, x)$. Let $t = d_\infty(e, x)$, then $n \le \frac{1+\varepsilon}{s}\rho(e, x)$, hence $n \le \frac{C}{s(\varepsilon)} t$ where C is a constant depending only on ρ (see (23)). We may then apply Lemma 6.11 (and the remark following it) to get, as $t \ge n$ as soon as $s(\varepsilon) \ge C$,

$$d_\infty(\delta_{\frac{1}{t}}(x), \delta_{\frac{1}{t}}(\pi_1(x_1) * \cdots * \pi_1(x_n))) \le c_1' t^{-\frac{1}{r}}$$

But for each i we have $\|\pi_1(x_i)\|_0 \leq \rho(e, x_i)$ by definition of the norm, hence

$$\begin{aligned} t &= d_\infty(e, x) \\ &\leq \sum \|\pi_1(x_i)\|_0 + d_\infty(x, \pi_1(x_1) * \cdots * \pi_1(x_n)) \\ &\leq (1+\varepsilon)\rho(e, x) + c_1' t^{1-\frac{1}{r}}. \end{aligned}$$

Since ε was arbitrary, letting $t \to \infty$ we obtain

$$\underline{\lim} \frac{\rho(e, x)}{d_\infty(e, x)} \geq 1.$$

We now turn to the upper bound. Let $t = d_\infty(e, x)$ and $\varepsilon > 0$. According to Lemma 6.10, there is a horizontal piecewise linear path $\{\xi(u)\}_{u \in [0,1]}$ with unit speed such that $d_\infty(\delta_{\frac{1}{t}}(x), \xi(1)) \leq C_2\varepsilon$ and no interval of length $\geq \frac{\varepsilon^r}{N_\varepsilon}$ contains more than one change of direction. Let $s_0(\varepsilon)$ be given by Lemma 6.13 and assume $t > s_0(\varepsilon^r) N_\varepsilon / \varepsilon^r$. We split $[0, 1]$ into n subintervals of length $u_1, \cdots, u_n$ such that ξ' is constant equal to y_i on the i-th subinterval and $s_0(\varepsilon^r) \leq tu_i \leq 2s_0(\varepsilon^r)$. We have $\xi(1) = u_1 y_1 * \cdots * u_n y_n$. Lemma 6.13 yields points $x_i \in H$ such that

$$\left\| y_i - \frac{\pi_1(x_i)}{tu_i} \right\| \leq \varepsilon^r$$

and $\rho(e, x_i) \in [(1-\varepsilon^r)tu_i, (1+\varepsilon^r)tu_i]$ (note that $tu_i > s_0(\varepsilon^r)$). Let $\bar{\xi}$ be the piecewise linear path $[0, 1] \to N_\infty$ with the same discontinuities as ξ and where the value y_i is replaced by $\frac{\pi_1(x_i)}{tu_i}$. Then according to Lemma 6.7, $d_\infty(\xi(1), \bar{\xi}(1)) \leq C\varepsilon$. Since $\rho(e, x_i) \leq 4s_0(\varepsilon^r)$ for each i, we may apply Lemma 6.11 (and the remark following it) and see that if $y = x_1 \cdot \cdots \cdot x_n$,

$$d_\infty(\bar{\xi}(1), \delta_{\frac{1}{t}}(y)) \leq c_1'(\varepsilon) t^{-\frac{1}{r}}.$$

Hence

$$d_\infty(\delta_{\frac{1}{t}}(x), \delta_{\frac{1}{t}}(y)) \leq (C_2 + C)\varepsilon + c_1'(\varepsilon) t^{-\frac{1}{r}}$$

and

$$\rho(e, y) \leq \sum \rho(e, x_i) \leq (1+\varepsilon^r)t,$$

while

$$\rho(x, y) \leq C' t d_\infty(e, \delta_{\frac{1}{t}}(x^{-1}y)) + C' \leq t(C d_\infty(\delta_{\frac{1}{t}}(x), \delta_{\frac{1}{t}}(y)) + o_\varepsilon(1)).$$

Hence

$$\rho(e, x) \leq t + o_\varepsilon(t).$$

Remark 6.14. In the last argument we used the fact that

$$\|\delta_{\frac{1}{t}}(xu) - \delta_{\frac{1}{t}}(x * u)\| = O\Big(\frac{1}{t^{\frac{1}{r}}}\Big)$$

if $\delta_{\frac{1}{t}}(x)$ and $\delta_{\frac{1}{t}}(u)$ are bounded, in order to get for $y = xu$,

$$\begin{aligned} d_\infty(e, \delta_{\frac{1}{t}}(u)) &\leq d_\infty(\delta_{\frac{1}{t}}(x), \delta_{\frac{1}{t}}(xu)) + d_\infty(\delta_{\frac{1}{t}}(xu), \delta_{\frac{1}{t}}(x * u)) \\ &\leq d_\infty(\delta_{\frac{1}{t}}(x), \delta_{\frac{1}{t}}(y)) + o(1). \end{aligned}$$

7. Locally compact G and proofs of the main results

In this section, we prove Theorem 1.2 and complete the proof of Theorem 1.4 and its corollaries. We begin with the latter.

Proof of Theorem 1.4. It is the combination of Proposition 5.1, which reduces the problem to nilpotent Lie groups, and Theorem 6.2, which treats the nilpotent case. It only remains to justify the last assertion that d_∞ is invariant under $T(H)$.

Since $K = \overline{T(H)}$ stabilizes m_1 (see Lemma 3.12 for the definition of m_1) and acts by automorphisms of the nilpotent (nilshadow) structure (Lemma 3.5), given any $k \in K$, the metric $d_\infty(k(x), k(y))$ is nothing else but the left invariant subFinsler metric on the nilshadow associated to the norm $\|k(v)\|$ for $v \in m_1$ (if $\|\cdot\|$ denotes the norm associated to d_∞).

However, d_∞ is asymptotically invariant under K, because of Proposition 5.1. Namely $d_\infty(e, k(x))/d_\infty(e, x)$ tends to 1 as x tends to infinity. Finally $d_\infty(e, v) = \|v\|$ and $d_\infty(e, k(v)) = \|k(v)\|$ for all $v \in m_1$. Two asymptotic norms on a vector space are always equal. It follows that the norms $\|\cdot\|$ and $\|k(\cdot)\|$ on m_1 coincide. Hence $d_\infty(e, k(x)) = d_\infty(e, k(x))$ for all $x \in S$ as claimed. □

Proof of Corollary 1.8. First some initial remark (see also Remark 2.1). If d is a left-invariant subFinsler metric on a simply connected nilpotent Lie group N induced by a norm $\|\cdot\|$ on a supplementary subspace m_1 of the commutator subalgebra, then it follows from the very definition of subFinsler metrics (see §2.1) that π_1 is 1-Lipschitz between the Lie group and the abelianization of it endowed with the norm $\|\cdot\|$, namely $\|\pi_1(x)\| \leq d(e, x)$, with equality if $x \in m_1$. From this and considering the definition of the limit norm in (33), we conclude that $\|\cdot\|$ coincides with the limit norm of d. In particular Theorem 6.2 implies that d is asymptotic to the $*$-left invariant subFinsler metric d_∞ induced by the same norm $\|\cdot\|$ on the graded Lie group $(N_\infty, *)$.

We can now prove Corollary 1.8. By the above remark, the limit metric d_∞ on the graded nilshadow of S is asymptotic to the subFinsler metric d induced by the same norm $\|\cdot\|$ on the same (K-invariant) supplementary subspace m_1 of the commutator subalgebra of the nilshadow, and which is left invariant for the nilshadow structure on S. However, it follows from Theorem 1.4 that d_∞ and the norm $\|\cdot\|$ are K-invariant. This implies that d is also left-invariant with respect to the original Lie group structure of S. Indeed, by (1), we can write

$$d(gx, gy) = d(g * (T(g)x), g * (T(g)y)) = d(T(g)x, T(g)y) = d(x, y),$$

where $*$ denotes this time the nilshadow product structure. We are done. □

Proof of Corollary 1.7. This follows immediately from Theorem 1.4, when $*$ denotes the graded nilshadow product. If $*$ denotes the nilshadow group structure, then it follows from Theorem 6.2 and the remark we just made in the proof of Corollary 1.8 (see also Remark 2.1). □

7.1. Proof of Theorem 1.2. Let G be a locally compact group of polynomial growth. We will show that G has a compact normal subgroup K such that G/K contains a closed co-compact subgroup, which can be realized as a closed co-compact subgroup of a connected and simply connected solvable Lie group of type (R), i.e. of polynomial growth. The proof will follow in several steps.

(a) First we show that *up to moding out by a normal compact subgroup, we may assume that G is a Lie group whose connected component of the identity has no compact normal subgroup.* Indeed, it follows from Losert's refinement of Gromov's theorem ([24], Theorem 2) that there exists a normal compact subgroup K of G such that G/K is a Lie group. So we may now assume that G is a Lie group (not necessarily connected) of polynomial growth. The connected component G_0 of G is a connected Lie group of polynomial growth. Recall the following classical fact.

Lemma 7.1. *Every connected Lie group has a unique maximal compact normal subgroup. By uniqueness it must be a characteristic Lie subgroup.*

Proof. Clearly if K_1 and K_2 are compact normal subgroups, then K_1K_2 is again a compact normal subgroup. Considering G/K, where K is a compact normal subgroup of maximal dimension, we may assume that G has no compact normal subgroup of positive dimension. But every finite normal subgroup of a connected group is central. Hence the closed group generated by all finite normal subgroups is contained in the center of G. The center is an abelian Lie subgroup, i.e. isomorphic to a product of a vector space $\mathbb{R}^n$, a torus $\mathbb{R}^m/\mathbb{Z}^m$, a free abelian group $\mathbb{Z}^k$ and a finite abelian group. In such a group, there clearly is a unique maximal compact subgroup (namely the product of the finite group and the torus). It is also normal, and maximal in G. □

The maximal compact normal subgroup of G_0 is a characteristic Lie subgroup of G_0. It is therefore normal in G and we may mod out by it. We therefore have shown that every locally compact (compactly generated) group with polynomial growth admits a quotient by a compact normal subgroup, which is a Lie group G whose connected component of the identity G_0 has polynomial growth and contains no compact normal subgroup. We will now show that a certain co-compact subgroup of G has the embedding property of Theorem 1.2.

(b) Second we show that, *up to passing to a co-compact subgroup, we may assume that the connected component G_0 is solvable.* For this purpose, let Q be the solvable radical of G_0, namely the maximal connected normal Lie subgroup of G_0. Note that it is a characteristic subgroup of G_0 and therefore normal in G. Moreover G_0/Q is a semisimple Lie group. Since G_0 has polynomial growth, it follows that G_0/Q must be compact. Consider the action of G by conjugation on G_0/Q, namely the map $\varphi: G \to \mathrm{Aut}(G_0/Q)$. Since G_0/Q is compact semisimple, its group of automorphisms is also a compact Lie group. In particular, the kernel $\ker \varphi$ is a co-compact subgroup of G.

The connected component of the identity of $\mathrm{Aut}(G_0/Q)$ is itself semisimple and hence has finite center. However the image of the connected component $(\ker \varphi)_0$ of $\ker \varphi$ in G_0/Q modulo Q is central. Therefore it must be trivial. We have shown that $(\ker \varphi)_0$ is contained in Q and hence is solvable. Moreover $(\ker \varphi)_0$ has no compact normal subgroup, because otherwise its maximal normal compact subgroup, being characteristic in $(\ker \varphi)_0$, would be normal in G (note that $(\ker \varphi)_0$ is normal in G).

Changing G into the co-compact subgroup $\ker \varphi$, we can therefore assume that G_0 is solvable, of polynomial growth, and has no non trivial compact normal subgroup. The group G/G_0 is discrete, finitely generated, and has polynomial growth. By Gromov's theorem, it must be virtually nilpotent, in particular virtually polycyclic.

(c) We finally prove the following proposition.

Proposition 7.2. *Let G be a Lie group such that its connected component of the identity G_0 is solvable, admits no compact normal subgroup, and with G/G_0 virtually polycyclic. Then G has a closed co-compact subgroup, which can be embedded as a closed co-compact subgroup of a connected and simply connected solvable Lie group.*

The proof of this proposition is mainly an application of a theorem of H. C. Wang, which is a vast generalization of Malcev's embedding theorem for torsion free finitely generated nilpotent groups. Wang's theorem [36] states that any $\mathcal{S}$-group can be embedded as a closed co-compact subgroup of a simply connected real linear solvable Lie group with only finitely many connected components. Wang defines a $\mathcal{S}$-group to be any real Lie group G, which admits a normal subgroup A such that G/A is finitely generated abelian and A is a torsion-free nilpotent Lie group whose connected components group is finitely generated. In particular any $\mathcal{S}$-group has a finite index (hence co-compact) subgroup which embeds as a co-compact subgroup in a connected and simply connected solvable Lie group. In order to prove Proposition 7.2, it therefore suffices to establish that G has a co-compact $\mathcal{S}$-group.

We first recall the following simple fact.

Lemma 7.3. *Every closed subgroup F of a connected solvable Lie group S is topologically finitely generated.*

Proof. We argue by induction on the dimension of S. Clearly there is an epimorphism $\pi: S \to \mathbb{R}$. By induction hypothesis $F \cap \ker \pi$ is topologically finitely generated. The image of F is a subgroup of $\mathbb{R}$. However every subgroup of $\mathbb{R}$ contains either one or two elements, whose subgroup they generate has the same closure as the original subgroup. We are done. □

Next we show the existence of a nilradical.

Lemma 7.4. *Let G be as in Proposition 7.2. Then G has a unique maximal normal nilpotent subgroup G_N.*

Proof. The subgroup generated by any two normal nilpotent subgroups of any given group is itself nilpotent (Fitting's lemma, see e.g. [30], 5.2.8). Let G_N be the closure of the subgroup generated by all nilpotent subgroups of G. We need to show that G_N is nilpotent. For this it is clearly enough to prove that it is topologically finitely generated (because any finitely generated subgroup of G_N is nilpotent by the remark we just made). Since G/G_0 is virtually polycyclic, every subgroup of it is finitely generated ([29], 4.2). Hence it is enough to prove that $G_N \cap G_0$ is topologically finitely generated. This follows from Lemma 7.3. □

Incidently, we observe that the connected component of the identity $(G_N)_0$ coincides with the nilradical N of G_0 (it is the maximal normal nilpotent connected subgroup of G_0).

We now claim the following lemma.

Lemma 7.5. *The quotient group G/G_N is virtually abelian.*

The proof of this lemma is inspired by the proof of the fact, due to Malcev, that polycyclic groups have a finite index subgroup with nilpotent commutator subgroup (see e.g. [30], 15.1.6).

Proof. We will show that G has a finite index normal subgroup whose commutator subgroup is nilpotent. This clearly implies the lemma, for this nilpotent subgroup will be normal, hence contained in G_N.

First we observe that the group G admits a finite normal series $G_m \leq G_{m-1} \leq \ldots \leq G_1 = G$, where each G_i is a closed normal subgroup of G such that G_i/G_{i+1} is either finite, or isomorphic to either $\mathbb{Z}^n$, $\mathbb{R}^n$ or $\mathbb{R}^n/\mathbb{Z}^n$. This see it pick one of the G_i's to be the connected component G_0 and then treat G/G_0 and G_0 separately. The first follows from the definition of a polycyclic group (G/G_0 has a normal polycyclic subgroup of finite index). While for G_0, observe that its nilradical N is a connected and simply connected nilpotent Lie group and it admits such a series of characteristic subgroups (pick the central descending series), and G_0/N is an abelian connected Lie group, hence isomorphic to the direct product of a torus $\mathbb{R}^n/\mathbb{Z}^n$ and a vector group $\mathbb{R}^n$. The torus part is characteristic in G_0/N, hence its preimage in G_0 is normal in G.

The group G acts by conjugation on each partial quotient $Q_i := G_i/G_{i+1}$. This yields a map $G \to \mathrm{Aut}(Q_i)$. Now note that in order to prove our lemma, it is enough to show that for each i, there is a finite index subgroup of G whose commutator subgroup maps to a nilpotent subgroup of $\mathrm{Aut}(Q_i)$. Indeed, taking the intersection of those finite index subgroup, we get a finite index normal subgroups whose commutator subgroup acts nilpotently on each Q_i, hence is itself nilpotent (high enough commutators will all vanish).

Now $\mathrm{Aut}(Q_i)$ is either finite (if Q_i is finite), or isomorphic to $\mathrm{GL}_n(\mathbb{Z})$ (in case Q_i is either $\mathbb{Z}^n$ or $\mathbb{R}^n/\mathbb{Z}^n$) or to $\mathrm{GL}_n(\mathbb{R})$ (when $Q_i \simeq \mathbb{R}^n$). The image of G in $\mathrm{Aut}(Q_i)$ is a solvable subgroup. However, every solvable subgroup of $\mathrm{GL}_n(\mathbb{R})$ contains a finite index subgroup, whose commutator subgroup is unipotent (hence nilpotent). This follows from Kolchin's theorem for example, that a connected solvable algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})$ is triangularizable. We are done. □

In the sequel we assume that G/G_0 is torsion-free polycyclic. It is legitimate to do so in the proof of Proposition 7.2, because every virtually polycyclic group has a torsion-free polycyclic subgroup of finite index (see e.g. [29], Lemma 4.6).

We now claim the following lemma.

Lemma 7.6. *G_N is torsion-free.*

Proof. Since G/G_0 is torsion-free, it is enough to prove that $G_N \cap G_0$ is torsion-free. However the set of torsion elements in G_N forms a subgroup of G_N (if x, y are torsion, then xy is too because $\langle x, y\rangle$ is nilpotent). Clearly it is a characteristic subgroup of G_N. Hence its intersection with G_0 is normal in G_0. Taking the closure, we obtain a nilpotent closed normal subgroup T of G_0 which contains a dense set of torsion elements. Recall that G_0 has no normal compact subgroup. From this it quickly follows that T is trivial, because first it must be discrete (the connected component T_0 is compact and normal in G_0), hence finitely generated (by Lemma 7.3), hence made of torsion elements. But a finitely generated torsion nilpotent group is finite. Again since G_0 has no compact normal subgroup, T must be trivial, and G_N is torsion-free. □

Now observe that the group of connected components of G_N, namely $G_N/(G_N)_0$ is finitely generated. Indeed, since G/G_0 is finitely generated (as any polycyclic group), it is enough to prove that $(G_0 \cap G_N)/(G_N)_0$ is finitely generated, but this follows from the fact that $G_0 \cap G_N$ is topologically finitely generated (Lemma 7.3).

Now we are almost done. Note that G is topologically finitely generated (again by Lemma 7.3), therefore so is G/G_N. By Lemma 7.5 G/G_N is virtually abelian, hence has a finite index normal subgroup isomorphic to $\mathbb{Z}^n \times \mathbb{R}^m$. It follows that G/G_N

has a co-compact subgroup isomorphic to a free abelian group $\mathbb{Z}^{n+m}$. Hence after changing G by a co-compact subgroup, we get that G is an extension of G_N (a torsion-free nilpotent Lie group with finitely generated group of connected components) by a finitely generated free abelian group. Hence it is an $\mathcal{S}$-group in the terminology of Wang [36]. We apply Wang's theorem and this ends the proof of Proposition 7.2.

(d) We can now conclude the proof of Theorem 1.2. By (a) and (b), G has a quotient by a compact group which admits a co-compact subgroup satisfying the assumptions of Proposition 7.2. Hence to conclude the proof it only remains to verify that the simply connected solvable Lie group in which a co-compact subgroup of G/K embeds has polynomial growth (i.e. is of type (R)). But this follows from the following lemma (see [21], Theorem I.2).

Lemma 7.7. *Let G be a locally compact group. Then G has polynomial growth if and only if some* (*resp. any*) *co-compact subgroup of it has polynomial growth.*

Proof. First one checks that G is compactly generated if and only if some (resp. any) co-compact subgroup is. This is by the same argument which shows that finite index subgroups of a finitely generated group are finitely generated. In particular, if Ω is a compact symmetric generating set of G and H is a co-compact subgroup, then there is $n_0 \in \mathbb{N}$ such that $\Omega^{n_0} H = G$. Then $H \cap \Omega^{3n_0}$ generates H.

If G has polynomial growth and H is any compactly generated closed subgroup, then H has polynomial growth. Indeed (see [21], Theorem I.2), if Ω_H denotes a compact generating set for H, and K a compact neighborhood of the identity in G, then

$$\mathrm{vol}_G(K)\mathrm{vol}_H(\Omega_H^n) \leq \mathrm{vol}_H(KK^{-1} \cap H)\mathrm{vol}_G(\Omega_H^n K).$$

This inequality follows by integrating over a left Haar measure of G the function

$$\varphi(x) := \int_{\Omega_H^n} 1_K(h^{-1}x)dh,$$

where dh is a left Haar measure on H. This integral equals the left hand side of the above displayed equation, while it is pointwise bounded by $\mathrm{vol}_H(xK^{-1} \cap H)$ inside HK and by zero outside HK.

In the other direction, if H has polynomial growth, then G also has, because one can write $\Omega^n \subset \Omega_H^n K$ for some compact generating set Ω_H of H and some compact neighborhood K of the identity in G (see Proposition 4.4). Then the result follows from the following inequality

$$\mathrm{vol}_H(\Omega_H)\mathrm{vol}_G(\Omega_H^n K) \leq \mathrm{vol}_H(\Omega_H^{n+1})\mathrm{vol}_G(\Omega_H^{-1} K),$$

which itself is a direct consequence of the fact that the function

$$\psi(x) := \int_{\Omega_H^{n+1}} 1_{\Omega_H^{-1} K}(h^{-1}x)dh,$$

where dh is a left Haar measure on H, satisfies

$$\int_G \psi(x)dx = \mathrm{vol}_H(\Omega_H^{n+1})\mathrm{vol}_G(\Omega_H^{-1}K)$$

on the one hand and is bounded below by $\mathrm{vol}_H(\Omega_H)$ for every $x \in \Omega_H^n K$ on the other hand. □

Note that the above proof would be slightly easier if we already knew that both G and H were unimodular, in which case G/H has an invariant measure. But we know this only a posteriori, because the polynomial growth condition implies unimodularity; see [21].

Similar considerations show that G has polynomial growth if and only if G/K has polynomial growth, given any normal compact subgroup K; see e.g. [21].

We end this paragraph with a remark and an example, which we mentioned in the Introduction.

Remark 7.8 (Discrete subgroups are virtually nilpotent). Suppose Γ is a discrete subgroup of a connected solvable Lie group of type (R), i.e. of polynomial growth. Then Γ is virtually nilpotent. Indeed, a similar argument as in Lemma 7.3 shows that every subgroup of Γ is finitely generated. It follows that Γ is polycyclic. However Wolf [37] proved that polycyclic groups with polynomial growth are virtually nilpotent.

Example 7.9 (A group with no nilpotent co-compact subgroup). Let G be the connected solvable Lie group $G = \mathbb{R} \ltimes (\mathbb{R}^2 \times \mathbb{R}^2)$, where $\mathbb{R}$ acts as a dense one-parameter subgroup of $SO(2,\mathbb{R}) \times SO(2,\mathbb{R})$. Then G is of type (R). It has no compact subgroup. And it has no nilpotent co-compact subgroup. Indeed suppose H is a closed co-compact nilpotent subgroup. Then it has a non-trivial center. Hence there is a non identity element whose centralizer is co-compact in G. However a simple examination of the possible centralizers of elements of G shows that none of them is co-compact.

7.2. Proof of Corollary 1.6 and Theorem 1.1. Let G be an arbitrary locally compact group of polynomial growth and ρ a periodic pseudodistance on G.

Claim 1. *Corollary 1.6 holds for a co-compact subgroup H of G, if and only if it holds for G.*

By Lemma 7.7, the groups G and H are unimodular, and hence G/H bears a G-invariant Radon measure $\mathrm{vol}_{G/H}$, which is finite since H is co-compact. Now let F be a bounded Borel fundamental domain for H inside G. And let $\bar{\rho}$ be the periodic pseudodistance on G induced by the restriction of ρ to H, that is $\bar{\rho}(x,y) := \rho(h_x, h_y)$

where h_x is the unique element of H such that $x \in h_x F$. By 4.2(1) and 4.2(4), ρ and $\bar{\rho}$ are at a bounded distance from each other. In particular, $B_{\bar{\rho}}(r-C) \subset B_\rho(r) \subset B_{\bar{\rho}}(r+C)$. Hence if the limit (3) holds for $\bar{\rho}$, it also holds for ρ with the same limit. However, $B_{\bar{\rho}}(r) = \{x \in G, \rho(e,h_x) \leq r\} = B_{\rho_H}(r)F$ where ρ_H is the restriction of ρ to H. Hence $\mathrm{vol}_G(B_{\bar{\rho}}(r)) = \mathrm{vol}_H(B_{\rho_H}(r)) \cdot \mathrm{vol}_{G/H}(F)$. By 4.2(4), ρ_H is a periodic pseudodistance on H. So the result holds for (H,ρ_H) if and only if it holds for (G,ρ). Conversely, if ρ_0 is a periodic pseudodistance on H, then $\overline{\rho_0}(x,y) := \rho_0(h_x,h_y)$ is a periodic pseudodistance on G, hence again $\mathrm{vol}_G(B_{\bar{\rho}_0}(r)) = \mathrm{vol}_H(B_{\rho_0}(r)) \cdot \mathrm{vol}_G(F)$ and the result will hold for (H,ρ_0) if and only if it holds for $(G,\overline{\rho_0})$.

Claim 2. *If Corollary 1.6 holds for G/K, where K is some compact normal subgroup, then it holds for G as well.*

Indeed, if ρ is a periodic pseudodistance on G, then the K-average ρ^K, as defined in (26), is at a bounded distance from G according to Lemma 4.7. Now ρ^K induces a periodic pseudodistance $\overline{\rho^K}$ on G/K and $B_{\rho^K}(r) = B_{\overline{\rho^K}}(r)K$. Hence, $\mathrm{vol}_G(B_{\rho^K}(r)) = \mathrm{vol}_{G/K}(B_{\overline{\rho^K}}(r)) \cdot \mathrm{vol}_K(K)$. And if the limit (3) holds for $\overline{\rho^K}$, it also holds for ρ^K, hence for ρ too.

Thus the discussion above combined with Theorem 1.2 reduces Corollary 1.6 to the case when G is simply connected and solvable, which was treated in Section 5 and 6.

7.3. Proof of Proposition 1.3 and Corollary 1.9

Proof of Proposition 1.3. We say that two metric spaces (X,d_X) and (Y,d_Y) are at a bounded distance if they are $(1,C)$-quasi-isometric for some finite C. This is an equivalence relation. Now if ρ is H-periodic with H co-compact, then (G,ρ) is at a bounded distance from $(H,\rho|H)$. Hence we may assume that $H=G$, i.e. that ρ is left invariant on G.

Now Theorem 1.2 gives the existence of a normal compact subgroup K, a co-compact subgroup H containing K and a simply connected solvable Lie group S such that H/K is isomorphic to a co-compact subgroup of S.

Lemma 4.7 shows that (G,ρ) is at a bounded distance from (G,ρ^K), where ρ^K is defined as in (26). Now ρ^K induces a left invariant periodic metric on G/K, and $(G/K,\rho^K)$ is clearly at a bounded distance from (G,ρ^K). Now by 4.2, its restriction to H/K is at a bounded distance and is left invariant. Now we set $\rho_S(s_1,s_2) = \rho^K(h_1,h_2)$, where (given a bounded fundamental domain F for the left action of H/K on S) h_i is the unique element of H/K such that $s_i \in h_i F$. Clearly then (S,ρ_S) is at a bounded distance from $(H/K,\rho^K)$. We are done. □

We note that our construction of S here depends on the stabilizer of ρ in G. Certainly not every choice of Lie shadow can be used for all periodic metrics (think that $\mathbb{R}^3$ is a Lie shadow of the universal cover of the group of motions of the plane). Perhaps a single one can be chosen for all, but we have not checked that.

Proof of Corollary 1.9. Proposition 1.3 reduces the proof to a periodic metric ρ on a simply connected solvable Lie group S. Let d_∞ the subFinsler metric on S (left invariant for the graded nilshadow group structure S_N) as given by Theorem 1.4. Let $\{\delta_t\}_t$ is the group of dilations in the graded nilshadow S_N of S as defined in Section 3. By definition of the pointed Gromov–Hausdorff topology (see [18]), it is enough to prove the following statement.

Claim. *The following quantity*

$$\left|\frac{1}{n}\rho(s_1,s_2) - d_\infty(\delta_{\frac{1}{n}}(s_1),\delta_{\frac{1}{n}}(s_2))\right|$$

converges to zero as n tends to $+\infty$ uniformly for all s_1, s_2 in a ball of radius $O(n)$ for the metric ρ.

Now this follows in three steps. First ρ is at a bounded distance from its restriction to the (co-compact) stabilizer H of ρ (cf. 4.2(1) and 4.2(4)). Then for $h_1, h_2 \in H$, we can write $\rho(h_1,h_2) = \rho(e, h_1^{-1}h_2)$. However Proposition 5.1 implies the existence of another periodic distance ρ_K on S, which is invariant under left translations by elements of H for both the original Lie structure and the nilshadow Lie structure on S, such that $\frac{\rho(e,x)}{\rho_K(e,x)}$ tends to 1 as x tends to ∞. Hence $\rho_K(e, h_1^{-1}h_2) = \rho_K(h_1,h_2) = \rho_K(e, h_1^{*-1}h_2)$, where $*$ is the nilshadow product on S. Hence $|\frac{1}{n}\rho(h_1,h_2) - \frac{1}{n}\rho_K(e, h_1^{*-1}h_2)|$ tends to zero uniformly as h_1 and h_2 vary in a ball of radius $O(n)$ for ρ.

Finally Theorem 6.2 implies that $|\frac{1}{n}\rho_K(e, h_1^{*-1}h_2) - \frac{1}{n}d_\infty(e, h_1^{*-1}h_2)|$ tends to zero and the claim follows, as one verifies from the Campbell Hausdorff formula by comparing (11) and (12)) as we did in (35), that

$$|d_\infty(\delta_{\frac{1}{n}}(h_1), \delta_{\frac{1}{n}}(h_2)) - d_\infty(e, \delta_{\frac{1}{n}}(h_1^{*-1}h_2))|$$

converges to zero.

The fact that the graded nilpotent Lie group does not depend (up to isomorphism) on the periodic metric ρ but only on the locally compact group G follows from Pansu's theorem [28] that if two Carnot groups (i.e. a graded simply connected nilpotent Lie group endowed with left-invariant subRiemannian metric induced by a norm on a supplementary subspace to the commutator subalgebra) are bi-Lipschitz, the underlying Lie groups must be isomorphic. This deep fact relies on Pansu's generalized Rademacher theorem, see [28]. Indeed, two different periodic metrics ρ_1 and ρ_2 on G are quasi-isometric (see Proposition 4.4), and hence their asymptotic cones are bi-Lipschitz (and bi-Lipschitz to any Carnot group metric on the same graded group, by (13)). □

8. Coarsely geodesic distances and speed of convergence

Under no further assumption on the periodic pseudodistance ρ, the speed of convergence in the volume asymptotics can be made arbitrarily small. This is easily seen if we consider examples of the following type: define $\rho(x,y) = |x-y| + |x-y|^\alpha$ on $\mathbb{R}$ where $\alpha \in (0,1)$. It is periodic and $\mathrm{vol}(B_\rho(t)) = t - t^\alpha + o(t^\alpha)$.

However, many natural examples of periodic metrics, such as word metrics or Riemannian metrics, are in fact coarsely geodesic. A pseudodistance on G is said to be *coarsely geodesic*, if there is a constant $C > 0$ such that any two points can be connected by a C-coarse geodesic, that is, for any $x, y \in G$ there is a map $g\colon [0,t] \to G$ with $t = \rho(x,y)$, $g(0) = x$ and $g(t) = y$, such that

$$|\rho(g(u), g(v)) - |u-v|| \leq C$$

for all $u, v \in [0,t]$.

This is a stronger requirement than to say that ρ is asymptotically geodesic; see (21). This notion is invariant under coarse isometry. In the case when G is abelian, D. Burago [6] proved the beautiful fact that any coarsely geodesic periodic metric on G is at a bounded distance from its asymptotic norm. In particular $\mathrm{vol}_G(B_\rho(t)) = c \cdot t^d + O(t^{d-1})$ in this case. In the remarkable paper [32], M. Stoll proved that such an error term in $O(t^{d-1})$ holds for any finitely generated 2-step nilpotent group. Whether $O(t^{d-1})$ is the right error term for any finitely generated nilpotent group remains an open question.

The example below shows on the contrary that in an arbitrary Lie group of polynomial growth no universal error term can be expected.

Theorem 8.1. *Let $\varepsilon_n > 0$ be an arbitrary sequence of positive numbers tending to 0. Then there exists a group G of polynomial growth of degree 3 and a compact generating set Ω in G and $c > 0$ such that*

$$\frac{\mathrm{vol}_G(\Omega^n)}{c \cdot n^3} \leq 1 - \varepsilon_n \tag{36}$$

holds for infinitely many n, although $\frac{1}{c \cdot n^3}\mathrm{vol}_G(\Omega^n) \to 1$ as $n \to +\infty$.

The example we give below is a semi-direct product of $\mathbb{Z}$ by $\mathbb{R}^2$ and the metric is a word metric. However, many similar examples can be constructed as soon as the map $T\colon G \to K$ defined in §5.1 in not onto. For example, one can consider left invariant Riemannian metrics on $G = \mathbb{R} \cdot (\mathbb{R}^2 \times \mathbb{R}^2)$ where $\mathbb{R}$ acts by via a dense one-parameter subgroup of the 2-torus $S^1 \times S^1$. Incidently, this group G is known as the *Mautner group* and is an example of a *wild* group in representation theory.

8.1. An example with arbitrarily small speed. In this paragraph we describe the example of Theorem 8.1. Let $G_\alpha = \mathbb{Z} \cdot \mathbb{R}^2$ where the action of $\mathbb{Z}$ is given by the rotation R_α of angle $\pi\alpha$, $\alpha \in [0,1)$. The group G_α is quasi-isometric to $\mathbb{R}^3$ and hence of polynomial growth of order 3 and it is co-compact in the analogously defined Lie group $\widetilde{G_\alpha} = \mathbb{R} \ltimes \mathbb{R}^2$. Its nilshadow is isomorphic to $\mathbb{R}^3$. The point is that if α is a suitably chosen Liouville number, then the balls in G_α will not be well approximated by the limit norm balls.

Elements of G_α are written (k,x) where $k \in \mathbb{Z}$ and $x \in \mathbb{R}^2$. Let $\|x\|^2 = \frac{1}{4}x_1^2 + x_2^2$ be a Euclidean norm on $\mathbb{R}^2$, and let Ω be the symmetric compact generating set given by $\{(\pm 1, 0)\} \cup \{(0,x), \|x\| \leq 1\}$. It induces a word metric ρ_Ω on G. It follows from Theorem 1.4 and the definition of the asymptotic norm that $\rho_\Omega(e,(k,x))$ is asymptotic to the norm on $\mathbb{R}^3$ given by $\rho_0(e,(k,x)) := |k| + \|x\|_0$ where $\|x\|_0$ is the rotation invariant norm on $\mathbb{R}^2$ defined by $\|x\|_0^2 = \frac{1}{4}(x_1^2 + x_2^2)$. The unit ball of $\|\cdot\|_0$ is the convex hull of the union of all images of the unit ball of $\|\cdot\|$ under all rotations $R_{k\alpha}$, $k \in \mathbb{Z}$.

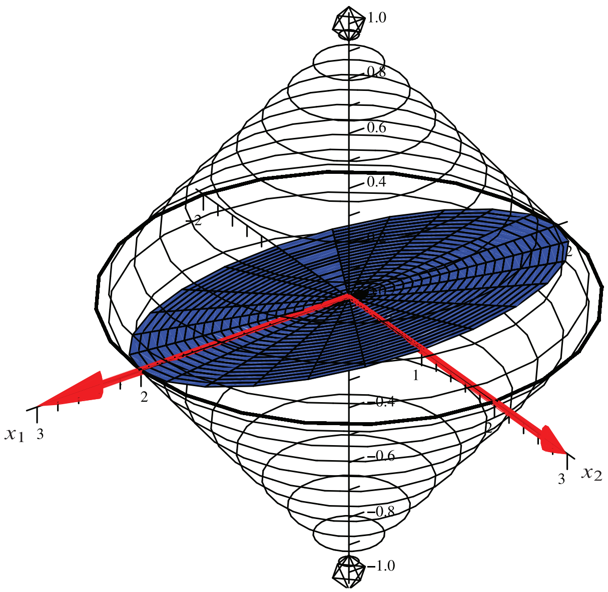

Figure 2. The union of the two cones, with basis the disc of radius 2, represents the limit shape of the balls Ω^n in the group $\mathbb{Z} \ltimes \mathbb{R}^2$, where $\mathbb{Z}$ acts by an irrational rotation, with generating set $\Omega = \{(\pm 1, 0, 0)\} \cup \{(0, x_1, x_2), \frac{1}{4}x_1^2 + x_2^2 \leq 1\}$.

We are going to choose α as a suitable Liouville number so that (36) holds. Let $\delta_n = (4\varepsilon_n)^{1/3}$ and choose α so that the following holds for infinitely many n's:

$$d(k\alpha, \mathbb{Z} + \frac{1}{2}) \geq 2\delta_n \tag{37}$$

for all $k \in \mathbb{Z}$, $|k| \leq n$. This is easily seen to be possible if we choose α of the form $\sum 1/3^{n_i}$ for some suitable lacunary increasing sequence of $(n_i)_i$.

Note that, since $\|x\|_0 \geq \|x\|$, we have $\rho_\Omega \geq \rho_0$. Let S_n be the piece of $\mathbb{R}^2$ defined by $S_n = \{|\theta| \leq \delta_n\}$ where θ is the angle between the point x and the vertical axis $\mathbb{R}e_2$. We *claim* that if $x \in S_n$, $\rho_0(e,(k,x)) \leq n$ and n satisfies (37), then

$$\rho_\Omega(e,(k,x)) \geq |k| + \Big(1 + \frac{\delta_n^2}{4}\Big)\|x\|_0$$

It follows easily from the claim that $\mathrm{vol}_G(\Omega^n) \leq (1-\varepsilon_n)\cdot \mathrm{vol}_G(B_{\rho_0}(n))$. Moreover $\mathrm{vol}_G(B_{\rho_0}(n)) = c\cdot n^3 + O(n^2)$, where $c = \frac{4\pi}{3}$ if vol_G is given by the Lebesgue measure.

Proof of Claim. Here is the idea to prove the claim. To find a short path between the identity and a point on the vertical axis, we have to rotate by a $R_{k\alpha}$ such that $k\alpha$ is close to $\frac{1}{2}$, hence go up from $(0,0)$ to $(k,0)$ first, thus making the vertical direction shorter. However if (37) holds, the vertical direction cannot be made as short as it could after rotation by any of the $R_{k\alpha}$ with $|k| \leq n$.

Note that if $\rho_0(e,(k,x)) \leq n$ then $|k| \leq n$ and

$$\rho_\Omega(e,(k,x)) \geq |k| + \inf \sum \|R_{k_i\alpha}x_i\|,$$

where the infimum is taken over all paths $x_1,\ldots,x_N$ such that $x = \sum x_i$ and all rotations $R_{k_i\alpha}$ with $|k_i| \leq n$. Note that if δ_n is small enough and (37) holds then for every $x \in S_n$ we have $\|R_{k\alpha}x\| \geq (1+\delta_n^2)\|x\|_0$. On the other hand $\|x\|_0 = \sum \|x_i\|_0 \cos(\theta_i)$ where θ_i is the angle between x_i and the x. Hence

$$\begin{aligned}\sum \|R_{k_i\alpha}x_i\| &\geq \sum_{|\theta_i|\leq\delta_n} \|R_{k_i\alpha}x_i\| + \sum_{|\theta_i|>\delta_n} \|R_{k_i\alpha}x_i\| \\ &\geq (1+\delta_n^2)\sum_{|\theta_i|\leq\delta_n} \|x_i\|_0\cos(\theta_i) + \frac{1}{\cos(\delta_n)}\sum_{|\theta_i|>\delta_n}\|x_i\|_0\cos(\theta_i) \\ &\geq \Big(1+\frac{\delta_n^2}{4}\Big)\cdot\|x\|_0\end{aligned}$$

□

8.2. Limit shape for more general word metrics on solvable Lie groups of polynomial growth. The determination of the limit shape of the word metric in §8.1 was possible due to the rather simple nature of the generating set. In general, using the identity, see (1),

$$\omega_1\cdot\dots\cdot\omega_m = \omega_1 * (T(\omega_1)\omega_2) * \dots * (T(\omega_{m-1}\cdot\dots\cdot\omega_1)\omega_m) \tag{38}$$

it is easy to check that the unit ball of the limit norm $\|\cdot\|_\infty$ inducing the limit subFinsler metric d_∞ on the nilshadow associated to a given word metric with generating set Ω

is contained in the K-orbit of the convex hull of the projection of Ω to the abelianized nilshadow, namely the convex hull of $K \cdot \pi_1(\Omega)$.

In the example of §8.1, we even had equality between the two. However this is not the case in general. For example, the limit shape is always K-invariant, but clearly the limit shape associated to a generating set Ω coincides with the one associated with a conjugate $g\Omega g^{-1}$ of it, while the convex hull of the respective K-orbits may not be the same.

Of course if the generating set Ω is K-invariant to begin with, then $\Omega^n = \Omega^{*n}$ and we are back in the nilpotent case, where we know that the unit ball of the limit norm is just the convex hull of the projection of the generating set to the abelianization. In general however it is a challenging problem to determine the precise asymptotic shape of a word metric on a general solvable Lie group with polynomial growth, and there seems to be no simple description analogous to what we have in the nilpotent case.

Even in the above example $G_\alpha = \mathbb{Z} \ltimes_\alpha \mathbb{R}^2$, or in the universal cover of the group of motions of the plane (in which G_α embeds co-compactly), it is not that simple. In general the shape is determined by solving an optimization problem in which one has to find the path which maximizes the coordinates of the endpoint. In order to illustrate this, we treat without proof the following simple example.

Suppose Ω is a symmetric compact neighborhood of the identity in $G_\alpha = \mathbb{Z} \ltimes_\alpha \mathbb{R}^2$ of the form $\Omega = (0, \Omega_0) \cup (1, \Omega_1) \cup (1, \Omega_1)^{-1}$, where $\Omega_0, \Omega_1 \subset \mathbb{R}^2$. Then the limit shape of the word metric ρ_Ω associated to Ω is the solid body (rotationally symmetric around the vertical axis as in Figure 8.1) made of two copies (upper and lower) of a truncated cone with base a disc on $(0, \mathbb{R}^2)$ of radius $\max\{r_0, r_1\}$ and top (resp. bottom) a disc on the plane $(1, \mathbb{R}^2)$ (resp. $(-1, \mathbb{R}^2)$) of radius r_2, where the radii are given by

$$r_0 = \max\{\|x\|, x \in \Omega_0\}, r_1 = \frac{1}{2}\operatorname{diam}(\Omega_1),$$

where $\operatorname{diam}(\Omega_1)$ is the diameter of Ω_1 and r_2 is given by the integral

$$r_2 = \int_0^{2\pi} \max\{\pi_\theta(\Omega_1)\} \frac{d\theta}{2\pi}, \tag{39}$$

where $\pi_\theta(\Omega_1)$ is the orthogonal projection on the x-axis of image of $\Omega_1 \subset \mathbb{R}^2$ by a rotation of angle θ around the origin. It is indeed convex (note that $r_2 \leq r_1$).

For example if Ω_1 is made of only one point, then the limit shape is the same as in the previous paragraph and as in Figure 8.1, namely two copies of a cone. However if Ω_1 is made of two points $\{a, b\}$, then the upper part of the limit shape will be a truncated cone with an upper disc of radius $r_2 = \frac{\|a-b\|}{\pi}$ (which is the result of the computation of the above integral).

Let us briefly explain formula (39). A path of length n reaching the highest z-coordinate in G_α is a word of the form $(1,\omega_1)\cdot\cdots\cdot(1,\omega_n)$, with $\omega_i\in\Omega_1$. By (38) this word equals

$$\Big(n,\sum_1^n R_\alpha^{i-1}\omega_i\Big).$$

Here ω_i can take any value in Ω_1. In order to maximize the norm of the second coordinate, or equivalently (by rotation invariance) its x-coordinate, one has to choose $\omega_i\in\Omega_1$ at each stage in such a way that the x-coordinate of $R_\alpha^{i-1}\omega_i$ is maximized. Formula (39) now follows from the fact that $\{R_\alpha^{i-1}\}_{1\le i\le n}$ becomes equidistributed in $\mathrm{SO}(2,\mathbb{R})$ as n tends to infinity.

In order to show that $\max\{r_0,r_1\}$ is the radius of the base disc and more generally that the limit shape is no bigger than this double truncated cone, one needs to argue further by considering all possible paths of the form $(\varepsilon_1,\omega_1)\cdot\cdots\cdot(\varepsilon_n,\omega_n)$ where $\varepsilon_i\in\{0,\pm1\}$ and $\sum\varepsilon_i$ is prescribed.

8.3. Bounded distance versus asymptotic metrics. In this paragraph we answer a question of D. Burago and G. Margulis (see [7]). Based on the abelian case and the reductive case (Abels and Margulis [1]), Burago and Margulis had conjectured that every two asymptotic word metrics should be at a bounded distance. We give below a counterexample to this. We first give an example (A) of a nilpotent Lie group endowed with two left invariant subFinsler metrics d_∞ and d'_∞ that are asymptotic to each other, i.e. $d_\infty(e,x)/d'_\infty(e,x)\to 1$ as $x\to\infty$ but such that $|d_\infty(e,x)-d'_\infty(e,x)|$ is not uniformly bounded. Then we exhibit (B) a word metric that is not at a bounded distance from any homogeneous quasi-norm. Finally these examples also yield (C) two word metrics ρ_1 and ρ_2 on the same finitely generated nilpotent group which are asymptotic but not at a bounded distance.

Note that the group G_α with ρ_0 and ρ_Ω from the last paragraph also provides an example of asymptotic metrics which are not at a bounded distance (but this group was not discrete).

(A) Let $N=\mathbb{R}\times H_3(\mathbb{R})$ where H_3 is classical Heisenberg group and $\Gamma=\mathbb{Z}\times H_3(\mathbb{Z})$ a lattice in N. In the Lie algebra $\mathfrak{n}=\mathbb{R}V\oplus\mathfrak{h}_3$ we pick two different supplementary subspaces of $[\mathfrak{n},\mathfrak{n}]=\mathbb{R}Z$, i.e. $m_1=\mathrm{span}\{V,X,Y\}$ and $m'_1=\mathrm{span}\{V+Z,X,Y\}$, where $\mathfrak{h}_3$ is the Lie algebra of $H_3(\mathbb{R})$ spanned by X,Y and $Z=[X,Y]$. We consider the L^1-norm on m_1 (resp. m'_1) corresponding to the basis (V,X,Y) (resp. $(V+Z,X,Y)$). Both norms induce the same norm on $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$. They give rise to left invariant Carnot–Caratheodory Finsler metrics on N, say d_∞ (resp. d'_∞). We use the coordinates $(v,x,y,z)=\exp(vV+xX+yY+zZ)$.

According to Remark (2) after Theorem 6.2, d_∞ and d'_∞ are asymptotic. Let us show that they are not at a bounded distance. First observe that, since V is central, $d_\infty(e,(v;(x,y,z)))=|v|+d_{H_3}(e,(x,y,z))$ where d_{H_3} is the Carnot–Caratheodory Finsler metric on $H_3(\mathbb{R})$ defined by the standard L^1-norm on the $\mathrm{span}\{X,Y\}$. Similarly $d'_\infty(e,(v;(x,y,z))) = |v| + d_{H_3}(e,(x,y,z-v)))$. If d_∞ and d'_∞ were at a bounded distance, we would have a $C > 0$ such that for all $t > 0$

$$|d_\infty(e,(t;(0,0,t))) - t| \le C.$$

Hence $|d_{H_3}(e,(0,0,t))| \le C$, which is a contradiction.

(B) Now let

$$\Omega = \{(1;(0,0,1))^{\pm1}, (1;(0,0,-1))^{\pm1}, (0;(1,0,0))^{\pm1}, (0;(0,1,0))^{\pm1}\}$$

be a generating set for Γ and ρ_Ω the word metric associated to it. Let $|\cdot|$ be a homogeneous quasi-norm on N which is at a bounded distance from ρ_Ω, i.e. $|\rho_\Omega(e,g) - |g||$ is bounded. Then $|\cdot|$ is asymptotic to ρ_Ω, hence is equal to the Carnot–Caratheodory Finsler metric d asymptotic to ρ_Ω and homogeneous with respect to the same one parameter group of dilations $\{\delta_t\}_{t>0}$. Let $m_1 = \{v \in \mathfrak{n}, \delta_t(v) = tv\}$. Then d is induced by some norm $\|\cdot\|_0$ on m_1, whose unit ball is given, according to Theorem 1.4 by the convex hull of the projections to m_1 of the generators in Ω. There is a unique vector in m_1 of the form $V + z_0 Z$. Its $\|\cdot\|_0$-norm is 1 and $d(e,(1;(0,0,z_0))) = 1$. However $d(e,(v;(x,y,z))) = |v| + d_{H_3}(e,(x,y,z-vz_0))$. Since $\rho_\Omega(e,(n;(0,0,n))) = n$, we get

$$d(e,(n;(0,0,n))) - \rho_\Omega(e,(n;(0,0,n))) = d_{H_3}(e,(0,0,n(1-z_0))).$$

If this is bounded, this forces $z_0 = 1$. But we can repeat the same argument with $(n;(0,0,-n))$ which would force $z_0 = -1$. A contradiction.

(C) Let now $\Omega_2 := \{(1;(0,0,0))^{\pm1}, (0;(1,0,0))^{\pm1}, (0;(0,1,0))^{\pm1}\}$ and ρ_{Ω_2} the associated word metric on Γ. Then again ρ_Ω and ρ_{Ω_2} are asymptotic by Theorem 6.2 because the convex hull of their projection modulo the z-coordinate coincide. However ρ_{Ω_2} is a product metric, namely we have $\rho_{\Omega_2}(e,(v;(x,y,z))) = |v| + \rho(e,(x,y,z))$, where ρ is the word metric on the discrete Heisenberg group $H_3(\mathbb{Z})$ with standard generators $\{(1,0,0)^{\pm1}, (0,1,0)^{\pm1}\}$. In particular

$$\rho_\Omega(e,(n;(0,0,n))) - \rho_{\Omega_2}(e,(n;(0,0,n))) = \rho(e,(0,0,n))$$

which is unbounded.

Remark 8.2 (An abnormal geodesic). We refer the reader to [9] for more on these examples. In particular we show there that ρ_1 and ρ_2 above are not $(1,C)$-quasi-isometric for any $C > 0$. The key phenomenon behind this example is the presence of an *abnormal geodesic* (see [25]), namely the one-parameter group $\{(t;(0,0,0))\}_t$.

Remark 8.3 (Speed of convergence in the nilpotent case). The slow speed phenomenon in Theorem 8.1 relied crucially on the presence of a non-trivial semisimple part in G_α; this does not occur in nilpotent groups. In [9], we show that for word metrics on finitely generated nilpotent groups, the convergence in Theorem 6.2 has a polynomial speed with an error term at least as good as $O(d_\infty(e,x)^{-\frac{2}{3r}})$, where r is the nilpotency class. We conjecture there that the optimal exponent is $\frac{1}{2}$. This involves refining quantitatively the estimates of the above proof of Theorem 6.2.

9. Appendix: the Heisenberg groups

Here we show how to compute the asymptotic shape of balls in the Heisenberg groups $H_3(\mathbb{Z})$ and $H_5(\mathbb{Z})$ and their volume, thus giving another approach to the main result of Stoll [33]. The leading term for the growth of $H_3(\mathbb{Z})$ is rational for all generating sets (Proposition 9.1 below), whereas in $H_5(\mathbb{Z})$ with its standard generating set, it is transcendental. This explains how our Figure 1.2 was made (compare with the odd [22], Figure 2).

9.1. 3-dim Heisenberg group. Let us first consider the Heisenberg group

$$H_3(\mathbb{Z}) = \langle a, b | [a,[a,b]] = [b,[a,b]] = 1 \rangle .$$

We see it as the lattice generated by $a = \exp(X)$ and $b = \exp(Y)$ in the real Heisenberg group $H_3(\mathbb{R})$ with Lie algebra $\mathfrak{h}_3$ generated by X, Y and spanned by $X, Y, Z = [X, Y]$. Let ρ_Ω be the standard word metric on $H_3(\mathbb{Z})$ associated to the generating set $\Omega = \{a^{\pm 1}, b^{\pm 1}\}$. According to Theorem 1.4, the limit shape of the n-ball Ω^n in $H_3(\mathbb{Z})$ coincides with the unit ball $\mathcal{C}_3 = \{g \in H_3(\mathbb{R}), d_\infty(e,g) \leq 1\}$ for the Carnot–Caratheodory metric d_∞ induced on $H_3(\mathbb{R})$ by the ℓ^1-norm $\|xX + yY\|_0 = |x| + |y|$ on $m_1 = \text{span}\{X, Y\} \subset \mathfrak{h}_3$.

Computing this unit ball is a rather simple task. Exchanging the roles of X and Y, we see that $\mathcal{C}_3$ is invariant under the reflection $z \mapsto -z$. Then clearly $\mathcal{C}_3$ is of the form $\{xX + yY + zZ, \text{ with } |x| + |y| \leq 1 \text{ and } |z| \leq z(x,y)\}$. Changing X to $-X$ and Y to $-Y$, we get the symmetries $z(x,y) = z(-x,y) = z(x,-y) = z(y,x)$. Hence when determining $z(x,y)$, we may assume $0 \leq y \leq x \leq 1$, $x + y \leq 1$.

The following well known observation is crucial for computing $z(x,y)$. If $\xi(t)$ is a horizontal path in $H_3(\mathbb{R})$ starting from id, then $\xi(t) = \exp(x(t)X + y(t)Y + z(t)Z)$, where $\xi'(t) = x(t)X + y(t)Y$ and $z(t)$ is the "balayage" area of the between the path $\{x(s)X + y(s)Y\}_{0 \leq s \leq t}$ and the chord joining 0 to $x(t)X + y(t)Y$.

Therefore, $z(x,y)$ is given by the solution to the "Dido isoperimetric problem" (see [25]): find a path in the X, Y-plane between 0 and $xX + yY$ of $\|\cdot\|_0$-length 1 that maximizes the "balayage area". Since $\|\cdot\|_0$ is the ℓ^1-norm in the X, Y-plane, as is well-known (see [8]), such extremal curves are given by arcs of square with sides parallel to the X, Y-axes. There is therefore a dichotomy: the arc of square has either 3 or 4 sides (it may have 1 or 2 sides, but these are included are limiting cases of the previous ones).

If there are 3 sides, they have length ℓ, x and $y+\ell$ with $y+\ell \leq x$. Hence $1=\ell+x+y+\ell$ and $z(x,y)=\ell x+\frac{1}{2}xy$. Therefore this occurs when $y \leq 3x-1$ and we then have $z(x,y)=\frac{x(1-x)}{2}$.

If there are 4 sides, they have length $\ell, x+u, y+\ell$ and u, with $\ell+y=x+u$. Hence $1=2\ell+2u+x+y$ and $z(x,y)=(\ell+y)(x+u)-\frac{xy}{2}$. This occurs when $y \geq 3x-1$ and we then have $z(x,y)=\frac{(1+x+y)^2}{16}-\frac{xy}{2}$.

Hence if $0 \leq y \leq x \leq 1$ and $x+y \leq 1$

$$z(x,y)=1_{y\leq 3x-1}\frac{x(1-x)}{2}+1_{y>3x-1}\frac{(1+x+y)^2}{16}-\frac{xy}{2}. \tag{40}$$

The unit ball $\mathcal{C}_3$ drawn in Figure 1.2 is the solid body $\mathcal{C}_3=\{xX+yY+zZ$, with $|x|+|y|\leq 1$ and $|z|\leq z(x,y)\}$.

A simple calculation shows that $\mathrm{vol}(\mathcal{C}_3)=\frac{31}{72}$ in the Lebesgue measure $dxdydz$. Since $H_3(\mathbb{Z})$ is easily seen to have co-volume 1 for this Haar measure on $H_3(\mathbb{R})$ (actually $\{xX+yY+zZ, x\in[0,1), y\in[0,1), z\in[0,1)\}$ is a fundamental domain), it follows that

$$\lim_{n\to\infty}\frac{\#(\Omega^n)}{n^4}=\mathrm{vol}(\mathcal{C}_3)=\frac{31}{72}.$$

We thus recover a well-known result (see [4] and [31] where even the full growth series is computed and shown to be rational).

One can also determine exactly which points of the sphere $\partial\mathcal{C}_3$ are joined to id by a unique geodesic horizontal path. The reader will easily check that uniqueness fails exactly at the points $(x,y,\pm z(x,y))$ with $|x|<\frac{1}{3}$ and $y=0$, or $|y|<\frac{1}{3}$ and $x=0$, or else at the points (x,y,z) with $|x|+|y|=1$ and $|z|<z(x,y)$.

The above method also yields the following result.

Proposition 9.1. *Let Ω be any symmetric generating set for $H_3(\mathbb{Z})$. Then the leading coefficient in $\#(\Omega^n)$ is rational, i.e.*

$$\lim_{n\to\infty}\frac{\#(\Omega^n)}{n^4}=r$$

is a rational number.

Proof. We only sketch the proof here. We can apply the method above and compute r as the volume of the unit CC-ball $\mathcal{C}(\Omega)$ of the limit CC-metric d_∞ defined in Theorem 1.4. Since we know what is the norm $\|\cdot\|$ in the (x,y)-plane $m_1 = span\langle X,Y\rangle$ that generates d_∞ (it is the polygonal norm given by the convex hull of the points of Ω), we can compute $\mathcal{C}(\Omega)$ explicitly. We need to know the solution to Dido's isoperimetric problem for $\|\cdot\|$ in m_1, and as is well known (see [8]) it is given by polygonal lines from the dual polygon rotated by 90°. Since the polygon defining $\|\cdot\|$ is made of rational lines (points in Ω have integer coordinates), any vector with rational coordinates has rational $\|\cdot\|$-length, and the dual polygon is also rational. The equations defining $z(x, y)$ will therefore have only rational coefficients, and $z(x, y)$ will be piecewisely given by a rational quadratic form in x and y, where the pieces are rational triangles in the (x, y)-plane. The total volume of $\mathcal{C}(\Omega)$ will therefore be rational. □

9.2. 5-dim Heisenberg group. The Heisenberg group $H_5(\mathbb{Z})$ is the group generated by a_1, b_1, a_2, b_2, c with relations $c = [a_1, b_1] = [a_2, b_2]$, a_1 and b_1 commute with a_2 and b_2 and c is central. Let $\Omega = \{a_i^{\pm 1}, b_i^{\pm 1}, i = 1, 2\}$. Let us describe the limit shape of Ω^n. Again, we see $H_5(\mathbb{Z})$ as a lattice of co-volume 1 in the group $H_5(\mathbb{R})$ with Lie algebra $\mathfrak{h}_5$ spanned by $X_1, Y_1 X_2, Y_2$ and $Z = [X_i, Y_i]$. By Theorem 1.4, the limit shape is the unit ball $\mathcal{C}_5$ for the Carnot–Caratheodory metric on $H_5(\mathbb{R})$ induced by the ℓ^1-norm $\|x_1 X_1 + y_1 Y_1 + x_2 X_2 + y_2 Y_2\|_0 = |x_1| + |y_1| + |x_2| + |y_2|$.

Since X_1, Y_1 commute with X_2, Y_2, in any piecewise linear horizontal path in $H_5(\mathbb{R})$, we can swap the pieces tangent to X_1 or Y_1 with those tangent to X_2 or Y_2 without changing the end point of the path. Therefore if $\xi(t) = \exp(x_1(t)X_1 + y_1(t)Y_1 + x_2(t)X_2 + y_2(t)Y_2 + z(t)Z)$ is a horizontal path, then $z(t) = z_1(t) + z_2(t)$, where $z_i(t)$, $i = 1, 2$, is the "balayage area" of the plane curve $\{x_i(s)X_i + y_i(s)Y_i\}_{0\le s\le t}$.

Since, just like for $H_3(\mathbb{Z})$, we know the curve maximizing this area, we can compute the unit ball $\mathcal{C}_5$ explicitly. In exponential coordinates it will take the form $\mathcal{C}_5 = \{\exp(x_1X_1 + y_1Y_1 + x_2X_2 + y_2Y_2 + zZ), |x_1| + |y_1| + |x_2| + |y_2| \le 1 \text{ and } |z| \le z(x_1, y_1, x_2, y_2)\}$. Then $z(x_1, y_1, x_2, y_2) = \sup_{0\le t\le 1}\{z_t(x_1, y_1) + z_{1-t}(x_2, y_2)\}$, where $z_t(x, y)$ is the maximum "balayage area" of a path of length t between 0 and $xX + yY$. It is easy to see that $z_t(x, y) = t^2 z(x/t, y/t)$ where z is given by (40). Hence z_t is a piecewise quadratic function of t. Again $z(x_1, y_1, x_2, y_2)$ is invariant under changing the signs of the x_i,y_i's, and swapping x and y, or else swapping 1 and 2. We may thus assume that the x_i,y_i's lie in $D = \{0 \le y_i \le x_i \le 1$ and $x_1 + y_1 + x_2 + y_2 \le 1$, and $x_2 - y_2 \ge x_1 - y_1\}$. We may therefore determine explicitly the supremum $z(x_1, y_1, x_2, y_2)$, which after some straightforward calculations takes on D the form

$$z(x_1, y_1, x_2, y_2) = 1_A \max\{d_1, d_2\} + 1_B \max\{d_1, c_1\} + 1_C \max\{c_1, c_2\}$$

where $d_1 = \frac{x_1 y_1}{2} + \frac{x_2}{2}(1 - x_1 - y_1 - x_2)$, $c_1 = \frac{1}{16}(1 + x_1 + y_1 - x_2 - y_2)^2 + \frac{x_2 y_2 - x_1 y_1}{2}$, and d_2 and c_2 are obtained from d_1 and c_1 by swapping the indices 1 and 2. The sets A, B and C form the following partition of D : $A = D \cap \{m \le x_1 - y_1\}$, $B = D \cap \{x_1 - y_1 < m < x_2 - y_2\}$ and $C = D \cap \{x_2 - y_2 \le m\}$, where $m = (1 - x_1 - x_2 - y_1 - y_2)/2$.

Since $\mathcal{C}_5$ has such an explicit form, it is possible to compute its volume. The fact that $z(x_1, y_1, x_2, y_2)$ is piecewisely given by the maximum of two quadratic forms makes the computation of the integral somewhat cumbersome but tractable. Our equations coincide (fortunately!) with those of Stoll (appendix of [33]), where he computed the main term of the asymptotics of $\#(\Omega^n)$ by a different method. Stoll did calculate that integral and obtained

$$\lim_{n\to\infty}\frac{\#(\Omega^n)}{n^6}=\mathrm{vol}(\mathcal{C}_5)=\frac{2009}{21870}+\frac{\log(2)}{32805}$$

which is transcendental. It is also easy to see by this method that if we change the generating set to $\Omega_0=\{a_1^{\pm1}b_1^{\pm1}a_2^{\pm1}b_2^{\pm1}\}$, then we get a rational volume. Hence the rationality of the growth series of $H_5(\mathbb{Z})$ depends on the choice of generating set, which is Stoll's theorem.

One advantage of our method is that it can also apply to fancier generating sets. The case of Heisenberg groups of higher dimension with the standard generating set is analogous: the function $z(\{x_i\},\{y_i\})$ is again piecewisely defined as the maximum of finitely many explicit quadratic forms on a linear partition of the ℓ^1-unit ball $\sum|x_i|+|y_i|\leq 1$.

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