

## A normal subgroup theorem for commensurators of lattices

Darren Creutz and Yehuda Shalom<sup>1</sup>

*Dedicated with admiration and affection to Pierre de la Harpe*

**Abstract.** We establish a general normal subgroup theorem for commensurators of lattices in locally compact groups. While the statement is completely elementary, its proof, which rests on the original strategy of Margulis in the case of higher rank lattices, relies heavily on analytic tools pertaining to amenability and Kazhdan’s property (T). It is a counterpart to the normal subgroup theorem for irreducible lattices of Bader and the second named author, and may also be used to sharpen that result when one of the ambient factors is totally disconnected.

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### 1. Introduction

An interesting feature of Margulis’ rigidity theory of lattices in semisimple algebraic groups is the parallelism between results in the higher rank setting, and those for dense commensurators of lattices with no rank assumption. Most notable here is Margulis’ superrigidity, which was fundamental to the proof of the arithmeticity theorem of higher rank lattices. Along with it was established an analogous rigidity result, key to the striking arithmeticity of *all* lattices with dense commensurators (see [22]).

Over the years Margulis’ rigidity theory has been widely extended to lattices in a general setting, first of more geometric character (e.g. [20], [3], and [4]) and then, following [28], to the very abstract “higher rank” framework of (irreducible lattices in) products of at least two locally compact groups – cf. [23], [10], [7], [9], [2], [24], and [25]. The parallelism between the theories of “higher rank lattices” and “rank free dense commensurators” remained (see also [26]), and essentially all rigidity results in the former framework found a counterpart in the latter. All but one – the celebrated normal subgroup theorem (abbreviated NST hereafter).

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In [5] a general NST was established by Bader and the second author for irreducible lattices in products of at least two locally compact, compactly generated groups (see also [4] in the case of tree lattices). The main purpose of this paper is to establish an analogous “rank free” result for dense commensurators of lattices in one such group. This theorem also allows for a sharpening of [5] when one of the ambient factors is totally disconnected. As in [5], and following the original spirit of Margulis’ NST, the precise statement remains of entirely elementary and purely group theoretic nature, while the proof employs heavy analytic techniques pertaining to Kazhdan’s property (T) and amenability.

Here is the main result of the paper.

**Theorem 1.1** (Theorem 3.1 and Proposition 4.1). *Let  $G$  be a locally compact, second countable compactly generated group, which is not a compact extension of an abelian group.*

*Let  $\Gamma < G$  be a discrete co-compact subgroup, or more generally, a finitely generated square integrable lattice (see Section 3.2 below), and  $\Lambda < G$  be a dense subgroup which contains and commensurates  $\Gamma$  (i.e.  $\lambda\Gamma\lambda^{-1} \cap \Gamma$  has finite index in both  $\lambda\Gamma\lambda^{-1}$  and  $\Gamma$ , for all  $\lambda \in \Lambda$ ).*

*If every closed, normal, non-cocompact subgroup of  $G$  has finite intersection with  $\Lambda$ , then every infinite normal subgroup  $N \triangleleft \Lambda$  contains a finite index subgroup of  $\Gamma$ . (The converse holds as well, and is easily verified).*

*Consequently, if  $\varphi: \Lambda \rightarrow H$  is a dense homomorphism into a locally compact totally disconnected group  $H$ , such that*

$$K := \overline{\varphi(\Gamma)} < H$$

*is compact open and  $\varphi^{-1}(K) = \Gamma$ , then there is a natural bijection between commensurability classes of infinite normal subgroups of  $\Lambda$ , and commensurability classes of open normal subgroups of  $H$ .*

It turns out that a group  $H$  as in the second part of the Theorem always exists, and in fact there is a unique minimal one – the relative profinite completion  $\Lambda // \Gamma$  of  $\Lambda$  with respect to  $\Gamma$ , akin to the quotient by a normal subgroup (see [30] for details). Note that it immediately follows from the Theorem that given any such  $H$ , every infinite normal subgroup of  $\Lambda$  is of finite index when (and precisely when)  $H$  has the same property for its own open subgroups. A good illustration is that of  $\Gamma = \mathrm{SL}_n(\mathbb{Z}) < \Lambda = \mathrm{SL}_n(\mathbb{Z}[\frac{1}{p}]) < \mathrm{SL}_n(\mathbb{R})$ . One can take here  $H = \mathrm{SL}_n(\mathbb{Q}_p)$  to deduce the well known normal subgroup theorem for  $\Lambda$  (the square integrability condition requires that  $n > 2$ ).

The restriction of  $G$  not being a compact extension of abelian is necessary. Indeed, consider the chain  $\Gamma = \mathbb{Z}^n < \Lambda = \mathrm{SO}(n, \mathbb{Q}) \ltimes (\mathbb{Q}[\sqrt{2}])^n < G = \mathrm{SO}(n, \mathbb{R}) \ltimes \mathbb{R}^n$ . Here  $\Gamma < G$  sits co-compactly (in the abelian part) and the latter is “just non-compact”, having only one non-trivial normal subgroup –  $\mathbb{R}^n$  – which is co-compact.

Hence the main condition of the Theorem is automatically satisfied for all  $\Lambda < G$ . However, the normal subgroup  $N = \sqrt{2} \mathbb{Q}^n \triangleleft \Lambda$  (again embedded in the abelian part of  $\Lambda$ ) intersects  $\Gamma$  only trivially.

It is perhaps worth remarking how our main Theorem serves as the natural commensurator analogue of the NST for lattices. Recall that in this theory counterpart results for homomorphisms of dense commensurators yield the same conclusion as those for higher rank lattices, under the crucial assumption that the image of the homomorphism, when restricted to the lattice, is “large” in an appropriate sense (depending on the setting in question). Our main theorem is equivalent to the statement that abstract group epimorphisms of  $\Lambda$  with infinite kernel, must have finite image – precisely as in the conclusion of the NST – once their restriction to the lattice is “large”. This “largeness” property should be interpreted as co-finiteness in our entirely abstract setting (see the proof of the Theorem in Section 3, where this equivalent formulation is being established and used).

**1.1. Dense commensurators.** By their nature, rigidity results for homomorphisms of commensurators are not as sharp as those for higher rank lattices, being conditioned on the “largeness” of the restriction of the homomorphism to the lattice. However, the advantage of this setting is that to date there are many more concrete examples of dense commensurators (in well understood groups  $G$ , which are essentially simple). One could hope that our main theorem shall motivate further investigation of the exotic locally compact groups  $H = \Lambda // \Gamma$  which naturally arise in this setting. In some cases, this should lead to a full normal subgroup theorem for the commensurators. The outstanding example here is that when  $G$  is the full automorphism group of a regular tree. Then the full commensurator of any uniform lattice is known to be dense [19], and in that case its simplicity is still an open question, first proposed by Lubotzky, Mozes, and Zimmer [20]. Exploring the group  $\Lambda // \Gamma$  for such  $\Lambda$ , or even for “smaller” (but still dense)  $\Lambda$ , would be of significant interest. Other exotic settings where density of the commensurator is established and to which the Main Theorem applies appear in [11], [26], [12], and [18]. We remark that at least the “amenability half” of the Main Theorem applies in various known cases of non-uniform lattices with dense commensurators, which are not finitely generated, as in [1].

**1.2. Irreducible lattices in products of groups.** As mentioned earlier, our main theorem yields a sharpening of the NST for irreducible lattices in product of groups due to Bader and the second author [5], once one of the ambient factors is totally disconnected. The resulting theorem becomes an “if and only if” statement.

**Corollary 1.2.** *Let  $G$  be a locally compact, second countable compactly generated group, which is not a compact extension of an abelian group, and  $H$  be any totally disconnected locally compact group. Let  $\Lambda < G \times H$  be an irreducible (i.e. has dense projections to the factors) co-compact discrete subgroup. Then each one of the*

following three conditions is necessary, and combined together they are sufficient, for the property that every infinite normal subgroup of  $\Lambda$  has finite index:

- (i) the intersection of  $\Lambda$  with  $H$  is finite;
- (ii) the group  $H$  has no infinite index open normal subgroups;
- (iii) the projection of  $\Lambda$  to  $G$  intersects finitely its closed normal non-cocompact subgroups.

Of course, in typical applications of this result the normal subgroup structure of  $G$  and  $H$  would be so (simple and) well understood, that the verification of these conditions is immediate. A somewhat more technical version of the result when  $\Lambda$  is non-uniform is discussed following its proof, in Section 5. Note that here (unlike [5]), we do not assume that  $H$  is compactly generated, and  $\Lambda$  may not be finitely generated. This variant is relevant e.g. in the adelic setting, giving a NST for groups of the form  $\mathbf{G}(K)$  where  $\mathbf{G}$  is a simple, simply connected algebraic group defined over a global field  $K$ ; see the discussion in Section 5. Finally, note that it is easy to turn the above result into a “just infinite” property of  $\Lambda$  once the necessary “no finite normal subgroups” condition is imposed further on  $G$  and  $H$ .

**1.3. On the approach.** As indicated, the general strategy of the proof of the Main Theorem follows the original one introduced by Margulis, and consists of two entirely independent “halves”: one pertaining to property (T) and the other to amenability, of an appropriate quotient group, which together yield finiteness. The property (T) half rests on considerations regarding reduced cohomology of unitary representations, primarily relying on results from [28], as in the property (T) half of the normal subgroup theorem for irreducible lattices. Here, however, the *irreducibility* of the cohomological representation plays a crucial role, and an additional significant input is given by a result of Gelernder, Karlsson, and Margulis [10].

A key notion in the proof of the amenability half is that of contractive (or SAT) group actions. A result of independent interest on which the proof is based is our Contractive Factor Theorem along the lines of Margulis’ original Factor Theorem (used in the amenability half of his NST; see [21]), and of the Factor Theorem in [5].

**Theorem** (Contractive factor theorem – Theorem 2.8). *Let  $G$  be a locally compact second countable group,  $\Gamma$  a lattice in  $G$ , and  $\Lambda < G$  a dense subgroup which contains and commensurates  $\Gamma$ .*

*Let  $(X, \nu)$  be a probability  $G$ -space such that the restriction of the  $G$ -action to  $\Gamma$  is contractive, and let  $(Y, \eta)$  be a  $\Lambda$ -space such that there exists a  $\Gamma$ -map*

$$\varphi: (X, \nu) \longrightarrow (Y, \eta).$$

*Then the  $\Lambda$ -action on  $(Y, \eta)$  extends measurably to  $G$ , in such a way that  $\varphi$  becomes a  $G$ -map. More precisely, there exists a  $G$ -space  $(Y', \eta')$ , a  $G$ -map*

$$\varphi': (X, \nu) \longrightarrow (Y', \eta')$$

and a  $\Lambda$ -isomorphism

$$\rho: (Y, \eta) \longrightarrow (Y', \eta')$$

such that

$$\varphi' = \rho \circ \varphi \quad a.e.$$

Contractive spaces, introduced by Jaworski [14] and [15] under the name strongly approximately transitive (SAT), are the extreme opposite of measure-preserving:  $G$  acts on  $(X, \nu)$  in such a way that for any measurable set  $B$  that is not conull there exists a sequence  $\{g_n\}$  in  $G$  along which  $\nu(g_n B) \rightarrow 0$ . Jaworski introduced this property to study the Choquet–Deny property on groups and showed that Poisson boundaries are contractive.

The topological analogue of this measurable phenomenon is the following (see Furstenberg and Glasner [8]): a continuous action of a group  $G$  on a compact metric space  $X$  with a quasi-invariant Borel probability measure  $\nu$  on  $X$  is contractible when for every  $x \in X$  there exists  $g_n \in G$  such that  $g_n \nu \rightarrow \delta_x$  weakly. In [8] it is shown that an action is measurably contractive if and only if every continuous compact model of it is contractible; it is natural to refer to  $G$ -spaces for which all models are contractible as contractive and for this reason we adopt the somewhat more descriptive terminology “contractive”. It seems that this interesting notion deserves considerably more attention; see also the recent [6].

## 2. Contractive actions

Contractive actions, introduced by Jaworski [14] and [15], with the idea going back to [13], have been studied by Kaimanovich [16] and by Furstenberg and Glasner [8]. Our aim here is to prove the Contractive Factor Theorem (Theorem 2.8) which plays the role of Margulis’ Factor Theorem for boundary actions of semisimple groups.

In this paper all probability measure spaces are assumed standard. Recall that such a space  $(X, \nu)$  is a  $G$ -space when  $G$  acts measurably on (a measure one subset of)  $X$  such that  $\nu$  is quasi-invariant under the  $G$ -action (meaning the measure class is preserved). A measurable map  $\varphi: (X, \nu) \rightarrow (Y, \eta)$  between  $G$ -spaces is a  $G$ -map when  $\varphi$  is  $G$ -equivariant ( $\varphi(gx) = g\varphi(x)$  for all  $g \in G$  and  $\nu$ -almost every  $x \in X$ ), and  $\varphi_* \nu = \eta$ .

**Definition 2.1** (Jaworski [14]). A  $G$ -space  $(X, \nu)$  is *contractive*, also called *SAT* (*strongly approximately transitive*), when for every measurable  $B \subseteq X$  with  $\nu(B) < 1$  and every  $\epsilon > 0$  there exists  $g \in G$  such that  $\nu(gB) < \epsilon$ .

The primary source of examples of contractive spaces is (quotients of) Poisson boundaries, and in the case of measure classes admitting a stationary measure we are not familiar with others.

**Definition 2.2** (Furstenberg and Glasner [8]). Let  $G$  be a locally compact second countable group acting continuously on a compact metric space  $X$ , and let  $\nu$  be a  $G$ -quasi-invariant Borel probability measure on  $X$ . The action is *contractible* if for every  $x \in X$  there exists a sequence  $g_n \in G$  such that  $\lim_n g_n \nu = \delta_x$  weakly (where  $\delta_x =$  point mass at  $x$ ).

**Theorem** (Furstenberg and Glasner [8]). *The action on a probability space is contractive if and only if every continuous compact model of it is contractible.*

**2.1. Lattices act contractively.** The following observation is Proposition 3.3 in [15]. For completeness we give a (different) proof.

**Proposition 2.3.** *Let  $G$  be a locally compact second countable group and  $\Gamma$  a cocompact subgroup of  $G$ . Let  $(X, \nu)$  be a contractive  $G$ -space. Then restricting the action to  $\Gamma$  makes  $(X, \nu)$  a contractive  $\Gamma$ -space.*

*Proof.* Let  $K \subseteq G$  be a compact set such that  $K\Gamma = G$ . Let  $B \subseteq X$  such that  $\nu(B) < 1$ . Then there exists  $g_n \in G$  such that  $\nu(g_n B) \rightarrow 0$  since the  $G$ -action is contractive. Write  $g_n = k_n \gamma_n$  for  $k_n \in K$  and  $\gamma_n \in \Gamma$ . Since  $K$  is compact there is a convergent subsequence  $k_{n_j} \rightarrow k_\infty$ . It is a standard fact (the proof is left to the reader) that if  $A_j$  is a sequence of measurable sets such that  $\nu(A_j) \rightarrow 0$  and  $\ell_j \rightarrow \ell_\infty$  is a convergent sequence then  $\nu(\ell_j A_j) \rightarrow 0$ . Applying this to  $A_j = g_{n_j} B$  and  $\ell_j = k_{n_j}^{-1}$  gives  $\nu(\gamma_{n_j} B) = \nu(k_{n_j}^{-1} g_{n_j} B) \rightarrow 0$ .  $\square$

The case of non-uniform lattices turns out to be more difficult, but for our purposes it will suffice to establish the result for actions on (quotients of) the Poisson boundary of the ambient group. However, the first author and J. Peterson [6] have, subsequent to our work, shown that the existence of a stationary measure in the class suffices: if  $\Gamma < G$  is a lattice, and  $G$  acts contractively on a stationary  $G$ -space, then so does  $\Gamma$ . The general case remains conjectured.

**Proposition 2.4.** *Let  $G$  be a locally compact second countable group and  $\Gamma$  a lattice in  $G$ . Then the action of  $\Gamma$  on any (quotient of the) Poisson boundary of  $G$  relative to a symmetric measure with support generating  $G$  is contractive.*

*Proof.* Let  $(X, \nu)$  be any compact model of (a quotient of) the Poisson boundary for  $(G, \mu)$ , where  $\mu$  is a symmetric Borel probability measure on  $G$  with support generating  $G$ . The key feature of the Poisson boundary of use to us is that for any  $L^\infty$ -function  $f$  on  $X$  (and in particular for any continuous  $f$ ), the maps  $\varphi_n: G^{\mathbb{N}} \rightarrow \mathbb{R}$  defined by  $\varphi_n(\omega_1, \omega_2, \dots) = \int f(\omega_1 \cdots \omega_n x) d\nu(x)$  form a martingale (by the stationarity of  $\nu$ ) and hence converge  $\mu^{\mathbb{N}}$ -almost surely. Therefore, for  $\mu^{\mathbb{N}}$ -almost every  $\omega \in G^{\mathbb{N}}$  the measures  $\omega_1 \cdots \omega_n \nu$  converge weakly to some limit measure  $\nu_\omega$ . In fact,  $\nu_\omega$  is a point mass almost surely (which characterizes quotients of the Poisson boundary), thus showing that a boundary action is contractive. The stationarity

of  $\nu$  implies that for any measurable set  $A$ ,  $\int \nu_\omega(A) d\mu^{\mathbb{N}}(\omega) = \nu(A)$ , and therefore  $\mu^{\mathbb{N}}\{\omega: \nu_\omega(A) = 1\} = \nu(A)$ .

Let  $m$  be the Haar measure on  $G/\Gamma$  normalized to be a probability measure. Recall that Kakutani's random ergodic theorem [17] states that for any  $f \in L^\infty(G/\Gamma)$ ,  $m$ -almost every  $z \in G/\Gamma$  and  $\mu^{\mathbb{N}}$ -almost every  $\omega \in G^{\mathbb{N}}$ ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(\omega_n \cdots \omega_1 z) = \int f d\nu.$$

Let  $K_0$  be any open bounded subset of  $G$ . Write  $K$  for the finite (positive) measure subset of  $G/\Gamma$  which is the image of  $K_0$  under the quotient map. Let  $\mathbb{1}_K$  be the characteristic function of  $K$ . By the Random Ergodic Theorem, for  $m$ -almost every  $z$  and  $\mu^{\mathbb{N}}$ -almost every  $\omega \in G^{\mathbb{N}}$ ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mathbb{1}_K(\omega_n \cdots \omega_1 z) = m(K) > 0.$$

Pick  $z \in G/\Gamma$  such that the above holds for  $\mu^{\mathbb{N}}$ -almost every  $\omega$ . Then  $\omega_n \cdots \omega_1 z \in K$  infinitely often  $\mu^{\mathbb{N}}$ -almost surely. As  $\mu$  is symmetric, also  $\omega_n^{-1} \cdots \omega_1^{-1} z \in K$  infinitely often  $\mu^{\mathbb{N}}$ -almost surely.

Fix an arbitrary measurable set  $B \subseteq X$  with  $\nu(B) < 1$ . Let  $z_0 \in G$  be a representative of  $z \in G/\Gamma$ . Set  $A = z_0 B$  and note that  $\nu(A) < 1$  (by quasi-invariance). As recalled above,

$$\mu^{\mathbb{N}}\{\omega: \nu_\omega(A) = 0\} = 1 - \nu(A) > 0.$$

Pick  $\omega$  such that  $\nu_\omega(A) = 0$  and  $\omega_n^{-1} \cdots \omega_1^{-1} z \in K$  infinitely often (the intersection of a positive measure set with a full measure set is nonempty). Let  $\{n_j\}$  be the times such that  $\omega_{n_j}^{-1} \cdots \omega_1^{-1} z \in K$  (which is happening in  $G/\Gamma$ ). Then

$$\begin{aligned} 0 &= \nu_\omega(A) \\ &= \lim_n \omega_1 \cdots \omega_n \nu(A) \\ &= \lim_{j \rightarrow \infty} \nu(\omega_{n_j}^{-1} \cdots \omega_1^{-1} A) \\ &= \lim_{j \rightarrow \infty} \nu(\omega_{n_j}^{-1} \cdots \omega_1^{-1} z_0 B) \end{aligned}$$

and  $\omega_{n_j}^{-1} \cdots \omega_1^{-1} z_0 \in K_0 \Gamma$  for each  $j$  (since  $z_0 \in z\Gamma$  and  $\omega_{n_j}^{-1} \cdots \omega_1^{-1} z \in K$ ).

Write  $\omega_{n_j}^{-1} \cdots \omega_1^{-1} z_0 = k_j \gamma_j$  for  $k_j \in K_0$  and  $\gamma_j \in \Gamma$ . Then

$$\lim_{j \rightarrow \infty} \nu(k_j \gamma_j B) = 0.$$

Choose a subsequence  $j_\ell$  such that  $k_{j_\ell} \rightarrow k_\infty$  for some  $k_\infty \in K$  and set  $B_\ell = k_{j_\ell} \gamma_{j_\ell} B$ . Then  $\nu(B_\ell) \rightarrow 0$  and  $k_{j_\ell}^{-1} \rightarrow k_\infty^{-1}$ , so,  $\nu(k_{j_\ell}^{-1} B_\ell) \rightarrow 0$  (an easy exercise on the continuity of the  $G$ -action on  $L^1$ ). Hence  $\lim_{\ell \rightarrow \infty} \nu(\gamma_{j_\ell} B) = 0$ , and the  $\Gamma$ -action is contractive.  $\square$

**2.2. Uniqueness of quotients of contractive spaces.** We begin by observing the following simple basic fact.

**Lemma 2.5.** *Let  $(X, \nu)$  be a  $G$ -space and  $\nu'$  a Borel probability measure on  $X$  in the same measure class as  $\nu$ . Let  $\{g_n\}$  be a sequence in  $G$  and  $B \subseteq X$  a measurable set. If  $\nu(g_n B) \rightarrow 0$  then  $\nu'(g_n B) \rightarrow 0$ . In particular, if  $(X, \nu)$  is contractive then so is  $(X, \nu')$ .*

*Proof.* Suppose that  $\limsup \nu'(g_n B) = \delta > 0$ . Let  $\{n_j\}$  be the sequence attaining this limit. Then  $\nu(g_{n_j} B) \rightarrow 0$  and  $\nu'(g_{n_j} B) \rightarrow \delta$ . Pick a further subsequence  $\{n_{j_t}\}$  such that  $\nu(g_{n_{j_t}} B) < 2^{-t}$ . Define  $B_k = \bigcup_{t=k}^\infty g_{n_{j_t}} B$  and observe that  $\nu(B_k) \leq \sum_{t=k}^\infty \nu(g_{n_{j_t}} B) \leq \sum_{t=k}^\infty 2^{-t} = 2^{-k+1} \rightarrow 0$  but  $\nu'(B_k) \geq \nu'(g_{n_{j_k}} B) \rightarrow \delta$ . As the  $B_k$  are decreasing,  $\nu(\bigcap_k B_k) = 0$  but  $\nu'(\bigcap_k B_k) \geq \delta$  contradicting that the measures are in the same class.  $\square$

We will need a basic fact about the existence of compact models. This result does not seem to appear explicitly in the literature, but the proof is essentially contained in [32] and follows from [31].

**Lemma 2.6** (Varadarajan [31]). *Let  $G$  be a locally compact second countable group, let  $(X, \nu)$  and  $(Y, \eta)$  be  $G$ -spaces, let  $\nu'$  be a Borel probability measure in the same measure class as  $\nu$ , let  $\eta' \in P(Y)$  be in the same measure class as  $\eta$  and let  $\pi: (X, \nu) \rightarrow (Y, \eta)$  and  $\pi': (X, \nu') \rightarrow (Y, \eta')$  be  $G$ -maps.*

*Then there exists a continuous compact model for  $\pi$  and  $\pi'$ , i.e. there exist compact metric spaces  $X_0$  and  $Y_0$  on which  $G$  acts continuously, fully supported Borel probability measures  $\nu_0, \nu'_0$  on  $X$  and  $\eta_0, \eta'_0$  on  $Y$ , continuous  $G$ -equivariant maps*

$$\pi_0: X_0 \longrightarrow Y_0 \quad \text{and} \quad \pi'_0: X_0 \rightarrow Y_0$$

*and measurable  $G$ -isomorphisms*

$$\Phi: (X, \nu) \longrightarrow (X_0, \nu_0) \quad \text{and} \quad \Psi: (Y, \eta) \rightarrow (Y_0, \eta_0),$$

*which are also  $G$ -isomorphisms  $\Phi: (X, \nu') \rightarrow (X_0, \nu'_0)$  and  $\Psi: (Y, \eta') \rightarrow (Y_0, \eta'_0)$ , such that the resulting diagrams commute:*

$$\Psi^{-1} \circ \pi_0 \circ \Phi = \pi \quad \text{and} \quad (\Psi')^{-1} \circ \pi'_0 \circ \Phi = \pi'.$$



*Proof.* Let  $\mathcal{X}$  be a countable collection of functions in  $L^\infty(X, \nu) = L^\infty(X, \nu')$  that separates points and let  $\mathcal{Y}$  be a countable collection in  $L^\infty(Y, \eta)$  that separates points. Let  $\mathcal{F} = \mathcal{X} \cup \{f \circ \pi : f \in \mathcal{Y}\} \cup \{f \circ \pi' : f \in \mathcal{Y}\}$ . Then  $\mathcal{F}$  is a countable collection. Let  $B$  be the unit ball in  $L^\infty(G, \text{Haar})$  which is a compact metric space in the weak- $*$  topology.

Define  $X_{00} = \prod_{f \in \mathcal{F}} B$  and  $Y_{00} = \prod_{f \in \mathcal{Y}} B$ , both of which are compact metric spaces using the product topology. Define  $\pi_{00}: X_{00} \rightarrow Y_{00}$  to be the restriction map: for  $f \in \mathcal{Y}$  take the  $f^{\text{th}}$  coordinate of  $\pi_{00}(x_{00})$  to be the  $(f \circ \pi)^{\text{th}}$  coordinate of  $x_{00}$ . Then  $\pi_{00}$  is continuous. Likewise, define  $\pi'_{00}: X_{00} \rightarrow Y_{00}$  by setting the  $f^{\text{th}}$  coordinate of  $\pi'_{00}(x_{00})$  to be the  $(f \circ \pi')^{\text{th}}$  coordinate of  $x_{00}$ .

Define the map  $\Phi: X \rightarrow X_{00}$  by  $\Phi(x) = (\varphi_f(x))_{f \in \mathcal{F}}$  where  $(\varphi_f(x))(g) = f(gx)$ . Then  $\Phi$  is an injective map (since  $\mathcal{F}$  separate points). Observe that we have  $(\varphi_f(hx))(g) = f(ghx) = (\varphi_f(x))(gh)$ . Consider the  $G$ -action on  $X_{00}$  given by the right action on each coordinate. Then  $G$  acts on  $X_{00}$  continuously (and likewise on  $Y_{00}$  continuously) and  $\Phi$  is  $G$ -equivariant. Similarly, define  $\Psi: Y \rightarrow Y_{00}$  by  $\Psi(y) = (\psi_f(y))_{f \in \mathcal{Y}}$  where  $(\psi_f(y))(g) = f(gy)$ .

Let  $X_0 = \overline{\Phi(X)}$ , let  $\nu_0 = \Phi_*\nu$ , let  $Y_0 = \overline{\Psi(Y)}$ , let  $\eta_0 = \Psi_*\eta$  and let  $\pi_0$  be the restriction of  $\pi_{00}$  to  $X_0$ . Then both the map  $\Phi: (X, \nu) \rightarrow (X_0, \nu_0)$  and the map  $\Psi: (Y, \eta) \rightarrow (Y_0, \eta_0)$  are  $G$ -isomorphisms. Since  $(\psi_f(\pi(x)))(g) = f(g\pi(x)) = f \circ \pi(gx) = (\varphi_{f \circ \pi}(x))(g)$ ,  $\pi_0(X_0) = Y_0$  and  $\Psi^{-1} \circ \pi_0 \circ \Phi = \pi$ . Likewise, letting  $\nu'_0 = \Phi_*\nu'$ ,  $\eta'_0 = \Psi_*\eta'$ , and  $\pi'_0$  be the restriction of  $\pi'_{00}$  to  $X_0$ , one readily observes that  $\Psi^{-1} \circ \pi'_0 \circ \Phi = \pi'$ . □

**Remark.** *The previous result extends to a countable collection of  $G$ -maps*

$$\pi_n: (X, \nu_n) \longrightarrow (Y, \eta_n)$$

when the  $\nu_n$  are all in the same measure class.

**Proposition 2.7.** *Let  $G$  be a locally compact second countable group. Let  $(X, \nu)$  be a contractive  $G$ -space and  $(Y, \eta)$  be a  $G$ -space. Let  $\varphi: (X, \nu) \rightarrow (Y, \eta)$  and  $\varphi': (X, \nu) \rightarrow (Y, \eta')$  be  $G$ -maps such that  $\eta$  and  $\eta'$  are in the same measure class. Then  $\varphi = \varphi'$  almost everywhere.*

*Proof.* By Lemma 2.6, take  $X$  and  $Y$  to be compact metric spaces where  $G$  acts continuously and such that  $\varphi, \varphi': X \rightarrow Y$  are continuous maps. Since  $(X, \nu)$  is a contractive  $G$ -space, the model is contractible. Let  $x_0 \in X$  be arbitrary. Then there exists a sequence  $g_n \in G$  such that  $g_n \nu \rightarrow \delta_{x_0}$  weakly.

Since  $\varphi$  is continuous so is the pushforward map  $\varphi_*$  and therefore we obtain  $\varphi_*(g_n \nu) \rightarrow \varphi_*(\delta_{x_0})$ . By the  $G$ -equivariance of  $\varphi$  this means

$$g_n \eta = g_n(\varphi_*\nu) \rightarrow \varphi_*(\delta_{x_0}) = \delta_{\varphi(x_0)}.$$

Of course the same reasoning gives that  $g_n \eta' \rightarrow \delta_{\varphi'(x_0)}$ .

Let  $B \subseteq Y$  be any open set containing  $\varphi(x_0)$ . Then  $g_n \eta(B^C) \rightarrow \delta_{\varphi(x_0)}(B^C) = 0$  since  $B^C$  is a continuity set for  $\delta_{\varphi(x_0)}$  (the portmanteau Theorem). By Lemma 2.5,  $g_n \eta'(B^C) \rightarrow 0$  also so  $\varphi'(x_0) \in B$ . As this holds for all open sets  $B$  containing  $\varphi(x_0)$ , it follows that  $\varphi'(x_0) = \varphi(x_0)$ . Since  $x_0$  was arbitrary this means that  $\varphi = \varphi'$  as maps between the compact models. So  $\varphi = \varphi'$  measurably.  $\square$

### 2.3. The contractive factor theorem

**Theorem 2.8.** *Let  $G$  be a locally compact second countable group,  $\Gamma < G$  a lattice, and  $\Lambda < G$  a dense subgroup that contains and commensurates  $\Gamma$ .*

*Let  $(X, \nu)$  be a  $G$ -space such that the restriction of the action to  $\Gamma$  is contractive, and let  $(Y, \eta)$  be a  $\Lambda$ -space such that there exists a  $\Gamma$ -map  $\varphi: (X, \nu) \rightarrow (Y, \eta)$ .*

*Then the  $\Lambda$ -action on  $(Y, \eta)$  extends measurably to  $G$ , in such a way that  $\rho$  becomes a  $G$ -map. More precisely, there exists a  $G$ -space  $(Y', \eta')$ , a  $G$ -map*

$$\varphi': (X, \nu) \longrightarrow (Y', \eta')$$

*and a  $\Lambda$ -isomorphism  $\rho: (Y, \eta) \rightarrow (Y', \eta')$  such that  $\varphi' = \rho \circ \varphi$  a.e.*

*Proof.* Fix  $\lambda \in \Lambda$ . Since  $\Lambda$  commensurates  $\Gamma$ , the subgroup  $\Gamma_0 = \Gamma \cap \lambda^{-1}\Gamma\lambda$  is also a lattice in  $G$ . Consider the map  $\varphi_\lambda: X \rightarrow Y$  given by

$$\varphi_\lambda(x) := \lambda^{-1}\varphi(\lambda x)$$

Since  $\varphi$  is  $\Gamma$ -equivariant,  $\varphi_\lambda$  is  $\Gamma_0$ -equivariant: for  $\gamma_0 \in \Gamma_0$  we have  $\lambda\gamma_0\lambda^{-1} \in \Gamma$  and so

$$\begin{aligned} \varphi_\lambda(\gamma_0 x) &= \lambda^{-1}\varphi(\lambda\gamma_0 x) \\ &= \lambda^{-1}\varphi((\lambda\gamma_0\lambda^{-1})\lambda x) \\ &= \lambda^{-1}(\lambda\gamma_0\lambda^{-1})\varphi(\lambda x) \\ &= \gamma_0\varphi_\lambda(x). \end{aligned}$$

Let  $\eta = \varphi_*\nu$  be the pushforward of  $\nu$  to  $Y$  over  $\varphi$  and  $\eta' = (\varphi_\lambda)_*\nu$  be the pushforward over  $\varphi_\lambda$ . Then  $\eta$  and  $\eta'$  are in the same measure class: if  $\eta(A) = 0$  then  $\eta(\lambda A) = 0$  by the  $\Lambda$ -quasi-invariance of  $\eta$ , and therefore  $\nu(\varphi^{-1}(\lambda A)) = 0$ . But  $\eta'(A) = \lambda\nu(\varphi^{-1}(\lambda A))$  so by the  $\Lambda$ -quasi-invariance of  $\nu$  this is zero, hence the measures are in the same class.

By Proposition 2.3, the action of  $\Gamma_0$  on  $(X, \nu)$  is contractive since the  $\Gamma$ -action is. Since  $\varphi$  and  $\varphi_\lambda$  are both  $\Gamma_0$ -equivariant maps, one relative to a contractive  $\Gamma_0$ -space, and one relative to another measure in the class of the contractive measure, by Proposition 2.7,  $\varphi_\lambda = \varphi$  a.e. Hence for each  $\lambda$  we have that  $\lambda^{-1}\varphi(\lambda x) = \varphi(x)$  for almost every  $x$ , making  $\varphi$  a  $\Lambda$ -map.

Treating  $L^\infty(Y, \eta)$  as a  $\Lambda$ -invariant sub- $\sigma$ -algebra of the  $G$ -invariant  $\sigma$ -algebra  $L^\infty(X, \nu)$ , the density of  $\Lambda$  in  $G$  means that as a  $\sigma$ -algebra  $L^\infty(Y, \eta)$  is  $G$ -invariant. Then by Mackey's point realization there exists a  $G$ -space  $(Y', \eta')$  measurably  $\Lambda$ -isomorphic to  $(Y, \eta)$ , and a  $G$ -map  $(X, \nu) \rightarrow (Y', \eta')$  such that this map composed with the  $\Lambda$ -isomorphism is  $\varphi$ . □

### 3. Proof of the main theorem

**Theorem 3.1.** *Let  $G$  be a locally compact, second countable, compactly generated group that is not a compact extension of an abelian group.*

*Let  $\Gamma < G$  be a finitely generated square integrable lattice and let  $\Lambda < G$  be a dense subgroup of  $G$  that contains and commensurates  $\Gamma$ .*

*Then every infinite normal subgroup  $N \triangleleft \Lambda$  has the property that  $N \cap \Gamma$  has finite index in  $\Gamma$ , if and only if  $\Lambda$  intersects finitely every closed normal non-cocompact subgroup of  $G$ .*

Theorem 3.1 will be a consequence of the following Proposition. We shall first state it and prove that Theorem 3.1 follows from it, and then turn to proving this result.

**Proposition 3.2** (The Reduction Step). *Let  $\Gamma <_c \Lambda < G$  be as in Theorem 3.1, but with no structural restriction on  $G$ . Let  $N$  be a normal subgroup of  $\Lambda$  such that  $\Gamma$  maps onto  $\Lambda/N$  via the coset map  $\Lambda \rightarrow \Lambda/N$ , and  $\overline{[N, N]}$  is co-compact in  $G$  (hence  $\overline{N}$  is as well). Then  $\Lambda/N$  is finite.*

*Proof of Theorem 3.1 assuming Proposition 3.2.* Assume that  $\Lambda$  intersects finitely every closed normal non-cocompact subgroup of  $G$ , and let  $N \triangleleft \Lambda$  be any infinite normal subgroup. Then  $\overline{N} \triangleleft \overline{\Lambda} = G$  and since  $N$  is infinite and contained in  $\Lambda \cap \overline{N}$  it follows from the assumption of the Theorem that  $\overline{N}$  is co-compact in  $G$ . Now  $[N, N]$  is a characteristic normal subgroup of  $N$ , hence also  $[N, N] \triangleleft \Lambda$ . Then either  $[N, N]$  is finite, or it's infinite, in which case the exact same argument as before (with  $N$  replaced by  $[N, N]$ ) shows that  $\overline{[N, N]}$  is co-compact in  $G$ . We now observe that the first possibility cannot occur.

Indeed, it is a general fact that  $\overline{[N, N]} = \overline{[\overline{N}, \overline{N}]}$ , hence the assumed finiteness property of  $[N, N]$  implies that property for the left, hence also for the right hand side. Now  $\overline{N} < G$  is co-compact so it inherits compact generation from  $G$ . By the general Lemma 3.3 below it then follows from this finiteness property that the center  $Z(\overline{N})$  has finite index in  $\overline{N}$ , hence is co-compact as well in  $G$ . Being a characteristic normal subgroup of  $\overline{N}$ , it is also normal in  $G$ . Hence  $G$  is a compact extension of the abelian group  $Z(\overline{N})$ , contradicting the hypothesis of the Theorem. We conclude that the second possibility holds:  $\overline{[N, N]}$  is co-compact in  $G$ .

Let  $\Lambda' = \Gamma \cdot N$ . Then  $\Lambda'$  is a subgroup of  $\Lambda$  that contains and commensurates  $\Gamma$ . Clearly  $\Gamma$  maps onto  $\Lambda'/N$  via the coset map  $\gamma \mapsto \gamma N$ . We are now in position to apply Proposition 3.2 to the groups  $\Gamma < \Lambda' < \overline{\Lambda'}$  with  $N \triangleleft \Lambda'$  (the closure of  $[N, N]$ , being co-compact in  $G$ , is so in  $\overline{\Lambda'}$  as well). It follows from this Proposition that  $\Lambda'/N$  is finite. Then  $\Gamma/(\Gamma \cap N) \simeq (\Gamma \cdot N)/N$  is finite as well, so  $N$  contains a finite index subgroup of  $\Gamma$ , as required.

The reverse direction of Theorem 3.1 is easy, and we prove it for completeness. Assume that for every infinite normal subgroup  $N \triangleleft \Lambda$  it holds that  $N \cap \Gamma$  has finite index in  $\Gamma$ . We need to show that every closed  $M \triangleleft G$  which intersects  $\Lambda$  infinitely, is co-compact. Given such  $M$ , set  $N = M \cap \Lambda \triangleleft \Lambda$ , noting that here by the reverse assumption of the Theorem  $N \cap \Gamma = M \cap \Gamma$  has finite index in  $\Gamma$ . Since (every finite index subgroup of)  $\Gamma$  has co-finite Haar measure in  $G$ , it follows that so does the normal subgroup  $M \triangleleft G$ . Hence the group  $G/M$  has finite Haar measure, and is therefore compact, as required. This completes the reduction of the proof of Theorem 3.1 to Proposition 3.2, modulo the following general (and probably well known) Lemma. □

**Lemma 3.3.** *Let  $H$  be a compactly generated, second countable locally compact group, for which  $[H, H]$  is finite. Then the center  $Z(H)$  has finite index in  $H$ .*

*Proof.* Let  $K \subseteq H$  be a compact generating set. For  $x \in K$  consider the orbit of  $x$  under conjugation by  $H$ :  $h \mapsto h x h^{-1}$ . Since  $[H, H]$  is finite,  $h x h^{-1} x^{-1}$  takes on only finitely many values, so for each  $x$ , the orbit  $\{h x h^{-1} : h \in H\}$  is finite. Therefore  $H_x = \{h \in H : h x h^{-1} = x\}$  has finite index in  $H$ .

Each  $H_x$  is compactly generated since  $H$  is. Let  $Q_x \subseteq H_x$  be a compact generating set. For  $q \in Q_x$  observe that  $q x q^{-1} x^{-1} = e$ . By the continuity of the action of  $H$  on itself there is then an open neighborhood  $U_x$  of  $x$  such that  $q y q^{-1} y^{-1} = e$  for all  $q \in Q_x$  and all  $y \in U_x$ . This can be seen as follows: if no such neighborhood exists then there exists  $x_n \rightarrow x$  and  $q_n \in Q_x$  such that  $q_n x_n q_n^{-1} x_n^{-1} \neq e$ . Since  $q_n x_n q_n^{-1} x_n^{-1} \in [H, H]$  is a finite set there is a subsequence on which  $q_n x_n q_n^{-1} x_n^{-1} = z \neq e$  is constant. Take a further subsequence along which  $q_n \rightarrow q \in Q_x$  (compactness of  $Q_x$ ). Then  $q_n x_n q_n^{-1} x_n^{-1} \rightarrow q x q^{-1} x^{-1}$  and  $q_n x_n q_n^{-1} x_n^{-1} = z$  hence  $q x q^{-1} x^{-1} = z \neq e$  contradicting that  $q \in H_x$ .

Therefore, for all  $x \in K$  there is an open neighborhood  $U_x$  of  $x$  such that for all  $q \in Q_x$  and all  $y \in U_x$  we have  $q y q^{-1} y^{-1} = e$ . Since  $Q_x$  generates  $H_x$  this means that  $U_x$  commutes with  $H_x$ . Now  $K \subseteq \bigcup_{x \in K} U_x$  is an open cover of a compact set hence there is a finite subcover:  $K \subseteq \bigcup_{j=1}^{\ell} U_{x_j}$  for some  $x_1, \dots, x_{\ell} \in K$ . Let  $H_0 = \bigcap_{j=1}^{\ell} H_{x_j}$ . Then  $H_0$  commutes with  $U_{x_1}, \dots, U_{x_{\ell}}$  hence it commutes with  $K$  and therefore  $H_0$  commutes with all of  $H$ . Now  $H_0$  has finite index in  $H$  since it is a finite intersection of finite index subgroups of it, hence  $H_0 \subseteq Z(H)$  and the latter has finite index, as claimed. □

In the rest of this section we prove Proposition 3.2. This is done in two independent parts: the “amenability half” and the “property (T) half”, which are Propositions 3.4 and 3.6 below.

*Proof of Proposition 3.2 from Propositions 3.4 and 3.6 below.* The group  $\Lambda/N$  has property (T) by Proposition 3.6 and is amenable by Proposition 3.4, hence it is finite.  $\square$

### 3.1. The amenability half

**Proposition 3.4.** *Let  $G$  be a locally compact second countable group and let  $\Gamma < G$  be a lattice in  $G$ . Let  $\Lambda < G$  be a dense subgroup that contains and commensurates  $\Gamma$ .*

*Let  $N$  be a normal subgroup of  $\Lambda$  such that  $\bar{N}$  is co-compact in  $G$ , and such that  $\Gamma$  maps onto  $\Lambda/N$  via the coset map. Then  $\Lambda/N$  is amenable.*

*Proof.* Since  $\Lambda/N$  is (second) countable, it is amenable if for any compact metric space on which  $\Lambda/N$  acts continuously, there is a  $\Lambda/N$  invariant probability measure. Let  $Z$  be such a space, viewed as a  $\Lambda$ -space with trivial action of  $N$ .

Let  $(X, \nu)$  be the Poisson boundary of  $G$  (with respect to any symmetric measure with support generating  $G$ ). By Proposition 2.4, the action of  $\Gamma$  on  $(X, \nu)$  is contractive. The  $G$ -action on  $(X, \nu)$  is amenable, hence also that of its closed subgroup  $\Gamma$ ; see [32]. Let then  $\varphi: X \rightarrow P(Z)$  be a measurable  $\Gamma$ -equivariant map. Let  $Y = P(Z)$  and  $\eta = \varphi_* \nu \in P(Y)$  so that  $\varphi: (X, \nu) \rightarrow (Y, \eta)$  is a  $\Gamma$ -map.

By hypothesis,  $\Gamma$  maps onto  $\Lambda/N$  via the coset map  $\gamma \mapsto \gamma N$  so for any  $\lambda \in \Lambda$  there is some  $\gamma \in \Gamma$  such that  $\gamma N = \lambda N$ . Since  $N$  acts trivially on  $Z$ , we have  $\lambda \eta = \gamma \eta$  and therefore the  $\Gamma$ -quasi-invariance of  $\eta$  implies  $\Lambda$ -quasi-invariance, so  $(Y, \eta)$  is a  $\Lambda$ -space.

By the Contractive Factor Theorem (Theorem 2.8),  $\varphi$  extends to a  $G$ -map to a  $\Lambda$ -isomorphic  $G$ -space  $(Y', \eta')$ . Since  $N$  acts trivially on  $Z$  the same is true on  $Y = P(Z)$  and therefore  $\bar{N}$  acts trivially on  $Y'$ . As  $\eta$  is invariant under  $N$ ,  $\eta'$  is  $\bar{N}$ -invariant.

Let  $Q = G/\bar{N}$ . Then  $Q$  is a compact group. Since  $\eta'$  is quasi-invariant under  $G$  it also is under  $Q$ . Let  $m$  be the Haar measure on  $Q$  normalized to be a probability measure and set  $\eta'' = m * \eta'$ . Then  $\eta''$  is in the same measure class as  $\eta'$ , and  $\eta''$  is  $Q$ -invariant. Therefore  $\eta''$  is  $G$ -invariant since  $\bar{N} \triangleleft G$  and  $\eta'$  is  $\bar{N}$ -invariant.

Let  $\eta'''$  be the isomorphic image of  $\eta''$  on  $Y$ . So  $\eta'''$  is a  $\Lambda$ -invariant probability measure on  $Y = P(Z)$ . Take  $\rho$  to be the barycenter of  $\eta'''$ :  $\rho = \int_{P(Z)} \xi \, d\eta'''(\xi)$ . Then  $\rho \in P(Z)$  is  $\Lambda$ -invariant since

$$\lambda \rho = \int_{P(Z)} \lambda \xi \, d\eta'''(\xi) = \int_{P(Z)} \xi \, d\lambda \eta'''(\xi) = \rho.$$

Hence  $\rho$  is a  $\Lambda/N$ -invariant probability measure on  $Z$  and the proof is complete.  $\square$

**3.2. The property (T) half.** The requirement that  $\Gamma$  be square-integrable in the main Theorem is only necessary for the property (T) half of the proof. Recall that if  $\Gamma$  is a finitely generated lattice in a locally compact group  $G$  then  $\Gamma$  is *square integrable* when there exists a fundamental domain  $F$  for  $G/\Gamma$  such that

$$\int_F |\alpha(g, x)|^2 dm(x) < \infty$$

where  $\alpha: G \times F \rightarrow \Gamma$  is the cocycle given by  $\alpha(g, x) = \gamma$  if and only if  $gx\gamma \in F$ ,  $|\cdot|$  denotes the word length in  $\Gamma$  (the choice of generating set will not affect the finiteness of the integral), and  $m$  is the finite Haar measure on  $F$ . This requirement is crucial in order to be able to define a natural ( $L^2$ -)induction map (going from  $\Gamma$  to  $G$ ) on the first cohomology with unitary coefficients, and is imposed in order to utilize the rigidity results of [28] and [10]. Of course, all uniform lattices are square integrable. Non-uniform lattices are known to be square integrable in higher-rank semisimple groups [28], rank-one simple Lie groups not locally isomorphic to  $SL_2(\mathbb{R})$  or  $SL_2(\mathbb{C})$ , see [29], and in the Kac-Moody case, see [27].

Before stating the property (T) half, we derive a consequence of a result of Gelfander, Karlsson, and Margulis [10]:

**Proposition 3.5.** *Let  $G$  be a locally compact second countable group and let  $\Gamma < G$  be a square-integrable lattice in  $G$ . Let  $\Lambda < G$  be a dense subgroup that contains and commensurates  $\Gamma$ . Let  $\pi: \Lambda \rightarrow \mathcal{H}$  be an irreducible unitary representation of  $\Lambda$  on a Hilbert space  $\mathcal{H}$  such that every finite index subgroup of  $\Gamma$  does not admit almost invariant vectors for  $\pi$ . Then any affine action of  $\Lambda$  on  $\mathcal{H}$  with no fixed points, whose linear part is given by  $\pi$ , extends to a continuous affine isometric  $G$ -action on  $\mathcal{H}$ .*

*Proof.* By density of  $\Lambda$ , the full set of points on which its action extends continuously to  $G$  is a closed,  $\Lambda$ -invariant affine subspace  $\mathcal{H}_0$ , on which the extended  $G$ -action is affine isometric as well. The main point of the argument is that this set is not empty. Indeed, by Lemma 2.10 of [10] there exists some minimal, closed, convex,  $\Lambda$ -invariant subset of  $\mathcal{H}$ . On this minimal subset the main condition of Theorem 8.1 in [10] applies, since in complete generality it holds that for any affine isometric action of a finitely generated group  $\Gamma_0$  on a Hilbert space  $\mathcal{H}$ , the displacement function relative to a generating set of  $\Gamma_0$  is proper (or goes to infinity in the terminology of [10]) precisely when the linear part of the action of  $\Gamma_0$  does not admit almost invariant vectors. Thus, by Theorem 8.1 of [10] the isometric  $\Lambda$ -action on this minimal subset extends continuously to  $G$ , and  $\mathcal{H}_0$  is not empty. Moreover,  $\mathcal{H}_0$  cannot consist of one point either, as there are no fixed points for  $\Lambda$ . By irreducibility of the linear  $\Lambda$ -representation on  $\mathcal{H}$  it then immediately follows that  $\mathcal{H}_0 = \mathcal{H}$ , as claimed.  $\square$

We can now state and prove the property (T) half of the main theorem.

**Proposition 3.6.** *Let  $\Gamma <_c \Lambda < G$  be as in Proposition 3.4, with  $G$  compactly generated and  $\Gamma$  (uniform or) square integrable. Let  $N$  be a normal subgroup of  $\Lambda$  such that  $\Gamma$  maps onto  $\Lambda/N$  via the coset map, and such that  $[\overline{N}, \overline{N}]$ , hence also  $\overline{N}$ , is co-compact in  $G$ . Then  $\Lambda/N$  has property (T).*

*Proof.* Recall that the reduced cohomology of  $G$  with coefficients in  $\pi$  (for a unitary representation  $\pi: G \rightarrow \mathcal{U}(\mathcal{H})$  of  $G$  on a Hilbert space) is

$$\overline{H^1}(G, \pi) = Z^1(G, \pi) / \overline{B^1(G, \pi)},$$

the space of 1-cocycles modulo the closure of the subspace of 1-coboundaries (with respect to the topology of uniform convergence on compact sets). The second author has shown [28] that property (T) is equivalent to the vanishing of reduced cohomology for every *irreducible* unitary representation, provided the group is finitely (compactly) generated. Note that while  $\Lambda$  may not be finitely generated, since  $\Gamma$  maps onto  $\Lambda/N$  and  $\Gamma$  is finitely generated,  $\Lambda/N$  is finitely generated. Let  $\pi: \Lambda/N \rightarrow \mathcal{U}(\mathcal{H})$  be an irreducible unitary representation and suppose that  $\overline{H^1}(\Lambda/N, \pi) \neq 0$ . We will obtain a contradiction to the existence of such a representation and therefore conclude that  $\Lambda/N$  has property (T). Our method is to first show that such irreducible cohomological representation must be finite-dimensional, then show that in a cohomological finite-dimensional representation  $\pi(\Gamma)$  must be finite, and finally show that finite representations support no cohomology. These three steps appear as the three Lemmas below. □

**Lemma 3.7.** *Let  $\Gamma <_c \Lambda < G$  and  $N$  be as in Proposition 3.6. Let*

$$\pi: \Lambda/N \longrightarrow \mathcal{U}(\mathcal{H})$$

*be an irreducible unitary representation such that  $\overline{H^1}(\Lambda/N, \pi) \neq 0$ . Then  $\pi$  is finite-dimensional.*

*Proof.* Treat  $\pi$  as a representation of  $\Lambda$  with  $\pi(N)$  being trivial. Let  $b \in Z^1(\Lambda, \pi)$  such that  $b \neq [0]$  in  $\overline{H^1}(\Lambda, \pi)$ . Then  $b|_\Gamma \neq [0]$  in  $\overline{H^1}(\Lambda, \pi)$  since  $\Gamma$  maps onto  $\Lambda/N$ . The second author has shown (Theorem 10.3 in [28]) that the latter implies that there exists a nonzero  $\Lambda$ -invariant subspace on which the *linear* (but not necessarily the isometric – this is why the Gelfand, Karlsson, and Margulis result will be needed below)  $\Lambda$ -action  $\pi$  extends continuously to a unitary representation of  $G$ . As  $\pi$  is irreducible for  $\Lambda$ , the action extends on the entire space. Continue to denote this new  $G$ -representation by  $\pi$ . Since  $\pi(N)$  is trivial,  $\pi(\overline{N})$  is too. Therefore  $\pi$  is in fact an irreducible representation of the compact group  $G/\overline{N}$  (as  $\overline{N}$  is co-compact), and is therefore finite-dimensional. □



**Lemma 3.8.** *Let  $\Gamma <_c \Lambda < G$  and  $N$  be as in Proposition 3.6. Let*

$$\pi: \Lambda/N \longrightarrow \mathcal{H}$$

*be a finite-dimensional, irreducible, unitary representation, such that  $\overline{H^1}(\Lambda/N, \pi) \neq 0$ . Then  $\pi(\Gamma) = \pi(\Lambda)$  is finite.*

*Proof.* Treat  $\pi$  as a  $\Lambda$ -representation, and let  $b$  be the assumed non-(cohomologically-)zero cocycle on  $\Lambda/N$ . As usual,  $\pi$  and  $b$  determine an affine, fixed point free isometric  $\Lambda$ -action via  $\lambda v := \pi(\lambda)v + b(\lambda)$ , having  $N$  in its kernel. Since  $\Gamma$  maps onto  $\Lambda/N$ ,  $\pi$  is an irreducible  $\Gamma$ -representation. We first show that *some* finite index subgroup  $\Gamma_0 < \Gamma$  has almost invariant vectors for  $\pi$ . Indeed, if this is not the case then we are exactly in position to invoke Proposition 3.5, which yields that the affine action of  $\Lambda$  extends to a continuous affine isometric action of  $G$ . Now  $N$  is in the kernel of the isometric  $\Lambda$ -action, hence  $\overline{N}$  is in the kernel of the  $G$ -action. But  $\overline{N}$  is co-compact so the  $G$ -action factors through an action of a compact group and hence has a fixed point. Thus the  $\Lambda$ -action has a fixed point, which contradicts our assumption on the cocycle  $b$ .

Let then  $\Gamma_0 < \Gamma$  be a finite index subgroup having almost invariant vectors for  $\pi$ . By passing to a finite index subgroup of it we may of course assume it is also normal in  $\Gamma$ . As  $\pi$  is finite-dimensional there is then a non-zero invariant vector for  $\Gamma_0$  (compactness). The normality of  $\Gamma_0$  implies that for any  $\Gamma$ -action the set of  $\Gamma_0$ -fixed points is  $\Gamma$ -invariant. In our setting it follows from irreducibility that the  $\Gamma$ -invariant subspace of  $\Gamma_0$ -fixed vectors is (non-zero and hence) all of  $\mathcal{H}$ . Thus  $\Gamma_0$  acts trivially and  $\pi(\Gamma) = \pi(\Lambda)$  is finite, as claimed. □

**Lemma 3.9.** *Let  $\Gamma <_c \Lambda < G$  and  $N$  be as in Proposition 3.6. Let*

$$\pi: \Lambda/N \longrightarrow \mathcal{H}$$

*be a finite unitary representation. Then  $\overline{H^1}(\Lambda/N, \pi) = 0$ .*

*Proof.* Assume to the contrary that there exists a non-(cohomologically-)zero cocycle  $b$  for the finite representation  $\pi$  of the group  $\Lambda/N$ . Let  $\Lambda_0 < \Lambda$  be the finite index kernel of  $\pi$ , and  $\Gamma_0 = \Lambda_0 \cap \Gamma < \Gamma$ . Since  $\Lambda_0$  has finite index,  $b|_{\Lambda_0}$  remains non-zero. Being a 1-cocycle for the trivial  $\Lambda_0$ -representation, it defines a non-zero homomorphism of  $\Lambda_0$  to the additive vector group of the representation space, from which we get a nontrivial homomorphism  $\varphi: \Lambda_0 \rightarrow \mathbb{R}$ . The second author has shown (Theorem 0.8 in [28]) that, since  $\Gamma_0$  is a finitely generated square integrable lattice in  $G$ ,  $\varphi|_{\Gamma_0}$  then extends to a homomorphism on  $G_0 = \overline{\Lambda_0}$  (note that the extension may not agree with  $\varphi$  on all of  $\Lambda_0$ , just on  $\Gamma_0$ ). Call this extension  $\psi: G_0 \rightarrow \mathbb{R}$ . It is of course still non-zero as  $\psi|_{\Gamma_0} = \varphi \neq 0$  (noting that  $\varphi(\Gamma_0)$  has finite index in  $\varphi(\Lambda_0)$ , thus cannot be zero).

Now,  $\psi$  vanishes on  $[N, N]$  hence also on  $\overline{[N, N]}$ . But  $\overline{[N, N]}$  is co-compact in  $G_0$ , hence  $\psi = 0$ , a contradiction. This completes the proof of the Lemma, and with it the proof of Proposition 3.6, and all of Theorem 3.1. □



#### 4. Proof of the bijection of commensurability classes

The second part of the Main Theorem follows immediately from Theorem 3.1 and the following general result.

**Proposition 4.1.** *Let  $\Gamma < \Lambda$  be countable discrete groups such that  $\Lambda$  commensurates  $\Gamma$ . Let  $\varphi: \Lambda \rightarrow H$  be a dense homomorphism into a locally compact totally disconnected group  $H$  such that  $K = \overline{\varphi(\Gamma)}$  is compact open, and  $\varphi^{-1}(K) = \Gamma$ . Then the map  $N \mapsto \overline{\varphi(N)}$  induces a bijection between commensurability classes of normal subgroups  $N \triangleleft \Lambda$  with  $[\Gamma : N \cap \Gamma] < \infty$ , and commensurability classes of open normal subgroups of  $H$ .*

*Proof.* Let  $N$  be a normal subgroup of  $\Lambda$  with  $[\Gamma : N \cap \Gamma] < \infty$ . Then we have  $[K : \overline{\varphi(\Gamma \cap N)}] < \infty$  and  $\overline{\varphi(\Gamma \cap N)}$  is a compact open subgroup of  $H$ . Since  $\overline{\varphi(N)}$  contains this group,  $\overline{\varphi(N)}$  is an open normal subgroup of  $H$ .

Let  $N_1$  and  $N_2$  be commensurate normal subgroups of  $\Lambda$  with  $[\Gamma : N_1 \cap \Gamma], [\Gamma : N_2 \cap \Gamma] < \infty$ . Then  $N_1 \cap N_2$  is a normal subgroup of  $\Lambda$  that has finite index in both  $N_1$  and  $N_2$ . Therefore  $\overline{\varphi(N_1 \cap N_2)}$  is an open normal subgroup of  $H$  that has finite index in both  $\overline{\varphi(N_1)}$  and  $\overline{\varphi(N_2)}$  meaning that  $N_1$  and  $N_2$  are mapped to the same commensurability class of open normal subgroups. Therefore the induced map on the commensurability classes is well defined.

Surjectivity is obvious: given an open normal subgroup  $M$  of  $H$ , set  $N = \varphi^{-1}(M)$ . Then  $N$  is normal in  $\Lambda$  and  $[\Gamma : N \cap \Gamma] < \infty$  since  $M$  contains a finite index subgroup of  $\overline{\varphi(\Gamma)}$ . Of course,  $\overline{\varphi(N)} = \overline{\varphi(\Lambda)} \cap M = M$ , as  $M$  is open and  $\overline{\varphi(\Lambda)}$  is dense.

Injectivity: take  $N_1$  and  $N_2$  to be normal subgroups of  $\Lambda$  with  $[\Gamma : N_1 \cap \Gamma], [\Gamma : N_2 \cap \Gamma] < \infty$  such that  $\overline{\varphi(N_1)}$  and  $\overline{\varphi(N_2)}$  are commensurate (open normal) subgroups. Since  $\varphi$  is a homomorphism,  $\varphi^{-1}(\overline{\varphi(N_1)})$  and  $\varphi^{-1}(\overline{\varphi(N_2)})$  are commensurate subgroups of  $\Lambda$ . Once we show that  $[\varphi^{-1}(\overline{\varphi(N_1)}) : N_1] < \infty$  and  $[\varphi^{-1}(\overline{\varphi(N_2)}) : N_2] < \infty$  we would get immediately that  $N_1$  and  $N_2$  are commensurate, implying injectivity. So, we are only left with proving that any  $N$  as in the Proposition has finite index in  $\varphi^{-1}(\overline{\varphi(N)})$ .

Indeed, as  $\overline{\varphi(N)}$  is dense in  $\overline{\varphi(N)}$  and  $K = \overline{\varphi(\Gamma)}$  is open,  $\overline{\varphi(N)} \subseteq K\overline{\varphi(N)}$ . Set  $Q = \overline{\varphi(N)} \cap \varphi(\Lambda)$ . For  $h \in Q$ , write  $h = kn$  for some  $k \in K$  and  $n \in \varphi(N)$ . Then  $hn^{-1} = k \in K$  and  $hn^{-1} \in \varphi(\Lambda)\varphi(N) = \varphi(\Lambda)$  so  $hn^{-1} \in K \cap \varphi(\Lambda) = \varphi(\Gamma)$  (since  $\varphi^{-1}(K) = \Gamma$ ). Therefore  $Q \subseteq \varphi(\Gamma)\varphi(N) = \varphi(\Gamma N)$ . Since  $\varphi$  is a homomorphism,

$$\begin{aligned} [\varphi^{-1}(Q) : \varphi^{-1}(\overline{\varphi(N)})] &\leq [Q : \varphi(N)] \\ &\leq [\varphi(\Gamma N) : \varphi(N)] \\ &\leq [\Gamma N : N] \\ &= [\Gamma : \Gamma \cap N] \\ &< \infty. \end{aligned}$$

Because  $\Gamma = \varphi^{-1}(K)$ ,  $\ker \varphi < \Gamma$  and we also have

$$[\varphi^{-1}(\varphi(N)) : N] = [N \ker \varphi : N] \leq [N\Gamma : N] < \infty$$

These two finiteness results yield  $[\varphi^{-1}(Q) : N] < \infty$ , precisely what needed to be proved. □

### 5. Irreducible lattices in products of groups

*Proof of Corollary 1.2.* Assume first that all three conditions hold. The first implies that  $\text{proj}_G : \Lambda \rightarrow G$  has a finite kernel (contained in  $H$ ), which by density of  $\Lambda$  is a normal subgroup of  $H$ . Replacing  $\Lambda$  and  $H$  with their quotient by  $\ker(\text{proj}_G)$ , we may and shall assume hereafter that this projection is faithful, so that we can naturally identify elements of  $\Lambda$  with their image in  $G$  (it is immediate to verify that all the other assumptions remain intact, and that the conclusion for  $\Lambda / \ker(\text{proj}_G)$  implies it for  $\Lambda$  itself).

Set  $\Gamma = \Lambda \cap (G \times K)$  where  $K$  is a compact open subgroup of  $H$ . As  $K$  is open,  $\text{proj}_K \Gamma$  is dense in  $K$  (using that the projection of  $\Lambda$  to  $H$  is dense). It is a general fact that when  $L$  is co-compact (a lattice) in a locally compact group  $M$ , and  $U$  is an open subgroup of  $M$ ,  $L \cap U$  is co-compact (a lattice) in  $U$ . Applying this to  $\Lambda$  we find that  $\Gamma$  is co-compact in  $G \times K$ . Since  $K$  is compact, the projection of  $\Gamma$  to  $G$  is discrete and co-compact in  $G$ .

As  $K$  is commensurated by  $H$ , being a compact open subgroup of it,  $\Gamma$  is commensurated by  $\Lambda$ . The projection of  $\Lambda$  to  $G$  is dense and so  $\text{proj}_G \Gamma < \text{proj}_G \Lambda < G$  satisfy the general setup of our Main Theorem, where condition (iii) ensures that its main assumption is satisfied. It follows that every infinite  $N \triangleleft \Lambda$  contains a finite index subgroup of  $\Gamma$ .

Finally, observe that  $\varphi = \text{proj}_H$  satisfies the assumption in the second part of the conclusion of the main Theorem. Condition (ii) in the Corollary says that there is only one commensurability class of open normal subgroups of  $H$  – that of  $H$  itself, thus the Main Theorem implies that there is only one for  $\Lambda$  as well, which must be the class of those (infinite) normal subgroups having finite index. This proves the main direction of the Corollary.

Now assume in the reverse direction that every infinite normal subgroup of  $\Lambda$  has finite index. Observe that  $N = \Lambda \cap (\{e\} \times H)$  is normal in  $\Lambda$  so it is either finite, or has finite index. If the latter holds then the projection of  $\Lambda$  to  $G$  is finite, hence so is  $G$ , contradicting the assumption that it is not a compact extension of an abelian. Thus condition (i) must hold.

Next, suppose towards a contradiction that  $H$  has an infinite index open normal subgroup  $M$  (in particular,  $H$  is infinite). Then  $\text{proj}_H \Lambda \cap M$  is an infinite index normal subgroup of  $\text{proj}_H \Lambda$  (which is dense in  $M$ ), hence its inverse image has the same property back in  $\Lambda$ . By our assumption on  $\Lambda$  that inverse image is finite, thus

$M$  itself must be finite, and  $H$  is discrete. It follows that  $\Lambda$  must project onto  $H$ , and as  $H$  is infinite the kernel of that projection cannot have finite index, so by our assumption on  $\Lambda$  it is finite. But this kernel is a lattice in  $G$  (in order for  $\Lambda$  to be a lattice in  $G \times H$ ), hence  $G$  is compact, a contradiction. This proves the necessity of (ii).

Finally, let  $M \triangleleft G$  be a closed non-cocompact normal subgroup of  $G$ . A previous argument showed that our assumption implies that the map  $\text{proj}_G: \Lambda \rightarrow G$  has finite kernel, so  $\Lambda_0 = \text{proj}_G \Lambda$  also has the property that every infinite normal subgroup has finite index. Set  $N = \Lambda_0 \cap M$ . Then  $\overline{N} \triangleleft \overline{\Lambda_0}$ , so  $N$  is finite or has finite index in  $\Lambda_0$ . If  $N$  had finite index then so did  $\overline{N} \triangleleft \overline{\Lambda_0} = G$ , hence  $M$  was co-compact. Therefore the projection of  $\Lambda$  to  $G$  intersects every closed non-cocompact normal subgroup of  $G$  finitely, which proves the necessity of condition (iii) and completes the proof of the whole Corollary.  $\square$

The case where  $\Lambda$  is a non-uniform lattice is more involved due to the complication in the property (T) half of the proof, and requires a modification of both the statement and the argument. Even if  $\Lambda$  were assumed square integrable, the lattice  $\Gamma < G$  may not have this property (e.g., when  $\Lambda = \text{SL}_2(\mathbb{Z}[\frac{1}{p}]) < (G = \text{SL}_2(\mathbb{R})) \times (H = \text{SL}_2(\mathbb{Q}_p))$ , the lattice  $\Gamma = \text{SL}_2(\mathbb{Z}) < G$  is not square integrable, even though  $\Lambda$  is [28]). One can even construct such examples where  $\Gamma$  is not finitely generated (taking  $\Lambda$  as an irreducible non-uniform lattice in a product of rank one simple algebraic groups over a local field of positive characteristics). The simplest and most direct way to deal with this issue is to assume that  $\Gamma$  itself be square integrable. In that case the whole argument (and result) goes through as is. However, with a bit more structural assumptions one can do by assuming the square integrability of  $\Lambda$ . Here one has to modify the whole property (T) argument so that instead of inducing unitary representations from  $\Gamma$  to  $G$ , they are induced from  $\Lambda$  to  $G \times H$ , and then restricted to  $G$ . This requires getting into the proofs of some results from [28] and since we anyway do not currently have any concrete applications in mind, we prefer not to elaborate further on this matter.

Finally, it is perhaps worth illustrating here how the lack of compact generation assumption on  $H$  yields useful additional flexibility. Let  $K$  be a global field, and  $\mathbf{G}$  be a simply connected, simple algebraic group defined over  $K$ . Then  $\mathbf{G}(K)$  is a lattice in  $\mathbf{G}(\mathbb{A})$ , where  $\mathbb{A}$  is the ring of adèles over  $K$ . One may then decompose  $\mathbf{G}(\mathbb{A}) = G \times H$  where  $G$  is a product over finitely many places including all the Archimedean ones, so that its  $S$ -rank is at least 2, and  $H$  is the (restricted) direct product over all other places. The fact that  $\Lambda$  has dense projections follows from strong approximation, while the fact that  $H$  has no open normal subgroups follows from the Kneser–Tits conjecture over local fields. Here  $\Gamma < G$  is square integrable by [28], so one deduces that every infinite normal subgroup of  $\mathbf{G}(K)$  has finite index. When  $\mathbf{G}$  is  $K$ -isotropic this can then be easily upgraded to simplicity (modulo the center), but in the anisotropic case the latter no longer holds in general. Of course, such normal subgroup theorems can be similarly proved for  $S$ -arithmetic groups when

$S$  is any infinite set of places (including the Archimedean ones) – see also Section 2 in Chapter VIII of [22]. Note that the assumption that  $\mathbf{G}$  be simply connected is crucial when  $S$  infinite. One can still invoke our strategy when it isn't, but here  $H$  may contain open normal subgroups which classify, using the Main Theorem, the abstract normal subgroups of the corresponding  $S$ -arithmetic group.

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Darren Creutz, Department of Mathematics, Vanderbilt University, 2201 West End Ave, Nashville, TN, 37235, U.S.A.

E-mail: [darren.creutz@vanderbilt.edu](mailto:darren.creutz@vanderbilt.edu)

Yehuda Shalom, Department of Mathematics,  
University of California Los Angeles (UCLA), Los Angeles, CA, 90095, U.S.A.  
School of Mathematical Sciences, Tel-Aviv University, Ramat Aviv, Tel Aviv 6997801,  
Israel

E-mail: [shalom@math.ucla.edu](mailto:shalom@math.ucla.edu)