Fixed-point property of random quotients by plain words

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Abstract. We show a fixed-point property of certain random groups for a wide class of CAT(0) spaces. The model of random groups under consideration is given as the set of presentations (S, R), where S is a generating set and the set of relations R is a subset of the set of all plain words of the same length with suitably fixed density. Our main theorem says that, with high probability, groups obtained by such presentations have the fixed-point property for all CAT(0) spaces having bounded singularities.

Contents

1. Introduction

Let Y be a metric space and $\mathrm{Isom}(Y)$ the group of isometries of Y. We say a group Γ has the fixed-point property for Y if any isometric action of Γ on Y admits a fixed point; namely, for any homomorphism $\rho\colon \Gamma \to \mathrm{Isom}(Y)$, there is a point $p \in Y$ such that $\rho(\gamma)p = p$ for all $\gamma \in \Gamma$. When we take a Hilbert space $\mathcal{H}$ as Y,

according to theorems due to Delorme and Guichardet, the fixed-point property for $\mathcal{H}$ is equivalent to Kazhdan's Property (T). Groups with Kazhdan's Property (T) have interesting and mysterious features, as described in [1]. On the other hand, a group with the fixed-point property for trees cannot be decomposed into an amalgamated free product, as explained in [20].

Looking at these examples, one may suspect that having such a fixed-point property imposes a strong restriction on groups. Recent progress suggests, however, that groups with the fixed-point property are distributed densely in a certain class of finitely generated groups, as shown in [8], [21], [23], [9], [10], and [6]. In these papers, we see that certain random groups have a strong fixed-point property: random groups in the triangle model ([23], [9]) and the graph model ([8], [21], [10], and [6]). Here, models of random groups are provided as certain sets of group presentations, and the phrase "random groups in the model admit the fixed-point property" means that groups given by presentations in the model admit the fixed-point property with high probability. In a slightly different direction, there is a result on so-called marked groups presented in [13], which claims that groups with the fixed-point property is dense in the closure of the set of marked groups which are torsion-free, infinite, and non-elementary hyperbolic with respect to a certain topology. Also we should mention that the existence of a group with a very strong fixed-point property is shown in [5] and [17]. Along the same lines as these results, our main theorems in the present paper say that groups with the fixed-point property for certain metric spaces in fact form an extensive class among finitely presented groups; random quotients by plain words with certain densities admit a strong fixed-point property as we explain below.

Let m, η be natural numbers and $0 < d < 1$ a real number. Denote by S a set consisting of m letters and their inverses: $S = \{s_1, s_1^{-1}, \ldots, s_m, s_m^{-1}\}$. We call a sequence of letters belonging to S a *word*, and the number of letters forming a word the *length* of the word. Thus, any word with length η takes the form of $s_{i_1}^{\epsilon_1} s_{i_2}^{\epsilon_2} \ldots s_{i_\eta}^{\epsilon_\eta}$, where $\epsilon_j = \pm 1$ and $s_{i_j} \in S$. We denote by W_η the set of words (plain words) of length η. We consider a group generated by S with a presentation $P = (S, R)$, where the set of relations R is a subset of W_η. We note that a relation $r \in R$ may be a reducible one, namely, r may contain a sequence of the form $s_i s_i^{-1}$ or $s_j^{-1} s_j$. Fix a real number $c > 1$ and set

$$P(m, \eta, d) = \{P = (S, R) \mid R \subset W_\eta \text{ and } c^{-1}(2m)^{d\eta} \leq \#R \leq c(2m)^{d\eta}\}.$$

Let Γ be the free group generated by S: $\Gamma = \langle s_1 \rangle * \cdots * \langle s_m \rangle$. Then each element in W_η determines an element in Γ in an obvious way. Denote by Γ_P the group given by a presentation $P = (S, R)$: $\Gamma_P = \Gamma / \bar{R}$, where $\bar{R}$ is the normal closure

of R in Γ. It is an easy exercise to show that, for any finitely presented group Γ, there exists $m, \eta \in \mathbb{N}$ and $0 < d < 1$ such that $\Gamma = \Gamma_P$ for some $P \in P(m, \eta, d)$; that is, our model contains all finitely presented groups.

Let Y be a CAT(0) space, a metric space with nonpositive curvature (see §3 for the definition). Complete, simply connected Riemannian manifolds with nonpositive sectional curvature, trees, and Hilbert spaces are typical and important examples of CAT(0) spaces. When Γ has the fixed-point property for Y, we say Γ has $F(Y)$ in what follows. Note that if Γ is a finite group, then Γ has the fixed-point property for any CAT(0) space. In fact, an orbit of any isometric action of Γ on Y consists of finite points, and hence, in CAT(0) space Y, we can find the barycenter of the orbit that must be invariant under the action of Γ. (See Proposition 3.2 for the definition and the existence of the barycenter.) Thus, in what follows, our interest is in an infinite group Γ with the fixed-point property for CAT(0) spaces.

If we take d in $P(m, \eta, d)$ larger, then producing homomorphism from Γ_P for $P \in P(m, \eta, d)$ into $\mathrm{Isom}(Y)$ becomes more difficult, which means that isometric actions of Γ_P on Y must carry a strong restriction in general. This leads us to a naive consideration that the larger d becomes, the more we can expect Γ_P for $P \in P(m, \eta, d)$ to have the fixed-point property. On the other hand, if we take d too large, then R becomes large; it may force Γ_P to be a finite group, which is the case we want to avoid. Our main theorems tell us that the truth is exactly as this consideration describes, and there does exist d which provides infinite groups with the fixed-point property for a certain class of CAT(0) spaces with high probability. For Hilbert spaces, we will show the following theorem.

Theorem 1.1. *Let $\mathcal{H}$ be a Hilbert space. For $1/3 < d < 1 - \frac{1}{2}\log_{2m}(8m - 4)$,*

$$\lim_{\eta \to \infty} \frac{\#\{P \in P(m, \eta, d) \mid \Gamma_P \text{ is an infinite group having } F(\mathcal{H})\}}{\#P(m, \eta, d)} = 1$$

holds.

If we only allow irreducible words to be relations in the definition of $P(m, \eta, d)$, we obtain the density model of Gromov. One can expect a result for the density model similar to Theorem 1.1, and an idea of the proof, a reduction to Zuk's theorem ([23, Theorem 4]), is suggested in [18, I.3.g][1].

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[1] After the present paper was submitted, a complete proof was given by Marcin Kotowski and Michał Kotowski [16].

In order to state our result for general CAT(0) spaces, we need to introduce an invariant of CAT(0) spaces defined in [11, Definition 6.1]: Let μ be a measure on a CAT(0) space Y, whose support supp μ consists of finite points. Denote by $\bar{\mu}$ the barycenter of μ (see Proposition 3.2), and let

$$\delta(\mu)$$
$$= \inf \left\{ \frac{\left| \int_Y \varphi(p) \, d\mu(p) \right|^2}{\int_Y |\varphi(p)|^2 \, d\mu(p)} \middle| \begin{array}{c} \varphi \colon \operatorname{supp} \mu \to \mathcal{H}, \quad \varphi(\bar{\mu}) = \mathbf{0}, \\ |\varphi(p)| = d(\bar{\mu}, p) \text{ for any } p \in \operatorname{supp} \mu, \\ |\varphi(p) - \varphi(q)| \le d(p, q) \text{ for any } p, q \in \operatorname{supp} \mu \end{array} \right\},$$

where $\mathcal{H}$ is a Hilbert space, and $\mathbf{0}$ denotes its origin. Then $\delta(Y)$ is defined to be

$$\delta(Y) = \sup \{ \delta(\mu) \mid \mu \text{ is a measure on } Y \text{ with } \#\operatorname{supp} \mu < \infty \}.$$

This invariant $\delta(Y)$ measures the singularity of Y, and obviously it takes values in $[0, 1]$. If $\delta(Y) = 0$, then Y is considered to be almost regular; CAT(0) Riemannian manifolds and trees satisfy $\delta(Y) = 0$. For general CAT(0) spaces we will show:

Theorem 1.2. *Take $0 \le \delta < 1$ and an integer k so that $k > 1/(1 - \delta)$. Let $\mathcal{Y}_\delta$ be the class of* CAT(0) *spaces with $\delta(Y) \le \delta$. Then, for $k/(2k + 1) < d < 1 - \frac{1}{2} \log_{2m}(8m - 4)$,*

$$\lim_{\eta \to \infty} \frac{\#\{P \in P(m, \eta, d) \mid \Gamma_P \text{ is an infinite group having } F(Y) \text{ for all } Y \in \mathcal{Y}_\delta\}}{\#P(m, \eta, d)}$$

$$= 1$$

holds.

A part of the theorems above is due to Ollivier; Ollivier [19] proved that, by choosing $d < 1 - \frac{1}{2} \log_{2m}(8m - 4)$,

$$\lim_{\eta \to \infty} \frac{\#\{P \in P(m, \eta, d) \mid \Gamma_P \text{ is non-elementary hyperbolic}\}}{\#P(m, \eta, d)} = 1,$$

where Γ being non-elementary hyperbolic means that Γ is an infinite hyperbolic group (in the sense of Gromov) which does not contain $\mathbb{Z}$ as a finite index subgroup. Since $\frac{1}{2} \log_{2m}(8m - 4) \to 1/2$ as $m \to \infty$, by taking m sufficiently large, Theorem 1.2 tells us that, for any $0 \le \delta < 1$, we can choose d so that infinite hyperbolic groups with $F(\mathcal{Y}_\delta)$ form an extensive class among the groups coming from $P(m, \eta, d)$. Here a group Γ has $F(\mathcal{Y}_\delta)$ means that Γ has $F(Y)$ for all

$Y \in \mathcal{Y}_\delta$. We note that, in [6], it is shown that any CAT(0) cube complex Y satisfies $\delta(Y) \le 1/2$. Therefore $\mathcal{Y}_{1/2}$ contains all CAT(0) Riemannian manifolds and CAT(0) cube complexes. On the other hand, there does exist a CAT(0) space Y with $\delta(Y) = 1$ ([14]).

This paper is organized as follows. In §2, we introduce a finite graph associated to each element of $P(m, \eta, d)$ and state Proposition 2.1, which concerns with the eigenvalue of random graphs associated to $P(m, \eta, d)$. In §3, we introduce an energy of equivariant maps from Γ to Y, and give a criterion for an action of Γ on Y to have a fixed point in terms of the ratio of certain energies, which is a modification of [10, Lemma 2.7(2)]. We also state Proposition 3.7, which connects the ratio of energies and the eigenvalue of the graph introduced in §2. In §4, we give the proofs of Theorem 1.1 and 1.2, assuming Propositions 2.1 and 3.7. After a brief review of some facts in probability theory in §5, we give the proofs of Propositions 2.1 and 3.7 in §6. As a matter of fact, our proofs give results on random quotients of any group, which will be explained in §7.

2. A graph associated to a presentation

Fix a natural number k. For each natural number $\eta \ge k$, we take l so that l is the largest integer satisfying $\eta \equiv l \pmod 2$ and $l \le \eta/(2k + 1)$. We set $2n = \eta - l$. Therefore we have

$$(2k + 1)l \le \eta = (2n + l) \le (2k + 1)l + 2(2k + 1), \qquad (2.1\text{a})$$

$$kl \le n \le kl + (2k + 1). \qquad (2.1\text{b})$$

Note that $\eta \to \infty$ implies both $l \to \infty$ and $n \to \infty$.

Let $0 < d < 1$ and $P \in P(m, \eta, d)$. We associate a finite graph G_P to $P = (S, R)$ as follows: the vertex set V of G_P is identified with W_n and a pair $\{v_1, v_2\} \in V \times V$ is joined by an edge if there is a word $v_1 u v_2^{-1} \in R \subset W_\eta$, where $u \in W_l$ and v^{-1} denotes the inverse word $s_{i_n}^{-\epsilon_n} \dots s_{i_1}^{-\epsilon_1}$ of $v = s_{i_1}^{\epsilon_1} \dots s_{i_n}^{\epsilon_n}$, $\epsilon_j = \pm 1$. We denote the set of (unoriented) edges of G_P by E_P. Thus $G_P = (V, E_P)$, where

$$\begin{cases} V = W_n, \\ E_P = \{\{v_1, v_2\} \mid v_1 u v_2^{-1} \in R \text{ for some } u \in W_l\}. \end{cases}$$

If there are two words $u, u' \in W_l$ such that $r = v_1 u v_2^{-1}, r' = v_1 u' v_2^{-1} \in R$, then r and r' define distinct edges with the same endpoints $v_1, v_2 \in V$; thus G_P may have multiple edges. Also G_P may have self loops, since $r = v u v^{-1}$ may appear in R. Let ν be a measure on $V \times V$ defined by

$$\nu(v_1, v_2) = \frac{\#\{u \in W_l \mid v_1 u v_2^{-1} \in R\} + \#\{u' \in W_l \mid v_2 u' v_1^{-1} \in R\}}{2\#R}.$$

By definition, ν is a symmetric probability measure on $V \times V$ and can be regarded as a probability measure on E_P. By a slight abuse of notation, we denote by ν a probability measure (or simply a funtion) on V defined as

$$\nu(v) = \sum_{v'} \nu(v, v') = \frac{\#\{u \in W_{n+l} \mid v u \in R\} + \#\{u' \in W_{n+l} \mid u' v^{-1} \in R\}}{2\#R}.$$

Note that

$$\nu(v) = \frac{\deg(v)}{2\#R} = \frac{\deg(v)}{2\#E_P},$$

where $\deg(v)$ is the *degree* of v, which is defined to be the cardinality (counted with multiplicity) of edges having v as one of the endpoints. Since a self loop comes from a relation of the form $v u v^{-1}$, it is counted twice in the definition of our degree. For real valued functions f, g defined on V, their inner product is defined by

$$(f, g)_{L^2} = \sum_{v \in V} \nu(v) f(v) g(v).$$

The *Laplacian* Δ_P of G_P is defined by

$$\Delta_P f(v) = f(v) - \sum_{v'} \frac{\nu(v, v')}{\nu(v)} f(v'),$$

for a funtion $f : V \to \mathbb{R}$ on V. It is easy to see that Δ_P is a symmetric and nonnegative operator with respect to the inner product $(\cdot, \cdot)_{L^2}$ defined above. The eigenvalues of Δ_P are distributed in the interval $[0, 2]$, and it is clear that constant functions are eigenfunctions corresponding to the smallest eigenvalue 0. We denote by $\lambda_1(G_P)$ the second smallest eigenvalue of Δ_P counted with multiplicity, which is positive if and only if G_P is connected. The *Rayleigh quotient* $RQ(f)$ of $f : V \to \mathbb{R}$ is defined as

$$RQ(f) = \frac{\|df\|_{L^2}^2}{F(f)},$$

where

$$\begin{cases} \|df\|_{L^2}^2 = \dfrac{1}{2} \sum_{v \in V} \sum_{v' \in V} \nu(v, v')|f(v) - f(v'))|^2 \\ F(f) = \dfrac{1}{2} \sum_{v, v' \in V} \nu(v)\nu(v')|f(v) - f(v')|^2. \end{cases}$$

Easy computations show that $F(f)$ is equal to the L^2-norm of the component of f which is perpendicular to constant functions and that $\|df\|_{L^2}^2 = (f, \Delta_P f)_{L^2}$. Therefore we obtain

$$\lambda_1(G_P) = \inf\{RQ(f) \mid f \text{ is a nonconstant function on } V\}.$$

In §6.2, We will show the following proposition on the eigenvalue of the Laplacian of G_P.

Proposition 2.1. *Fix a natural number k. If $k/(2k + 1) < d$, then, for any $\varepsilon > 0$,*

$$\lim_{\eta \to \infty} \frac{\#\{P \in P(m, \eta, d) \mid \lambda_1(G_P) > 1 - \varepsilon\}}{\#P(m, \eta, d)} = 1$$

holds.

3. Energy of equivariant maps and fixed points

In this section, we review and improve some results obtained in [9] and [10], which concern the energy of equivariant maps from a finitely generated group to a CAT(0) space. A complete metric space Y is called CAT(0) space if any geodesic triangle is thinner than that in the Euclidean plane in the following sense.

Let $Y = (Y, d_Y)$ be a complete metric space and $p, q \in Y$. A *geodesic* joining p to q is an isometric embedding c of a closed interval $[0, l]$ into Y such that $c(0) = p$ and $c(l) = q$, where $l = d_Y(p, q)$. If any pair of points in Y are joined by a geodesic, then Y is called a *geodesic space*. Consider a triangle in a geodesic space Y with vertices $p_1, p_2, p_3 \in Y$ and three geodesic segments $p_1 p_2, p_2 p_3, p_3 p_1$ joining them. We denote this triangle by $\Delta(p_1, p_2, p_3)$ and call such a triangle a *geodesic triangle*. Note that we can take a triangle $\Delta(\overline{p_1}, \overline{p_2}, \overline{p_3})$ in $\mathbb{R}^2$ so that $d_Y(p_i, p_j) = d_{\mathbb{R}^2}(\overline{p_i}, \overline{p_j})$, since the side lengths of $\Delta(p_1, p_2, p_3)$ satisfy the triangle inequality. We call $\Delta(\overline{p_1}, \overline{p_2}, \overline{p_3})$ a *comparison triangle* for $\Delta(p_1, p_2, p_3)$. And we call a point $\bar{q} \in \overline{p_i p_j}$ a *comparison point* for $q \in p_i p_j$ if $d_Y(p_i, q) = d_{\mathbb{R}^2}(\overline{p_i}, \bar{q})$. Then we say that $\Delta(p_1, p_2, p_3)$ satisfies the CAT(0) *inequality*, if $d_Y(q_1, q_2) \le d_{\mathbb{R}^2}(\overline{q_1}, \overline{q_2})$ for any pair of points q_1, q_2 on the sides of

$\Delta(p_1, p_2, p_3)$ and their comparison points $\overline{q_1}, \overline{q_2}$. If every geodesic triangle in Y satisfies the CAT(0) inequality, then Y is called a CAT(0) *space*.

If Y is a CAT(0) space, it is easy to verify that Y has two important properties: the uniqueness of a geodesic connecting given two points and the contractibility. A complete, simply connected Riemannian manifold of nonpositive sectional curvature is a typical example of CAT(0) space. Also trees and Hilbert spaces are CAT(0) spaces. For an detailed exposition on CAT(0) space, we refer the reader to [3].

Now we recall some definitions which will be necessary later.

Definition 3.1. Let Y be a CAT(0) space.

(1) Let c and c' be two nontrivial geodesics in Y starting from $p \in Y$. The *angle* $\angle_p(c, c')$ between c and c' is defined by

$$\angle_p(c, c') = \lim_{t,t' \to 0} \angle_{\bar{p}}(\overline{c(t)}, \overline{c'(t')}),$$

where $\angle_{\bar{p}}(\overline{c(t)}, \overline{c'(t')})$ denotes the angle between the sides $\bar{p}\overline{c(t)}$ and $\bar{p}\overline{c'(t')}$ of the comparison triangle $\Delta(\bar{p}, \overline{c(t)}, \overline{c'(t')}) \subset \mathbb{R}^2$.

(2) Let $p \in Y$. For a pair of nontrivial geodesics c, c' starting from p, we define a relation $\sim$ by $c \sim c'$ if and only if $\angle_p(c, c') = 0$. Then $\sim$ becomes an equivalence relation on the set of all nontrivial geodesics starting from p denoted by $\widetilde{(S_p Y)}^\circ$. Then the angle $\angle_p$ induces a distance on the quotient $(S_p Y)^\circ = \widetilde{(S_p Y)}^\circ / \sim$, which we denote by the same symbol $\angle_p$. The completion $(S_p Y, \angle_p)$ of the metric space $((S_p Y)^\circ, \angle_p)$ is called the *space of directions* at p.

(3) Let $TC_p Y$ be the cone over $S_p Y$, namely,

$$TC_p Y = (S_p Y \times \mathbb{R}_{\geq 0})/(S_p Y \times \{0\}).$$

Let $W, W' \in TC_p Y$. We may write $W = (V, t)$ and $W' = (V', t')$, where $V, V' \in S_p Y$ and $t, t' \in \mathbb{R}_{\geq 0}$. Then

$$d_{TC_p Y}(W, W') = t^2 + t'^2 - 2tt' \cos \angle_p(V, V')$$

defines a distance on $TC_p Y$. The metric space $(TC_p Y, d_{TC_p Y})$ is known to be a CAT(0) space and is called the *tangent cone* at p. We define an "inner product" on $TC_p Y$ by

$$\langle W, W' \rangle = tt' \cos \angle_p(V, V').$$

We often denote the length t of W by $|W|$; thus we have $|W| = \sqrt{\langle W, W \rangle} = d_{TC_p Y}(0_p, W)$, where 0_p denotes the origin of $TC_p Y$.

(4) Define a map $\pi_p \colon Y \to TC_pY$ by $\pi_p(q) = ([c], d_Y(p, q))$, where c is the geodesic joining p to q and $[c] \in S_pY$ is the equivalence class of c. Then π_p is distance non-increasing.

We often use the following important fact.

Proposition 3.2 ([15, Lemma 2.5.1]). *Let Y be a* CAT(0) *space. Suppose that μ is a finitely supported measure on Y; that is, μ is a measure of the form $\mu = \sum_{i=1}^{m} \mu_i \mathrm{Dirac}_{p_i}$, where Dirac_p is the Dirac measure supported on a point $p \in Y$, and $p_1, \ldots, p_m \in Y$. Then there exists a unique point $p_0 \in Y$ which minimizes the function*

$$F(q) = \int_Y d(p, q)^2 \, d\mu(p) = \sum_{i=1}^{m} \mu_i d(p_i, q)^2, \quad q \in Y.$$

We call the point p_0 the *barycenter of* $\{p_1, \ldots, p_m\}$ *with respect to a measure* μ, or simply *barycenter of* μ.

Let Y be a CAT(0) space, and denote by $\mathrm{Isom}(Y)$ the group of isometries of Y. Let Γ be a finitely generated group, and $\rho \colon \Gamma \to \mathrm{Isom}(Y)$ a homomorphism. We call a map $f \colon \Gamma \to Y$ ρ-*equivariant* if f satisfies $f(\gamma\gamma') = \rho(\gamma)f(\gamma')$ for all $\gamma, \gamma' \in \Gamma$. Suppose that a Γ-invariant, symmetric random walk with finite-support property on Γ is given. In other words, we are given $\mu \colon \Gamma \times \Gamma \to [0, 1]$ with the following properties.

- FINITE-SUPPORT PROPERTY. For any $\gamma \in \Gamma$, $\mu(\gamma, \gamma') = 0$ for all but finitely many $\gamma' \in \Gamma$,
- For any $\gamma \in \Gamma$, $\sum_{\gamma' \in \Gamma} \mu(\gamma, \gamma') = 1$,
- Γ-INVARIANCE. $\mu(\gamma\gamma', \gamma\gamma'') = \mu(\gamma', \gamma'')$ for any γ, γ', and $\gamma'' \in \Gamma$,
- SYMMETRY. $\mu(\gamma, \gamma') = \mu(\gamma', \gamma)$ for any $\gamma, \gamma' \in \Gamma$.

Then we define the *energy* $E_\mu(f)$ of a ρ-equivariant map f by

$$E_\mu(f) = \frac{1}{2} \sum_{\gamma' \in \Gamma} \mu(\gamma, \gamma') d_Y(f(\gamma), f(\gamma'))^2. \tag{3.1}$$

Note that, due to the Γ-invariance of $\mu(\cdot, \cdot)$ and the ρ-equivariance of f, this value does not depend on the choice of γ. A ρ-equivariant map f is said to be *harmonic* if f minimizes E_μ among all ρ-equivariant maps. Note that the image of a ρ-equivariant map $f \colon \Gamma \to Y$ is the $\rho(\Gamma)$-orbit of the point $f(e)$, and f is

determined by the choice of $f(e)$. Therefore, the set of all ρ-equivariant maps from Γ to Y can be identified with Y. Then the energy functional E_μ becomes a convex continuous function on Y, since Y is a CAT(0) space. Let $-\text{grad}\,E_\mu(f)$ be the negative gradient of the energy functional E_μ at f. When Y is a Riemannian manifold this should be understood as the negative of the ordinary gradient. In general, one can give a reasonable definition of $-\text{grad}\,E_\mu(f)$ as an element of the tangent cone of Y, identified with the space of ρ-equivariant maps, at a point f. In fact, $-\text{grad}\,E_\mu(f)$ should be $2(-\Delta f(e))$, where $-\Delta f(e)$ is the barycenter of $\{F_{f(e)}(\gamma) = \pi_{f(e)} \circ f(\gamma) \mid \mu(e, \gamma) \neq 0\}$ with respect to a measure $\sum_\gamma \mu(e, \gamma)\text{Dirac}_{F_{f(e)}(\gamma)}$. The following proposition gives a sufficient condition for the existence of a fixed point of $\rho(\Gamma)$ in terms of the energy functional.

Proposition 3.3 ([9, §1], see also [8, §3]). *Let Γ be a finitely generated group equipped with a Γ-invariant, finitely supported, symmetric random walk. Let Y be a CAT(0) space and $\rho \colon \Gamma \to \text{Isom}(Y)$ a homomorphism. Suppose there is a positive constant C such that $|-\Delta f(e)|^2 \geq C E_\mu(f)$ holds for every ρ-equivariant map f. Then there is a ρ-equivariant map f_0 such that $E_\mu(f_0) = 0$.*

Recall that μ is called *irreducible* if μ satisfies the following: for any $\gamma, \gamma' \in \Gamma$, there exists $\gamma_0, \gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\gamma = \gamma_0$, $\gamma' = \gamma_n$, and $\mu(\gamma_i, \gamma_{i+1}) \neq 0$, $i = 0, \ldots, n - 1$. If $E_\mu(f_0) = 0$, then $d(f(\gamma), f(\gamma')) = 0$ for any pair γ, γ' whenever there exists a sequence $\gamma_0, \gamma_1, \ldots, \gamma_n \in \Gamma$ as above. Therefore, if μ is irreducible, then f_0 with $E_\mu(f_0) = 0$ must be a constant map, and the image of f_0 is a fixed-point of the action of $\rho(\Gamma)$.

Let μ be the transition probability of the standard random walk on Γ with generating set $S = \{s_1, \ldots, s_m, s_1^{-1}, \ldots, s_m^{-1}\}$, namely,

$$\mu(\gamma, \gamma') = \frac{\#\{s \in S \mid \gamma' = \gamma s\}}{2m}.$$

We note that $s \in S$ may become the identity element in Γ, and if this is the case, then $\mu(\gamma, \gamma) \neq 0$ for any $\gamma \in \Gamma$. It is clear that so defined μ satisfies the four properties listed above and the irreducibility.

Denote by μ^n the nth convolution of μ:

$$\mu^n(\gamma, \gamma') = \sum_{\gamma_1 \in \Gamma} \cdots \sum_{\gamma_{n-1} \in \Gamma} \mu(\gamma, \gamma_1) \ldots \mu(\gamma_{n-1}, \gamma').$$

We define *n-step energy* $E_{\mu^n}(f)$ of a ρ-equivariant map f by

$$E_{\mu^n}(f) = \frac{1}{2} \sum_{\gamma' \in \Gamma} \mu^n(\gamma, \gamma') d_Y(f(\gamma), f(\gamma'))^2.$$

Since μ^n is Γ-invariant as μ, again this does not depend on the choice of γ.

In [8], Gromov gave an interesting criterion for an action of Γ on Y to have a fixed point in terms of the ratio of $E_\mu(f)$ and $E_{\mu^n}(f)$ (see also [10] and [21]). We need a variant of this criterion, stated as Theorem 3.6 below, in order to prove our main theorems.

A slight modification of the proof of [10, Lemma 2.7(2)] gives the following lemma, which we state in a way convenient for our use.

Lemma 3.4. *Let Γ be a finitely generated group equipped with the standard random walk, and Y a CAT(0) space. Let n, k, l be natural numbers satisfying $n \geq kl$. Then, for any natural number $i \leq 2k$, we have*

$$E_{\mu^{2n-il}}(f) \leq (2k - i + 1)^2 \left(E_{\mu^l}(f) + E_{\mu^{2n-2kl}}(f) \right) \tag{3.2}$$

and

$$|-\Delta_{2n-il} f(e)| \leq (2k - i + 1) \sqrt{2 \left(E_{\mu^l}(f) + E_{\mu^{2n-2kl}}(f) \right)},$$

where $-\Delta_j f(e)$ is the barycenter of

$$\{F_{f(e)}(\gamma) = \pi_{f(e)} \circ f(\gamma) \mid \mu^j(e, \gamma) \neq 0\}$$

with respect to a measure $\sum_\gamma \mu^j(e, \gamma) \mathrm{Dirac}_{F_{f(e)}(\gamma)}$.

Proof. Noting

$$\mu^{2n-il} = (\mu^l)^{2k-i} * \mu^{2n-2kl},$$

we get

$$
\begin{aligned}
&E_{\mu^{2n-il}}(f) \\
&= \frac{1}{2} \sum_{\gamma_1, \dots, \gamma_{2k-i+1}} \mu^l(e, \gamma_1) \cdots \mu^l(\gamma_{2k-i-1}, \gamma_{2k-i}) \mu^{2n-2kl}(\gamma_{2k-i}, \gamma_{2k-i+1}) \\
&\qquad \times d_Y(f(e), f(\gamma_{2k-i+1}))^2 \\
&\leq \frac{1}{2} \sum_{\gamma_1, \dots \gamma_{2k-i+1}} \mu^l(e, \gamma_1) \cdots \mu^l(\gamma_{2k-i-1}, \gamma_{2k-i}) \mu^{2n-2kl}(\gamma_{2k-i}, \gamma_{2k-i+1}) \\
&\qquad \times (2k - i + 1)(d_Y(f(e), f(\gamma_1))^2 + \cdots + d_Y(f(\gamma_{2k-i}), f(\gamma_{2k-i+1}))^2).
\end{aligned}
\tag{3.3}
$$

Note that

$$\frac{1}{2} \sum_{\gamma_1,\ldots,\gamma_{2k-i+1}} \mu^l(e,\gamma_1)\ldots\mu^{2n-2kl}(\gamma_{2k-i},\gamma_{2k-i+1})d_Y(f(\gamma_{2k-i}),f(\gamma_{2k-i+1}))^2$$

$$= \frac{1}{2} \sum_{\gamma_1,\ldots,\gamma_{2k-1}} \mu^l(e,\gamma_1)\ldots\mu^l(\gamma_{2k-i-1},\gamma_{2k-i})$$

$$\times \sum_{\gamma_{2k-i+1}} \mu^{2n-2kl}(\gamma_{2k-i},\gamma_{2k-i+1})d_Y(f(\gamma_{2k-i}),f(\gamma_{2k-i+1}))^2$$

$$= \sum_{\gamma_1,\ldots,\gamma_{2k-i}} \mu^l(e,\gamma_1)\ldots\mu^l(\gamma_{2k-i-1},\gamma_{2k-i}) \times E_{\mu^{2n-2kl}}(f)$$

$$= E_{\mu^{2n-2kl}}(f),$$

and, for $j < 2k - i$,

$$\frac{1}{2} \sum_{\gamma_1,\ldots,\gamma_{2k-i+1}} \mu^l(e,\gamma_1)\ldots\mu^{2n-kl}(\gamma_{2k-i},\gamma_{2k-i+1})d_Y(f(\gamma_j),f(\gamma_{j+1}))^2$$

$$= \frac{1}{2} \sum_{\gamma_1,\ldots,\gamma_{2k-i}} \mu^l(e,\gamma_1)\ldots\mu^l(\gamma_j,\gamma_{j+1})\ldots\mu^l(\gamma_{2k-i-1},\gamma_{2k-i})$$

$$\times \sum_{\gamma_{2k-i+1}} \mu^{2n-2kl}(\gamma_{2k-i},\gamma_{2k-i+1})d_Y(f(e),f(\gamma_j^{-1}\gamma_{j+1}))^2$$

$$= \frac{1}{2} \sum_{\gamma_1,\ldots,\gamma_{2k-i-1}} \mu^l(e,\gamma_1)\ldots\mu^l(\gamma_j,\gamma_{j+1})\ldots\mu^l(\gamma_{2k-i-2},\gamma_{2k-i-1})$$

$$\times \sum_{\gamma_{2k-i}} \mu^l(\gamma_{2k-i-1},\gamma_{2k-i})d_Y(f(\gamma_j),f(\gamma_{j+1}))^2$$

$$= \frac{1}{2} \sum_{\gamma_1,\ldots,\gamma_j} \mu^l(e,\gamma_1)\ldots\mu^l(\gamma_{j-1},\gamma_j) \times \sum_{\gamma_{j+1}} \mu^l(\gamma_j,\gamma_{j+1})d_Y(f(\gamma_j),f(\gamma_{j+1}))^2$$

$$= \sum_{\gamma_1}\cdots\sum_{\gamma_j} \mu^l(e,\gamma_1)\ldots\mu^l(\gamma_{j-1},\gamma_j)E_{\mu^l}(f)$$

$$= E_{\mu^l}(f)$$

holds. Together with (3.3), this implies (3.2):

$$E_{\mu^{2n-il}}(f) \le (2k-i+1)((2k-i)E_{\mu^l}(f) + E_{\mu^{2n-2kl}}(f))$$

$$\le (2k-i+1)^2(E_{\mu^l}(f) + E_{\mu^{2n-2kl}}(f)).$$

Since $|-\Delta_{2n-il}f(e)| \le \sqrt{2E_{\mu^{2n-il}}(f)}$ follows from [10, Lemma 2.7(1)], combining with (3.2) implies the other inequality. $\qquad\square$

The following is a variant of [10, Proposition 2.8].

Proposition 3.5. *Let Γ be a finitely generated group equipped with the standard random walk, and Y a CAT(0) space. If natural numbers n, k and l satisfy $n \geq kl$, we have*

$$E_{\mu^{2n}}(f) \geq E_{\mu^{2n-2kl}}(f) + 2k E_{\mu^l}(f)$$
$$- \sum_{i=1}^{2k}(2k - i + 1)\sqrt{2\left(E_{\mu^l}(f) + E_{\mu^{2n-2kl}}(f)\right)}\,|-\Delta_l f(e)|. \tag{3.4}$$

Proof. By using [10, Proposition 2.5] $2k$ times, we obtain

$$\begin{aligned} E_{\mu^{2n}}(f) &\geq E_{\mu^{2n-l}}(f) + E_{\mu^l}(f) - \langle -\Delta_{2n-l} f(e), -\Delta_l f(e)\rangle \\ &\geq \left(E_{\mu^{2n-2l}}(f) + E_{\mu^l}(f) - \langle -\Delta_{2n-2l} f(e), -\Delta_l f(e)\rangle\right) \\ &\quad + E_{\mu^l}(f) - \langle -\Delta_{2n-l} f(e), -\Delta_l f(e)\rangle \\ &\vdots \\ &\geq E_{\mu^{2n-2kl}}(f) + 2k E_{\mu^l}(f) - \sum_{i=1}^{2k}\langle -\Delta_{2n-il} f(e), -\Delta_l f(e)\rangle \\ &\geq E_{\mu^{2n-2kl}}(f) + 2k E_{\mu^l}(f) - \sum_{i=1}^{2k}|-\Delta_{2n-il} f(e)||-\Delta_l f(e)|. \end{aligned}$$

Now Lemma 3.4 implies the desired inequality. $\qquad\square$

Theorem 3.6. *Let Γ be a finitely generated group equipped with the standard random walk, and Y a CAT(0) space. Suppose that natural numbers n, k and l satisfy $n \geq kl$, and that there exists $\varepsilon > 0$ such that, for any ρ-equivariant map f, the following inequality holds*:

$$E_{\mu^{2n}}(f) \leq (2k - \varepsilon)E_{\mu^l}(f).$$

Then there is a ρ-equivariant map f_0 with $E_{\mu^l}(f_0) = 0$. Moreover, $\rho(\Gamma)$ admits a fixed point.

Proof. By (3.4) and the assumption, we get

$$(2k - \varepsilon)E_{\mu^l}(f) \geq E_{\mu^{2n-2kl}}(f) + 2k E_{\mu^l}(f)$$
$$- \sum_{i=1}^{2k}(2k - i + 1)\sqrt{2\left(E_{\mu^l}(f) + E_{\mu^{2n-2kl}}(f)\right)}\,|-\Delta_l f(e)|,$$

and hence

$$c(k) \left| -\Delta_l f(e) \right| \sqrt{E_{\mu^l}(f) + E_{\mu^{2n-2kl}}(f)}$$
$$\geq \varepsilon E_{\mu^l}(f) + E_{\mu^{2n-2kl}}(f)$$
$$\geq \varepsilon \left(E_{\mu^l}(f) + E_{\mu^{2n-2kl}}(f) \right),$$

since we may assume $\varepsilon \leq 1$. Hence

$$\left| -\Delta_l f(e) \right| \geq \frac{\varepsilon}{c(k)} \sqrt{E_{\mu^l}(f) + E_{\mu^{2n-2kl}}(f)}$$
$$\geq \frac{\varepsilon}{c(k)} \sqrt{E_{\mu^l}(f)}.$$

By Proposition 3.3, this implies the existence of f_0 with $E_{\mu^l}(f_0) = 0$. If l is odd, then $\mu^l(e, s) \neq 0$ for any $s \in S$. Thus $f_0(s) = f_0(e)$ for any $s \in S$. Since f_0 is ρ-equivariant, this implies that $f_0(\gamma) = f_0(e)$ for any $\gamma \in \Gamma$, which means that f_0 is a constant map and $f_0(e)$ is a fixed point. Suppose l is even. Let Γ' be a subgroup of Γ generated by $S' = \{ \gamma \in \Gamma \mid \mu^2(e, \gamma) \neq 0 \}$. Then $\mu^l(e, \gamma) \neq 0$ for any $\gamma \in S'$, and $f_0(e)$ is fixed by $\rho(\Gamma')$. On the other hand,

$$\varphi(\gamma) = \begin{cases} 0 & \text{if } \gamma \text{ can be expressed by word with even length,} \\ 1 & \text{otherwise,} \end{cases}$$

defines a homomorphism $\varphi : \Gamma \to \mathbb{Z}/2\mathbb{Z}$, and the kernel is exactly Γ'. Therefore, Γ' is a subgroup of index at most 2. Since $f_0(\Gamma')$ is a point and the image $f_0(\Gamma)$ is the $\rho(\Gamma)$-orbit of $f_0(\Gamma')$, $f_0(\Gamma)$ consists of at most two points. In other words, we have an orbit consisting of at most two points. If the orbit consists of a single point, then the point is a fixed point for the action of $\rho(\Gamma)$. Suppose the orbit consists of two points. In this case, the action $\rho(\Gamma)$ is simply a permutation of these two points; if $\gamma \in \Gamma'$, then $\rho(\gamma)$ must fix both points, while if $\gamma \notin \Gamma'$, then $\rho(\gamma)$ exchanges these two points. Since the action of $\rho(\Gamma)$ is by isometry, their unique midpoint must be fixed by $\rho(\Gamma)$. (The uniqueness of the midpoint follows from the uniqueness of a geodesic joining given two points, which is true because Y is a CAT(0) space.) This completes the proof. ☐

According to Theorem 3.6, in order to prove the fixed-point property of Γ for Y, we need to give a bound on the ratio $E_{\mu^{2n}}(f)/E_{\mu^l}(f)$, which is less than $2k$ and independent of ρ and f. The ratio $E_{\mu^{2n}}(f)/E_{\mu^l}(f)$ has an interesting connection with the Rayleigh quotient of a map from a graph into Y as we explain below.

Let Γ be the free group generated by S: $\Gamma = \langle s_1 \rangle * \cdots * \langle s_m \rangle$. Let Γ_P be the group given by a presentation $P = (S, R) \in P(m, \eta, d)$. Let $G_P = (V, E_P)$ be

the graph associated to P defined in §2. Let Y be a CAT(0) space with a metric $d = d_Y$, and $f \colon V \to Y$ a map into Y. Then the *Rayleigh quotient* $RQ^G(f)$ of f, introduced by Gromov, is defined as

$$RQ^G(f) = \frac{\|df\|_{L^2}^2}{F(f)},$$

where

$$\begin{cases} \|df\|_{L^2}^2 = \dfrac{1}{2} \sum_{v \in V} \sum_{v' \in V} \nu(v, v') d_Y(f(v), f(v'))^2, \\ F(f) = \dfrac{1}{2} \sum_{v,v' \in V} \nu(v)\nu(v') d_Y(f(v), f(v'))^2. \end{cases}$$

When $Y = \mathbb{R}$ this is nothing but the Rayleigh quotient introduced in §2. For a general CAT(0) space Y, using the triangle inequality, one sees that

$$F(f) \le \frac{1}{2} \sum_{v,v' \in V} \nu(v)\nu(v')(2d_Y(\mathrm{bar}(f), f(v))^2 + 2d_Y(\mathrm{bar}(f), f(v'))^2)$$

$$= 2 \sum_{v \in V} d_Y(\mathrm{bar}(f), f(v)),$$

where $\mathrm{bar}(f)$ is the barycenter of $f(V)$. Let $\delta(Y)$ be the invariant introduced in [11, Definition 6.1], and $\lambda_1(G, Y)$ be Wang's invariant (see [22] or [11, Definition 4.2]). Then by [11, Proposition 6.3], for any nonconstant map $f \colon V \to Y$, we obtain

$$\begin{aligned} RQ^G(f) &\ge \frac{1}{2} \frac{\|df\|_{L^2}^2}{\displaystyle\sum_{v \in V} d_Y(\mathrm{bar}(f), f(v))} \\ &\ge \frac{1}{2}\lambda_1(G, Y) \ge \frac{1}{2}(1 - \delta(Y))\lambda_1(G). \end{aligned} \tag{3.5}$$

Let $\rho \colon \Gamma_P \to \mathrm{Isom}(Y)$ be a homomorphism, and take any ρ-equivariant map $f \colon \Gamma_P \to Y$. Then we can consider $\tilde{\rho}$-equivariant map $\tilde{f} = f \circ \pi_P \colon \Gamma \to Y$, where π_P denotes the projection from Γ onto Γ_P and $\tilde{\rho} = \rho \circ \pi_P \colon \Gamma \to \mathrm{Isom}(Y)$. Regarding $v \in V$ as an element $\pi_P(\bar{v}) \in \Gamma_P$ in a natural way, we get a map $\bar{f} \colon V \to Y$ induced from f, where $\bar{v}$ denotes the element in Γ determined by a word v.

Regarding a ρ-equivariant map as a map defined on V as above, we will be able to show that the Rayleigh quotient approximates the ratio of l-step and $2n$-step energies with high probability.

Proposition 3.7. *Fix a natural number k, and let $k/(2k + 1) < d < 1/2$. Set l and n as described in* (2.1)*. Then, for any $\varepsilon > 0$, we have*

$$\lim_{\eta \to \infty} \frac{\#\left\{ P \in P(m, \eta, d) \left| \begin{array}{l} (3.6) \text{ holds for all } Y, \ \rho \colon \Gamma_P \to \mathrm{Isom}(Y), \\ \text{for all nonconstant } \rho\text{-equivariant map } f \end{array} \right. \right\}}{\#P(m, \eta, d)} = 1,$$

where

$$(1 - \varepsilon)RQ^G(\bar{f}) \leq \frac{E_{\mu^l}(\tilde{f})}{E_{\mu^{2n}}(\tilde{f})} \leq (1 + \varepsilon)RQ^G(\bar{f}), \tag{3.6}$$

and Y denotes a CAT(0) *space.*

The proof will be given in §6.2.

Remark 1. We are assuming $d < 1/2$ in order to regard $P(m, \eta, d)$ as the probability space describing an experiment given by choosing a word $c_0(2m)^{d\eta}$ times randomly from W_η, where $c^{-1} \leq c_0 \leq c$. However, the assumption $d < 1/2$ is not essential as we will explain in Remark 2.

4. Proofs of Theorem 1.1 and 1.2

We first give the proof of Theorem 1.2, assuming Proposition 2.1 and 3.7 and then point out a necessary change in order to show Theorem 1.1.

Note that $1 - \frac{1}{2}\log_{2m}(8m - 4) < 1/2$ holds, and that we have $RQ^G(f) \geq \lambda_1(G_P, Y) \geq \frac{1}{2}(1 - \delta(Y))\lambda_1(G_P) \geq \frac{1}{2}(1 - \delta)\lambda_1(G_P)$ for any $Y \in \mathcal{Y}_\delta$ by (3.5) and our assumption. By Propositions 2.1 and 3.7, we see that if $k/(2k + 1) < d < 1 - \frac{1}{2}\log_{2m}(8m - 4)$, with probability tending to 1 as $\eta \to \infty$, Γ_P satisfies

$$\frac{E_{\mu^l}(\tilde{f})}{E_{\mu^{2n}}(\tilde{f})} \geq \frac{1 - \delta}{2}(1 - \varepsilon) \tag{4.1}$$

for any $Y \in \mathcal{Y}_\delta$, $\rho \colon \Gamma_P \to \mathrm{Isom}(Y)$, and $\tilde{\rho}$-equivariant map $\tilde{f}$, where $\tilde{\rho} = \rho \circ \pi_P \colon \Gamma \to \mathrm{Isom}(Y)$. Rewriting this, we see that, for any $\varepsilon' > 0$, $P \in P(m, \eta, d)$ satisfies the following property with high probability, that is, probability tending to 1 as $\eta \to \infty$: For any $Y \in \mathcal{Y}_\delta$, $\rho \colon \Gamma_P \to \mathrm{Isom}(Y)$, and for any $\tilde{\rho}$-equivariant map $\tilde{f} \colon \Gamma \to Y$,

$$E_{\mu^{2n}}(\tilde{f}) \leq \frac{2}{1 - \delta}(1 + \varepsilon')E_{\mu^l}(\tilde{f})$$

holds. Since we have $k > 1/(1 - \delta)$, by taking ε' to be sufficiently small, we may assume

$$\frac{2}{1 - \delta}(1 + \varepsilon') \leq (2k - \varepsilon'')$$

for some $\varepsilon'' > 0$. Therefore, we obtain, for any $\rho \colon \Gamma_P \to \mathrm{Isom}(Y)$ and $\tilde{\rho}$-equivariant map $\tilde{f}$,

$$E_{\mu^{2n}}(\tilde{f}) \leq (2k - \varepsilon'')E_{\mu^l}(\tilde{f})$$

and hence $\rho(\Gamma_P) = \tilde{\rho}(\Gamma)$ admits a fixed point by Theorem 3.6. This implies fixed-point property of Γ_P with high probability. This completes the proof of Theorem 1.2.

If $Y = \mathcal{H}$, then we have $RQ^G(f) \geq \lambda_1(G_P)$. Therefore we have

$$\frac{E_{\mu^l}(\tilde{f})}{E_{\mu^{2n}}(\tilde{f})} \geq 1 - \varepsilon$$

instead of (4.1), and hence

$$E_{\mu^{2n}}(\tilde{f}) \leq (1 + \varepsilon')E_{\mu^l}(\tilde{f}).$$

Now we need $1 + \varepsilon' \leq (2k - \varepsilon'')$ in order to apply Theorem 3.6, and it suffices to take $k = 1$. Therefore taking $d > 1/(2+1) = 1/3$ is sufficient, and this completes the proof of Theorem 1.1.

5. Preliminaries on probability theory

In this section, we collect some facts from probability theory, which will be used in the proof of Propositions 2.1 and 3.7.

5.1. Notation and some inequalities. Let X be a random variable defined on a probability space (Ω, P). We denote the probability of X taking its value in a set A by $\mathbb{P}(X \in A)$:

$$\mathbb{P}(X \in A) = \int_{\{\omega \colon X(\omega) \in A\}} dP(\omega).$$

Suppose X is nonnegative and let μ be the mean (expected value $\mathbb{E}(X)$) of X. Then, for $t > 0$,

$$\mu = \int X \, dP \geq \int_{X \geq t\mu} X \, dP \geq \int_{X \geq t\mu} t\mu \, dP = \mathbb{P}(X \geq t\mu)t\mu.$$

Dividing the both ends by $t\mu$ yields *Markov's inequality*:

$$\mathbb{P}(X \geq t\mu) \leq \frac{1}{t}.$$

Setting $\varepsilon = t\mu$ gives

$$\mathbb{P}(X \geq \varepsilon) \leq \frac{\mu}{\varepsilon}. \tag{5.1}$$

If X is a real valued random variable with mean μ and variance σ^2. Then, for any $d > 0$,

$$\sigma^2 = \mathbb{E}((X - \mu)^2) \geq \int_{|X-\mu| \geq d} (X - \mu)^2 \, dP \geq \mathbb{P}(|X - \mu| \geq d)d^2.$$

Dividing the both ends by d^2, we obtain *Chebyshev's inequality*:

$$\mathbb{P}(|X - \mu| \geq d) \leq \frac{\sigma^2}{d^2}.$$

By taking $d = \varepsilon\mu$, we get

$$\mathbb{P}(|X - \mu| \geq \varepsilon\mu) \leq \frac{\sigma^2}{\varepsilon^2 \mu^2}.$$

Let X be a binomially distributed random variable with size n and parameter p; namely, X satisfies

$$\mathbb{P}(X = x) = {}_nC_x p^x (1 - p)^{n-x}, \quad x = 0, 1, \ldots, n.$$

Then we say that X has the binomial distribution $\mathrm{Bi}(n, p)$. Such an X has mean $\mu = np$ and variance $\sigma^2 = np(1 - p)$. Thus Chebyshev's inequality yields

$$\mathbb{P}(|X - \mu| \geq \varepsilon\mu) \leq \frac{np(1 - p)}{\varepsilon^2 n^2 p^2} = \frac{1 - p}{\varepsilon^2 \mu} \leq \frac{1}{\varepsilon^2 \mu}.$$

In particular, we see that if $\{X_i\}$ is a sequence of binomially distributed random variable with mean $\mu_i \to \infty$, then $\mathbb{P}(|X_i - \mu_i| \geq \varepsilon\mu_i) \to 0$. Actually, there are finer and more useful bounds called *Chernoff bounds*:

$$\mathbb{P}(X \geq \mu + t) \leq \exp\left(-\frac{t^2}{2(\mu + t/3)}\right)$$

$$\mathbb{P}(X \leq \mu - t) \leq \exp\left(-\frac{t^2}{2\mu}\right).$$

See, for example, [12, § 2.1] for proofs. In particular, taking $t = \varepsilon\mu$ with $0 < \varepsilon < 1$, we obtain

$$\mathbb{P}(|X - \mu| > \varepsilon\mu) \leq 2\exp\left(-\frac{\varepsilon^2 \mu}{3}\right). \tag{5.2}$$

5.2. Moment generating function for binomially distributed random variables.

Let X be a random variable on a probability space. Then

$$\mathbb{E}(\exp(tX)) = \mathbb{E}\left(\sum_{k=0}^{\infty} \frac{1}{k!} t^k X^k\right) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \mathbb{E}(X^k)$$

is called the moment generating function of X. We can compute α*th moment* $\mathbb{E}(X^\alpha)$ of X by

$$\mathbb{E}(X^\alpha) = \frac{d^\alpha}{dt^\alpha} \mathbb{E}(\exp(tX))\bigg|_{t=0}.$$

Let X be a random variable with distribution $\mathrm{Bi}(n, p)$. Then the mean of X is np, and by computing the average, we get

$$\mathbb{E}(\exp t(X - np)) = \sum_{x=0}^{n} \exp t(x - np) \, {}_nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} ({}_nC_x (e^t p)^x q^{n-x}) e^{-tnp}$$

$$= (e^t p + q)^n e^{-tnp},$$

where $q = 1 - p$. A computation shows the following.

Lemma 5.1. *Let X be a random variable with distribution* $\mathrm{Bi}(n, p)$*, and set*

$$M(t) = \mathbb{E}(\exp t(X - np)).$$

Then, αth derivative $M^{(\alpha)}(t)$ of $M(t)$ can be computed as

$$M^{(\alpha)}(t) = \sum_{\beta=0}^{\alpha} {}_\alpha C_\beta [(e^t p + q)^n]^{(\beta)} [e^{-tnp}]^{(\alpha-\beta)},$$

where $[\cdot]^{(a)}$ denotes the ath derivative of $[\cdot]$ and they turned out to be

$$[(e^t p + q)^n]^{(\beta)} = \sum_{j=1}^{\beta} c(\beta, j) \frac{n!}{(n-j)!} (e^t p + q)^{n-j} e^{jt} p^j \quad (\beta > 0),$$

$$[e^{-tnp}]^{(\alpha-\beta)} = (-np)^{\alpha-\beta} e^{-tnp}.$$

Here $c(k, j)$ is determined by

$$c(j, j) = 1, \quad c(j, 0) = 0, \quad c(k+1, j) = c(k, j-1) + jc(k, j).$$

We will use the following estimate in the proof of Proposition 2.1.

Lemma 5.2. *For a random variable X with distribution $\mathrm{Bi}(n, p)$ and with mean $\mu = np < 1$, we have*

$$|\mathbb{E}((X - np)^{\alpha})| \leq c(\alpha)np.$$

Proof. Since $\mathbb{E}((X-np)^{\alpha}) = M^{(\alpha)}(0)$ as we have seen above, we need to estimate $M^{(\alpha)}(0)$. By Lemma 5.1, and $np < 1$, if $\beta > 0$, then

$$|[(e^t p + q)^n]^{(\beta)}|_{t=0}| \leq \sum_{j=1}^{\beta} c(\beta, j) \frac{n!}{(n-j)!} p^j$$

$$\leq \sum_{j=1}^{\beta} c(\beta, j) n^j p^j$$

$$\leq \sum_{j=1}^{\beta} c(\beta, j) np,$$

and if $\alpha > \beta$

$$|[e^{-tnp}]^{(\alpha-\beta)}|_{t=0}| \leq (np)^{\alpha-\beta} \leq np.$$

Thus we obtain the desired estimate. $\qquad\square$

5.3. Remarks on our model. In order to prove Propositions 2.1 and 3.7, it is convenient to fix the cardinality of R, the set of relations, and consider

$$P(m, \eta, d, c_0) = \{P = (S, R) \in P(m, \eta, d) \mid \#R = c_0(2m)^{d\eta}\},$$

where $c^{-1} \leq c_0 \leq c$. Then the model $P(m, \eta, d, c_0)$ can be regarded as the set of subsets of W_{η} with cardinality $c_0(2m)^{d\eta}$ equipped with the uniform counting measure. Of course, this is not the same as the probability space $P'(m, \eta, d, c_0)$ which describes an experiment given by choosing a word $c_0(2m)^{d\eta}$ times randomly from W_{η}; there may be multiply chosen words and there is a natural action of the permutation group given by changing the order of the choice. If we take $d < 1/2$, then the probability of an elements in $P'(m, \eta, d, c_0)$ to have multiply chosen words tends to 0 as $\eta \to \infty$ as the following computation shows. Note that, on the subset $P'_0(m, \eta, d, c_0)$ of $P'(m, \eta, d, c_0)$ consisting of elements without multiply chosen words, the action of the permutation group mentioned above is free. Thus, taking the quotient by the action of the permutation group, we can regard $P'_0(m, \eta, d, c_0)$ as $P(m, \eta, d, c_0)$.

If X is a random variable with distribution $\mathrm{Bi}(n, p)$, then

$$\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X = 0 \text{ or } 1) = 1 - ((1-p)^n + n(1-p)^{n-1}p).$$

Since

$$(1-p)^n + n(1-p)^{n-1}p$$

$$= \sum_{\alpha=0}^{n} {}_nC_\alpha 1^{n-\alpha}(-p)^\alpha + np \sum_{\alpha=0}^{n-1} {}_{n-1}C_\alpha 1^{n-\alpha}(-p)^\alpha$$

$$\geq 1 - np - n^2p^2 - \cdots - n^np^n$$
$$+ np(1 - (n-1)p - (n-1)^2p^2 - \cdots - (n-1)^{n-1}p^{n-1})$$

$$\geq 1 - 2(np)^2 - \cdots - 2(np)^n,$$

we obtain

$$\mathbb{P}(X \geq 2) \leq 2\sum_{\alpha=2}^{n}(np)^\alpha.$$

Take any $w \in W_\eta$, and denote by $X_w(P)$ the number of times that w is chosen in an element $P \in P'(m, \eta, d, c_0)$. Then X_w is a random variable on $P'(m, \eta, d, c_0)$ with distribution $\text{Bi}(c_0(2m)^{d\eta}, (2m)^{-\eta})$. By the computation above, we have

$$\mathbb{P}(X_w \geq 2) \leq 2\sum_{\alpha=2}^{c_0(2m)^{d\eta}} (c_0(2m)^{d\eta}(2m)^{-\eta})^\alpha$$

Since $\#W_\eta = (2m)^\eta$, we see that

$$\mathbb{P}(X_w \geq 2 \text{ for some } w \in W_\eta) \leq (2m)^\eta \times 2\sum_{\alpha=2}^{c_0(2m)^{d\eta}} c_0^\alpha(2m)^{\alpha d\eta}(2m)^{-\alpha\eta}$$

$$= 2\sum_{\alpha=2}^{c_0(2m)^{d\eta}} c_0^\alpha(2m)^{\alpha d\eta}(2m)^{-(\alpha-1)\eta}$$

$$= 2c_0^2(2m)^{(2d-1)\eta} \sum_{\alpha=2}^{c_0(2m)^{d\eta}} c_0^{\alpha-2}(2m)^{(\alpha-2)(d-1)\eta}$$

$$\leq 2c_0^2(2m)^{(2d-1)\eta} \frac{1}{1 - c_0(2m)^{-(1-d)\eta}}$$

holds for large η, since $c_0(2m)^{-(1-d)\eta} < 1$ when η is large enough. By taking $d < 1/2$ and suitable constant c', we see that

$$\mathbb{P}(X_w \geq 2 \text{ for some } w \in W_\eta) \leq c'(2m)^{(2d-1)\eta} \longrightarrow 0 \quad (\eta \to \infty).$$

Note that c' can be taken so that depending only on c taken in order to define $P(m, \eta, d)$ but independent of c_0. In other words, the measure of $P_0'(m, \eta, d, c_0)$ in the probability space $P'(m, \eta, d, c_0)$ uniformly tends to 1. Hence, assuming $d < 1/2$, we may identify $P(m, \eta, d, c_0)$ with $P'(m, \eta, d, c_0)$. We will use this identification in the proof of Proposition 3.7.

In order to prove Proposition 2.1, we need another model $P''(m, \eta, d, c_0)$ of random groups. In this model $P''(m, \eta, d, c_0)$, each word in W_η is independently chosen to be a relation of $P = (S, R)$ with probability $(c_0(2m)^{d\eta})/(2m)^\eta$. Therefore any subset of W_η can occur as R of $P = (S, R)$ in $P''(m, \eta, d, c_0)$, and the probability of the occurrence is given by

$$\mathbb{P}(P = (S, R)) = \left(\frac{c_0(2m)^{d\eta}}{(2m)^\eta} \right)^{\#R} \left(1 - \frac{c_0(2m)^{d\eta}}{(2m)^\eta} \right)^{(2m)^\eta - \#R}.$$

This model corresponds to a model of random graphs often denoted by $G(n, p)$ in the literature, where $n = (2m)^\eta$ and $p = (c_0(2m)^{d\eta})/(2m)^\eta$ in our case; while our original model $P(m, \eta, d, c_0)$ corresponds to another model of random graphs often denoted by $G(n, M)$, where $n = (2m)^\eta$ and $M = c_0(2m)^{d\eta}$ in our case. In our setting, a well-known relation between $G(n, p)$ and $G(n, M)$ can be read as follows.

Lemma 5.3 (cf. [2, p. 38] and [12, p. 17]). *Let $\mathfrak{Q}$ be any property of a group, and denote by $\mathbb{P}_P(G_P \in \mathfrak{Q})$ (resp. $\mathbb{P}_{P''}(G_P \in \mathfrak{Q})$) the probability of G_P given by $P \in P(m, \eta, d, c_0)$ (resp. $P \in P''(m, \eta, d, c_0)$) having property $\mathfrak{Q}$. Then we have*

$$\mathbb{P}_P(G_P \in \mathfrak{Q}) \leq 3\sqrt{c_0(2m)^{d\eta}}\,\mathbb{P}_{P''}(G_P \in \mathfrak{Q}).$$

Proof. Set $N = (2m)^\eta$, $M = c_0(2m)^{d\eta}$, $p = (c_0(2m)^{d\eta})/(2m)^\eta$ and $q = 1 - p$. By the law of total probability, we have

$$\mathbb{P}_{P''}(G_P \in \mathfrak{Q}) = \sum_{s=0}^{N} \mathbb{P}_{P,s}(G_P \in \mathfrak{Q}) \,_N C_s\, p^s q^{N-s}$$

$$\geq \mathbb{P}_{P,M}(G_P \in \mathfrak{Q}) \,_N C_M\, p^M q^{N-M},$$

where $P_{P,s}(G_P \in \mathfrak{Q})$ denotes the probability of $G_P \in \mathfrak{Q}$ in the model $P(m, \eta, d, c_0')$ with $s = c_0'(2m)^{d\eta}$. Now [2, p. 4] gives

$$_N C_M\, p^M q^{N-M} \geq e^{-1/(6M)} \frac{1}{\sqrt{2\pi}} (pqN)^{-1/2}.$$

Noting $M = pN$ implies the desired inequality. $\qquad\square$

6. Proofs of Proposition 2.1 and 3.7

We first prove Proposition 3.7 and then proceed to Proposition 2.1.

6.1. Proof of Proposition 3.7. Let c_0 be a positive real number such that $c^{-1} \leq c_0 \leq c$, where c is the constant fixed in order to define $P(m, \eta, d)$, and set

$$P(m, \eta, d, c_0) = \{P = (S, R) \in P(m, \eta, d) \mid \#R = c_0(2m)^{d\eta}\}$$

as above. Recall that $\eta = 2n + l \geq (2k + 1)l$, take $d > k/(2k + 1)$, and set $\varepsilon_0 = d(2k + 1) - k$. Because of (2.1), we see that

$$\begin{cases} d\eta \geq d(2k + 1)l = \varepsilon_0 l + kl, \\ d\eta - n \geq d(2k + 1)l - (kl + (2k + 1)) = \varepsilon_0 l - (2k + 1), \\ d\eta - n \leq d((2k + 1)l + 2(2k + 1)) - kl = \varepsilon_0 l + 2d(2k + 1). \end{cases} \tag{6.1}$$

We will show that, for any $\varepsilon' > 0$, there exist constants a_1 and a_2 depending only on m, k, c, and ε' (and independent of l and c_0) such that

$$\frac{\left\{ P \in P(m, \eta, d, c_0) \middle| \begin{matrix} (3.6)' \text{ holds for all } Y, \rho \colon \Gamma_P \to \mathrm{Isom}(Y), \\ \text{for all nonconstant } \rho\text{-equivariant map } f \end{matrix} \right\}}{\#P(m, \eta, d, c_0)}$$

$$\geq 1 - a_1 \exp(-a_2(2m)^{\varepsilon_0 l})$$

holds for large l (and hence for large η), where

$$(1 - \varepsilon')RQ^G(\bar{f}) \leq \frac{E_{\mu^l}(\tilde{f})}{E_{\mu^{2n}}(\tilde{f})} \tag{3.6}'$$
$$\leq (1 + \varepsilon')RQ^G(\bar{f}).$$

It is clear that this implies Proposition 3.7.

Since we are assuming $d < 1/2$, we may deal with $P'(m, \eta, d, c_0)$ instead of $P(m, \eta, d, c_0)$ as we have just explained in §5.3. Thus what we will actually show is that, for any $\varepsilon' > 0$, there exist constants a_1 and a_2 depending only on m, k, c, and ε' (and independent of l and c_0) such that

$$\mathbb{P}\left(\begin{matrix} P \in P'(m, \eta, d, c_0) \text{ satisfies } (3.6)' \\ \text{for all } Y, \rho \colon \Gamma_P \to \mathrm{Isom}(Y), \\ \text{for all nonconstant } \rho\text{-equivariant map } f \end{matrix} \right) \tag{6.2}$$

$$\geq 1 - a_1 \exp(-a_2(2m)^{\varepsilon_0 l})$$

holds for large l (and hence for large η). Then this implies Proposition 3.7.

Let $w \in W_\eta$. We denote the word given by the first n letters of w by w_1, the word formed by the next l letters by u, and the inverse word of the last n letters by w_2; thus w can be written as $w = w_1 u w_2^{-1}$. Take $P = (S, R) \in P'(m, \eta, d, c_0)$. Let $\rho \colon \Gamma_P \to \mathrm{Isom}(Y)$ be a homomorphism, and take any ρ-equivariant map $f \colon \Gamma_P \to Y$. Then we can consider $\tilde{\rho}$-equivariant map $\tilde{f} \colon \Gamma \to Y$, where $\tilde{\rho} = \rho \circ \pi_P \colon \Gamma \to \mathrm{Isom}(Y)$. Regarding w_i, $i = 1, 2$, as an element $\pi_P(\overline{w_i}) \in \Gamma_P$ in a natural way, we get a map $\bar{f} \colon V \to Y$ induced from f. Since $V = W_n$, we have

$$
\begin{aligned}
\|d\bar{f}\|_{L^2}^2 &= \frac{1}{2} \sum_{v,v' \in W_n} \nu(v,v') d_Y(\bar{f}(v), \bar{f}(v'))^2 \\
&= \frac{1}{2} \sum_{v,v' \in W_n} \nu(v,v') d_Y(\tilde{f}(\bar{v}), \tilde{f}(\bar{v}'))^2 \\
&= \frac{1}{2} \sum_{v,v' \in W_n} \nu(v,v') d_Y(\tilde{f}(e), \tilde{f}(\bar{v}^{-1}\bar{v}'))^2 \\
&= \frac{1}{2} \sum_{\gamma \in \Gamma} \sum_{v,v' \in W_n \,:\, \bar{v}^{-1}\bar{v}' = \gamma} \nu(v,v') d_Y(\tilde{f}(e), \tilde{f}(\gamma))^2 \\
&= \frac{1}{2} \sum_{\gamma_P \in \Gamma_P} \sum_{\gamma \in \Gamma \,:\, \pi_P(\gamma) = \gamma_P} \sum_{v,v' \in W_n \,:\, \bar{v}^{-1}\bar{v}' = \gamma} \nu(v,v') d_Y(f(e), f(\gamma_P))^2.
\end{aligned}
$$

Note that $w = w_1 u w_2^{-1} \in R$ means that $\pi_P(\overline{w_1 u}) = \pi_P(\overline{w_2})$ in Γ_P. Hence $\pi_P(\overline{w_1}^{-1}\overline{w_2}) = \pi_P(\bar{u})$ and $\pi_P(\overline{w_2}^{-1}\overline{w_1}) = \pi_P(\bar{u}^{-1})$. Thus, recalling the definition of $\nu(v, v')$ and fixing $\gamma_P \in \Gamma_P$, we can rewrite as

$$
\begin{aligned}
&\sum_{\gamma \in \Gamma \,:\, \pi_P(\gamma) = \gamma_P} \sum_{v,v' \in W_n \,:\, \bar{v}^{-1}\bar{v}' = \gamma} \nu(v,v') \\
&= \sum_{\gamma \in \Gamma \,:\, \pi_P(\gamma) = \gamma_P} \sum_{v,v' \in W_n \,:\, \bar{v}^{-1}\bar{v}' = \gamma} \frac{\#\{w \in R \mid w_1 = v \text{ and } w_2 = v'\}}{2\#R} \\
&\quad + \frac{\#\{w \in R \mid w_1 = v' \text{ and } w_2 = v\}}{2\#R} \\
&= \frac{\#\{w \in R \mid \pi_P(\bar{u}) = \gamma_P \text{ or } \pi_P(\bar{u}^{-1}) = \gamma_P\}}{2\#R}.
\end{aligned}
$$

Therefore we get

$$
\|d\bar{f}\|_{L^2}^2 = \frac{1}{2} \sum_{\gamma_P \in \Gamma_P} \frac{\#\{w \in R \mid \pi_P(\bar{u}) = \gamma_P \text{ or } \gamma_P^{-1}\}}{2\#R} d_Y(f(e), f(\gamma_P))^2, \quad (6.3)
$$

Since

$$\#\{w \in R \mid \pi_P(\bar{u}) = \gamma_P \text{ or } \gamma_P^{-1}\} = \sum_{\gamma \in \Gamma : \pi_P(\gamma) = \gamma_P} \#\{w \in R \mid \bar{u} = \gamma \text{ or } \gamma^{-1}\},$$

lifting (6.3) to Γ yields

$$\|d\bar{f}\|_{L^2}^2 = \frac{1}{2} \sum_{\gamma \in \Gamma} \frac{\#\{w \in R \mid \bar{u} = \gamma \text{ or } \gamma^{-1}\}}{2\#R} d_Y(\tilde{f}(e), \tilde{f}(\gamma))^2.$$

Note that if $\bar{u} = e$, then $d_Y(\tilde{f}(e), \tilde{f}(\gamma)) = d_Y(\tilde{f}(e), \tilde{f}(e)) = 0$, and hence we may ignore this case in what follows. Then we have

$$\#\{w \in R \mid \bar{u} = \gamma \text{ or } \gamma^{-1}\}$$

$$= \#\{w \in R \mid \bar{u} = \gamma\} + \#\{w \in R \mid \bar{u} = \gamma^{-1}\}$$

$$= \sum_{v \in W_l : \bar{v} = \gamma} \#\{w \in R \mid u = v\} + \sum_{v' \in W_l : \bar{v'} = \gamma^{-1}} \#\{w \in R \mid u = v'\}$$

for $\gamma \neq e$. Now let $X_{v,l} : P'(m, \eta, d, c_0) \to \mathbb{Z}$ be a random variable defined by

$$X_{v,l}(P) = \#\{w \in R \mid u = v\},$$

where $v \in W_l$. By the definition of $P'(m, \eta, d, c_0)$, $X_{v,l}$ has the binomial distribution $\text{Bi}(\#R, \frac{1}{\#W_l})$ with mean $\frac{\#R}{\#W_l}$. Note that, for $P = (S, R) \in P'(m, \eta, d, c_0)$ under consideration, we can write as

$$\|d\bar{f}\|_{L^2}^2 = \frac{1}{2} \sum_{\gamma \in \Gamma} \frac{\sum_{v \in W_l : \bar{v} = \gamma} X_{v,l}(P) + \sum_{v' \in W_l : \bar{v'} = \gamma^{-1}} X_{v',l}(P)}{2\#R}$$

$$\times d_Y(\tilde{f}(e), \tilde{f}(\gamma))^2. \tag{6.4}$$

Then, since $\mu^l(e, \gamma) = \#\{w \in W_l \mid \bar{w} = \gamma\}/\#W_l$, the expected value of the coefficient of $d_Y(\tilde{f}(e), \tilde{f}(\gamma))$ in (6.4) becomes

$$\mathbb{E}\left(\frac{\sum_{v \in W_l : \bar{v} = \gamma} X_{v,l} + \sum_{v' \in W_l : \bar{v'} = \gamma^{-1}} X_{v',l}}{2\#R} \right)$$

$$= \frac{1}{2\#R} \left(\sum_{v \in W_l : \bar{v} = \gamma} \mathbb{E}(X_{v,l}) + \sum_{v' \in W_l : \bar{v'} = \gamma^{-1}} \mathbb{E}(X_{v',l}) \right)$$

$$= \frac{1}{2\#R} \left(\sum_{v \in W_l : \bar{v} = \gamma} \frac{\#R}{\#W_l} + \sum_{v' \in W_l : \bar{v'} = \gamma^{-1}} \frac{\#R}{\#W_l} \right)$$

$$= \frac{1}{2} (\mu^l(e, \gamma) + \mu^l(e, \gamma^{-1}))$$

$$= \mu^l(e, \gamma).$$

Therefore, if $X_{v,l}(P) = \#\{w \in R \mid u = v\}$ of $P = (S, R)$ is close to the expected value for all $v \in W_l$, then $\|d\bar{f}\|_{L^2}^2$ must be close to

$$\frac{1}{2} \sum_{\gamma \in \Gamma} \mu^l(e, \gamma) d_Y(\tilde{f}(e), \tilde{f}(\gamma))^2 = E_{\mu^l}(\tilde{f}).$$

More precisely speaking, if $P = (S, R)$ satisfies

$$(1 - \varepsilon)\frac{\#R}{\#W_l} \le X_{v,l}(P) \le (1 + \varepsilon)\frac{\#R}{\#W_l}$$

for all $v \in W_l$, then we see that, for any Y, $\rho\colon \Gamma_P \to \mathrm{Isom}(Y)$ and ρ-equivariant map f,

$$(1 - \varepsilon)E_{\mu^l}(\tilde{f}) \le \|d\bar{f}\|_{L^2}^2 \le (1 + \varepsilon)E_{\mu^l}(\tilde{f}) \tag{6.5}$$

holds.

On the other hand, we have

$$\begin{aligned}
F(\bar{f}) &= \frac{1}{2} \sum_{v,v' \in W_n} \nu(v)\nu(v') d_Y(\bar{f}(v), \bar{f}(v'))^2 \\
&= \frac{1}{2} \sum_{v,v' \in W_n} \nu(v)\nu(v') d_Y(f(\pi_P(\bar{v})), f(\pi_P(\bar{v}')))^2 \\
&= \frac{1}{2} \sum_{v,v' \in W_n} \nu(v)\nu(v') d_Y(f(e), f(\pi_P(\bar{v}^{-1}\bar{v}')))^2 \\
&= \frac{1}{2} \sum_{\gamma,\gamma' \in \Gamma} \sum_{v,v' \in W_n \,:\, \bar{v}=\gamma, \bar{v}'=\gamma'} \nu(v)\nu(v') d_Y(\tilde{f}(e), \tilde{f}(\gamma^{-1}\gamma'))^2.
\end{aligned}$$

Let $X_{i,v,n}\colon P(m, \eta, d, c_0) \to \mathbb{Z}$, $i = 1, 2$, be a random variable defined by

$$X_{i,v,n}(P) = \#\{w \in R \mid w_i = v\},$$

where $v \in W_n$. Using this random variable and recalling the definition of $\nu(v)$, we see that, for $P = (S, R)$,

$$\begin{aligned}
\nu(v) &= \sum_{v' \in W_n} \nu(v, v') = \frac{\#\{w \in R \mid w_1 = v\} + \#\{w \in R \mid w_2 = v\}}{2\#R} \\
&= \frac{X_{1,v,n}(P) + X_{2,v,n}(P)}{2\#R}
\end{aligned}$$

holds. Therefore we get

$$F(\bar{f}) = \frac{1}{2} \sum_{\gamma,\gamma'\in\Gamma} \frac{\sum_{v\in W_n\,:\,\bar{v}=\gamma}\left(X_{1,v,n}(P) + X_{2,v,n}(P)\right)}{2\#R}$$

$$\times \frac{\sum_{v\in W_n\,:\,\bar{v}=\gamma'}\left(X_{1,v,n}(P) + X_{2,v,n}(P)\right)}{2\#R}$$

$$\times d_Y(\tilde{f}(e), \tilde{f}(\gamma^{-1}\gamma'))^2.$$

Since $X_{i,v,n}$ has the binomial distribution $\mathrm{Bi}(\#R, \frac{1}{\#W_n})$ for $i = 1, 2$, we get

$$\mathbb{E}\left(\frac{\sum_{v\in W_n\,:\,\bar{v}=\gamma} X_{1,v,n} + X_{2,v,n}}{2\#R}\right) = \frac{1}{2\#R} \sum_{v\in W_n\,:\,\bar{v}=\gamma} \left(\mathbb{E}(X_{1,v,n}) + \mathbb{E}(X_{2,v,n})\right)$$

$$= \frac{1}{\#R}\#\{v \in W_n \mid \bar{v} = \gamma\}\frac{\#R}{\#W_n}$$

$$= \mu^n(e, \gamma).$$

Therefore, if both $\#\{w \in R \mid w_1 = v\}$ and $\#\{w \in R \mid w_2 = v\}$ are close to the expected value for all $v \in W_n$, then $F(\bar{f})$ must be close to

$$\frac{1}{2} \sum_{\gamma,\gamma'\in\Gamma} \mu^n(e, \gamma)\mu^n(e, \gamma')d_Y(\tilde{f}(e), \tilde{f}(\gamma^{-1}\gamma'))^2$$

$$= \frac{1}{2} \sum_{\gamma,\gamma'\in\Gamma} \mu^n(\gamma^{-1}, e)\mu^n(\gamma^{-1}, \gamma^{-1}\gamma')d_Y(\tilde{f}(e), \tilde{f}(\gamma^{-1}\gamma'))^2$$

$$= \frac{1}{2} \sum_{\gamma} \mu^n(\gamma^{-1}, e) \sum_{\gamma'} \mu^n(\gamma^{-1}, \gamma^{-1}\gamma')d_Y(\tilde{f}(e), \tilde{f}(\gamma^{-1}\gamma'))^2$$

$$= \frac{1}{2} \sum_{\gamma} \mu^n(e, \gamma^{-1}) \sum_{\gamma''} \mu^n(\gamma^{-1}, \gamma'')d_Y(\tilde{f}(e), \tilde{f}(\gamma''))^2$$

$$= \frac{1}{2} \sum_{\gamma''} \sum_{\gamma} \mu^n(e, \gamma^{-1})\mu^n(\gamma^{-1}, \gamma'')d_Y(\tilde{f}(e), \tilde{f}(\gamma''))^2$$

$$= \frac{1}{2} \sum_{\gamma''} \mu^{2n}(e, \gamma'')d_Y(\tilde{f}(e), \tilde{f}(\gamma''))^2$$

$$= E_{\mu^{2n}}(\tilde{f}).$$

Actually, if $P = (S, R)$ satisfies

$$(1 - \varepsilon)\frac{\#R}{\#W_n} \leq X_{i,v,n}(P) \leq (1 + \varepsilon)\frac{\#R}{\#W_n}, \quad i = 1, 2$$

for all $v \in W_n$, then we see that, for any Y, $\rho \colon \Gamma_P \to \mathrm{Isom}(Y)$, and ρ-equivariant map f,

$$(1 - \varepsilon)^2 E_{\mu^{2n}}(\tilde{f}) \leq F(\bar{f}) \leq (1 + \varepsilon)^2 E_{\mu^{2n}}(\tilde{f}) \tag{6.6}$$

holds.

Since our random variables $X_{v,l}$ and $X_{i,v,n}$ are binomially distributed with distribution $\mathrm{Bi}(\#R, \frac{1}{\#W_l})$ and $\mathrm{Bi}(\#R, \frac{1}{\#W_n})$ respectively, we have

$$\mathbb{E}(X_{v,l}) = \frac{\#R}{\#W_l} = c_0(2m)^{d\eta - l} \geq c_0(2m)^{\varepsilon_0 l + (k-1)l},$$

$$\mathbb{E}(X_{i,v,n}) = \frac{\#R}{\#W_n} = c_0(2m)^{d\eta - n} \geq c_0(2m)^{\varepsilon_0 l - (2k+1)}$$

by (2.1) and (6.1). Since each expected value does not depend on the choice of v, we denote it by μ_l and $\mu_{i,n}$ respectively. Using (5.2), we see that, for any given $\varepsilon > 0$,

$$\mathbb{P}(|X_{v,l} - \mu_l| > \varepsilon\mu_l) \leq 2\exp\left(-\frac{\varepsilon^2 c_0(2m)^{(\varepsilon_0 + k - 1)l}}{3}\right),$$

$$\mathbb{P}(|X_{i,v,n} - \mu_{i,n}| > \varepsilon\mu_{i,n}) \leq 2\exp\left(-\frac{\varepsilon^2 c_0(2m)^{\varepsilon_0 l - (2k+1)}}{3}\right)$$

hold. Since $\#W_n = (2m)^n$, $n \leq kl + (2k+1)$, and $d(2k+1) = \varepsilon_0 + k$, we have

$$\mathbb{P}(|X_{i,v,n} - \mu_{i,n}| > \varepsilon\mu_{i,n} \text{ for some } v \in W_n)$$

$$\leq (2m)^n 2\exp\left(-\frac{\varepsilon^2 c_0(2m)^{\varepsilon_0 l - (2k+1)}}{3}\right)$$

$$\leq 2\max_l\left\{(2m)^{kl+(2k+1)}\exp\left(-\frac{\varepsilon^2 c_0(2m)^{\varepsilon_0 l - (2k+1)}}{6}\right)\right\}$$

$$\times \exp\left(-\frac{\varepsilon^2 c_0(2m)^{\varepsilon_0 l - (2k+1)}}{6}\right).$$

Letting

$$a = 2\max_l\left\{(2m)^{kl+(2k+1)}\exp\left(-\frac{\varepsilon^2 c^{-1}(2m)^{\varepsilon_0 l - (2k+1)}}{6}\right)\right\},$$

$$a' = \frac{\varepsilon^2 c^{-1}(2m)^{-(2k+1)}}{6},$$

we obtain

$$\mathbb{P}(|X_{i,v,n} - \mu_{i,n}| > \varepsilon\mu_{i,n} \text{ for some } v \in W_n) \leq a \exp(-a'(2m)^{\varepsilon_0 l})$$

for any $P'(m, \eta, d, c_0)$ for $c^{-1} \leq c_0 \leq c$. Since $k \geq 1$,

$$\mathbb{P}(|X_{v,l} - \mu_l| > \varepsilon\mu_l \text{ for some } v \in W_l) \leq a \exp(-a'(2m)^{\varepsilon_0 l})$$

holds for the same a and a' as above. Recalling (6.5) and (6.6) and letting $a_1 = 3a$ and $a_2 = a'$, we see that these inequalities imply

$$\mathbb{P}\begin{pmatrix} P \in P'(m, \eta, d, c_0) \text{ satisfies } (6.7) \\ \text{for all } Y, \rho\colon \Gamma_P \to \mathrm{Isom}(Y), \\ \text{for all nonconstant } \rho\text{-equivariant map } f \end{pmatrix} \geq 1 - a_1 \exp(-a_2(2m)^{\varepsilon_0 l}),$$

where

$$\frac{1-\varepsilon}{(1+\varepsilon)^2} RQ^G(\bar{f}) \leq \frac{E_{\mu^l}(\tilde{f})}{E_{\mu^{2n}}(\tilde{f})} \leq \frac{1+\varepsilon}{(1-\varepsilon)^2} RQ^G(\bar{f}). \tag{6.7}$$

Thus, by taking ε suitably, this implies (6.2). This completes the proof of Proposition 3.7.

Remark 2. We should mention that it is also possible to prove Proposition 3.7 using the model $P''(m, \eta, d, c_0)$ introduced in §5.3 with the help of Lemma 5.3; thus we can drop the assumption $d < 1/2$ in Proposition 3.7. However, the appearance of μ^l and μ^n in the proof becomes less natural compared with the proof presented above.

6.2. Proof of Proposition 2.1. Our proof follows the idea of [4], although our model is different from theirs and a certain modification is needed.

As we have mentioned in §5.3, we will deal with $P''(m, \eta, d, c_0)$ in order to prove Proposition 2.1. Let $P \in P''(m, \eta, d, c_0)$ and $G_P = (V, E_P)$ the graph associated to P. Recall that G_P has measures on $V \times V$ and V both denoted by ν, which are used to define the inner product on the space of functions on V and the Laplacian Δ_P. We are going to deal with the eigenvalues of Δ_P, and it is convenient to give an expression of Δ_P by matrices. Let us denote $V = \{v_1, \ldots, v_{(2m)^n}\}$, and $\mathbf{e}_i$ be a function defined by

$$\mathbf{e}_i(v_j) = \begin{cases} \frac{1}{\sqrt{\nu(v_i)}} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Then $\{\mathbf{e}_1, \ldots, \mathbf{e}_{(2m)^n}\}$ forms an orthonormal basis of the space of functions on V with respect to the inner product $(\cdot, \cdot)_{L^2}$. Note that

$$\Delta_P \mathbf{e}_i(v_j) = \mathbf{e}_i(v_j) - \sum_{v' \in V} \frac{\nu(v_j, v')}{\nu(v_j)} \mathbf{e}_i(v') = \frac{\delta_{ij}}{\sqrt{\nu(v_i)}} - \frac{\nu(v_j, v_i)}{\nu(v_j)\sqrt{\nu(v_i)}},$$

therefore

$$\Delta_P \mathbf{e}_i = \mathbf{e}_i - \sum_j \frac{\nu(v_j, v_i)}{\sqrt{\nu(v_j)\nu(v_i)}} \mathbf{e}_j,$$

and Δ_P can be expressed as

$$\Delta_P = I - A',$$

where I denotes the identity matrix, and A' is expressed by means of its entries as

$$A' = \left(\frac{\nu(v_i, v_j)}{\sqrt{\nu(v_i)\nu(v_j)}} \right).$$

Note that, by definition,

$$\frac{\nu(v_i, v_j)}{\sqrt{\nu(v_i)\nu(v_j)}} = \frac{\text{multiplicity of } \{v_i, v_j\}}{\sqrt{\deg(v_i)\deg(v_j)}}.$$

Let A be the multiplicity matrix (the adjacency matrix taking the multiplicity in account) and D the square root of the inverse of the degree matrix:

$$A = \left(\text{multiplicity of } \{v_i, v_j\}\right) \quad \text{and} \quad D = \left(\frac{\delta_{ij}}{\sqrt{\deg(v_i)}} \right).$$

Then we see that $A' = DAD$.

Let π_0 be the orthogonal projection onto the space of constant functions on V. Note that

$$M = (I - \Delta_P)(I - \pi_0) = I - \Delta_P - \pi_0 = DAD - \pi_0$$

has eigenvalues 0 and $1 - \lambda_i(G_P)$, $i \neq 0$, where $0 = \lambda_0(G_P) \leq \lambda_1(G_P) \leq \cdots \leq \lambda_m(G_P)$ are the eigenvalues of Δ_P, and eigenfunctions corresponding to 0 are constant functions. (Here we regard M is given as a matrix form with respect the orthonormal basis $\{\mathbf{e}_i\}$. Note that the eigenfunctions of M are the same as that of Δ_P.) Thus we have

$$\max_{i \neq 0} |1 - \lambda_i(G_P)| \leq \|M\|, \tag{6.8}$$

where $\|M\|$ denotes the operator norm of M.

We regard $\|M\| = \|M_P\|$ as a random variable on $P''(m, \eta, d, c_0)$ and will show that, for any $\varepsilon' > 0$, there exist positive constants a_1 and a_2 depending only on m, k, c, and ε' (and, in particular, independent of c_0) and a_3 depending on m, k, ε', and α such that

$$\mathbb{P}_{P''}(\|M\| > \varepsilon') \leq a_1 \exp(-a_2(2m)^{\varepsilon_0 l}) + a_3(2m)^{-(\alpha\varepsilon_0 - k)l} \qquad (6.9)$$

holds, where α is an arbitrary natural number and $\varepsilon_0 = d(2k + 1) - k$. Once (6.9) is proven, recalling Lemma 5.3 and $d\eta \leq (k + \varepsilon_0)l + 2(k + \varepsilon_0)$, we obtain

$$\mathbb{P}_P(\|M\| > \varepsilon')$$

$$\leq 3\sqrt{c_0(2m)^{d\eta}}\,\mathbb{P}_{P''}(\|M\| > \varepsilon')$$

$$\leq 3c^{1/2}(2m)^{(k+\varepsilon_0)l + 2(k+\varepsilon_0)}(a_1\exp(-a_2(2m)^{\varepsilon_0 l}) + a_3(2m)^{-(\alpha\varepsilon_0 - k)l})$$

$$\leq a_1'(2m)^{(k+\varepsilon_0)l}\exp(-a_2(2m)^{\varepsilon_0 l}) + a_3'(2m)^{-(\alpha\varepsilon_0 - (2k+\varepsilon_0))l}.$$

By choosing $\alpha > (2k + \varepsilon_0)/\varepsilon_0$ and setting $\alpha_0 = \alpha - (2k + \varepsilon_0)/\varepsilon_0$, we see that

$$\mathbb{P}_P(\|M\| \leq \varepsilon') \geq 1 - a_1'(2m)^{(k+\varepsilon_0)l}\exp(-a_2(2m)^{\varepsilon_0 l}) - a_3''(2m)^{-(\alpha_0\varepsilon_0)l} \longrightarrow 1$$

as $\eta \to \infty$, where a_1', a_3'', and α_0 depends only on m, k, c, ε', and α chosen as above (and independent of c_0). Together with (6.8), this implies Proposition 2.1. In what follows we will deal only with $P''(m, \eta, d, c_0)$, and will drop subscript P'' of $\mathbb{P}_{P''}$, since there will be no chance of confusion.

The first step of the proof of (6.9) is to estimate $\|B\|$ of an operator B defined below, which approximates M with high probability. Recall our random graph has $\#V = (2m)^n$ vertices and, for each pair of vertices, the multiplicity of the edge joining them is at most $(2m)^l = \#W_l$. Since each word in W_η is chosen with probability $(c_0(2m)^{d\eta})/(2m)^\eta$, the multiplicity of an edge has the binomial distribution $\text{Bi}((2m)^l, (c_0(2m)^{d\eta})/(2m)^\eta)$, and the expected multiplicity μ_e of each edge is $(2m)^l \times (c_0(2m)^{d\eta})/(2m)^\eta$. Thus, by recalling (2.1) and (6.1), the expected multiplicity μ_e can be estimated as

$$\mu_e = c_0(2m)^{l + (d-1)(2n+l)}$$

$$\leq 2c_0(2m)^{d(2k+1)l + 2d(2k+1) - (2k+1)l + l}$$

$$= 2c_0(2m)^{-(k-\varepsilon_0)l + 2d(2k+1)}.$$

On the other hand, since a vertex is given as the first (or the last) n letters of elements of W_η, it is followed by (or follows) the elements of W_{n+l}. Therefore, for any $v \in W_n$, the random variable $Y_{i,v,n}$ on $P''(m, \eta, d, c_0)$ defined by

$$Y_{i,v,n}(P) = \#\{w \in R \mid w_i = v\}$$

has the binomial distribution $\mathrm{Bi}((2m)^{n+l}, (c_0(2m)^{d\eta})/(2m)^\eta)$. Since the degree of a vertex v is given by $\deg(v) = Y_{1,v,n} + Y_{2,v,n}$, the expected degree μ_v of a vertex is given by $\mu_v = 2(2m)^{n+l} \times (c_0(2m)^{d\eta})/(2m)^\eta$. Thus, again by (2.1) and (6.1), μ_v can be estimated as

$$\mu_v = 2c_0(2m)^{d\eta}(2m)^{-n} \geq 2c_0(2m)^{\varepsilon_0 l - (2k+1)}.$$

Let a matrix B be

$$B = \frac{1}{\mu_v}(A - \mu_e K),$$

where K is a matrix all of whose entries are 1. Note that each entries of the matrix $A - \mu_e K$ is equal to

$$(\text{multiplicity of } \{v_i, v_j\}) - \mu_e,$$

and hence, writing $B = (b_{ij})$, we see that $\mathbb{E}(b_{ij}) = 0$ for each (i, j). We first show that the eigenvalue of B approaches to 0 as $l \to \infty$. Note that, since B is symmetric

$$\mathrm{trace}(B^{2\alpha}) = \lambda_1(B)^{2\alpha} + \ldots \lambda_{(2m)^n}(B)^{2\alpha}$$

for any positive integer α, where $\lambda_i(B)$'s are eigenvalues of B (counted with multiplicity). Let $\lambda_1(B)$ be the eigenvalue of B with maximum absolute value, which is equal to the operator norm of B: $\|B\| = |\lambda_1(B)|$. Then we have

$$\mathbb{E}(\|B\|^{2\alpha}) = \mathbb{E}(\lambda_1(B)^{2\alpha}) \leq \mathbb{E}(\mathrm{trace}(B^{2\alpha})).$$

The ith diagonal entry of $B^{2\alpha}$ is a sum of terms of the form $b_{ii_1}b_{i_1i_2}\ldots b_{i_{2\alpha-1}i}$, which corresponds to a closed path of length 2α in the complete graph (with self-loops) having V as the vertex set with basepoint v_i.

Note that b_{ij}'s are independent random variables, and that the mean of b_{ij} is 0 as we have mentioned above. Therefore the expected value of the diagonal entries $B^{2\alpha}$ involves only closed paths of length 2α all of whose edges are visited at least twice. Such a closed path is called a *closed good walk* in [7]. Now we see that $\mathbb{E}(\mathrm{trace}(B^{2\alpha}))$ is a sum of terms of the form

$$\mathbb{E}\left(\prod b_{ij}^{\alpha_{ij}}\right) = \prod \mathbb{E}(b_{ij}^{\alpha_{ij}}),$$

where α_{ij}'s are positive integers greater than 1 and their sum is equal to 2α. The multiplicity of each edge is distributed binomially with mean μ_e and, by taking l suitably large, we may assume $\mu_e \leq 2c_0(2m)^{-(k-\varepsilon_0)l + 2d(2k+1)} < 1$. Therefore, we can apply Lemma 5.2 to obtain

$$\mathbb{E}(b_{ij}^{\alpha_{ij}}) = \frac{1}{\mu_v^{\alpha_{ij}}}\mathbb{E}\left((\text{multiplicity of } \{v_i, v_j\} - \mu_e)^{\alpha_{ij}}\right) \leq c(\alpha_{ij})\frac{1}{\mu_v^{\alpha_{ij}}}\mu_e.$$

Let $W(\alpha, \beta)$ be the set of all closed good walks of length 2α which pass through $\beta + 1$ distinct vertices. If $b_{ii_1} b_{i_1 i_2} \ldots b_{i_{2\alpha-1} i}$ corresponds to a walk in $W(\alpha, \beta)$, then there appear at least β distinct b_{ij}'s. Since we may assume $\mu_e < 1$, we get

$$
\prod \mathbb{E}(b_{ij}^{\alpha_{ij}}) \le c(\alpha) \frac{1}{\mu_v^{2\alpha}} \mu_e^\beta
$$
$$
\le 2^{-2\alpha} c(\alpha) c_0^{\beta-2\alpha} (2m)^{-2\alpha\varepsilon_0 l + 2\alpha(2k+1)} (2m)^{-\beta(k-\varepsilon_0)l + 2d\beta(2k+1)}
$$
$$(6.10)$$

for some constant $c(\alpha)$. Note that

$$
\#W(\alpha, \beta)
$$

$$
\le \#(\text{choice of } \beta + 1 \text{ vertices from } V)
$$
$$
\quad \times \#(\text{good walks with length } 2\alpha \text{ in the complete graph with } \beta + 1 \text{ vertices})
$$
$$
\le \#V^{\beta+1} c(\alpha, \beta)
$$
$$
= (2m)^{(\beta+1)n} c(\alpha, \beta)
$$
$$
\le c(\alpha, \beta)(2m)^{(\beta+1)(kl+(2k+1))}
$$

holds, where, clearly, $c(\alpha, \beta)$ is a constant depending only on α and β. (See [7] for a sharp bound of $W(\alpha, \beta)$.)

Thus, noting $c^{-1} \le c_0 \le c$ and summing up (6.10) over all good walks of length 2α gives

$$
\mathbb{E}(\text{trace}(B^{2\alpha}))
$$

$$
\le 2^{-2\alpha} c(\alpha) \sum_{\beta=0}^{\alpha} c^{2\alpha-\beta} \#W(\alpha, \beta)(2m)^{-2\alpha\varepsilon_0 l}(2m)^{-\beta(k-\varepsilon_0)l}(2m)^{(2\alpha+2d\beta)(2k+1)}
$$

$$
\le \left(\frac{c}{2}\right)^{2\alpha} c(\alpha) \sum_{\beta=0}^{\alpha} c(\alpha, \beta)(2m)^{(\beta+1)(kl+(2k+1))}(2m)^{(2\alpha+2d\beta)(2k+1)}
$$
$$
\quad \times (2m)^{-2\alpha\varepsilon_0 l - \beta(k-\varepsilon_0)l}
$$

$$
\le \left(\frac{c}{2}\right)^{2\alpha} c(\alpha) \sum_{\beta=0}^{\alpha} c(\alpha, \beta)(2m)^{(\beta+1+2\alpha+2d\beta)(2k+1)}(2m)^{(\beta+1)kl-2\alpha\varepsilon_0 l - \beta(k-\varepsilon_0)l}
$$

$$
\le \left(\frac{c}{2}\right)^{2\alpha} c(\alpha)(2m)^{(5\alpha+1)(2k+1)} \sum_{\beta=0}^{\alpha} c(\alpha, \beta)(2m)^{kl-(2\alpha-\beta)\varepsilon_0 l},
$$

where we have used $\beta \leq \alpha$ to deduce the last inequality. By $\beta \leq \alpha$ again, and by setting $c'(\alpha) = 2^{-2\alpha} c(\alpha) c^{2\alpha} (2m)^{(5\alpha+1)(2k+1)} \sum_\beta c(\alpha, \beta)$, we get

$$\mathbb{E}(\text{trace}(B^{2\alpha})) \leq c'(\alpha)(2m)^{kl-\alpha\varepsilon_0 l}.$$

Thus, by Markov's inequality (5.1), for any $\varepsilon > 0$, we get

$$\mathbb{P}(\|B\|^{2\alpha} \geq \varepsilon^{2\alpha}) \leq \frac{\mathbb{E}(\|B\|^{2\alpha})}{\varepsilon^{2\alpha}} \leq \frac{\mathbb{E}(\text{trace}(B^{2\alpha}))}{\varepsilon^{2\alpha}} = \frac{c'(\alpha)}{\varepsilon^{2\alpha}} (2m)^{kl-\alpha\varepsilon_0 l},$$

and hence

$$\mathbb{P}(\|B\| \geq \varepsilon) \leq \frac{c'(\alpha)}{\varepsilon^{2\alpha}} (2m)^{-(\alpha\varepsilon_0 - k)l}. \tag{6.11}$$

Next we will show that B actually approximates M with high probability. Now take a matrix M' to be

$$M' = D\,(A - \mu_e K)\,D.$$

Writing $M' = (m'_{ij})$, we see that

$$m'_{ij} = \frac{\mu_v}{\sqrt{\deg(v_i)\deg(v_j)}} b_{ij}.$$

Let $f : V \to \mathbb{R}$ and write $f = \sum_i f_i \mathbf{e}_i$. Since M' is symmetric,

$$\begin{aligned}
\|M'\| &= \max_{|f|=1} \left|(f, M'f)\right| \\
&= \max_{|f|=1} \left| \sum_i f_i \left(\sum_j \frac{\mu_v}{\sqrt{\deg(v_i)\deg(v_j)}} b_{ij} f_j \right) \right| \\
&= \max_{|f|=1} \left| \sum_i \frac{\sqrt{\mu_v}}{\sqrt{\deg(v_i)}} f_i \left(\sum_j b_{ij} \frac{\sqrt{\mu_v}}{\sqrt{\deg(v_j)}} f_j \right) \right| \\
&= \max_{|f|=1} \left|(f', Bf')\right| \\
&\leq \max_{|f|=1} \|B\||f'|^2,
\end{aligned}$$

where $f' = \sum_i f'_i \mathbf{e}_i$ is defined by

$$f'_i = \frac{\sqrt{\mu_v}}{\sqrt{\deg(v_i)}} f_i.$$

By $\deg(v) = Y_{1,v,n} + Y_{2,v,n}$ and (5.2), we know that, for any $\varepsilon > 0$, there is a constant a, and

$$
\begin{aligned}
\mathbb{P}(|\deg(v) - \mu_v| \\
> \varepsilon\mu_v \text{ for some } v \in V) \\
< 2\#V \times 2\exp\left(-\frac{\varepsilon^2\mu_v}{3}\right) \\
= 4(2m)^n \exp\left(-\frac{\varepsilon^2 2c_0(2m)^{\varepsilon_0 l - (2k+1)}}{3}\right) \\
\leq 2a\exp(-a'(2m)^{\varepsilon_0 l}),
\end{aligned}
\tag{6.12}
$$

where a, a' can be taken as the same constants as the preceding subsection. Thus we have, for any given $\varepsilon > 0$,

$$
\|M'\| \leq \|B\|(1 - \varepsilon)^{-1}
\tag{6.13}
$$

with probability $1 - 2a\exp\left(-a'(2m)^{\varepsilon_0 l}\right)$, which is independent of c_0 (but depends on m, k, c, and ε).

On the other hand, since $M = DAD - \pi_0$, we see that

$$
M - M' = \mu_e DKD - \pi_0.
$$

With respect to the orthonormal basis $\{\mathbf{e}_i\}$, π_0 can be expressed as

$$
\pi_0 = \begin{pmatrix} \sqrt{\nu(v_1)} \\ \vdots \\ \sqrt{\nu(v_{(2m)^n})} \end{pmatrix} \begin{pmatrix} \sqrt{\nu(v_1)} & \cdots & \sqrt{\nu(v_{(2m)^n})} \end{pmatrix} = \left(\frac{\sqrt{\deg(v_i)\deg(v_j)}}{2\#R}\right),
$$

and hence

$$
\begin{aligned}
M - M' &= \left(\frac{\mu_e}{\sqrt{\deg(v_i)\deg(v_j)}} - \frac{\sqrt{\deg(v_i)\deg(v_j)}}{2\#R}\right) \\
&= \frac{1}{2\#R}\left(\frac{2\#R\mu_e - \deg(v_i)\deg(v_j)}{\sqrt{\deg(v_i)\deg(v_j)}}\right).
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
&\|M - M'\| \\
&= \max_{|f|=1} \left| \left(f, (M - M')f \right) \right| \\
&= \frac{1}{2\#R} \max_{|f|=1} \left| \sum_i f_i \left(\sum_j \frac{2\#R\mu_e - \deg(v_i)\deg(v_j)}{\sqrt{\deg(v_i)\deg(v_j)}} f_j \right) \right| \\
&= \frac{1}{2\#R} \max_{|f|=1} \left| \sum_i \sqrt{\deg(v_i)} f_i \left(\sum_j \left(\frac{2\#R\mu_e}{\deg(v_i)\deg(v_j)} - 1 \right) \sqrt{\deg(v_j)} f_j \right) \right|.
\end{aligned}
$$

Recall that, by definition, $\#R$ is a binomially distributed random variable with distribution $\mathrm{Bi}((2m)^\eta, (c_0(2m)^{d\eta})/(2m)^\eta)$. Thus the expected value of $\#R$ is equal to $c_0(2m)^{d\eta}$. Noting that

$$
\begin{aligned}
2c_0(2m)^{d\eta}\mu_e &= 2c_0(2m)^{d\eta} \times 2c_0(2m)^{d\eta-2n} \\
&= 4c_0^2(2m)^{2d\eta}(2m)^{-2n} \\
&= \mu_v^2
\end{aligned}
$$

and (5.2), we see that, for any $\varepsilon > 0$,

$$
\begin{aligned}
\mathbb{P}(|2\#R\mu_e - \mu_v^2| > \varepsilon\mu_v^2) &= \mathbb{P}(|\#R - c_0(2m)^{d\eta}| > \varepsilon c_0(2m)^{d\eta}) \\
&\leq 2\exp\left(-\frac{\varepsilon^2 c_0(2m)^{d\eta}}{3} \right) \\
&\leq 2\exp\left(-\frac{\varepsilon^2 c_0}{3}(2m)^{kl+\varepsilon_0 l} \right) \\
&\leq b\exp(-b'(2m)^{\varepsilon_0 l})
\end{aligned}
$$

holds, where

$$
b = 2, \quad b' = \frac{\varepsilon^2 c^{-1}}{3}.
$$

Note that we may assume $2a > b$ and $a' < b'$, which implies

$$
\mathbb{P}(|2\#R\mu_e - \mu_v^2| > \varepsilon\mu_v^2) < 2a\exp(-a'(2m)^{\varepsilon_0 l}). \tag{6.14}
$$

By (6.12) and (6.14), we see that, for any given $\varepsilon > 0$,

$$\|M - M'\|$$

$$\leq \frac{1}{2\#R} \max_{|f|=1} \left| \sum_i \sqrt{\deg(v_i)} f_i \left(\left(\frac{1+\varepsilon}{(1-\varepsilon)^2} - 1 \right) \sum_j \sqrt{\deg(v_j)} f_j \right) \right|$$

$$\leq \frac{1}{2\#R} \max_{|f|=1} \sqrt{\sum_i \deg(v_i)} |f| \left(\frac{3\varepsilon - \varepsilon^2}{(1-\varepsilon)^2} \sqrt{\sum_j \deg(v_j)} |f| \right)$$

$$= \frac{1}{2\#R} \left(\frac{\varepsilon}{1-\varepsilon} + \frac{2\varepsilon}{(1-\varepsilon)^2} \right) \sum_i \deg(v_i)$$

$$= \frac{\varepsilon}{1-\varepsilon} + \frac{2\varepsilon}{(1-\varepsilon)^2}$$

with probability $1 - 4a \exp(-a'(2m)^{\varepsilon_0 l})$, where Cauchy-Schwarz inequality is used in the second inequality and $\sum_j \deg(v_j) = 2\#E_P = 2\#R$ implies the last equality. Thus, together with (6.13), for any $\varepsilon > 0$, we see that

$$\|M\| \leq \|M'\| + \|M - M'\|$$

$$\leq \frac{\|B\|}{1-\varepsilon} + \frac{\varepsilon}{1-\varepsilon} + \frac{2\varepsilon}{(1-\varepsilon)^2}$$

holds with probability greater than $1 - 6a \exp\left(-a'(2m)^{\varepsilon_0 l}\right)$, which is independent of c_0 (but depends on m, k, c, and ε). Recall that (6.11) tells us that we may assume $\|B\| < \varepsilon$ with high probability. By taking $\varepsilon < 1/2$, we may assume

$$\frac{2\varepsilon}{1-\varepsilon} + \frac{2\varepsilon}{(1-\varepsilon)^2} < 12\varepsilon.$$

Thus we obtain, by taking (6.11) in account,

$$\mathbb{P}(\|M\| \leq 12\varepsilon) \geq 1 - a_1 \exp(-a_2(2m)^{\varepsilon_0 l}) - a_3(2m)^{-(\alpha\varepsilon_0 - k)l},$$

where

$$a_1 = 6a, \ a_2 = a', \ a_3 = \frac{c'(\alpha)}{\varepsilon^{2\alpha}}.$$

By taking $\varepsilon = \varepsilon'/12$, this completes the proof of (6.9).

7. Random quotients

In the proof of Propositions 2.1 and 3.7, we did not use the fact that Γ is a free group generated by S; Γ can be any group generated by S (see [18, IV.c] for related

comments). Therefore what we have actually proven is the following fixed-point theorem for random quotients:

Theorem 7.1. *Let $\mathcal{H}$ be a Hilbert space. Suppose that Γ is a group generated by $S = \{s_1, \ldots, s_m, s_1^{-1}, \ldots, s_m^{-1}\}$, and let $\Gamma'_P = \Gamma/\bar{R}$ for $P \in P(m, \eta, d)$, where $\bar{R}$ denotes the normal closure of the subset of Γ determined by R. For $1/3 < d$,*

$$\lim_{\eta \to \infty} \frac{\#\{P \in P(m, \eta, d) \mid \Gamma'_P \text{ has } F(\mathcal{H})\}}{\#P(m, \eta, d)} = 1$$

holds.

Theorem 7.2. *Let Y be a $\mathrm{CAT}(0)$ space. Suppose that Γ is a group generated by $S = \{s_1, \ldots, s_m, s_1^{-1}, \ldots, s_m^{-1}\}$, and let $\Gamma'_P = \Gamma/\bar{R}$ for $P \in P(m, \eta, d)$, where $\bar{R}$ denotes the normal closure of the subset of Γ determined by R. Take $0 \le \delta < 1$ and an integer k so that $k > 1/(1 - \delta)$. Let $\mathcal{Y}_\delta$ be the class of $\mathrm{CAT}(0)$ spaces with $\delta(Y) \le \delta$. Then, for $k/(2k + 1) < d$,*

$$\lim_{\eta \to \infty} \frac{\#\{P \in P(m, \eta, d) \mid \Gamma'_P \text{ has } F(Y) \text{ for all } Y \in \mathcal{Y}_\delta\}}{\#P(m, \eta, d)} = 1$$

holds.

In the theorems above, we cannot say Γ is infinite in general. What is known in this direction is the following theorem for hyperbolic groups due to Ollivier:

Theorem 7.3 ([19], Theorem 4). *Let Γ be a hyperbolic group with harmless torsion generated by $S = \{s_1, \ldots, s_m, s_1^{-1}, \ldots, s_m^{-1}\}$ and θ the gross cogrowth of Γ. Suppose that $d < 1 - \theta$. Then we have*

$$\lim_{\eta \to \infty} \frac{\#\{P \in P(m, \eta, d) \mid \Gamma'_P \text{ is non-elementary hyperbolic}\}}{\#P(m, \eta, d)} = 1.$$

See [19] for the definitions of *gross cogrowth* and *harmless torsion*. We remark here that the gross cogrowth of the free group of rank *m* is equal to

$$\frac{1}{2} \log_{2m}(8m - 4).$$

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