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Invariant measures and orbit equivalence for generalized Toeplitz subshifts

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Abstract. We show that for every metrizable Choquet simplex K and for every group G which is infinite, countable, amenable and residually finite, there exists a Toeplitz G-subshift whose set of shift-invariant probability measures is affinely homeomorphic to K. Furthermore, we get that for every integer $d > 1$ and every Toeplitz flow (X, T), there exists a Toeplitz $\mathbb{Z}^d$-subshift which is topologically orbit equivalent to (X, T).

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Contents

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1. Introduction

The *Toeplitz subshifts* are a rich class of symbolic systems introduced by Jacobs and Keane in [21] in the context of $\mathbb{Z}$-actions. Since then, they have been extensively studied and used to provide series of examples with interesting dynamical properties (see for example [7, 8, 17, 27]). Generalizations of Toeplitz subshifts and some of their properties to more general group actions can be found in [3, 5, 9, 22, 23]. For instance, in [5] Toeplitz subshifts are characterized as the minimal symbolic almost 1–1 extensions of odometers (see [13] for this result in the context of $\mathbb{Z}$-actions). In this paper, we give an explicit construction that generalizes the result of Downarowicz in [7], to Toeplitz subshifts given by actions of groups which are amenable, countable and residually finite. The following is our main result.

Theorem A. *Let G be an infinite, countable, amenable and residually finite group. For every metrizable Choquet simplex K and any G-odometer O, there exists a Toeplitz G-subshift which is an almost 1–1 extension of O such that the set of its invariant probability measures is affinely homeomorphic to K.*

Typical examples of the groups G involved in this theorem are the finitely generated subgroups of upper triangular matrices in $GL(n, \mathbb{C})$.

The strategy of Downarowicz in [7] is to construct an affine homeomorphism between an arbitrary metrizable Choquet simplex K and a subset of the space of invariant probability measures of the full shift $\{0,1\}^{\mathbb{Z}}$. Then he shows it coincides with the space of invariant probability measures of a Toeplitz subshift $Y \subseteq \{0,1\}^{\mathbb{Z}}$. To do this, he uses the structure of metric space of the space of measures. In this paper we consider the representation of K as an inverse limit of finite dimensional simplices with linear transition maps $(M_n)_n$. Then we use this transition maps to construct Toeplitz G-subshifts having sequences of Kakutani–Rokhlin partitions with $(M_n)_n$ as the associated sequence of incidence matrices. Our approach is closer to the strategy used in [17] by Gjerde and Johansen, and deals with the combinatorics of Følner sequences.

We obtain furthermore some consequences for orbit equivalence. Two minimal Cantor systems are (topologically) orbit equivalent, if there exists an orbit-preserving homeomorphism between their phase spaces. Giordano, Matui, Putnam and Skau show in [15] that every minimal $\mathbb{Z}^d$-action on the Cantor set is orbit equivalent to a minimal $\mathbb{Z}$-action. It is still unknown if every minimal action of a countable amenable group on the Cantor set is orbit equivalent to a $\mathbb{Z}$-action. Nevertheless it is clear that the result in [15] can not be extended to any countable group. For instance, by using the notion of cost, Gaboriau [14] proves that if two free actions of free groups $\mathbb{F}_n$ and $\mathbb{F}_p$ are (even measurably) orbit equivalent then their rank are the same i.e. $n = p$. Another problem is to know which are the $\mathbb{Z}$-orbit equivalence classes that the $\mathbb{Z}^d$-actions (or more general group actions) realize. We give a partial answer for this question. As a consequence of the proof of Theorem A we obtain the following result.

Theorem B. *Let $(X, \sigma|_X, \mathbb{Z})$ be a Toeplitz $\mathbb{Z}$-subshift. Then for every $d \geq 1$ there exists a Toeplitz $\mathbb{Z}^d$-subshift which is orbit equivalent to $(X, \sigma|_X, \mathbb{Z})$.*

This paper is organized as follows. Section 2 is devoted to introduce the basic definitions. For an amenable discrete group G and a decreasing sequence of finite index subgroups of G with trivial intersection, we construct in Section 3 an associated sequence $(F_n)_{n\geq 0}$ of fundamental domains, so that it is Følner and each F_{n+1} is tileable by translated copies of F_n. In Section 4 we construct Kakutani–Rokhlin partitions for generalized Toeplitz subshifts, and in Section 5 we use the fundamental domains introduced in Section 3 to construct Toeplitz subshifts having sequences of Kakutani–Rokhlin partitions with a prescribed sequence of incidence matrices. This construction improves and generalizes that one given in [4] for $\mathbb{Z}^d$-actions, and moreover allows one to characterize the associated ordered group with unit. In Section 6 we give a characterization of any Choquet simplex as an inverse limit defined by sequences of matrices that we use in Section 5 (they are called “managed" sequences). Finally, in Section 7 we use the previous results to prove Theorems A and B.

2. Basic definitions and background

In this article, by a *topological dynamical system* we mean a triple (X, T, G), where T is a continuous left action of a countable group G on the compact metric space (X, d). For every $g \in G$, we denote T^g the homeomorphism that induces the action of g on X. The unit element of G will be called e. The system (X, T, G) or the action T is *minimal* if for every $x \in X$ the orbit $o_T(x) = \{T^g(x) \colon g \in G\}$ is dense in X. We say that (X, T, G) is a *minimal Cantor system* or a minimal Cantor G-system if (X, T, G) is a minimal topological dynamical system with X a Cantor set.

An *invariant probability measure* of the topological dynamical system (X, T, G) is a probability Borel measure μ such that $\mu(T^g(A)) = \mu(A)$, for every Borel set A. We denote by $\mathcal{M}(X, T, G)$ the space of invariant probability measures of (X, T, G).

2.1. Subshifts. For every $g \in G$, denote $L_g \colon G \to G$ the left multiplication by $g \in G$. That is, $L_g(h) = gh$ for every $h \in G$. Let Σ be a finite alphabet. Σ^G denotes the set of all the functions $x \colon G \to \Sigma$. The (left) *shift action* σ of G on Σ^G is given by $\sigma^g(x) = x \circ L_{g^{-1}}$, for every $g \in G$. Thus $\sigma^g(x)(h) = x(g^{-1}h)$. We consider Σ endowed with the discrete topology and Σ^G with the product topology. Thus every σ^g is a homeomorphism of the Cantor set Σ^G. The topological dynamical system (Σ^G, σ, G) is called the full G-shift on Σ. For every finite subset D of G and $x \in \Sigma^G$, we denote $x|_D \in \Sigma^D$ the restriction of x to D. For $F \in \Sigma^D$ (F is a function from D to Σ) we denote by $[F]$ the set of all $x \in \Sigma^D$ such that $x|_D = F$. The set $[F]$ is called the *cylinder* defined by F, and it is a clopen set (both open and closed). The collection of all the sets $[F]$ is a base of the topology of Σ^G.

Definition 1. A *subshift* or G-subshift of Σ^G is a closed subset X of Σ^G which is invariant under the shift action.

The topological dynamical system $(X, \sigma|_X, G)$ is also called subshift or G-subshift. See [2] for details.

2.1.1. Toeplitz G-subshifts. An element $x \in \Sigma^G$ is a *Toeplitz sequence* if for every $g \in G$ there exists a finite index subgroup Γ of G such that $\sigma^\gamma(x)(g) = x(\gamma^{-1}g) = x(g)$ for every $\gamma \in \Gamma$.

A subshift $X \subseteq \Sigma^G$ is a *Toeplitz subshift* or Toeplitz G-subshift if there exists a Toeplitz sequence $x \in \Sigma^G$ such that $X = \overline{o_\sigma(x)}$. It is shown in [5], [22] and [23] that a Toeplitz sequence x is *regularly recurrent*, *i.e.*, for every neighborhood V of x there exists a finite index subgroup Γ of G such that $\sigma^\gamma(x) \in V$, for every $\gamma \in \Gamma$. This condition is stronger than almost periodicity, which implies minimality of the closure of the orbit of x (see [1] for details about almost periodicity).

2.2. Inverse and direct limit. Given a sequence of continuous maps

$$f_n \colon X_{n+1} \longrightarrow X_n, \quad n \geq 0$$

on topological spaces X_n, we denote the associated *inverse limit* by

$$\begin{aligned}\varprojlim_n (X_n, f_n) &= X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \cdots \\ &:= \{(x_n)_n; x_n \in X_n,\ x_n = f_n(x_{n+1}), \text{for all } n \geq 0\}.\end{aligned}$$

Let us recall that this space is compact when all the spaces X_n are compact and the inverse limit spaces associated to any increasing subsequences $(n_i)_i$ of indices are homeomorphic.

In a similar way, we denote for a sequence of maps

$$g_n \colon X_n \longrightarrow X_{n+1}, n \geq 0$$

the associated *direct limit* by

$$\begin{aligned}\varinjlim_{n}(X_n, g_n) &= X_0 \xrightarrow{g_0} X_1 \xrightarrow{g_1} X_2 \xrightarrow{g_2} \cdots \\ &:= \{(x,n), x \in X_n,\ n \geq 0\}/\sim,\end{aligned}$$

where two elements are equivalent $(x,n) \sim (y,m)$ if and only if there exists $k \geq m, n$ such that $g_k \circ \ldots \circ g_n(x) = g_k \circ \ldots \circ g_m(x)$. We denote by $[x,n]$ the equivalence class of (x,n). When the maps g_n are homomorphisms on groups X_n, then the direct limit inherits a group structure.

2.3. Odometers. A group G is said to be *residually finite* if there exists a nested sequence $(\Gamma_n)_{n \geq 0}$ of finite index normal subgroups such that $\bigcap_{n \geq 0} \Gamma_n$ is trivial. For every $n \geq 0$, there exists then a canonical projection $\pi_n \colon G/\Gamma_{n+1} \to G/\Gamma_n$. The G-*odometer* or *adding machine* O associated to the sequence $(\Gamma_n)_n$ is the inverse limit

$$O := \varprojlim_{n}(G/\Gamma_n, \pi_n) = G/\Gamma_0 \xleftarrow{\pi_0} G/\Gamma_1 \xleftarrow{\pi_1} G/\Gamma_2 \xleftarrow{\pi_2} \cdots.$$

We refer to [5] for the basic properties of such a space. Let us recall that it inherits a group structure through the quotient groups G/Γ_n and it contains G as a subgroup thanks to the injection $G \ni g \mapsto ([g]_n) \in O$, where $[g]_n$ denotes the class of g in G/Γ_n. Thus the group G acts by left multiplication on O. When there is no confusion, we also call this action an odometer. It is equicontiuous, minimal and the left Haar measure is the unique invariant probability measure. Note that this action is free: the stabilizer of any point is trivial. The Toeplitz G-subshifts are characterized as the subshifts that are minimal almost 1–1 extensions of G-odometers [5].

2.4. Ordered groups. For more details about ordered groups and dimension groups we refer to [12] and [18].

An *ordered group* is a pair (H, H^+), such that H is a countable abelian group and H^+ is a subset of H verifying $(H^+)+(H^+) \subseteq H^+$, $(H^+)+(-H^+) = H$ and $(H^+) \cap (-H^+) = \{0\}$ (we use 0 as the unit of H when H is abelian). An ordered group (H, H^+) is a *dimension group* if for every $n \in \mathbb{Z}^+$ there exist $k_n \geq 1$ and a positive homomorphism $A_n \colon \mathbb{Z}^{k_n} \to \mathbb{Z}^{k_{n+1}}$, such that (H, H^+) is isomorphic to (J, J^+), where J is the direct limit

$$\varinjlim_{n}(\mathbb{Z}^{k_n}, A_n) = \mathbb{Z}^{k_0} \xrightarrow{A_0} \mathbb{Z}^{k_1} \xrightarrow{A_1} \mathbb{Z}^{k_2} \xrightarrow{A_2} \cdots,$$

and $J^+ = \{[v,n] \colon a \in (\mathbb{Z}^+)^{k_n}, n \in \mathbb{Z}^+\}$. The dimension group is *simple* if the matrices A_n can be chosen strictly positive.

An *order unit* in the ordered group (H, H^+) is an element $u \in H^+$ such that for every $g \in H$ there exists $n \in \mathbb{Z}^+$ such that $nu - g \in H^+$. If (H, H^+) is a simple dimension group then each element in $H^+ \setminus \{0\}$ is an order unit. A *unital ordered group* is a triple (H, H^+, u) such that (H, H^+) is an ordered group and u is an order unit. An isomorphism between two unital ordered groups (H, H^+, u) and (J, J^+, v) is an isomorphism $\phi \colon H \to J$ such that $\phi(H^+) = J^+$ and $\phi(u) = v$. A *state* of the unital ordered group (H, H^+, u) is a homomorphism $\phi \colon H \to \mathbb{R}$ so that $\phi(u) = 1$ and $\phi(H^+) \subseteq \mathbb{R}^+$. The *infinitesimal subgroup* of a simple dimension group with unit (H, H^+, u) is

$$\inf(H) = \{a \in H : \phi(a) = 0 \text{ for all state } \phi\}.$$

It is not difficult to show that $\inf(H)$ does not depend on the order unit.

The quotient group $H/\inf(H)$ of a simple dimension group (H, H^+) is also a simple dimension group with positive cone

$$(H/\inf(H))^+ = \{[a] \colon a \in H^+\}.$$

The next result is well known. The proof is left to the reader.

Lemma 1. *Let (H, H^+) be a simple dimension group equal to the direct limit*

$$\lim_{\to n}(\mathbb{Z}^{k_n}, M_n) = \mathbb{Z}^{k_0} \xrightarrow{M_0} \mathbb{Z}^{k_1} \xrightarrow{M_1} \mathbb{Z}^{k_2} \xrightarrow{M_2} \cdots.$$

Then for every $z = (z_n)_{n \geq 0}$ in the inverse limit

$$\lim_{\leftarrow n}((\mathbb{R}^+)^{k_n}, M_n^T) = (\mathbb{R}^+)^{k_0} \xleftarrow{M_0^T} (\mathbb{R}^+)^{k_1} \xleftarrow{M_1^T} (\mathbb{R}^+)^{k_2} \xleftarrow{M_2^T} \cdots,$$

the function $\phi_z \colon H \to \mathbb{R}$ given by

$$\phi([n,v]) = \langle v, z_n \rangle,$$

for every $[n,v] \in H$, is well defined and is a homomorphism of groups such that $\phi_z(H^+) \subseteq \mathbb{R}^+$. Conversely, for every group homomorphism $\phi \colon H \to \mathbb{R}$ such that $\phi(H^+) \subseteq \mathbb{R}^+$, there exists a unique $z \in \lim_{\leftarrow n}((\mathbb{R}^+)^{k_n}, M_n^T)$ such that $\phi = \phi_z$.

The following lemma is a preparatory lemma to prove Theorem A and B.

Lemma 2. *Let* (H, H^+, u) *be a simple dimension group with unit given by the following direct limit*

$$\lim_{\to n}(\mathbb{Z}^{k_n}, A_n) = \mathbb{Z} \xrightarrow{A_0} \mathbb{Z}^{k_1} \xrightarrow{A_1} \mathbb{Z}^{k_2} \xrightarrow{A_2} \cdots,$$

with unit $u = [1, 0]$*. Suppose that* $A_n > 0$ *for every* $n \geq 0$*. Then* (H, H^+, u) *is isomorphic to*

$$\mathbb{Z} \xrightarrow{\widetilde{A}_0} \mathbb{Z}^{k_1+1} \xrightarrow{\widetilde{A}_1} \mathbb{Z}^{k_2+1} \xrightarrow{\widetilde{A}_2} \cdots,$$

where $\widetilde{A}_0$ *is the* $(k_1 + 1) \times 1$*-dimensional matrix given by*

$$\widetilde{A}_0 = \begin{pmatrix} A_0(1, \cdot) \\ A_0(1, \cdot) \\ A_0(2, \cdot) \\ \vdots \\ A_0(k_1, \cdot) \end{pmatrix},$$

and $\widetilde{A}_n$ *is the* $(k_{n+1} + 1) \times (k_n + 1)$ *dimensional matrix given by*

$$\widetilde{A}_n = \begin{pmatrix} 1 & A_n(1,1) - 1 & A_n(1,2) & \cdots & A_n(1,k_n) \\ 1 & A_n(1,1) - 1 & A_n(1,2) & \cdots & A_n(1,k_n) \\ 1 & A_n(2,1) - 1 & A_n(2,2) & \cdots & A_n(2,k_n) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & A_n(k_{n+1},1) - 1 & A_n(k_{n+1},2) & \cdots & A_n(k_{n+1},k_n) \end{pmatrix},$$

for every $n \geq 0$*.*

Proof. For $n \geq 1$, consider M_n the $(k_n + 1) \times k_n$-dimensional matrix given by

$$M_n(\cdot, k) = \begin{cases} \vec{e}_{n,1} + \vec{e}_{n,2} & \text{if } k = 1, \\ \vec{e}_{k+1} & \text{if } 3 \leq k \leq k_n, \end{cases}$$

where $\vec{e}_{n,1}, \ldots, \vec{e}_{n,k_n+1}$ are the canonical vectors in $\mathbb{R}^{k_n+1}$. Let B_n be the $k_{n+1} \times (k_n + 1)$-dimensional matrix defined by

$$B_n(i,j) = \begin{cases} 1 & \text{if } j = 1, \\ A_n(i,1) - 1 & \text{if } j = 2, \\ A_n(i,j-1) & \text{if } 3 \leq j \leq k_n + 1. \end{cases}$$

We have $A_n = B_n M_n$ and $\widetilde{A}_n = M_{n+1} B_n$ for every $n \geq 1$, and $\widetilde{A}_0 = M_1 A_0$.

Thus the Bratteli diagrams defined by the sequences of matrices $(A_n)_{n\geq 0}$ and $(\widetilde{A}_n)_{n\geq 0}$ are contractions of the same diagram. This shows that the respective dimension groups with unit are isomorphic (see [16] or [10]). □

2.5. Associated ordered group and orbit equivalence. Let (X, T, G) be a topological dynamical system such that X is a Cantor set and T is minimal. The ordered group associated to (X, T, G) is the unital ordered group

$$\mathcal{G}(X, T, G) = (D_m(X, T, G), D_m(X, T, G)^+, [1]),$$

where

$$D_m(X, T, G) = C(X, \mathbb{Z})/\left\{ f \in C(X, \mathbb{Z})\colon \int f d\mu = 0, \text{ for all } \mu \in \mathcal{M}(X, T, G) \right\},$$

and

$$D_m(X, T, G)^+ = \{[f]\colon f \geq 0\},$$

and where $[1] \in D_m(X, T, G)$ is the class of the constant function 1.

Two topological dynamical systems (X_1, T_1, G_1) and (X_2, T_2, G_2) are (topologically) *orbit equivalent* if there exists a homeomorphism $F\colon X_1 \to X_2$ such that $F(o_{T_1}(x)) = o_{T_2}(F(x))$ for every $x \in X_1$.

In [15] the authors show the following algebraic caracterization of orbit equivalence.

Theorem 1 ([15], Theorem 2.5). *Let $(X, T, \mathbb{Z}^d)$ and $(X', T', \mathbb{Z}^m)$ be two minimal actions on the Cantor set. Then they are orbit equivalent if and only if there is an isomorphism*

$$\mathcal{G}(X, T, \mathbb{Z}^d) \simeq \mathcal{G}(X', T', \mathbb{Z}^m)$$

of unital ordered groups.

3. Suitable Følner sequences

Let G be a residually finite group, and let $(\Gamma_n)_{n\geq 0}$ be a nested sequence of finite index normal subgroups of G such that $\bigcap_{n\geq 0} \Gamma_n = \{e\}$.

For technical reasons it is important to notice that since the groups Γ_n are normal, we have $g\Gamma_n = \Gamma_n g$, for every $g \in G$.

To construct a Toeplitz G-subshift that is an almost 1–1 extension of the odometer defined by the sequence $(\Gamma_n)_n$, we need a suitable sequence $(F_n)_n$ of fundamental domains of G/Γ_n. More precisely, each F_{n+1} has to be tileable by translated copies of F_n. To control the simplex of invariant measures of the subshift, we need in addition the sequence $(F_n)_n$ to be Følner. We did not find in the specialized literature a result ensuring these conditions.

3.1. Suitable sequence of fundamental domains. Let Γ be a normal subgroup of G. By a *fundamental domain* of G/Γ, we mean a subset $D \subseteq G$ containing exactly one representative element of each equivalence class in G/Γ.

Lemma 3. *Let $(D_n)_{n\geq 0}$ be an increasing sequence of finite subsets of G such that for every $n \geq 0$, $e \in D_n$ and D_n is a fundamental domain of G/Γ_n. Let $(n_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ be an increasing sequence. Consider $(F_i)_{i\geq 0}$ defined by $F_0 = D_{n_0}$ and*

$$F_i = \bigcup_{v\in D_{n_i}\cap\Gamma_{n_{i-1}}} vF_{i-1}, \quad \textit{for every } i \geq 1.$$

Then for every $i \geq 0$ we have the following:

(1) *$F_i \subseteq F_{i+1}$ and F_i is a fundamental domain of G/Γ_{n_i}.*

(2) *$F_{i+1} = \bigcup_{v\in F_{i+1}\cap\Gamma_{n_i}} vF_i$.*

Proof. Since $e \in D_{n_i}$, the sequence $(F_i)_{i\geq 0}$ is increasing.

$F_0 = D_{n_0}$ is a fundamental domain of G/Γ_{n_0}. We will prove by induction on i that F_i is a fundamental domain of G/Γ_{n_i}. Let $i > 0$ and suppose that F_{i-1} is a fundamental domain of $G/\Gamma_{n_{i-1}}$.

Let $v \in D_{n_i}$. There exist then $u \in F_{i-1}$ and $w \in \Gamma_{n_{i-1}}$ such that $v = wu$. Let $z \in D_{n_i}$ and $\gamma \in \Gamma_{n_i}$ be such that $w = \gamma z$. Since $z \in \Gamma_{n_{i-1}} \cap D_{n_i}$ and $v = \gamma z u$, we conclude that F_i contains one representing element of each class in G/Γ_{n_i}.

Let $w_1, w_2 \in F_i$ be such that there exists $\gamma \in \Gamma_{n_i}$ verifying $w_1 = \gamma w_2$. By definition, $w_1 = v_1u_1$ and $w_2 = v_2u_2$, for some $u_1, u_2 \in F_{i-1}$ and $v_1, v_2 \in D_{n_i} \cap \Gamma_{n_{i-1}}$. This implies that u_1 and u_2 are in the same class of $G/\Gamma_{n_{i-1}}$. Since F_{i-1} is a fundamental domain, we have $u_1 = u_2$. From this we get $v_1 = \gamma v_2$, which implies that $v_1 = v_2$. Thus we deduce that F_i contains at most one representing element of each class in G/Γ_{n_i}. This shows that F_i is a fundamental domain of G/Γ_{n_i}.

To show that $D_{n_i} \cap \Gamma_{n_{i-1}} \subseteq F_i \cap \Gamma_{n_{i-1}}$, observe that the definition of F_i implies that for every $v \in D_{n_i} \cap \Gamma_{n_{i-1}}$ and $u \in F_{i-1}$, $vu \in F_i$. Then for $u = e \in F_{i-1}$ we get $v = ve \in F_i$. Now suppose that $v \in F_i \cap \Gamma_{n_{i-1}} \subseteq F_i$. The definition of F_i implies there exist $u \in F_{i-1}$ and $\gamma \in D_{n_i} \cap \Gamma_{n_{i-1}}$ such that $v = \gamma u$. Since v and γ are in $\Gamma_{n_{i-1}}$, we get that $u \in \Gamma_{n_{i-1}} \cap F_{i-1}$. This implies that $u = e$ because $\Gamma_{n_{i-1}} \cap F_{i-1} = \{e\}$. ☐

In this paper, by Følner sequences we mean right Følner sequences. That is, a sequence $(F_n)_{n\geq 0}$ of nonempty finite sets of G is a Følner sequence if for every $g \in G$

$$\lim_{n\to\infty} \frac{|F_n g \triangle F_n|}{|F_n|} = 0.$$

Observe that $(F_n)_{n\geq 0}$ is a right Følner sequence if and only if $(F_n^{-1})_{n\geq 0}$ is a left Følner sequence.

Lemma 4. *Suppose that G is amenable. There exists an increasing sequence $(n_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ and a Følner sequence $(F_i)_{i\in\mathbb{Z}^+}$, such that*

i) *$F_i \subseteq F_{i+1}$ and F_i is a fundamental domain of G/Γ_{n_i}, for every $i \geq 0$.*

ii) *$G = \bigcup_{i\geq 0} F_i$.*

iii) *$F_{i+1} = \bigcup_{v\in F_{i+1}\cap\Gamma_{n_i}} vF_i$, for every $i \geq 0$.*

Proof. From [26, Theorem 1] (see [22, Proposition 4.1] for a proof in our context), there exists an increasing sequence $(m_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ and a Følner sequence $(D_i)_{i\in\mathbb{Z}^+}$ such that for every $i \geq 0$, $D_i \subseteq D_{i+1}$, D_i is a fundamental domain of G/Γ_{m_i}, and $G = \bigcup_{i\geq 0} D_i$. Up to taking subsequences, we can assume that D_i is a fundamental domain of G/Γ_i for every $i \geq 0$, and that $e \in D_0$.

We will construct the sequences $(n_i)_{i\geq 0}$ and $(F_n)_{n\geq 0}$ as follows:

STEP 0. We set $n_0 = 0$ and $F_0 = D_0$.

STEP i. Let $i > 0$. We assume that we have chosen n_j and F_j for every $0 \leq j < i$. We take $n_i > n_{i-1}$ in order that the following two conditions are verified:

$$\frac{|D_{n_i} g \,\triangle\, D_{n_i}|}{|D_{n_i}|} < \frac{1}{i|F_{i-1}|}, \text{ for every } g \in F_{i-1}. \tag{1}$$

$$D_{n_{i-1}} \subseteq \bigcup_{v\in D_{n_i}\cap\Gamma_{n_{i-1}}} vF_{i-1}. \tag{2}$$

Such an integer n_i exists because $(D_n)_{n\geq 0}$ is a Følner sequence and F_{i-1} is a fundamental domain of $G/\Gamma_{n_{i-1}}$ (then $G = \bigcup_{v\in\Gamma_{n_{i-1}}} vF_{i-1}$).

We define

$$F_i = \bigcup_{v\in D_{n_i}\cap\Gamma_{n_{i-1}}} vF_{i-1}.$$

Lemma 3 ensures that $(F_i)_{i\geq 0}$ verifies *i*) and *iii*) of the lemma. Equation (2) implies that $(F_i)_{i\geq 0}$ verifies *ii*) of the lemma.

It remains to show that $(F_i)_{i\geq 0}$ is a Følner sequence.

By definition of F_i we have

$$\left(F_i \setminus D_{n_i}\right) \subseteq \bigcup_{g\in F_{i-1}} \left(D_{n_i}g \setminus D_{n_i}\right).$$

Then by equation (1) we get

$$\begin{aligned}\frac{|F_i \setminus D_{n_i}|}{|D_{n_i}|} &\leq \sum_{g\in F_{i-1}} \Big(\frac{|D_{n_i}g \setminus D_{n_i}|}{|D_{n_i}|}\Big)\\ &\leq \Big(|F_{i-1}|\frac{1}{i|F_{i-1}|}\Big)\\ &= \frac{1}{i}.\end{aligned}$$

Since

$$\left(|F_i \cap D_{n_i}| + |D_{n_i} \setminus F_i|\right) = |D_{n_i}| = |F_i| = |F_i \cap D_{n_i}| + |F_i \setminus D_{n_i}|,$$

we obtain

$$\frac{|D_{n_i} \setminus F_i|}{|D_{n_i}|} \leq \frac{1}{i}.$$

Let $g \in G$. Since

$$\begin{aligned}F_i g \setminus F_i &= [(F_i \cap D_{n_i})g \setminus F_i] \cup \left[(F_i \setminus D_{n_i})g \setminus F_i\right]\\ &\subseteq [(F_i \cap D_{n_i})g \setminus F_i] \cup (F_i \setminus D_{n_i})g\\ &\subseteq [D_{n_i}g \setminus (F_i \cap D_{n_i})] \cup (F_i \setminus D_{n_i})g,\end{aligned}$$

we have

$$\begin{aligned}\frac{|F_i g \setminus F_i|}{|F_i|} &\leq \frac{|D_{n_i}g \setminus (F_i \cap D_{n_i})|}{|D_{n_i}|} + \frac{|(F_i \setminus D_{n_i})g|}{|D_{n_i}|}\\ &\leq \frac{|D_{n_i}g \setminus (F_i \cap D_{n_i})|}{|D_{n_i}|} + \frac{1}{i}.\end{aligned} \tag{3}$$

On the other hand, the relation

$$\begin{aligned}D_{n_i}g \setminus D_{n_i} &= D_{n_i}g \setminus [(D_{n_i} \cap F_i) \cup (D_{n_i} \setminus F_i)]\\ &= [D_{n_i}g \setminus (D_{n_i} \cap F_i)] \setminus (D_{n_i} \setminus F_i),\end{aligned}$$

implies that

$$\begin{aligned} D_{n_i}g \setminus (F_i \cap D_{n_i}) &= [(D_{n_i}g \setminus (F_i \cap D_{n_i})) \cap (D_{n_i} \setminus F_i)] \\ &\quad \cup [(D_{n_i}g \setminus (F_i \cap D_{n_i})) \setminus (D_{n_i} \setminus F_i)] \\ &= [(D_{n_i}g \setminus (F_i \cap D_{n_i})) \cap (D_{n_i} \setminus F_i)] \\ &\quad \cup [D_{n_i}g \setminus D_{n_i}] \\ &\subseteq (D_{n_i} \setminus F_i) \cup (D_{n_i}g \setminus D_{n_i}), \end{aligned}$$

which ensures that

$$\frac{|D_{n_i}g \setminus (F_i \cap D_{n_i})|}{|D_{n_i}|} \leq \frac{|D_{n_i} \setminus F_i|}{|D_{n_i}|} + \frac{|D_{n_i}g \setminus D_{n_i}|}{|D_{n_i}|}. \tag{4}$$

From equations (3) and (4), we obtain

$$\frac{|F_i g \setminus F_i|}{|F_i|} \leq \frac{2}{i} + \frac{|D_{n_i}g \setminus D_{n_i}|}{|D_{n_i}|},$$

which implies

$$\lim_{i\to\infty} \frac{|F_i g \setminus F_i|}{|F_i|} = 0. \tag{5}$$

In a similar way we deduce that

$$F_i \setminus F_i g \subseteq [D_{n_i} \setminus (F_i \cap D_{n_i})g] \cup (F_i \setminus D_{n_i}),$$

$$D_{n_i} \setminus D_{n_i}g = [D_{n_i} \setminus (D_{n_i} \cap F_i)g] \setminus (D_{n_i} \setminus F_i),$$

and

$$D_{n_i} \setminus (F_i \cap D_{n_i})g \subseteq (D_{n_i} \setminus F_i) \cup (D_{n_i} \setminus D_{n_i}g).$$

Combining the last three equations we get

$$\frac{|F_i \setminus F_i g|}{|F_i|} \leq \frac{2}{i} + \frac{|D_{n_i} \setminus D_{n_i}g|}{|D_{n_i}|},$$

which implies

$$\lim_{i\to\infty} \frac{|F_i \setminus F_i g|}{|F_i|} = 0. \tag{6}$$

Equations (5) and (6) imply that $(F_i)_{i\geq 0}$ is Følner. □

The following result is a direct consequence of Lemma 4.

Lemma 5. *Let G be an amenable residually finite group and let $(\Gamma_n)_{n\geq 0}$ be a decreasing sequence of finite index normal subgroups of G such that $\bigcap_{n\geq 0}\Gamma_n = \{e\}$. There exists an increasing sequence $(n_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ and a Følner sequence $(F_i)_{i\geq 0}$ of G such that*

(1) *$\{e\} \subseteq F_i \subseteq F_{i+1}$ and F_i is a fundamental domain of G/Γ_{n_i}, for every $i \geq 0$;*

(2) *$G = \bigcup_{i\geq 0} F_i$;*

(3) *$F_j = \bigcup_{v\in F_j\cap\Gamma_{n_i}} vF_i$, for every $j > i \geq 0$.*

Proof. The existence of the sequence of subgroups of G and the Følner sequence verifying (1), (2) and (3) for $j = i + 1$ is direct from Lemma 4. Using induction, it is straightforward to show (3) for every $j > i \geq 0$. □

4. Kakutani–Rokhlin partitions for generalized Toeplitz subshifts

In this section G is an amenable, countable, and residually finite group.

Let Σ be a finite alphabet and let (Σ^G, σ, G) be the respective full G-shift.

For a finite index subgroup Γ of G, $x \in \Sigma^G$ and $a \in \Sigma$, we define

$$\mathrm{Per}(x, \Gamma, a) = \{g \in G : \sigma^\gamma(x)(g) = x(\gamma^{-1}g) = a, \text{ for all } \gamma \in \Gamma\},$$

and

$$\mathrm{Per}(x, \Gamma) = \bigcup_{a\in\Sigma} \mathrm{Per}(x, \Gamma, a).$$

It is straightforward to show that $x \in \Sigma^G$ is a Toeplitz sequence if and only if there exists an increasing sequence $(\Gamma_n)_{n\geq 0}$ of finite index subgroups of G such that

$$G = \bigcup_{n\geq 0} \mathrm{Per}(x, \Gamma_n);$$

see [5, Proposition 5].

A *period structure* of $x \in \Sigma^G$ is an increasing sequence of finite index subgroups $(\Gamma_n)_{n\geq 0}$ of G such that $G = \bigcup_{n\geq 0}\mathrm{Per}(x, \Gamma_n)$ and such that for every $n \geq 0$, Γ_n is an *essential group of periods*. This means that if $g \in G$ is such that $\mathrm{Per}(x, \Gamma_n, a) \subseteq \mathrm{Per}(\sigma^g(x), \Gamma_n, a)$ for every $a \in \Sigma$, then $g \in \Gamma_n$.

It is known that every Toeplitz sequence has a period structure (see for example [5, Corollary 6]). We construct in this section, thanks to the period structure, a Kakutani–Rokhlin partition, and we deduce a characterization of its ordered group.

4.1. Existence of Kakutani–Rokhlin partitions. In this subsection we suppose that $x_0 \in \Sigma^G$ is a non-periodic Toeplitz sequence ($\sigma^g(x_0) = x_0$ implies $g = e$) having a period structure $(\Gamma_n)_{n\geq 0}$ such that for every $n \geq 0$,

(i) Γ_{n+1} is a proper subset of Γ_n;

(ii) Γ_n is a normal subgroup of G.

Every non-periodic Toeplitz sequence has a period structure verifying (i) [5, Corollary 6]. Condition (ii) is satisfied for every Toeplitz sequence whose Toeplitz subshift is an almost 1–1 extension of an odometer (in the general case these systems are almost 1–1 extensions of subodometers. See [5] for the details).

By Lemma 5 we can assume there exists a Følner sequence $(F_n)_{n\geq 0}$ of G such that

(F1) $\{e\} \subseteq F_n \subseteq F_{n+1}$ and F_n is a fundamental domain of G/Γ_n, for every $n \geq 0$;

(F2) $G = \bigcup_{n\geq 0} F_n$;

(F3) $F_n = \bigcup_{v\in F_n\cap\Gamma_i} vF_i$, for every $n > i \geq 0$.

We denote by X the closure of the orbit of x_0. Thus $(X, \sigma|_X, G)$ is a Toeplitz subshift.

Definition 2. We say that a finite clopen partition $\mathcal{P}$ of X is a regular Kakutani–Rokhlin partition (r-K-R partition), if there exists a finite index subgroup Γ of G with a fundamental domain F containing e and a clopen C_k, such that

$$\mathcal{P} = \{\sigma^{u^{-1}}(C_k) : u \in F, 1 \leq k \leq N\}$$

and

$$\sigma^{\gamma}\Big(\bigcup_{k=1}^{N} C_k\Big) = \bigcup_{k=1}^{N} C_k \text{ for every } \gamma \in \Gamma.$$

To construct a regular Kakutani–Rokhlin partition of X, we need the following technical lemma.

Lemma 6. *Let $\mathcal{P}' = \{\sigma^{u^{-1}}(D_k) : u \in F, 1 \leq k \leq N\}$ be an r-K-R partition of X and $\mathcal{Q}$ any other finite clopen partition of X. Then there exists a r-K-R partition $\mathcal{P} = \{\sigma^{u^{-1}}(C_k) : u \in F, 1 \leq k \leq M\}$ of X such that*

(1) *$\mathcal{P}$ is finer than $\mathcal{P}'$ and $\mathcal{Q}$,*

(2) $\bigcup_{k=1}^{M} C_k = \bigcup_{k=1}^{N} D_k$.

Proof. Let $F = \{u_0, u_1, \ldots, u_{|F|-1}\}$, with $u_0 = e$.

We refine every set D_k with respect to the partition $\mathcal{Q}$. Thus we get a collection of disjoint sets

$$D_{1,1}, \ldots, D_{1,l_1}; \ldots; D_{N,1}, \ldots, D_{N,l_N},$$

such that each of these sets is in an atom of $\mathcal{Q}$ and $D_k = \bigcup_{j=1}^{l_k} D_{k,j}$ for every $1 \leq k \leq N$. Thus

$$\mathcal{P}_0 = \{\sigma^{u^{-1}}(D_{k,j}) \colon u \in F, 1 \leq j \leq l_k, 1 \leq k \leq N\}$$

is an r-K-R partition of X. For simplicity we write

$$\mathcal{P}_0 = \{\sigma^{u^{-1}}(D_k^{(0)}) \colon u \in F, 1 \leq k \leq N_0\}.$$

We have that $\mathcal{P}_0$ verifies (2) and every $D_k^{(0)}$ is contained in atoms of $\mathcal{P}'$ and $\mathcal{Q}$.

Let $0 \leq n < |F| - 1$. Suppose that we have defined a r-K-R partition of X

$$\mathcal{P}_n = \{\sigma^{u^{-1}}(D_k^{(n)}) \colon u \in F, 1 \leq k \leq N_n\},$$

such that $\mathcal{P}_n$ verifies (2) and such that for every $0 \leq j \leq n$ and $1 \leq k \leq N_n$ there exist $A \in \mathcal{P}'$ and $B \in \mathcal{Q}$ such that

$$\sigma^{u_j^{-1}}(D_k^{(n)}) \subseteq A, B.$$

Now we refine every set $\sigma^{u_{n+1}^{-1}}(D_k^{(n)})$ with respect to $\mathcal{Q}$. Thus we get a collection of disjoint sets

$$D_{1,1}, \ldots, D_{1,s_1}; \ldots; D_{N_n,1}, \ldots, D_{N_n,s_{N_n}}$$

such that each of these sets is in an atom of $\mathcal{Q}$ and

$$\sigma^{u_{n+1}^{-1}}(D_k^{(n)}) = \bigcup_{j=1}^{s_k} D_{k,j},$$

for every $1 \leq k \leq N_n$.

For every $1 \leq k \leq N_n$ and $1 \leq j \leq s_k$, let $C_{k,j} = \sigma^{u_{n+1}}(D_{k,j}) \subseteq D_k^{(n)}$. We have that

$$\mathcal{P}_{n+1} = \{\sigma^{u^{-1}}(C_{k,j}) \colon u \in F, 1 \leq j \leq s_k, 1 \leq k \leq N_n\}$$

is an r-K-R partition of X verifying (2) and such that for every $0 \leq i \leq n+1$, $1 \leq j \leq s_k$ and $1 \leq k \leq N_n$ there exist $A \in \mathcal{P}'$ and $B \in \mathcal{Q}$ such that

$$\sigma^{u_j^{-1}}(C_{k,j}) \subseteq A, B.$$

At the step $n = |F| - 1$ we get $\mathcal{P} = \mathcal{P}_{|F|-1}$ verifying (1) and (2). $\square$

Proposition 1. *There exists a sequence* $(\mathcal{P}_n = \{\sigma^{u^{-1}}(C_{n,k})\colon u \in F_n, 1 \leq k \leq k_n\})_{n\geq 0}$ *of r-K-R partitions of* X *such that for every* $n \geq 0$*,*

(1) $\mathcal{P}_{n+1}$ *is finer than* $\mathcal{P}_n$*,*

(2) $C_{n+1} \subseteq C_n = \bigcup_{k=1}^{k_n} C_{n,k}$*,*

(3) $\bigcap_{n\geq 1} C_n = \{x_0\}$*,*

(4) *The sequence* $(\mathcal{P}_n)_{n\geq 0}$ *spans the topology of* X*.*

Proof. For every $n \geq 0$, let define

$$C_n = \{x \in X\colon \operatorname{Per}(x, \Gamma_n, a) = \operatorname{Per}(x_0, \Gamma_n, a), \text{ for all } a \in \Sigma\}.$$

From [5, Proposition 6] we get

$$C_n = \overline{\{\sigma^\gamma(x_0)\colon \gamma \in \Gamma_n\}},$$

and that $\mathcal{P}'_n = \{\sigma^{u^{-1}}(C_n)\colon u \in F_n\}$ is a clopen partition of X such that $\sigma^\gamma(C_n) = C_n$ for every $\gamma \in \Gamma_n$. Thus $\mathcal{P}'_n$ is an r-K-R partition of X. Furthermore, the sequence $(\mathcal{P}'_n)_{n\geq 0}$ verifies (1), (2) and (3).

For every $n \geq 0$, let

$$\mathcal{Q}_n = \{[B] \cap X\colon B \in \Sigma^{F_n}, [B] \cap X \neq \emptyset\}.$$

This is a finite clopen partition of X and $(\mathcal{Q}_n)_{n\geq 0}$ spans the topology of X.

We define

$$\mathcal{P}_0 = \{\sigma^{u^{-1}}(C_{0,k})\colon u \in F_0, 1 \leq k \leq k_0\})$$

the r-K-R partition finer than $\mathcal{P}'_0$ and $\mathcal{Q}_0$ given by Lemma 6. Now we take $\mathcal{P}''_n$ the r-K-R partition finer that $\mathcal{P}_{n-1}$ and $\mathcal{Q}_n$ given by Lemma 6, and we define

$$\mathcal{P}_n = \{\sigma^{u^{-1}}(C_{n,k})\colon u \in F_n, 1 \leq k \leq k_n\},$$

the r-K-R partition finer than $\mathcal{P}' = \mathcal{P}'_n$ and $\mathcal{Q} = \mathcal{P}''_n$ given by Lemma 6. Thus $\mathcal{P}_n$ is finer than $\mathcal{P}_{n-1}$ and $\mathcal{Q}_n$. This implies that the sequence $(\mathcal{P}_n)_{n\geq 0}$ verifies (1) and (4). Since $\bigcup_{k=1}^{k_n} C_{n,k} = C_n$, we deduce that $(\mathcal{P}_n)_{n\geq 0}$ verifies (2) and (3). □

Remark 1. The sequence of partitions introduced in Proposition 1 is a generalization to Toeplitz G-subshifts of the sequences of Kakutani–Rokhlin partitions for Toeplitz $\mathbb{Z}$-subshifts introduced in [17]. See [19] for more details about Kakutani–Rokhlin partitions for minimal $\mathbb{Z}$-actions on the Cantor set

Definition 3. We say that a sequence $(\mathcal{P}_n)_{n\geq 0}$ of r-K-R partitions as in Proposition 1 is a nested sequence of r-K-R partitions of X.

Let

$$(\mathcal{P}_n = \{\sigma^{u^{-1}}(C_{n,k}) \colon u \in F_n, 1 \leq k \leq k_n\})_{n\geq 0}$$

be a sequence of nested r-K-R partitions of X.

For every $n \geq 0$ we define the matrix $M_n \in \mathcal{M}_{k_n \times k_{n+1}}(\mathbb{Z}^+)$ as

$$M_n(i,k) = |\{\gamma \in F_{n+1} \cap \Gamma_n \colon \sigma^{\gamma^{-1}}(C_{n+1,k}) \subseteq C_{n,i}\}|,$$

We call M_n the *incidence matrix* of the partitions $\mathcal{P}_{n+1}$ and $\mathcal{P}_n$.

Let p be a positive integer. For every $n \geq 1$ we denote by $\triangle(n, p)$ the closed convex hull generated by the vectors $\frac{1}{p}e_1^{(n)}, \dots, \frac{1}{p}e_n^{(n)}$, where $e_1^{(n)}, \dots, e_n^{(n)}$ is the canonical basis in $\mathbb{R}^n$. Thus $\triangle(n, 1)$ is the unitary simplex in $\mathbb{R}^n$.

Observe that for every $n \geq 0$ and $1 \leq k \leq k_{n+1}$,

$$\sum_{i=1}^{k_n} M_n(i,k) = \frac{|F_{n+1}|}{|F_n|}.$$

This implies that $M_n(\triangle(k_{n+1}, |F_{n+1}|)) \subseteq \triangle(k_n, |F_n|)$.

The next result characterizes the maximal equicontinuous factor, the space of invariant probability measures and the associated ordered group of $(X, \sigma|_X, G)$ in terms of the sequence of incidence matrices of a nested sequence of r-K-R partitions.

Proposition 2. *Let*

$$(\mathcal{P}_n = \{\sigma^{u^{-1}}(C_{n,k}) \colon u \in F_n, 1 \leq k \leq k_n\})_{n\geq 0}$$

be a nested sequence of r-K-R partitions of X with an associated sequence of incidence matrices $(M_n)_{n\geq 0}$. Then

(1) *$(X, \sigma|_X, G)$ is an almost 1–1 extension of the odometer $O = \varprojlim_n (G/\Gamma_n, \pi_n)$;*

(2) *there is an affine homeomorphism between the set of invariant probability measures of $(X, \sigma|_X, G)$ and the inverse limit $\varprojlim_n (\triangle(k_n, |F_n|), M_n)$;*

(3) *the ordered group $\mathcal{G}(X, \sigma|_X, G)$ is isomorphic to $(H/\inf(H), (H/\inf(H))^+, u + \inf(H))$, where (H, H^+) is given by*

$$\mathbb{Z} \xrightarrow{M^T} \mathbb{Z}^{k_0} \xrightarrow{M_0^T} \mathbb{Z}^{k_1} \xrightarrow{M_1^T} \mathbb{Z}^{k_2} \xrightarrow{M_2^T} \cdots,$$

where $M = |F_0|(1, \dots, 1)$ and $u = [M^T, 0]$.

Proof. (1) For every $x \in X$, $n \geq 0$, let $v_n(x) \in F_n$ be such that $x \in \sigma^{v_n(x)^{-1}}(C_n)$.

The map $\pi\colon X \to O$ given by $\pi(x) = (v_n(x)^{-1}\Gamma_n)_{n\geq 1}$ is well defined, is a factor map and verifies $\pi^{-1}(\pi(x_0)) = \{x_0\}$. This shows that $(X, \sigma|_X, G)$ is an almost 1–1 extension of O.

(2) It is clear that for any invariant probability measure μ of $(X, \sigma|_X, G)$, the sequence $(\mu_n)_{n\geq 0}$, with $\mu_n = (\mu(C_{n,k})\colon 1 \leq k \leq k_n)$, is an element of the inverse limit $\varprojlim_n (\triangle(k_n, |F_n|), M_n)$. Conversely, any element $(\mu_{n,k}\colon 1 \leq k \leq k_n)_{m\geq 0}$ of such an inverse limit defines a probability measure μ on the σ-algebra generated by $(\mathcal{P}_n)_{n\geq 0}$, which is equal to the Borel σ-algebra of X because $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of X and is countable. Since the sequence (F_n) is Følner, it is standard to check that the measure μ is invariant by the G-action.

The function $\mu \mapsto (\mu_n)_{n\geq 0}$ is thus an affine bijection between $\mathcal{M}(X, \sigma|_X, G)$ and the inverse limit $\varprojlim_n (\triangle(k_n, |F_n|), M_n)$. Observe that this function is a homeomorphism with respect to the weak topology in $\mathcal{M}(X, \sigma|_X, G)$ and the product topology in the inverse limit.

(3) We denote by $[k, -1]$ the class of the element $(k, -1) \in \mathbb{Z} \times \{-1\}$ in H. Let

$$\phi\colon H \longrightarrow D_m(X, \sigma|_X, G)$$

be the function given by

$$\phi([v, n]) = \sum_{k=1}^{k_n} v_i [1_{C_{n,k}}], \quad \text{for every } v = (v_1, \ldots, v_{k_n}) \in \mathbb{Z}^{k_n} \text{ and } n \geq 0,$$

and

$$\phi([k, -1]) = k 1_X \quad \text{for every } k \in \mathbb{Z}.$$

It is easy to check that ϕ is a well defined homomorphism of groups that verifies $\phi(H^+) \subseteq D_m(X, \sigma|_X, G)^+$. Since $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of X, every function $f \in C(X, \mathbb{Z})$ is constant on every atom of $\mathcal{P}_n$, for some $n \geq 0$. This implies that ϕ is surjective. Lemma 1 and (2) of Proposition 2, imply that $\mathrm{Ker}(\phi) = \inf(H)$. Finally, ϕ induces a isomorphism

$$\hat{\phi}\colon H/\inf(H) \longrightarrow D_m(X, \sigma|_X, G)$$

such that

$$\hat{\phi}((H/\inf(H))^+) = D_m(X, \sigma|_X, G)^+.$$

Since $[1, -1] = [M^T, 0]$, we get $\phi([M^T, 0]) = [1_X]$. □

5. Kakutani–Rokhlin partitions with prescribed incidence matrices

We say that a sequence of positive integer matrices $(M_n)_{n\geq 0}$ is *managed* by the increasing sequence of positive integers $(p_n)_{n\geq 0}$, if for every $n \geq 0$ the integer p_n divides p_{n+1}, and if the matrix M_n verifies the following properties:

(1) M_n has $k_n \geq 2$ rows and $k_{n+1} \geq 2$ columns;

(2) $\sum_{i=1}^{k_n} M_n(i,k) = \frac{p_{n+1}}{p_n}$, for every $1 \leq k \leq k_{n+1}$.

If $(M_n)_{n\geq 0}$ is a sequence of matrices managed by $(p_n)_{n\geq 0}$, then for each $n \geq 0$, $M_n(\triangle(k_{n+1}, p_{n+1})) \subseteq \triangle(k_n, p_n)$.

Observe that the sequences of incidence matrices associated to the nested sequences of r-K-R partitions defined in Section 4 are managed by $(|F_n|)_{n\geq 0}$.

In this Section we construct Toeplitz subshifts with nested sequences of r-K-R partitions whose sequences of incidence matrices are managed.

5.1. Construction of the partitions.

In the rest of this section G is an amenable and residually finite group. Let $(\Gamma_n)_{n\geq 0}$ be a decreasing sequence of finite index normal subgroup of G such that $\bigcap_{n\geq 0} \Gamma_n = \{e\}$, and let $(F_n)_{n\geq 0}$ be a Følner sequence of G such that

(F1) $\{e\} \subseteq F_n \subseteq F_{n+1}$ and F_n is a fundamental domain of G/Γ_n, for every $n \geq 0$;

(F2) $G = \bigcup_{n\geq 0} F_n$;

(F3) $F_n = \bigcup_{v\in F_n\cap\Gamma_i} vF_i$, for every $n > i \geq 0$.

Lemma 5 ensures the existence of a Følner sequence verifying conditions (F1), (F2) and (F3).

For every $n \geq 0$, we call R_n the set $F_n \cdot F_n^{-1} \cup F_n^{-1} \cdot F_n$. This will enable us to define a "border" of each domain F_{n+1}.

Let Σ be a finite alphabet. For every $n \geq 0$, let $k_n \geq 3$ be an integer. We say that the sequence of sets $(\{B_{n,1}, \dots, B_{n,k_n}\})_{n\geq 0}$ (where $\{B_{n,1}, \dots, B_{n,k_n}\} \subseteq \Sigma^{F_n}$, for any $n \geq 0$, is a collection of different functions) *verifies conditions* (C1)–(C4) if it verifies the following four conditions for any $n \geq 0$:

(C1) $\sigma^{\gamma^{-1}}(B_{n+1,k})|_{F_n} \in \{B_{n,i} : 1 \leq i \leq k_n\}$, for every $\gamma \in F_{n+1} \cap \Gamma_n$, and $1 \leq k \leq k_{n+1}$;

(C2) $B_{n+1,k}|_{F_n} = B_{n,1}$, for every $1 \leq k \leq k_{n+1}$;

(C3) For any $g \in F_n$ such that for some $1 \leq k, k' \leq k_n$, $B_{n,k}(gv) = B_{n,k'}(v)$ for all $v \in F_n \cap g^{-1}F_n$, then $g = e$;

(C4) $\sigma^{\gamma^{-1}}(B_{n+1,k})|_{F_n} = B_{n,k_n}$ for every $\gamma \in (F_{n+1} \cap \Gamma_n) \cap [F_{n+1} \setminus F_{n+1}g^{-1}]$, for some $g \in R_n$.

Example 1. To illustrate these conditions, let us consider the case $G = \mathbb{Z}$, with $\Sigma = \{1, 2, 3, 4\}$ and $\Gamma_n = 3^{2(n+1)}\mathbb{Z}$ for every $n \geq 0$. The set

$$F_n = \Big\{ -\Big(\frac{3^{2(n+1)} - 1}{2}\Big), -\Big(\frac{3^{2(n+1)} - 1}{2}\Big) + 1, \ldots, \Big(\frac{3^{2(n+1)} - 1}{2}\Big) \Big\}$$

is a fundamental domain of $\mathbb{Z}/\Gamma_n$. Furthermore we have

$$F_n = \bigcup_{v \in \{k3^{2n} :\, -4 \leq k \leq 4\}} (F_{n-1} + v),$$

for every $n \geq 1$. This shows that sequence $(F_n)_{n \geq 0}$ satisfies (F1), (F2) and (F3).

Now let us consider the case where $k_n = 4$ for every $n \geq 0$. We define $B_{0,k}(j) = k$ for every $j \in F_0$ and $1 \leq k \leq 4$, and for $n \geq 1$,

$$B_{n,k}|_{F_{n-1}} = B_{n-1,1},$$
$$B_{n,k}|_{F_{n-1}+v} = B_{n-1,4}$$

for $v \in \{-l \cdot 3^{2n}, l \cdot 3^{2n} : l = 3, 4\}$.

Thus they verify the conditions (C1) and (C4). We fill the rest of the $B_{n,k}|_{F_{n-1}+v}$ with $B_{n-1,3}$ and $B_{n-1,2}$ in order that $B_{n,1}, \ldots, B_{n,4}$ are different. They satisfy conditions (C2) and (C4). The limit in $\Sigma^{\mathbb{Z}}$ of the functions $B_{n,1}$ is a $\mathbb{Z}$-Toeplitz sequence x. If X denotes the closure of the orbit of x, then we prove in the next lemma (in a more general setting) that

$$(\mathcal{P}_n = \{\sigma^j([B_{n,k}] \cap X) :\, j \in F_n, 1 \leq k \leq 4\})_{n \geq 0}$$

is a sequence of nested Kakutani–Rokhlin partitions of the subshift X.

In the next lemma, we show that conditions (C1) and (C2) are sufficient to construct a Toeplitz sequence. The technical conditions (C3) (aperiodicity) and (C4) (also known as "forcing the border") will allow to construct a nested sequence of r-K-R partitions of X.

Lemma 7. *Let* $(\{B_{n,1},\ldots,B_{n,k_n}\})_{n\geq 0}$ *be a sequence that verifies* (C1)–(C4).

(1) *The set* $\bigcap_{n\geq 0}[B_{n,1}]$ *contains only one element* x_0 *which is a Toeplitz sequence.*

(2) *Let* X *be the orbit closure of* x_0 *with respect to the shift action. For every* $n \geq 0$*, let*

$$\mathcal{P}_n = \{\sigma^{u^{-1}}([B_{n,k}] \cap X) \colon 1 \leq k \leq k_n, u \in F_n\}.$$

Then $(\mathcal{P}_n)_{n\geq 0}$ *is a sequence of nested r-K-R partitions of* X.

Let $(M_n)_{n\geq 0}$ *be the sequence of incidence matrices of* $(\mathcal{P}_n)_{n\geq 0}$.

(3) *The Toeplitz subshift* $(X, \sigma|_X, G)$ *is an almost 1–1 extension of the odometer* $O = \varprojlim_n (G/\Gamma_n, \pi_n)$.

(4) *There is an affine homeomorphism between the set of invariant probability measures of* $(X, \sigma|_X, G)$ *and the inverse limit* $\varprojlim_n (\triangle(k_n, |F_n|), M_n)$.

(5) *The ordered group* $\mathcal{G}(X, \sigma|_X, G)$ *is isomorphic to* $(H/\inf(H), (H/\inf(H))^+, u + \inf(H))$*, where* (H, H^+) *is given by*

$$\mathbb{Z} \xrightarrow{M^T} \mathbb{Z}^{k_0} \xrightarrow{M_0^T} \mathbb{Z}^{k_1} \xrightarrow{M_1^T} \mathbb{Z}^{k_2} \xrightarrow{M_2^T} \cdots,$$

with $M = |F_0|(1,\ldots,1)$ *and* $u = [M^T, 0]$.

Proof. Condition (C2) implies that $\bigcap_{n\geq 0}[B_{n,1}]$ is non empty, and since $G = \bigcup_{n\geq 0} F_n$, there is only one element x_0 in this intersection. Let X be the orbit closure of x_0. For every $n \geq 0$ and $1 \leq k \leq k_n$, we denote $C_{n,k} = [B_{n,k}] \cap X$.

Claim: For every $m > n \geq 0$, $1 \leq k \leq k_m$ and $\gamma \in F_m \cap \Gamma_n$,

$$\sigma^{\gamma^{-1}}(B_{m,k})|_{F_n} \in \{B_{n,i} \colon 1 \leq i \leq k_n\}. \tag{7}$$

Condition (C1) implies that (7) holds when $n = m-1$. We will show the claim by induction on n.

Suppose that for every $1 \leq k \leq k_m$ and $\gamma \in F_m \cap \Gamma_{n+1}$,

$$\sigma^{\gamma^{-1}}(B_{m,k})|_{F_{n+1}} \in \{B_{n+1,i} \colon 1 \leq i \leq k_{n+1}\}.$$

Let $g \in \Gamma_n \cap F_m$. Condition (F3) implies there exist $v \in \Gamma_{n+1} \cap F_m$ and $u \in F_{n+1}$ such that $g = vu$. Thus we get

$$\sigma^{g^{-1}}(B_{m,k})|_{F_n} = \sigma^{u^{-1}v^{-1}}(B_{m,k}) = \sigma^{v^{-1}}(B_{m,k})|_{uF_n}.$$

Since $u \in \Gamma_n \cap F_{n+1}$, condition (F3) implies that $uF_n \subseteq F_{n+1}$. Then by hypothesis there exists $1 \leq l \leq k_{n+1}$ such that

$$\sigma^{v^{-1}}(B_{m,k})|_{uF_n} = B_{n+1,l}|_{uF_n},$$

which is equal to some $B_{n,s}$, by (C1). This shows the claim.

From (7) we deduce that $\sigma^{\gamma^{-1}}(x_0)|_{F_n} \in \{B_{n,i}: 1 \leq i \leq k_n\}$, for every $\gamma \in \Gamma_n$. Thus if g is any element in G, and $u \in F_n$ and $\gamma \in \Gamma_n$ are such that $g = \gamma u$, then $\sigma^{g^{-1}}(x_0) = \sigma^{u^{-1}}(\sigma^{\gamma^{-1}}(x_0)) \in \sigma^{u^{-1}}(C_{n,k})$, for some $1 \leq k \leq k_n$. It follows that

$$\mathcal{P}_n = \{\sigma^{u^{-1}}(C_{n,k}) \colon 1 \leq k \leq k_n, u \in F_n\}$$

is a clopen covering of X.

From condition (C2) and (7) we get that $\sigma^{\gamma^{-1}}(x_0)|_{F_{n-1}} = B_{n-1,1}$ for any $\gamma \in \Gamma_n$, which implies that $F_{n-1} \subseteq \mathrm{Per}(x_0, \Gamma_n)$. This shows that x_0 is Toeplitz.

Now we will show that $\mathcal{P}_n$ is a partition. Suppose that $1 \leq k, l \leq k_n$ and $u \in F_n$ are such that $\sigma^{u^{-1}}(C_{n,k}) \cap C_{n,l} \neq \emptyset$. Then there exist $x \in C_{n,k}$ and $y \in C_{n,l}$ such that $\sigma^{u^{-1}}(x) = y$. From this we have $x(uv) = y(v)$ for every $v \in G$. In particular, $x(uv) = y(v)$ for every $v \in F_n \cap u^{-1}F_n$, which implies $B_{n,k}(uv) = B_{n,l}(v)$ for every $v \in F_n \cap u^{-1}F_n$. From condition (C3) we get $u = e$ and $k = l$. This ensures that the *set of return times* of x_0 to $\bigcup_{k=1}^{k_n} C_{n,k}$, *i.e.* the set $\{g \in G \colon \sigma^{g^{-1}}(x_0) \in \bigcup_{k=1}^{k_n} C_{n,k}\}$, is Γ_n. From this it follows that $\mathcal{P}_n$ is an r-K-R partition. From (C1) we have that $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$ and that $C_{n+1} \subseteq \bigcup_{k=1}^{k_n} C_{n,k} = C_n$. By the definition of x_0 we have that $\{x_0\} = \bigcap_{n\geq 0} C_n$.

Now we will show that $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of X. Since every $\mathcal{P}_n$ is a partition, for every $n \geq 0$ and every $x \in X$ there are unique $v_n(x) \in F_n$ and $1 \leq k_n(x) \leq k_n$ such that

$$x \in \sigma^{v_n(x)^{-1}}(C_{n,k_n(x)}).$$

The collection $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of X if and only if $(v_n(x))_{n\geq 0} = (v_n(y))_{n\geq 0}$ and $(k_n(x))_{n\geq 0} = (k_n(y))_{n\geq 0}$ imply $x = y$.

Let $x, y \in X$ be two sequences such that $v_n(x) = v_n(y) = v_n$ and $k_n(x) = k_n(y)$ for every $n \geq 0$. Let $g \in G$ be such that $x(g) \neq y(g)$.

We have then for any $n \geq 0$

$$\sigma^{v_n}(x)|_{F_n} = \sigma^{v_n}(y)|_{F_n} \in \{B_{n,i}: 1 \leq i \leq k_n\},$$

and then

$$x|_{v_n^{-1}F_n} = y|_{v_n^{-1}F_n}.$$

Thus by definition, we get $g \notin v_n^{-1} F_n$ for any n. We can take n sufficiently large in order that $g \in F_{n-1}$.

Let $\gamma \in \Gamma_n$ and $u \in F_n$ such that $v_n(x)g = \gamma u$. Observe that $ug^{-1} \notin F_n$. Indeed, if $ug^{-1} \in F_n$, then the relation $v_n(x) = \gamma u g^{-1}$ implies $\gamma = e$, but in that case we get $v_n(x)g = u \in F_n$ which is not possible by hypothesis. By the condition (C1), there exists an index $1 \leq i \leq k_n$ such that $\sigma^{\gamma^{-1}}(\sigma^{v_n}(x))|_{F_n} = B_{n,i}$ and then

$$x(g) = \sigma^{\gamma^{-1}} \sigma^{v_n}(x)(\gamma^{-1} v_n g) = B_{n,i}(u).$$

Let $\gamma' \in \Gamma_{n-1} \cap F_n$ and $u' \in F_{n-1}$ such that $u = \gamma' u'$. Since $\gamma' u' g^{-1} = ug^{-1} \notin F_n$, we get $\gamma' \in F_n \setminus F_n g u'^{-1}$. This implies that $\gamma' \in F_n \setminus F_n w$, for $w = gu'^{-1} \in R_{n-1}$ and $B_{n,i}(u) = B_{n-1,k_{n-1}}(u')$ by the condition (C4). Thus $x(g) = B_{n-1,k_{n-1}}(u')$. The same argument implies that $y(g) = B_{n-1,k_{n-1}}(u') = x(g)$ and we obtain a contradiction.

This shows that $(\mathcal{P}_n)_{n\geq 0}$ is a sequence of nested r-K-R partitions of X.

The point (3), (4) and (5) follows from Propositions 2. □

The next result shows that, up to telescoping a managed sequence of matrices, it is possible to obtain a managed sequence of matrices with sufficiently large coefficients to satisfy the conditions of Lemma 7.

Lemma 8. *Let $(M_n)_{n\geq 0}$ be a sequence of matrices managed by $(|F_n|)_{n\geq 0}$. Let k_n be the number of rows of M_n, for every $n \geq 0$.*

Then there exists an increasing sequence $(n_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ such that for every $i \geq 0$ and every $1 \leq k \leq k_{n_{i+1}}$,

(i) $R_{n_i} \subseteq F_{n_{i+1}}$;

(ii) *for every* $1 \leq l \leq k_{n_i}$,

$$M_{n_i} M_{n_i+1} \cdots M_{n_{i+1}-1}(l,k) > 1 + \left| \bigcup_{g \in R_{n_i}} F_{n_{i+1}} \setminus F_{n_{i+1}} g^{-1} \right|.$$

If in addition there exists a constant $K > 0$ such that $k_{n+1} \leq K \frac{|F_{n+1}|}{|F_n|}$ for every $n \geq 0$, then the sequence $(n_i)_{i\geq 0}$ can be chosen in order that

(iii) $k_{n_{i+1}} < M_{n_i} \cdots M_{n_{i+1}-1}(i,k)$, *for every* $1 \leq i \leq k_{n_i}$.

Proof. We define $n_0 = 0$. Let $i \geq 0$ and suppose that we have defined n_j for every $0 \leq j \leq i$. Let $m_0 > n_i$ be such that for every $m \geq m_0$,

$$R_{n_i} \subseteq F_m.$$

Let $0 < \varepsilon < 1$ be such that $\varepsilon |R_{n_i}| < 1$. Since $(F_n)_{n \geq 0}$ is a Følner sequence, there exists $m_1 > m_0$ such that for every $m \geq m_1$,

$$\frac{|F_m \setminus F_m g^{-1}|}{|F_m|} < \frac{\varepsilon}{|F_{n_i+1}|}, \quad \text{for every } g \in R_{n_i}. \tag{8}$$

Since $\varepsilon |R_{n_i}| < 1$, there exists $m_2 > m_1$ such that for every $m \geq m_2$,

$$1 - \frac{|F_{n_i+1}|}{|F_m|} > \varepsilon |R_{n_i}|.$$

Then

$$\frac{|F_m|}{|F_{n_i+1}|} - 1 > \varepsilon |R_{n_i}| \frac{|F_m|}{|F_{n_i+1}|}.$$

Since the matrices M_n are positive, using induction on m and condition (2) for managed sequences, we get

$$M_{n_i} \cdots M_{m-1}(l, j) \geq \frac{|F_m|}{|F_{n_i+1}|}, \quad \text{for every } 1 \leq l \leq k_{n_i}, 1 \leq j \leq k_m.$$

Combining the last two equations we get

$$M_{n_i} \cdots M_{m-1}(l, j) - 1 > \varepsilon |R_{n_i}| \frac{|F_m|}{|F_{n_i+1}|},$$

and from equation (8), we obtain

$$M_{n_i} \cdots M_{m-1}(l, j) - 1 > |F_m \setminus F_m g^{-1}||R_{n_i}|, \quad \text{for every } g \in R_{n_i},$$

which finally implies that

$$M_{n_i} \cdots M_{m-1}(l, j) > \left| \bigcup_{g \in R_{n_i}} F_m \setminus F_m g^{-1} \right| + 1,$$

for every $1 \leq l \leq k_{n_i}$ and $1 \leq j \leq k_m$.

Now, suppose there exists $K > 0$ such that

$$k_{m+1} \leq K \frac{|F_{m+1}|}{|F_m|} \quad \text{for every } m \geq 0.$$

Property (2) for managed sequences of matrices implies

$$M_{n_i} \cdots M_m(l, j) \geq \frac{|F_{m+1}|}{|F_{n_i+1}|} \quad \text{for every } m > n_i.$$

Let $m_3 > m_2$ be such that

$$K < \frac{|F_m|}{|F_{n_i+1}|} \quad \text{for every } m \geq m_3.$$

Then for every $m \geq m_3$ we have

$$k_{m+1} \leq K \frac{|F_{m+1}|}{|F_{n_i}|} \leq M_{n_i} \cdots M_m(l, j)$$

for every $1 \leq l \leq k_{n_i}$ and $1 \leq j \leq k_{m+1}$.

By taking $n_{i+1} \geq m_3$ we get the desired subsequence $(n_i)_{i \geq 0} \subseteq \mathbb{Z}^+$. □

The following proposition shows that given a managed sequence, there exists a sequence of decorations verifying conditions (C1)-(C4). The aperiodicity condition (C3) is obtained by decorating the center of F_n in a unique way with respect to other places in F_n. A restriction on the number of columns of the matrices gives enough choices of coloring to ensure conditions (C3) and (C4).

Proposition 3. *Let $(M_n)_{n \geq 0}$ be a sequence of matrices which is managed by $(|F_n|)_{n \geq 0}$. For every $n \geq 0$, we denote by k_n the number of rows of M_n. Suppose in addition there exists $K > 0$ such that $k_{n+1} \leq K \frac{|F_{n+1}|}{|F_n|}$, for every $n \geq 0$. Then there exists a Toeplitz subshift $(X, \sigma|_X, G)$ verifying the following three conditions:*

1. *the set of invariant probability measures of $(X, \sigma|_X, G)$ is affinely homeomorphic to $\varprojlim_n (\triangle(k_n, |F_n|), M_n)$;*
2. *the ordered group $\mathcal{G}(X, \sigma|_X, G)$ is isomorphic to $(H/\mathrm{inf}(H), (H/\mathrm{inf}(H))^+, u + \mathrm{inf}(H))$, where (H, H^+) is given by*
$$\mathbb{Z} \xrightarrow{M^T} \mathbb{Z}^{k_0} \xrightarrow{M_0^T} \mathbb{Z}^{k_1} \xrightarrow{M_1^T} \mathbb{Z}^{k_2} \xrightarrow{M_2^T} \cdots,$$
with $M = |F_0|(1, \ldots, 1)$ and $u = [M^T, 0]$;
3. *$(X, \sigma|_X, G)$ is an almost 1–1 extension of the odometer $O = \varprojlim_n (G/\Gamma_n, \pi_n)$.*

Proof. Let $(n_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ be a sequence as in Lemma 8. Since $(M_n)_{n\geq 0}$ and the sequence $(M_{n_i} \cdots M_{n_{i+1}-1})_{i\geq 0}$ define the same inverse and direct limits, without loss of generality we can assume that, for every $n \geq 0$,

$$R_n \subseteq F_{n+1},$$

$$M_n(i,k) > 1 + \left| \bigcup_{g\in R_n} F_{n+1} \setminus F_{n+1}g^{-1} \right| \quad \text{for every } 1 \leq i \leq k_n,\ 1 \leq k \leq k_{n+1},$$

and

$$k_{n+1} < \min\{M_n(i,j) \colon 1 \leq i \leq k_n, 1 \leq j \leq k_{n+1}\}.$$

Let $\widetilde{M}$ be the $1 \times (k_0 + 1)$-dimensional matrix given by

$$\widetilde{M}(\cdot,1) = \widetilde{M}(\cdot,2) = M(\cdot,1),$$

and

$$\widetilde{M}(\cdot,k+1) = M(\cdot,k) \quad \text{for every } 2 \leq k \leq k_0.$$

For every $n \geq 0$, consider the $(k_n + 1) \times (k_{n+1} + 1)$-dimensional matrix given by

$$\widetilde{M}_n(\cdot,1) = \widetilde{M}_n(\cdot,2) = \begin{pmatrix} 1 \\ M_n(1,1) - 1 \\ M_n(2,1) \\ \vdots \\ M_n(k_n,1) \end{pmatrix}$$

and

$$\widetilde{M}_n(\cdot,k+1) = \begin{pmatrix} 1 \\ M_n(1,k) - 1 \\ M_n(2,k) \\ \vdots \\ M_n(k_n,k) \end{pmatrix}, \quad \text{for every } 2 \leq k \leq k_{n+1}.$$

Lemma 2 implies that the dimension groups with unit given by

$$\mathbb{Z} \xrightarrow{M^T} \mathbb{Z}^{k_0} \xrightarrow{M_0^T} \mathbb{Z}^{k_1} \xrightarrow{M_1^T} \mathbb{Z}^{k_2} \xrightarrow{M_2^T} \cdots,$$

and

$$\mathbb{Z} \xrightarrow{\widetilde{M}^T} \mathbb{Z}^{k_0+1} \xrightarrow{\widetilde{M}_0^T} \mathbb{Z}^{k_1+1} \xrightarrow{\widetilde{M}_1^T} \mathbb{Z}^{k_2+1} \xrightarrow{\widetilde{M}_2^T} \cdots,$$

are isomorphic.

Thus from Lemma 1 we get that both

$$\varprojlim_n (\triangle(k_n, |F_n|), M_n) \quad \text{and} \quad \varprojlim_n (\triangle(k_n+1, |F_n|), \widetilde{M}_n)$$

are affinely homeomorphic. Observe that $(\widetilde{M}_n)_{\geq 0}$ is managed by $(|F_n|)_{n\geq 0}$ and verifies, for every $n \geq 0$,

$$\widetilde{M}_n(i,k) \geq 1 + \left| \bigcup_{g\in R_n} F_{n+1} \setminus F_{n+1}g^{-1} \right|,$$

for every $2 \leq i \leq k_n + 1$ and $1 \leq k \leq k_{n+1} + 1$, and

$$3 \leq k_{n+1} + 1 \leq \min\{M_n(i,j) \colon 2 \leq i \leq k_n + 1, 1 \leq j \leq k_{n+1} + 1\}.$$

Thus, by Lemma 7, to prove the proposition it is enough to find a Toeplitz subshift having a sequence of r-K-R-partitions whose sequence of incidence matrices is $(\widetilde{M}_n)_{n\geq 0}$.

For every $n \geq 0$, we call l_n and l_{n+1} the number of rows and columns of $\widetilde{M}_n$ respectively.

For every $n \geq 0$, we will construct a collection of functions $B_{n,1}, \ldots, B_{n,l_n} \in \Sigma^{F_n}$ as in Lemma 7, where $\Sigma = \{1, \ldots, l_0\}$.

For every $1 \leq k \leq l_0$ we define $B_{0,k} \in \Sigma^{F_0}$ by $B_{0,k}(g) = k$, for every $g \in F_0$. Observe that the collection $\{B_{0,1}, \ldots, B_{0,l_0}\}$ verifies condition (C3).

Let $n \geq 0$. Suppose that we have defined $B_{n,1}, \ldots, B_{n,l_n} \in \Sigma^{F_n}$ verifying condition (C3). For $1 \leq k \leq l_{n+1}$, we define

$$B_{n+1,k}|_{F_n} = B_{n,1},$$

and

$$\sigma^{s^{-1}}(B_{n+1,k})|_{F_n} = B_{n,l_n} \text{ for every } s \in \bigcup_{g\in R_n} F_{n+1} \setminus F_{n+1}g^{-1} \cap \Gamma_n.$$

We fill the rest of the coordinates $v \in F_{n+1} \cap \Gamma_n$ in order that $\sigma^{v^{-1}}(B_{n+1,k})|_{F_n} \in \{B_{n,1}, \ldots, B_{n,l_n}\}$ and such that

$$|\{v \in F_{n+1} \cap \Gamma_n \colon \sigma^{v^{-1}}(B_{n+1,k})|_{F_n} = B_{n,i}\}| = \widetilde{M}_n(i,k),$$

for every $2 \leq i \leq l_n$.

Since $\widetilde{M}_n(1,k) = 1$, if $\sigma^{v^{-1}}(B_{n+1,k})|_{F_n} = B_{n,1}$ then $v = e$.

Note that the number of such v is at least $\widetilde{M}_n(2,k)+1$, because there are at least $\widetilde{M}_n(2,k)$ coordinates to be filled with $B_{n,2}$ and at least 1 coordinate to be filled with B_{n,l_n}. Thus we have at least $\widetilde{M}_n(2,k)+1 \geq l_{n+1}$ different ways to fill the coordinates such that the functions $B_{n+1,1},\dots,B_{n+1,l_{n+1}}$ are pairwise different (the number of columns of $\widetilde{M}_n$ which are equal to the k-column is at most the number of different functions that "respect the rules" of the k-column).

By construction, every function $B_{n+1,k}$ verifies (C1), (C2) and (C4). Let us assume there are $g \in F_{n+1}$ and $1 \leq k,k' \leq k_{n+1}$ such that $B_{n+1,k}(gv) = B_{n+1,k'}(v)$ for any v where it is defined, then by the induction hypothesis, $g \in \Gamma_n$. This implies $\sigma^{g^{-1}}(B_{n+1,k})|_{F_n} = B_{n+1,k'}|_{F_n} = B_{n,1}$ and then $g=e$. This shows that the collection $B_{n+1,1},\dots,B_{n+1,l_{n+1}}$ verifies (C3). We conclude applying Lemma 7. □

For positive integers $n_1,\dots,n_k$, we denote by $(n_1,\dots,n_k)!$ the corresponding multinomial coefficient. That is,

$$(n_1,\dots,n_k)! = \frac{(n_1+\cdots+n_k)!}{n_1!\cdots n_k!}.$$

Remark 2. To construct the collection of functions $(B_{n,1}\cdots,B_{n,l_n})_{n\geq 0}$ in Proposition 3 we just need that the number of columns of $\widetilde{M}_n$ which are equal to $\widetilde{M}_n(\cdot,k)$ does not exceed the number of possible ways to construct different functions $B \in \Sigma^{F_n}$ verifying

$$B|_{F_{n-1}} = B_{n-1,1} \quad \text{and} \quad B|_{vF_{n-1}} = B_{n-1,l_{n-1}},$$

for every

$$v \in \bigcup_{g\in R_{n-1}} F_n \setminus F_n g^{-1} \cap \Gamma_{n-1}.$$

In other words, it is possible to make this construction with $\widetilde{M}_n$ verifying the following property: for every $1 \leq k \leq l_{n+1}$ the number of $1 \leq l \leq l_{n+1}$ such that $\widetilde{M}_n(\cdot,l) = \widetilde{M}_n(\cdot,k)$ is not greater than

$$\left(\widetilde{M}_n(2,k),\dots,\widetilde{M}_n(l_n-1,k),\widetilde{M}_n(l_n,k) - \left|\bigcup_{g\in R_{n-1}} F_n \setminus F_n g^{-1} \cap \Gamma_{n-1}\right|\right)!$$

Among the hypothesis of Proposition 3, we ask a stronger condition on the number of columns of M_n which is stable under multiplication of matrices, unlike the condition that we mention in this remark.

6. Characterization of Choquet simplices

A compact, convex, and metrizable subset K of a locally convex real vector space is said to be a (metrizable) Choquet simplex, if for each $v \in K$ there is a unique probability measure μ supported on the set of extreme points of K such that

$$\int x d\mu(x) = v.$$

In this section we show that any metrizable Choquet simplex is affinely homeomorphic to the inverse limit defined by a managed sequence of matrices satisfying the additional restriction on the number of columns.

6.1. Finite dimensional Choquet simplices. For technical reasons, we have to separate the finite and the infinite dimensional cases.

Lemma 9. *Let K be a finite dimensional metrizable Choquet simplex with exactly $d \geq 1$ extreme points. Let $(p_n)_{n\geq 0}$ be an increasing sequence of positive integers such that for every $n \geq 0$ the integer p_n divides p_{n+1}, and let $k \geq \max\{2, d\}$. Then there exist an increasing subsequence $(n_i)_{i\geq 0}$ of indices and a sequence $(M_i)_{i\geq 0}$ of square k-dimensional matrices which is managed by $(p_{n_i})_{i\geq 0}$ such that K is affinely homeomorphic to $\varprojlim_n (\triangle(k, p_{n_i}), M_i)$.*

Proof. Let $k \geq \max\{3, d\}$, we will define the subsequence $(n_i)_{i\geq 0}$ by induction on i through a condition explained later. For every $i \geq 0$, we define M_i the k-dimensional matrix by

$$M_i(l, j) = \begin{cases} \dfrac{p_{n_{i+1}}}{p_{n_i}} - k(k-1) & \text{if } 1 \leq l = j \leq d, \\ k & \text{if } l \neq j, 1 \leq l \leq k \text{ and } 1 \leq j \leq d, \\ M_i(l, d) & \text{if } d < j \leq k. \end{cases}$$

We always suppose that n_{i+1} is sufficiently large in order to have

$$\frac{p_{n_{i+1}}}{p_{n_i}} - k(k-1) > 0.$$

By the very definition, M_i is a positive matrix having $k \geq 3$ rows and columns;

$$\sum_{l=1}^{k} M_i(l, j) = \frac{p_{n_{i+1}}}{p_{n_i}},$$

for every $1 \leq j \leq k$ and the range of M_i is at most d. Thus the convex set $\varprojlim_n (\triangle(k, p_{n_i}), M_i)$ has at most d extreme points.

If it has exactly d extreme points, it is affinely homeomorphic to K. We will choose the sequence $(p_{n_i})_{i\geq 0}$ in order that $P = \bigcap_{i\geq 0} M_0 \cdots M_i(\triangle(k, p_{n_{i+1}}))$ has d extreme points, which implies that $\varprojlim_n (\triangle(k, p_{n_i}), M_i)$ has exactly d extreme points.

For every $i \geq 0$, the set $P_i = M_0 \cdots M_i(\triangle(k, p_{n_{i+1}}))$ is the closed convex set generated by the vectors $v_{i,1}, \ldots, v_{i,d}$, where

$$v_{i,l} = \frac{1}{p_{n_{i+1}}} M_0 \cdots M_i(\cdot, l), \text{ for every } 1 \leq l \leq d.$$

Since every $v_{i,l}$ is in $\triangle(k, p_{n_0})$, there exists a sequence $(i_j)_{j\geq 0}$ such that for every $1 \leq l \leq d$, the sequence $(v_{i_j,l})_{j\geq 0}$ converges to an element v_l in $\triangle(k, p_{n_0})$. Observe that P is the closed convex set generated by $v_1, \ldots, v_d$. Thus if $v_1, \ldots, v_d$ are linearly independent then P has d extreme points.

Since for every $1 \leq l \leq d$ we have

$$\sum_{j=1}^{k} \frac{1}{p_{n_{i+1}}} M_0 \cdots M_i(j, l) = \frac{1}{p_{n_0}},$$

there exists a positive vector

$$\delta_l^{(i)} = (\delta_{1,l}^{(i)}, \ldots, \delta_{k,l}^{(i)})^T$$

such that

$$\sum_{j=1}^{k} \delta_{j,l}^{(i)} = 1$$

and such that, for each $1 \leq j \leq k$,

$$\frac{1}{p_{n_{i+1}}} M_0 \cdots M_i(j, l) = \delta_{j,l}^{(i)} \frac{1}{p_{n_0}}.$$

Thus if B_i is the matrix given by

$$B_i(\cdot, l) = \begin{cases} v_{i,l} & \text{if } 1 \leq l \leq d, \\ \dfrac{1}{p_{n_0}} e_l^{(k)} & \text{if } d+1 \leq l \leq k, \end{cases}$$

then $B_i = DA_i$, where D is the k-dimensional diagonal matrix given by

$$D_i(l, l) = \frac{1}{p_{n_0}}, \text{ for every } 1 \leq l \leq k,$$

and A_i is the k-dimensional matrix defined by

$$A_i(\cdot, l) = \begin{cases} \delta_l^{(i)} & \text{if } 1 \le l \le d, \\ e_l^{(k)} & \text{if } d+1 \le l \le k. \end{cases}$$

If $\lim_{j\to\infty} A_j = A$ is invertible (A is the k-dimensional matrix whose columns are the vectors $\lim_{j\to\infty} \delta_l^{(i_j)}$ and the canonical vectors $e_{d+1}^{(k)}, \ldots, e_k^{(k)}$), then $v_1, \ldots, v_l$ are linearly independent. For this it is enough to show that A is strictly diagonally dominant (see the Levy–Desplanques Theorem in [20]).

Now we will define $(n_i)_{i\ge 0}$ in order that A is strictly diagonally dominant.

Let $\varepsilon \in (0, \frac{1}{4})$. Let $n_0 = 0$ and $n_1 > n_0$ such that for every $1 \le l \le d$,

$$\delta_{l,l}^{(0)} = 1 - \frac{p_{n_0}}{p_{n_1}} \sum_{j=1, j\ne l}^{k} M_0(j,l) = 1 - \frac{p_{n_0}}{p_{n_1}} k(k-1) \ge \frac{3}{4} + \varepsilon.$$

For $i \ge 1$ we choose $n_{i+1} > n_i$ in order that

$$\frac{1}{p_{n_{i+1}}} M_0 \cdots M_{i-1}(l,l) < \varepsilon \frac{1}{p_{n_0} k(k-1) 2^i}, \quad \text{for every } 1 \le l \le d.$$

After a standard computation, for every $i \ge 1$ and $1 \le l \le d$ we get

$$\delta_{l,l}^{(i)} \ge \delta_{l,l}^{(i-1)} - \frac{p_{n_0}}{p_{n_{i+1}}} k(k-1) M_0 \cdots M_{i-1}(l,l),$$

which implies that

$$\delta_{l,l}^{(i)} \ge \delta_{l,l}^{(0)} - \varepsilon \sum_{j\ge 1} \frac{1}{2^j} \ge \frac{3}{4}.$$

It follows that $A(l,l) \ge \frac{3}{4}$ for every $1 \le l \le k$, and since the sum of the elements in a column of A is equal to 1, we deduce that A is strictly diagonally dominant. □

6.2. Infinite dimensional Choquet simplices. We use the following characterization of an infinite dimensional metrizable Choquet simplex.

Lemma 10 ([24], Corollary p.186). *For every infinite dimensional metrizable Choquet simplex K, there exists a sequence of matrices $(A_n)_{n\ge 1}$ such that, for every $n \ge 1$,*

(1) $A_n(\triangle(n+1, 1)) = \triangle(n, 1)$,

(2) *K is affinely homeomorphic to $\varprojlim_n (\triangle(n,1), A_n)$.*

Our strategy is to approximate the sequence of matrices $(A_n)_n$ by a managed sequence. Then we show that the associated inverse limits are affinely homeomorphic. For this, we need the following classical density result, whose proof follows from the fact that every non cyclic subgroup of $\mathbb{R}$ is dense.

Lemma 11. *Let $\mathbf{r} = (r_n)_{n\geq 0}$ be a sequence of integers such that $r_n \geq 2$ for every $n \geq 0$. Let $C_{\mathbf{r}}$ be the subgroup of $(\mathbb{R}, +)$ generated by $\{(r_0 \cdots r_n)^{-1} : n \geq 0\}$. Then*

$$(C_{\mathbf{r}})^p \cap \triangle(p, 1) \cap \{v \in \mathbb{R}^p : v > 0\}$$

is dense in $\triangle(p, 1)$, for every $p \geq 2$, where $(C_{\mathbf{r}})^p$ is the Cartesian product $\prod_{i=1}^p C_{\mathbf{r}}$.

Lemma 12. *Let K be an infinite dimensional metrizable Choquet simplex, and let $(p_n)_{n\geq 0}$ be an increasing sequence of positive integers such that for every $n \geq 0$ the integer p_n divides p_{n+1}. Then there exist an increasing subsequence $(n_i)_{i\geq 1}$ of indices and a sequence of matrices $(M_i)_{i\geq 1}$ managed by $(p_{n_i})_{i\geq 0}$ such that, for every $i \geq 0$,*

$$k_{i+1} \leq \min\{M_i(l, k) : 1 \leq l \leq k_i, 1 \leq k \leq k_{i+1}\},$$

and K is affinely homeomorphic to the inverse limit $\varprojlim_n (\triangle(k_i, p_{n_i}), M_i)$, where k_i is the number of rows of M_i, for every $i \geq 0$.

Proof. For every $n \geq 0$, let $r_n \geq 2$ be the integer such that $p_{n+1} = p_n r_n$.

Let $(A_n)_{n\geq 1}$ be the sequence of matrices given in Lemma 10. We can assume that $A_n : \triangle(n + 3, 1) \longrightarrow \triangle(n + 2, 1)$, for every $n \geq 1$. Now we define the subsequence $(n_i)_i$ by induction.

We set $n_1 = 0$.

Let $i \geq 1$ and suppose that we have defined $n_i \geq 0$. We set

$$\mathbf{r}^{(\mathbf{i})} = (r_n)_{n\geq n_i}.$$

For every $1 \leq j \leq i + 3$, Lemma 11 ensures the existence of

$$v^{(i,j)} \in (C_{\mathbf{r}^{(\mathbf{i})}})^{i+2} \cap \triangle(i + 2, 1) \cap \{v \in \mathbb{R}^{i+2} : v > 0\}$$

such that

$$\|v^{(i,j)} - A_i(\cdot, j)\|_1 < \frac{1}{2^i}. \tag{9}$$

Let B_i be the matrix given by

$$B_i(\cdot, j) = v^{(i,j)}, \text{ for every } 1 \leq j \leq i + 3.$$

Observe that (9) implies that

$$\sum_{n\geq 1} \sup\{\|A_n v - B_n v\|_1 : v \in \triangle_{n+3}\} < \infty.$$

It follows from [6, Lemma 9] that K is affine homeomorphic $\varprojlim_n (\triangle(i+2,1), B_i)$.

Let $n_{i+1} > n_i$ be such that $r_{n_i} \cdots r_{n_{i+1}-1} v^{(i,j)}$ is an integer vector and such that $r_{n_i} \cdots r_{n_{i+1}-1} v^{(i,j)} > i+3$, for every $1 \leq j \leq i+3$.

We define

$$M_i = \frac{p_{n_{i+1}}}{p_{n_i}} B_i.$$

Thus $M_i = P_i^{-1} B_i P_{i+1}$, where P_i is the diagonal matrix given by $P_i(j,j) = p_{n_i}$ for every $1 \leq j \leq i+2$ and $i \geq 1$. This shows that $\varprojlim_n (\triangle(i+2,1), B_i)$ is affinely homeomorphic to $\varprojlim_n (\triangle(i+2, p_{n_i}), M_i)$.

The proof conclude verifying that $(M_i)_{i\geq 0}$ is managed by $(p_{n_i})_{i\geq 0}$. □

7. Proofs of the main theorems

7.1. Proof of Theorem A. The proof of Theorem A is a corollary of previous results.

Proof of Theorem A. Let $\mathrm{ext}(K)$ be the set of extreme points of K. If $\mathrm{ext}(K)$ is finite, then the proof is direct from Proposition 3 and Lemma 9. If $\mathrm{ext}(K)$ is infinite, the proof follows from Proposition 3 and Lemma 12. □

7.2. Proof of Theorem B. We refer to [8] for definitions and properties about Toeplitz $\mathbb{Z}$-subshifts or Toeplitz flows. See [11] and [19] for details about ordered Bratteli diagrams, Kakutani–Rokhlin partitions and dimension groups associated to minimal $\mathbb{Z}$-actions on the Cantor set.

We denote by Σ a finite alphabet with at least two elements. For $x = (x_n)_{n\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$ and $n \leq m \in \mathbb{Z}$, we set

$$x[n,m] = x_n \cdots x_m.$$

In a similar way, if $w = w_0 \cdots w_{n-1}$ is a word in Σ^n, we set

$$w[k,l] = w_k \cdots w_l, \quad \text{for every } 0 \leq k \leq l < n.$$

The next result follows from the proof of [17, Theorem 8].

Lemma 13. *Let $x_0 \in \Sigma^{\mathbb{Z}}$ be a Toeplitz sequence and let $(X, \sigma|_X, \mathbb{Z})$ be the associated Toeplitz $\mathbb{Z}$-subshift. There exist a period structure $(p_n)_{n\geq 0}$ of x_0 and a sequence of matrices $(A_n)_{n\geq 0}$ managed by $(p_n)_{n\geq 0}$ such that the dimension group associated to $(X, \sigma|_X, \mathbb{Z})$ is isomorphic to*

$$\mathbb{Z} \xrightarrow{A_0^T} \mathbb{Z}^{k_1} \xrightarrow{A_1^T} \mathbb{Z}^{k_2} \xrightarrow{A_2^T} \cdots.$$

Furthermore, if k_n is the number of rows of A_n and $r_n = \frac{p_{n+1}}{p_n}$, then for every $m > n > 0$ and $1 \leq k \leq k_m$,

$$\begin{aligned}&|\{1 \leq l \leq k_m \colon A_{n,m-1}(\cdot, l) = A_{n,m-1}(\cdot, k)\}| \\ &\quad \leq (A_{n,m-1}(1,k) - r_{n+2}\cdots r_{m-1}, \ldots, A_{n,m-1}(k_n, k) - r_{n+2}\cdots r_{m-1})!,\end{aligned}$$

where $A_{n,m-1} = A_n \cdots A_{m-1}$.

Proof. In the proof of Theorem 8 in [17] the authors show there exist a period structure $(p_n)_{n\geq 1}$ of x_0 and a sequence $(\mathcal{P}_n)_{n\geq 0}$ of nested Kakutani–Rokhlin partitions of $(X, \sigma|_X, \mathbb{Z})$ such that

$$\mathcal{P}_0 = \{X\}$$

and

$$\mathcal{P}_n = \{T^j(C_{n,k}) \colon 0 \leq j < p_n, 1 \leq k \leq k_n\},$$

where

$$C_{n,k} = \{x \in X \colon x[0, p_n - 1] = w_{n,k}\}, \quad \text{for every } 1 \leq k \leq k_n,$$

with $W_n = \{w_{n,1}, \ldots, w_{n,k_n}\}$ the set of the words w of x_0 of length p_n verifying $w[0, p_{n-1} - 1] = x_0[0, p_{n-1} - 1]$, for every $n \geq 1$ (with $p_0 = 1$).

Thus the dimension group with unit associated to $(X, \sigma|_X, \mathbb{Z})$ is isomorphic to

$$\lim_{\to n}(\mathbb{Z}^{k_n}, A_n^T) = \mathbb{Z} \xrightarrow{A_0^T} \mathbb{Z}^{k_1} \xrightarrow{A_1^T} \mathbb{Z}^{k_2} \xrightarrow{A_2^T} \cdots,$$

where $A_n(i, j)$ is the number of times that the word $w_{n,i}$ appears in the word $w_{n+1,j}$, for every $1 \leq i \leq k_n$, $1 \leq j \leq k_{n+1}$ and $n \geq 1$, and the matrix A_0^T is the vector in $\mathbb{Z}^{k_1}$ whose coordinates are equal to p_1.

Since $w_{n+1,i} \neq w_{n+1,j}$ for $i \neq j$, equal columns of the matrix A_n produce different concatenations of words in W_n. This implies that for every $1 \leq k \leq k_{n+1}$, the number of columns of A_n which are equal to $A_n(\cdot, k)$ can not exceed the number of different concatenations of r_n words in W_n using exactly $A_n(j, k)$ copies of $w_{n,j}$, for every $1 \leq j \leq k_n$. This means that the number of columns which are equal to $A_n(\cdot, k)$ is smaller than or equal to $(A_n(1, k), \ldots, A_n(k_n, k))!$.

Now fix $n > 0$ and take $m > n$. The coordinate (i, j) of the matrix $A_{n,m-1}$ contains the number of times that the word $w_{n,i} \in W_n$ appears in $w_{m,j} \in W_m$. Observe that every word u in W_m is a concatenation of $r_{n+2} \cdots r_{m-1}$ words in W_{n+2}. In addition, each word in W_{n+2} starts with $x_0[0, p_{n+1} - 1] \in W_{n+1}$, which is a word containing every word in W_n (we can always assume that the matrices A_n are positive). Thus there exist $0 \leq l_1 < \cdots < l_{r_{n+1} \cdots r_{m-1}} < p_m$ such that $u[l_s, l_s + p_n - 1] = w[l_s, l_s + p_n - 1] \in W_n$, for every $1 \leq s \leq r_{n+2} \cdots r_{m-1}$ and $u, w \in W_m$.

This implies that the number of all possible concatenations of words in W_n producing a word in W_m according to the column k of the matrix $A_{n,m-1}$ is smaller than or equal to

$$(A_{n,m-1}(1,k) - r_{n+2} \cdots r_{m-1}, \ldots, A_{n,m-1}(k_n, k) - r_{n+2} \cdots r_{m-1})!. \qquad \square$$

Proof of Theorem B. Let $x_0 \in X$ be a Toeplitz sequence. Let $(p_n)_{n \geq 1}$ and $(A_n)_{n \geq 0}$ be the period structure of x_0 and the sequence of matrices given by Lemma 13 respectively. It is straightforward to check that Lemma 13 is also true if we take a subsequence of $(p_n)_{n \geq 0}$. Thus we can assume that for every $n \geq 1$, the matrix A_n has its coordinates strictly greater than 1 and that there exist positive integers $r_{n,1}, \ldots, r_{n,d} > 1$ such that

$$\frac{p_{n+1}}{p_n} = r_n = r_{n,1} \cdots r_{n,d}.$$

Define $q_{n+1,i} = r_{0,i} \cdots r_{n,i}$ for every $1 \leq i \leq d$, and $\Gamma_{n+1} = \prod_{i=1}^{d} q_{n+1,i}\mathbb{Z}$, for every $n \geq 0$. We have $\Gamma_{n+1} \subseteq \Gamma_n$, $\bigcap_{n \geq 1} \Gamma_n = \{0\}$ and $|\mathbb{Z}^d / \Gamma_n| = p_n$. Let $(F_n)_{n \geq 0}$ be a Følner sequence associated to $(\Gamma_n)_{n \geq 1}$ as in Lemma 5. We let R_n be as in Section 5 (the set that defines the “border").

Now, we define an increasing sequence $(n_i)_{i \geq 1}$ of integers as follows:

We set $n_1 = 1$. For $i \geq 1$, given n_i we chose $n_{i+1} > n_i + 1$ such that

$$\sum_{g \in R_{n_i}} \frac{|F_{n_{i+1}} \setminus F_{n_{i+1}} - g|}{|F_{n_{i+1}}|} < \frac{1}{|F_{n_i}| r_{n_i} r_{n_i+1}}.$$

Thus,

$$\begin{aligned} \frac{|F_{n_{i+1}}|}{|F_{n_i}|} - \sum_{g \in R_{n_i}} |F_{n_{i+1}} \setminus F_{n_{i+1}} - g| &> \frac{|F_{n_{i+1}}|}{|F_{n_i}|} - \frac{|F_{n_{i+1}}|}{|F_{n_i}| r_{n_i} r_{n_i+1}} \\ &= r_{n_i} \cdots r_{n_{i+1}-1} - r_{n_i+2} \cdots r_{n_{i+1}-1} \\ &> r_{n_i} \cdots r_{n_{i+1}-1} - k_{n_i} r_{n_i+2} \cdots r_{n_{i+1}-1}. \end{aligned}$$

Let $M_0 = A_0$ and $M_i = A_{n_i} \cdots A_{n_{i+1}-1}$ for every $i \geq 1$. For every $1 \leq k \leq k_{n_{i+1}}$ we get

$$M_i(k_{n_i}, k) - \sum_{g \in R_{n_i}} |F_{n_{i+1}} \setminus F_{n_{i+1}} - g| > M_i(k_{n_i}, k) - r_{n_i+2} \cdots r_{n_{i+1}-1},$$

which implies that

$$(M_i(1,k), \ldots, M_i(k_{n_i} - 1, k), M_i(k_{n_i}, k) - \sum_{g \in R_{n_i}} |F_{n_{i+1}} \setminus F_{n_{i+1}} - g|)!$$

is greater than

$$(M_i(1,k) - r_{n_i+2} \cdots r_{n_{i+1}-1}, \ldots, M_i(k_{n_i}, k) - r_{n_i+2} \cdots r_{n_{i+1}-1})!$$

Then by the previous inequality and Lemma 13, the number of columns of M_i which are equal to $M_i(\cdot, k)$ is smaller than

$$(M_i(1,k), \ldots, M_i(k_{n_i} - 1, k), M_i(k_{n_i}, k) - \sum_{g \in R_{n_i}} |F_{n_{i+1}} \setminus F_{n_{i+1}} - g|)!$$

As in the proof of Proposition 3, we define $\widetilde{M}_i$ and we call l_i and l_{i+1} the number of rows and columns of $\widetilde{M}_i$ respectively, for every $i \geq 0$. According to the notation of the proof of Proposition 3, in our case M_0 corresponds to the matrix M and $\widetilde{M}_0$ corresponds to the matrix $\widetilde{M}$. Observe that the bound on the number of columns which are equal to $M_i(\cdot, k)$ (and then to $\widetilde{M}_i(\cdot, k)$) ensures the existence of enough possibilities to fill the coordinates of F_{n_i} in order to obtain different function $B_{i,1} \cdots, B_{i,l_i} \in \{1, \ldots, l_1\}^{F_{n_i}}$ as in the proof of Proposition 3, for every $i \geq 1$ (see Remark 2).

The Toeplitz $\mathbb{Z}^d$-subshift $(Y, \sigma|_Y, \mathbb{Z}^d)$ given by $(B_{i,1}, \ldots, B_{i,l_i})_{i \geq 1}$ has an ordered group $\mathcal{G}(Y, \sigma|_Y, \mathbb{Z}^d)$ isomorphic to $(H/\inf(H), (H/\inf(H))^+, u + \inf(H))$, where (H, H^+) is given by

$$\mathbb{Z} \xrightarrow{\widetilde{M}_0^T} \mathbb{Z}^{l_0} \xrightarrow{\widetilde{M}_1^T} \mathbb{Z}^{l_2} \xrightarrow{\widetilde{M}_2^T} \mathbb{Z}^{l_3} \xrightarrow{\widetilde{M}_3^T} \cdots,$$

with $\widetilde{M}_0 = |F_1|(1, \ldots, 1)$ and $u = [1, 0]$ (Cf. Lemma 7).

Lemma 2 implies that (H, H^+, u) is isomorphic to the dimension group with unit (J, J^+, w) associated to $(X, \sigma|_X, \mathbb{Z})$. Thus, we deduce that the ordered group $(J/\inf(J), (J/\inf(J))^+, w + \inf(J))$ associated to $(X, \sigma|_X, \mathbb{Z})$ is isomorphic to $\mathcal{G}(Y, \sigma|_Y, \mathbb{Z}^d)$. We conclude the proof applying Theorem 1. □

In [25], the author shows that every minimal Cantor system $(Y, T, \mathbb{Z})$ having an associated Bratteli diagram which satisfies the equal path number property is strong orbit equivalent to a Toeplitz subshift $(X, \sigma|_X, \mathbb{Z})$. Thus the next result is immediate.

Corollary 1. *Let $(X, T, \mathbb{Z})$ be a minimal Cantor having an associated Bratteli diagram which satisfies the equal path number property. Then for every $d \geq 1$ there exists a Toeplitz subshift $(Y, \sigma|_Y, \mathbb{Z}^d)$ which is orbit equivalent to $(X, T, \mathbb{Z})$.*

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