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Variable Order Differential Equations with Piecewise Constant Order-Function and Diffusion with Changing Modes

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Abstract. In this paper diffusion processes with changing modes are studied involving the variable order partial differential equations. We prove the existence and uniqueness theorem of a solution of the Cauchy problem for fractional variable order (with respect to the time derivative) pseudo-differential equations. Depending on the parameters of variable order derivatives short or long range memories may appear when diffusion modes change. These memory effects are classified and studied in detail. Processes that have distinctive regimes of different types of diffusion depending on time are ubiquitous in the nature. Examples include diffusion in a heterogeneous media and protein movement in cell biology.

Keywords. Variable order differential equations, short memory, long memory, diffusion with changing modes, Cauchy problem, Mittag–Leffler function

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1. Introduction

Diffusion processes can be classified according to the asymptotic behaviour of their mean square displacement (MSD) as a function of time. If the dependence of the MSD on time is linear, then the process is classified as normal, otherwise as anomalous. For many processes, the MSD satisfies

$$\mathrm{MSD}(t) \sim K_\beta t^\beta , \quad t \to \infty, \tag{1}$$

where K_β is a constant. If $\beta = 1$ the diffusion is normal, if $\beta > 1$ the process is super-diffusive, while if $\beta < 1$ the process is sub-diffusive [19,36]. The ultra-slow diffusion processes studied in [5, 18, 20] lead to logarithmic behaviour of MSD

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for large t. The MSD of more complex processes with retardation (see [5, 12]) behaves like t^{β_2} for t small, and t^{β_1} for t large, where $\beta_1 < \beta_2$. Subdiffusive motion with $0.1 < \beta < 0.9$ was recorded in [10], and with $0.22 < \beta < 0.48$ in [28], depending on macromolecules and cells. In [22, 30] protein movement is studied in the cell membrane with a few types of compartments and made a conclusion that β depends on time scales. Our models are subdiffusive, but of variable order with *order function* $\beta = \beta(t)$.

It is well known that simple homogeneous subdiffusive processes can be modeled using a fractional order partial differential equation where only the time derivative has a constant fractional order [19]. Variable fractional order derivatives and operators were studied by N. Jacob et al. [15], S. G. Samko, et al. [26, 27], W. Hoh [14]. Recently A. V. Chechkin et al. [6] used a version of variable order derivatives to describe kinetic diffusion in heterogeneous media. In the recent paper [17], Lorenzo and Hartley introduced several types of fractional variable order derivatives and applied them to engineering problems. We will modify these operators, restrict them to order functions $\beta(t)$ that are piecewise constant and then apply the resulting variable order partial differential equations (VOPDE) to diffusion processes with changing diffusion modes. An important aspect of the modeling is that the VOPDEs provides a description of memory effects arising from a change of diffusion modes that are distinct from the "long range memory" connected with the non-Markovian character of diffusion. Thus, in the VOPDE based description of anomalous diffusion models, both non-Markovian long range memory and new type of memory may be present simultaneously.

The paper is organized as follows. In Section 2 we introduce background material. In Section 3 we study the memory effects arising in connection with a change of diffusion modes. In Sections 4 and 5 we study the mathematical model of diffusion processes with changing modes in terms of an initial value problem for VOPDE. Namely, we prove the theorem on the existence and uniqueness of a solution of the initial value problems for variable order differential equations and study some properties of a solution. The theorems are proved under the assumption that diffusion mode change times are known.

2. LH-parallelogram and variable order derivatives

Recently Lorenzo and Hartley [17] introduced three types of derivatives of variable fractional order $\beta(t)$, $t > 0$, $0 < \beta(t) \leq 1$, all of which are special cases of a more general fractional order derivative

$$\mathcal{D}_{\mu,\nu}^{\beta(t)} f(t) = \frac{d}{dt} \int_0^t \mathcal{K}_{\mu,\nu}^{\beta(t)}(t,\tau) f(\tau) d\tau, \tag{2}$$

where μ and ν are real parameters, $t > 0$, and

$$\mathcal{K}_{\mu,\nu}^{\beta(t)}(t,\tau) = \frac{1}{\Gamma(1-\beta(\mu t+\nu\tau))(t-\tau)^{\beta(\mu t+\nu\tau)}}, \quad 0 < \tau < t\,. \tag{3}$$

For convenience in studying of initial value problems, we prefer to use the closely related Caputo type operator

$$\mathcal{D}_{*\mu,\nu}^{\beta(t)} f(t) = \int_0^t \mathcal{K}_{\mu,\nu}^{\beta(t)}(t,\tau)\frac{df(\tau)}{d\tau}d\tau. \tag{4}$$

To describe the properties of the kernel (3) and the fractional derivative operators (2) and (4) we introduce the Lorenzo–Hartley (LH) *causality parallelogram* [17] $\Pi = \{(\mu,\nu) \in \mathbb{R}^2 : 0 \le \mu \le 1, -1 \le \nu \le +1, 0 \le \mu+\nu \le 1\}$. The kernel (3), and thus, both the operators (2) and (4) are weakly singular for $(\mu,\nu) \in \Pi$. Further, denote

$$\mathcal{K}(t,\tau,s) = \frac{1}{\Gamma(1-\beta(s))(t-\tau)^{\beta(s)}}, \quad t > 0,\ 0 < \tau < t,\ s \ge 0, \tag{5}$$

where $0 < \beta(s) \le 1$ is a given function[1], which is called an *order function.*

Our main goal is to model problems where for different time intervals there are different modes[2] of diffusion. To this end, let T_i be a partition of the interval $(0,\infty)$ into $N+1$ sub-intervals (T_k, T_{k+1}), where $0 = T_0 < T_1 < \cdots < T_N < T_{N+1} = \infty$. Then let $\beta(t)$ be the piecewise constant function

$$\beta(t) = \sum_{k=0}^{N} \beta_k \mathcal{I}_k(t)\,, \quad t \in (0,\infty),$$

where $\mathcal{I}_k$ is the indicator of the interval (T_k, T_{k+1}) and $0 < \beta_k \le 1$, $k = 0, \ldots, N$, are constants. Under these conditions, the function (5) becomes

$$K(t,\tau,s) = \sum_{k=0}^{N} \mathcal{I}_k(s)\frac{1}{\Gamma(1-\beta_k)(t-\tau)^{\beta_k}}, \quad t > 0,\ 0 < \tau < t,\ s \ge 0, \tag{6}$$

and the kernel of the fractional order operator (4) becomes

$$\mathcal{K}_{\mu,\nu}^{\beta(t)}(t,\tau) = K(t,\tau,\mu t+\nu\tau), \quad t > 0,\ 0 \le \tau < t. \tag{7}$$

with $K(t,\tau,s)$ defined in (6).

We think of the input to our model as the triplet (β_k, μ, ν), $0 \le k \le N$, while the output of our model is determined by the kernel (7). Correspondingly,

[1]If $\beta(t) = 1$, then we agree $\mathcal{D}_{*\mu,\nu}^{\beta(t)} f(t) = \frac{df(t)}{dt}$.
[2]See definition in Section 2.3.

we say that the triplet (β_k, μ, ν) determines the diffusion mode in the time interval (T_k, T_{k+1}). The output is determined by which values of $\beta(t)$ are used to compute the variable order derivative, that is, by which interval (T_k, T_{k+1}) the point $\mu t + \nu\tau$ belongs to. We always assume that $(\mu, \nu) \in \Pi$ and then note that $\tau \in (0, t)$ yields $\mu t + \nu\tau \in (\mu t, (\mu + \nu)t)$ and that $(\mu t, (\mu + \nu)t) \subset (0, t)$. This means that the operators $\mathcal{D}^{\beta(t)}_{\mu,\nu}$ and $\mathcal{D}^{\beta(t)}_{*\mu,\nu}$ use information taken in the time sub-interval $(\mu t, (\mu+\nu)t)$ if ν is positive and from the sub-interval $((\mu+\nu)t, \mu t)$ if ν is negative. In both cases, the length of this interval is $|\nu|\, t$. The condition $(\mu, \nu) \in \Pi$ predetermines the causality, since $0 \le \mu t + \nu\tau \le t$ for all $t > 0$ and $0 \le \tau \le t$.

2.1. Generalized function spaces $\Psi_{G,p}(\mathbb{R}^n)$, $\Psi'_{-G,q}(\mathbb{R}^n)$. Let $p > 1$, $q > 1$, $p^{-1} + q^{-1} = 1$, be two conjugate numbers. The generalized functions space $\Psi_{-G,q}(\mathbb{R}^n)$, which we are going to introduce is distinct from the classical spaces of generalized functions. In the particular case of $p = 2$ this space was first used by Yu. A. Dubinskiĭ [8] in the course of initial-value problems for pseudo-differential equations with analytic symbols. Later, the general case for all p was studied in [31, 32]. Here we briefly recall some basic facts related to these spaces, referring the interested reader to [11, 31] for details.

Let $G \subset \mathbb{R}^n$ be an open domain and a system $\mathcal{G} \equiv \{g_k\}_{k=0}^{\infty}$ of open sets be a locally finite covering of G, i.e., $G = \bigcup_{k=0}^{\infty} g_k$, $g_k \subset\subset G$. This means that any compact set $K \subset G$ has a nonempty intersection with a finite number of sets g_k. Denote by $\{\phi_k\}_{k=0}^{\infty}$ a smooth partition of unity for G. We set $G_N = \cup_{k=1}^{N} g_k$ and $\kappa_N(\xi) = \sum_{k=1}^{N} \phi_k(\xi)$. It is clear that $G_N \subset G_{N+1}$, $N = 1, 2, \ldots,$ and $G_N \to G$ for $N \to \infty$. Further, by Ff (or $\hat{f}(\xi)$) for a given function $f(x)$ we denote its Fourier transform, formally setting $Ff(\xi) = \int_{\mathbb{R}^n} f(x)e^{ix\xi}dx$, and by $F^{-1}\hat{f}$ the inverse Fourier transform, i.e., $F^{-1}\hat{f}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}(\xi)e^{-ix\xi}d\xi$. The support of a given f we denote by supp$f$.

Definition 2.1. Let $N \in \mathbb{N}$. Denote by $\Psi_{N,p}$ the set of functions $f \in L_p(\mathbb{R}^n)$ satisfying the conditions (1)–(3):

(1) supp $Ff \subset G_N$;
(2) supp $Ff \cap \text{supp}\, \phi_j = \emptyset$ for $j > N$;
(3) $p_N(f) = \|F^{-1}\kappa_N Ff\|_p < \infty$.

Lemma 2.2. *For $N = 1, 2, \ldots,$ the relations $\Psi_{N,p} \hookrightarrow \Psi_{N+1,p}$, $\Psi_{N,p} \hookrightarrow L_p(\mathbb{R}^n)$ hold, where $\hookrightarrow$ denotes the operation of continuous embedding.*

It follows from Lemma 2.2 that $\Psi_{N,p}$ form an increasing sequence of Banach spaces. Its limit with the inductive topology we denote by $\Psi_{G,p}$.

Definition 2.3. $\Psi_{G,p}(\mathbb{R}^n) = \text{ind}\lim_{N\to\infty} \Psi_{N,p}$.

The inductive limit topology of $\Psi_{G,p}(\mathbb{R}^n)$ is equivalent to the following convergence.

Definition 2.4. A sequence of functions $f_m \in \Psi_{G,p}(\mathbb{R}^n)$ is said to converge to an element $f_0 \in \Psi_{G,p}(\mathbb{R}^n)$ iff:

1. there exists a compact set $K \subset G$ such that $\operatorname{supp} \hat{f}_m \subset K$ for all $m \in \mathbb{N}$;
2. $\|f_m - f_0\|_p = (\int_{R^n} |f_m - f_0|^p dx)^{\frac{1}{p}} \to 0$ for $m \to \infty$.

Remark 2.5. According to the Paley–Wiener–Schwartz theorem, elements of $\Psi_{G,p}(\mathbb{R}^n)$ are entire functions of exponential type which, restricted to $\mathbb{R}^n$, are in the space $L_p(\mathbb{R}^n)$.

The space topologically dual to $\Psi_{G,p}(\mathbb{R}^n)$, which is the projective limit of the sequence of spaces conjugate to $\Psi_{N,p}$, is denoted by $\Psi'_{-G,q}(\mathbb{R}^n)$.

Definition 2.6. $\Psi'_{-G,q}(\mathbb{R}^n) = \text{pr} \lim_{N\to\infty} \Psi^*_{N,p}$.

In other words, $\Psi'_{-G,q}(\mathbb{R}^n)$ is the space of all linear bounded functionals defined on the space $\Psi_{G,p}(\mathbb{R}^n)$ endowed with the weak topology. Namely, a sequence of generalized functions $g_N \in \Psi'_{-G,q}(\mathbb{R}^n)$ converges to an element $g_0 \in \Psi'_{-G,q}(\mathbb{R}^n)$ in the weak sense if for all $f \in \Psi_{G,p}(\mathbb{R}^n)$ the sequence of numbers $\langle g_N, f\rangle$ converges to $\langle g_0, f\rangle$ as $N \to \infty$. We recall that the notation $\langle g, f\rangle$ means the value of $g \in \Psi'_{-G,q}(\mathbb{R}^n)$ on an element $f \in \Psi_{G,p}(\mathbb{R}^n)$.

2.2. Pseudo-differential operators with constant symbols. Now we recall some properties of pseudo-differential operators with symbols defined and continuous in a domain $G \subset \mathbb{R}^n$. Outside of G or on its boundary the symbol $A(\xi)$ may have singularities of arbitrary type. For a function $\varphi \in \Psi_{G,p}(\mathbb{R}^n)$ the operator $A(D)$ corresponding to $A(\xi)$ is defined by the formula

$$A(D)\varphi(x) = \frac{1}{(2\pi)^n} \int_G A(\xi) F\varphi(\xi) e^{ix\xi} d\xi. \tag{8}$$

Generally speaking, $A(D)$ does not make sense even for functions in the space $C_0^\infty(\mathbb{R}^n)$. In fact, let ξ_0 be a non-integrable singular point of $A(\xi)$ and denote by $O(\xi_0)$ some neighborhood of ξ_0. Let us take a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $F\varphi(\xi) > 0$ for $\xi \in O(\xi_0)$ and $F\varphi(\xi_0) = 1$. Then it is easy to verify that $A(D)\varphi(x) = \infty$. On the other hand, for $\varphi \in \Psi_{G,p}(\mathbb{R}^n)$ the integral in Eq. (8) is convergent due to the compactness of $\operatorname{supp} F\varphi \subset G$. We define the operator $A(-D)$ acting in the space $\Psi'_{-G,q}(\mathbb{R}^n)$ by the duality formula

$$\langle A(-D)f, \varphi\rangle = \langle f, A(D)\varphi\rangle, \quad f \in \Psi'_{-G,q}(\mathbb{R}^n),\ \varphi \in \Psi_{G,p}(\mathbb{R}^n).$$

Lemma 2.7. *The spaces $\Psi_{G,p}(\mathbb{R}^n)$ and $\Psi'_{-G,q}(\mathbb{R}^n)$ are invariant with respect to the action of an arbitrary pseudo-differential operator $A(D)$ whose symbol is continuous in G. Moreover, if $A(\xi)\kappa_N(\xi)$ is a multiplier on L_p for every $N \in \mathbb{N}$, then this operator acts continuously.*

Remark 2.8. In the case $p = 2$ an arbitrary pseudo-differential operator whose symbol is continuous in G acts continuously without the additional condition for $A(\xi)\kappa_N(\xi)$ to be multiplier in L_2 for every $N \in \mathbb{N}$.

2.3. Subdiffusion processes. As is known [11, 19], a (sub-)diffusion process is governed by the fractional order partial differential equation

$$D_*^\beta u(t,x) = \mathcal{A}(D)u(t,x), \quad t > 0, x \in \mathbb{R}^n, \tag{9}$$

where D_*^β is the Caputo fractional derivative of order $\beta \in (0,1]$, and $\mathcal{A}(D)$, $D = (D_1, \dots, D_n)$, $D_j = -i\frac{\partial}{\partial x_j}$, $j = 1, \dots, n$, some elliptic pseudo-differential operator.

Many diffusion processes driven by a Brownian motion can be described by equation (9) with a second order elliptic differential operator $\mathcal{A}(D)$ and $\beta = 1$. Lévy stochastic processes (which include jumps) also connected with (9) and an elliptic pseudo-differential operator $\mathcal{A}(D)$ (see, e.g., [2]). In particular, if particle jumps are given by a symmetric Lévy stable distribution with infinite mean square displacement, then $\mathcal{A}(D)$ is a hyper-singular integral, defined as the inverse to the Riesz–Feller fractional order ($0 < \alpha < 2$) operator (for details see [25]). A wide variety of non-Gaussian stochastic (subdiffusive) processes lead to equation (9) with $0 < \beta < 1$ (see [19, 20]). For diffusion governed by distributed order differential equations see [1, 34]. The parameter β determines the sub-diffusive mode, which is slower than the classical free diffusion.

Generalizing this approach we will say that the $\{\beta_k, \mu, \nu\}$-diffusion mode in the time interval (T_{k-1}, T_k) is governed by the equation

$$\mathcal{D}_{*\{\mu,\nu\}}^{\beta_k} u(t,x) = \mathcal{A}(D)u(t,x), \quad t \in (T_{k-1}, T_k),\ x \in \mathbb{R}^n.$$

The entire process then can be described by the equation

$$\mathcal{D}_{*\{\mu,\nu\}}^{\beta(t)} u(t,x) = \mathcal{A}(D)u(t,x), \quad t > 0,\ x \in \mathbb{R}^n, \tag{10}$$

where $\mathcal{D}_{*\{\mu,\nu\}}^{\beta(t)}$ is the variable fractional order operator with the kernel $K_{\{\mu,\nu\}}^{\beta(t)}(t,\tau)$ in (7).

In Section 4 we will prove the existence of solutions to the initial value problem defined by the differential equation (10).

3. Changing of modes: 'short-range' and 'long-range' memories

We call the triplet $\{\beta_k, \mu, \nu\}$ *admissible* if $0 < \beta_k \leq 1$ and $(\mu, \nu) \in \Pi$. Diffusion in complex heterogeneous media is accompanied by frequent changes of diffusion modes. It is known that a particle undergoing non-Markovian movement possesses a memory of past (see [19,36]). Protein diffusion in cell membrane, as is recorded in [23, 24] is anomalous diffusion. Descriptions of this process using random walks also shows the presence of non-Markovian type memory [1,13,16]. It turns out, there is another type of memory noticed first by Lorenzo and Hartley in their paper [17] in some particular cases of μ and ν. This kind of memory arises when the diffusion mode changes.

In this section we study a special case of this phenomenon where there is a single change of diffusion mode, that is, a sub-diffusion mode given by an admissible triplet $\{\beta_1, \mu, \nu\}$ changes to a sub-diffusion mode corresponding to another admissible triplet $\{\beta_2, \mu, \nu\}$ at some particular time T.

Definition 3.1. Let $\{\beta_1, \mu, \nu\}$ and $\{\beta_2, \mu, \nu\}$ be two admissible triplets. Assume the diffusion mode is changed at time $t = T$ from $\{\beta_1, \mu, \nu\}$-mode to $\{\beta_2, \mu, \nu\}$-mode. Then the process is said to have a 'short-range' (or short) memory, if there is a finite $T^* > T$ such that for all $t > T^*$ the $\{\beta_2, \mu, \nu\}$-mode holds. Otherwise, the process is said to have a 'long-range' (or long) memory.

Remark 3.2. According to Definition 3.1, a diffusion mode has a long memory if the influence of the old diffusion mode never vanishes, even though the diffusion mode is changed, i.e., the particle does not forget its past. In the case of short memory, the particle remembers the old mode for some critical time, and then forgets it fully, recognizing the new mode.

Theorem 3.3. *Let $\nu > 0$ and $\mu \neq 0$. Assume the $\{\beta_1, \mu, \nu\}$-diffusion mode is changed at time $t = T$ to the $\{\beta_2, \mu, \nu\}$-diffusion mode. Let $T^* = \frac{T}{\mu}$ and $t^* = \frac{T}{\mu+\nu}$. Then the process has a short memory. Moreover,*

- (i) *$\{\beta_1, \mu, \nu\}$-diffusion mode holds for all $0 < t < t^*$;*
- (ii) *$\{\beta_2, \mu, \nu\}$-diffusion mode holds for all $t > T^*$;*
- (iii) *a mix of both $\{\beta_1, \mu, \nu\}$ and $\{\beta_2, \mu, \nu\}$-diffusion modes holds for all $t^* < t < T^*$.*

Proof. Let $\beta(s) = \beta_1$ for $0 < s < T$ and $\beta(s) = \beta_2$ for $s > T$. Assume $\nu > 0$. Denote $s = \mu t + \nu\tau$. So, the $\{\beta_1, \mu, \nu\}$-diffusion mode holds if $\mu t + \nu\tau < T$. Let $0 < t < t^* = \frac{T}{\mu+\nu}$. Then for every $\tau \in (0, t)$ we have $\mu t + \nu\tau < (\mu + \nu)t < T$. This means that the order operator $\beta(s)$ in $\mathcal{D}^{\beta(t)}_{*\{\mu,\nu\}}$ takes the value β_1 giving (i). If $t > \frac{T}{\mu}$ then for all $\tau > 0$, $\mu t + \nu\tau > T$. Hence, $\beta(s) = \beta_2$, obtaining (ii).

Now assume $\frac{T}{\mu+\nu} < t < \frac{T}{\mu}$. Denote $\tau_0 = \frac{T-\mu t}{\nu}$. Obviously $\tau_0 > 0$. It follows from $(\mu+\nu)t > T$ dividing by ν that $t > \frac{T}{\nu} - \frac{t\mu}{\nu} = \tau_0$, i.e., $0 < \tau_0 < t$. It is easy to check that if $0 < \tau < \tau_0$ then $\mu t + \nu\tau \in (\mu t, T) \subset (0, T)$, giving $\beta(s) = \beta_1$, while if $\tau_0 < \tau < t$ then $\mu t + \nu\tau \in (T, (\mu+\nu)t) \subset (T, \infty)$, giving $\beta(s) = \beta_2$. Hence, in this case the mix of both $\{\beta_1, \mu, \nu\}$ and $\{\beta_2, \mu, \nu\}$-diffusion modes is present. □

Theorem 3.4. *Let $\nu < 0$ and $\mu + \nu \neq 0$. Assume the $\{\beta_1, \mu, \nu\}$-diffusion mode is changed at time $t = T$ to the $\{\beta_2, \mu, \nu\}$-diffusion mode. Let $t^{*'} = \frac{T}{\mu}$ and $T^{*'} = \frac{T}{\mu+\nu}$. Then the process has a short memory. Moreover,*

(i') *$\{\beta_1, \mu, \nu\}$-diffusion mode for all $0 < t < t^{*'}$;*
(ii') *$\{\beta_2, \mu, \nu\}$-diffusion mode holds for all $t > T^{*'}$;*
(iii') *a mix of both $\{\beta_1, \mu, \nu\}$ and $\{\beta_2, \mu, \nu\}$-diffusion modes holds for all $t^{*'} < t < T^{*'}$.*

Proof. Let $\nu < 0$. Assume again $\beta(s) = \beta_1$ for $0 < s < T$ and $\beta(s) = \beta_2$ for $s > T$. As in the previous theorem, denote $s = \mu t + \nu\tau$. First we notice that if $0 < t < \frac{T}{\mu}$ then $\mu t + \nu\tau < T$, which implies $\beta(s) = \beta_1$, giving (i'). Now let $t > \frac{T}{\mu+\nu}$ be any number. Then for $0 < \tau < t$ we have $\mu t + \nu\tau > T$, which yields $\beta(s) = \beta_2$. So, we get (ii'). Now assume $\frac{T}{\mu} < t < \frac{T}{\mu+\nu}$. Again denote $\tau_0 = \frac{T-\mu t}{\nu}$. Obviously $\tau_0 > 0$. It follows from $(\mu+\nu)t < T$ dividing by $\nu < 0$ that $t > \frac{T}{\nu} - \frac{t\mu}{\nu} = \tau_0$, i.e., $0 < \tau_0 < t$. It is easy to check that if $0 < \tau < \tau_0$ then $\mu t + \nu\tau \in (T, \mu t) \subset (T, \infty)$, giving $\beta(s) = \beta_2$, while if $\tau_0 < \tau < t$ then $\mu t + \nu\tau \in ((\mu+\nu)t, T) \subset (0, T)$, giving $\beta(s) = \beta_1$. Hence, in this case the mix of both $\{\beta_1, \mu, \nu\}$ and $\{\beta_2, \mu, \nu\}$-diffusion modes is present, obtaining (iii'). □

Corollary 3.5. *Let $\nu = 0$ and $\mu \neq 0$. Assume the $\{\beta_1, \mu, \nu\}$-diffusion mode is changed at time $t = T$ to the $\{\beta_2, \mu, \nu\}$-diffusion mode. Let $T^* = \frac{T}{\mu}$. Then the process has a short memory. Moreover,*

(a) *for all $0 < t < T^*$ there holds $\{\beta_1, \mu, \nu\}$-diffusion mode;*
(b) *for all $t > T^*$ there holds $\{\beta_2, \mu, \nu\}$-diffusion mode.*

Proof. If $\nu = 0$ then we have $\beta(s) = \beta(\mu t) = \beta_1$ for $t < \frac{T}{\mu}$ and $\beta(s) = \beta_2$ for $t > \frac{T}{\mu}$. □

Corollary 3.6. *Let $\mu = 0$ or $\mu + \nu = 0$. Assume the $\{\beta_1, \mu, \nu\}$-diffusion mode is changed at time $t = T$ to the $\{\beta_2, \mu, \nu\}$-diffusion mode. Then the process has the long memory.*

Proof. According to the structure of LH-parallelogram $\mu = 0$ implies $\nu > 0$. In this case $T^* = \infty$. If $\mu + \nu = 0$ then $\nu < 0$ and $t^* = \infty$. In both cases we a have long memory effect. □

Remark 3.7. Notice, that if $\nu = 0$, then there is no intervals of mix of modes. Moreover, if $\nu = 0$, $\mu = 1$, then $T^* = t^* = T$. In this sense we say that a process has no memory. For all points $\{(\mu, \nu)\}$ except $\{\mu = 0, 0 \leq \nu \leq 1\}$ and $\{\nu < 0, \mu + \nu = 0\}$, the operator $D^{\beta(t)}_{*\{\mu,\nu\}}$ has a short memory. The memory is stronger in the region $\nu < 0$ and weaker in $\nu > 0$. On the line $\mu + \nu = 1$ we have $t^* = T < T^*$. The lines $\mu = 0$, $\nu \geq 0$ and $\mu + \nu = 0$ identify the long range memory.

4. The Cauchy problem for variable order differential equations

In this section we study the Cauchy problem for variable order differential equations with a piecewise constant order function $\beta(t) = \sum_{k=0}^{N} \mathcal{I}_k \beta_k$, where $\mathcal{I}_k$ is the indicator function of $[T_k, T_{k+1})$. We assume that the diffusion mode change times $T_1, T_2, \ldots, T_N$ are known, and set $T_0 = 0$, $T_{N+1} = \infty$. We assume that the solution of the initial value problem for the VOPDE (10) is continuous[3] when the diffusion mode changes.

Thus, the Cauchy problem is formulated in the form

$$\mathcal{D}^{\beta(t)}_{*\{\mu,\nu\}} u(t,x) = \mathcal{A}(D) u(t,x), \quad t > 0,\, x \in \mathbb{R}^n \tag{11}$$

$$u(0,x) = \varphi(x) \tag{12}$$

$$u(T_k^* - 0, x) = u(T_k^* + 0, x), \quad k = 1, \ldots, N,\, x \in \mathbb{R}^n, \tag{13}$$

where $\mathcal{A}(D)$ is a pseudo-differential operator with a continuous symbol $A(\xi)$, $\xi \in \mathbb{R}^n$, and T_k^* are actual mode change times. It follows from Theorems 3.3 and 3.4 that T_k^* are defined through $\frac{T_k}{\mu+\nu}$ and $\frac{T_k}{\mu}$.

We note that, since the integration operator order β depends on the variable t, a variable order analog of the integration operator becomes

$$J^{\beta(t)}_{\{\mu,\nu\}} f(t) = \int_0^t \frac{(t-\tau)^{\beta(\mu t + \nu\tau) - 1} f(\tau)}{\Gamma(\beta(\mu t + \nu \tau))} d\tau,$$

which we call *a variable order integration operator*.

For further purpose we recall the definition of the Mittag–Leffler function [7, 21] in the power series form

$$E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad z \in C^1.$$

Obviously, $E_\beta(z) = e^z$, if $\beta = 1$. For all $\beta > 0$ $E_\beta(z)$ is an entire function of type 1 and order $\frac{1}{\beta}$. Note that $E_\beta(-t)$, $t > 0$, is completely monotone [21], and has asymptotics $E_\beta(-t) = O(t^{-1})$, $t \to \infty$.

[3] In the topology of $\Psi_{G,p}(\mathbb{R}^n)$ (or $\Psi'_{-G,p}$).

Lemma 4.1. *Assume* $0 < \beta_* = \min_{0\le j\le N}\{\beta_j\}$ *and* $[k\beta_*]$ *is the integer part of* $k\beta_*$. *Let* $v(t)$ *be a function continuous in* $[0,\infty)$. *Then for arbitrary* $T > 0$ *and every* $k = 1, 2, \ldots$ *the estimate*

$$\max_{0\le t\le T} |J^{\beta(t)k} v(t)| \le \frac{[\psi(T)]^k}{[k\beta_* + 1]!} \max_{0\le t\le T} |v(t)| \tag{14}$$

holds with $\psi(\tau) = \begin{cases} \tau^{\beta_*}, & 0 < \tau < 1 \\ \tau, & \tau \ge 1. \end{cases}$

Proof. Let $v(t)$ be a function continuous in $[0,\infty)$. For k large enough, so that $\beta_* k \ge 2$ we have $\min \Gamma(k\beta(\mu t + \nu\tau)) = \Gamma(k\beta_*)$. Taking this into account, for all such k and for all $t \in (0, T]$ we obtain the estimate

$$|J^{\beta(t)k} v(t)| = \left| \int_0^t \frac{(t-\tau)^{k\beta(\mu t+\nu\tau)-1} v(\tau)d\tau}{\Gamma(k\beta(\mu t + \nu\tau))} \right| \le \frac{[\psi(T)]^k}{\Gamma(k\beta_* + 1)} \max_{0\le t\le T} |v(t)|,$$

and hence, the estimate in Eq. (14). □

Let $t_{cr,j} = \frac{T_j}{\mu+\nu}$, $j = 1, \ldots, N$, be critical points corresponding to the mode change times $T_j, j = 1, \ldots, N$. We accept the conventions $t_{cr,0} = 0, t_{cr,N+1} = \infty$. Let $E_\beta(z)$ be the Mittag–Leffler function with parameter $\beta \in (0, 1]$. Now we introduce the symbols which play an important role in the representation of a solution. Let

$$S_j(t,\xi) = E_{\beta_j}((t - t_{cr,j})^{\beta_j} A(\xi)),\ t \ge t_{cr,j},\ j = 0, \ldots, N, \tag{15}$$

and

$$M_k(t,\xi) = S_k(t - t_{cr,k}, \xi) \prod_{j=0}^{k-1} S_j(t_{cr,j+1} - t_{cr,j}, \xi), \quad t \ge t_{cr,k},\ k = 1, \ldots, N. \tag{16}$$

Further, we define recurrently the symbols

$$\mathcal{R}_1(t,\xi) = -\frac{1}{\Gamma(1-\beta_1)} \int_0^{t_{cr,1}} \frac{\frac{\partial}{\partial\tau} S_0(\tau,\xi)}{(t-\tau)^{\beta_1}} d\tau = \frac{-\beta_0 A(\xi)}{\Gamma(1-\beta_1)} \int_0^{t_{cr,1}} \frac{E'_{\beta_0}(\tau^{\beta_0} A(\xi))d\tau}{\tau^{1-\beta_0}(t-\tau)^{\beta_1}},$$

$P_{-1}(t,\xi) \equiv 0$, $P_0(t,\xi) \equiv 1$, $P_1(t,\xi) = \int_{t_{cr_1}}^t S_j(t + t_{cr,1} - \tau, \xi)\, {}_{t_{cr,1}}D_\tau^{1-\beta_1} \mathcal{R}_1(\tau,\xi)d\tau$, and if $P_j(t,\xi) = \int_{t_{cr_j}}^t S_j(t + t_{cr,j} - \tau, \xi)\, {}_{t_{cr,j}}D_\tau^{1-\beta_j} \mathcal{R}_j(\tau,\xi)d\tau$ is defined for $t \ge t_{cr,j}$ and for all $j \le k-1$, then for $t \ge t_{cr,k}$,

$$\mathcal{R}_k(t,\xi) = -\frac{1}{\Gamma(1-\beta_k)} \sum_{j=0}^{k-1} \int_{t_{cr,j}}^{t_{cr,j+1}} \frac{\frac{\partial}{\partial\tau}[M_j(\tau,\xi) + S_j(\tau - t_{cr,j}, \xi) P_{j-1}(t_{cr,j},\xi) + P_j(\tau,\xi)]}{(t-\tau)^{\beta_k}} d\tau, \tag{17}$$

for $k = 2, \ldots, N$.

4.1. The case $\nu = 0$.

Theorem 4.2. *Assume $\nu = 0$ and $\varphi \in \Psi_{G,p}(\mathbb{R}^n)$. Then the Cauchy problem* (11)–(13) *has a unique solution $u(t,x) \in C([0,T], \Psi_{G,p}(\mathbb{R}^n))$, $T < \infty$, which is represented in the form $u(t,x) = \mathcal{S}(t,D)\varphi(x)$, where $\mathcal{S}(t,D)$ is the pseudo-differential operator with the symbol*

$$\mathcal{S}(t,\xi) = \mathcal{I}_0' S_0(t,\xi) + \sum_{k=1}^{N} \mathcal{I}_k'(t) \Bigg\{ M_k(t,\xi) + S_k(t,\xi) \int_{t_{cr,k-1}}^{t_{cr,k}} S_{k-1}(t_{cr,k} + t_{cr,k-1} - \tau, \xi)\ {}_{t_{cr,k-1}}D_\tau^{1-\beta_k}\, \mathcal{R}_k(\tau,\xi) d\tau + \int_{t_{cr,k}}^{t} S_k(t + t_{cr,k} - \tau, \xi)\ {}_{t_{cr,k}}D_\tau^{1-\beta_k}\, \mathcal{R}_k(\tau,\xi)\, d\tau \Bigg\}, \tag{18}$$

where $\mathcal{I}_k' = \mathcal{I}_{[t_{cr,k}, t_{cr,k+1})}(t)$, $k = 0,\ldots,N$, are indicator functions of the intervals $[t_{cr,k}, t_{cr,k+1})$, $k = 0,\ldots,N$; $S_j(t,\xi)$, $j = 0,\ldots,N$, $M_k(t,\xi)$ and $\mathcal{R}_k(t,\xi)$, $k = 1,\ldots,N$, are defined in (15), (16) *and* (17)*, respectively.*

Proof. It is not hard to verify that

$$\begin{aligned} J_{\{\mu,0\}}^{\beta(t)} \mathcal{D}_{*\{\mu,0\}}^{\beta(t)} u(t,x) &= \sum_{k=0}^{N} \mathcal{I}_k' J^{\beta_k} D_*^{\beta_k} u(t,x) \\ &= u(t,x) - \sum_{k=0}^{N} \mathcal{I}_k' u(t_{cr,k}, x) + g(t,x), \end{aligned} \tag{19}$$

where $g(t,x) = \sum_{k=1}^{N} \mathcal{I}_k' \frac{t^{\beta_k}}{\Gamma(1+\beta_k)} \sum_{j=0}^{k-1} {}_{t_{cr,j}}\mathcal{D}_{*\{\mu,0\}}^{\beta_k} u(t_{cr,j+1}, x)$. Multiplying both sides of equation (11) by $J_{\{\mu,0\}}^{\beta(t)}$ and applying the formula (19), we obtain

$$u(t,x) - \sum_{k=0}^{N} \mathcal{I}_k' J^{\beta_k} \mathcal{A}(D) u(t,x) = \sum_{k=0}^{N} \mathcal{I}_k' u(t_{cr,k}, x) - g(t,x). \tag{20}$$

Let $t \in (0, t_{cr,1})$. Then $\beta(\mu t) = \beta_0$ and $g(t,x) \equiv 0$. In this case taking into account the initial condition (12), we can rewrite equation (20) in the form

$$u(t,x) - J^{\beta_0} \mathcal{A}(D) u(t,x) = \varphi(x), \quad 0 < t < t_{cr,1}.$$

The obtained equation can be solved by using the iteration method. Determine the sequence of functions $\{u_0(t,x), \ldots, u_m(t,x)\}$ in the following way. Let $u_0(t,x) = \varphi(x)$ and by iteration

$$u_m(t,x) = J^{\beta_0} \mathcal{A}(D) u_{m-1}(t,x) + \varphi(x), \quad m = 1, 2, \ldots. \tag{21}$$

We show that this sequence is convergent in the topology of $C[0,T;\Psi(\mathbb{R}^n)]$ and its limit is a solution to the Cauchy problem (11),(12). Moreover, this solution can be represented in the form of functional series

$$u(t,x)=\sum_{k=0}^{\infty} J^{\beta_0 k}\mathcal{A}^k(D)\varphi(x). \tag{22}$$

Indeed, it follows from the iteration process (21) that

$$u_m(t,x)=J^{\beta(t)m}\mathcal{A}^m(D)\varphi(x)+J^{\beta(t)(m-1)}\mathcal{A}^{m-1}(D)\varphi(x)+\cdots+\varphi(x). \tag{23}$$

Now we estimate $u_m(t,x)$ applying Lemma 14 term by term in the right hand side of (23). Indeed, let $N\in\mathbb{N}$. Then taking into account the fact that the Fourier transform in x commutes with $J^{\beta(t)}$, we have

$$\max_{[0,T]} p_N\big(J^{\beta_0 k}\mathcal{A}^k(D)\varphi(x)\big)\le \frac{\psi^{k-1}(T)}{[k\beta_0]!}\,p_N(\mathcal{A}\varphi(x)).$$

Further, since $A(\xi)$ is continuous on G there exists a constant $C_N>0$ such that $\max_{\xi\in supp\,\kappa_N}|A(\xi)|\le C_N$, or, by induction $\max_{\xi\in supp\,\kappa_N}|A^k(\xi)|\le C_N^k$. Hence, for every $N\in\mathbb{N}$, we have

$$p_N\big(J^{\beta(t)k}\mathcal{A}^k(D)\varphi(x)\big)\le\|\varphi\|_p\frac{C_N^{k-1}\,\psi^{k-1}(T)}{[k\beta_0]!}. \tag{24}$$

It follows from (24) that, for $N=1,2,\dots,$

$$\max_{[0,T]} p_N(u_m(t,x))\le\|\varphi(x)\|_p\sum_{k=0}^{m}\frac{C_N^k\psi^k(T)}{\Gamma(\beta_0 k+1)}\le C\|\varphi(x)\|_p E_{\beta_0}(C_N\,\psi(T)),$$

where $E_{\beta_0}(\tau)$ is the Mittag–Leffler function corresponding to β_0. As far as the right hand side of the latter does not depend on m, we conclude that $u_m(t,x)$ defined in (23) is convergent. Again making use of estimate (14) in Lemma 4.1 we have

$$p_N(u(t,x)-u_m(t,x))\le\|\varphi(x)\|_p\sum_{k=m+1}^{\infty}\frac{C_N\psi^k(T)}{\Gamma(\beta_0 k+1)},\quad N=1,2,\dots. \tag{25}$$

The function $\mathcal{R}_m(\eta)=\sum_{k=m+1}^{\infty}\frac{\eta^k}{\Gamma(\beta_0 k+1)}$ on the right side of equation (25) is the residue in the power series representation of the Mittag–Leffler function $E_{\beta_0}(\eta)$, and, hence, $\mathcal{R}_m(\eta)\to 0$, when $m\to\infty$ for any real (or even complex) η. Consequently, $u_m(t,x)\to u(t,x)$ for every $N=1,2,\dots$, that is in the inductive topology of the space $C([0,\infty),\Psi_{G,p})$. Thus, $u(t,x)\in C([0,\infty),\Psi_{G,p})$ is a solution. Moreover, it is readily seen that $u(t,x)$ in (22) can be represented through the pseudo-differential operator $S(t,D)$ with the symbol $S_0(t,\xi) = E_{\beta_0}(t^{\beta_0}\mathcal{A}(\xi))$ in the form $u(t,x) = u_0(t,x) = S_0(t,D)\varphi(x)$, $t \in (0,t_{cr,1})$. By construction $u(t,x)$ is unique and continuous in t. So, $\lim_{t\to t_{cr,1}-0} u(t,x) = E_{\beta_0}(t_{cr,1}^{\beta_0}\mathcal{A}(D))\varphi(x)$ exists in $\Psi_{G,p}(\mathbb{R}^n)$. Further we extend $u(t,x)$ to $[t_{cr,1},t_{cr,2})$. Equation (11) in the interval $(t_{cr,1},t_{cr,2})$ reads $D_*^{\beta_1}u(t,x) = \mathcal{A}(D)u(t,x), \ \in (t_{cr,1},t_{cr,2})$. Splitting the integration interval $(0,t)$ on the left hand side of the last equation into subintervals $(0,t_{cr,1})$ and $(t_{cr,1},t)$, we can rewrite it in the form

$$ {}_{t_{cr,1}}D_*^{\beta_1}u(t,x) = \mathcal{A}(D)u(t,x) + F_1(t,x), \quad t \in (t_{cr,1},t_{cr,2}), $$

where $F_1(t,x) = -\frac{1}{\Gamma(1-\beta_1)}\int_0^{t_{cr,1}}\frac{\frac{\partial}{\partial\tau}u_0(\tau,x)}{(t-\tau)^{\beta_1}}d\tau$. Taking into account the fact $u_0(t,x) = S_0(t,D)\varphi(x)$, it is not hard to see that $F_1(t,x) = \mathcal{R}_1(t,D)\varphi(x)$, where

$$ \mathcal{R}_1(t,\xi) = \frac{-\beta_0 A(\xi)}{\Gamma(1-\beta_1)}\int_0^{t_{cr,1}}\frac{E'_{\beta_0}(\tau^{\beta_0}A(\xi))}{\tau^{1-\beta_0}(t-\tau)^{\beta_1}}d\tau. $$

Due to the continuity condition (13), we have also

$$ u(t_{cr,1},x) = u_0(t_{cr,1},x) = E_{\beta_0}\big(t_{cr,1}^{\beta_0}\mathcal{A}(D)\big)\varphi(x) = S_0(t_{cr,1},D)\varphi(x). $$

In the general case, assuming that solution are found in the intervals $[0,t_{cr,1}),\dots,[t_{cr,k-1},t_{cr,k})$, we have the following inhomogeneous Cauchy problem for the interval $(t_{cr,k},t_{cr,k+1})$:

$$ {}_{t_{cr,k}}D_*^{\beta_k}u(t,x) = \mathcal{A}(D)u(t,x) + F_k(t,x), \quad t \in (t_{cr,k},t_{cr,k+1}), \tag{26} $$

$$ u(t_{cr,k},x) = u_{k-1}(t_{cr,k},x), \tag{27} $$

where $F_k(t,x) = -\frac{1}{\Gamma(1-\beta_k)}\sum_{j=0}^{k-1}\int_{t_{cr,j}}^{t_{cr,j+1}}\frac{\frac{\partial}{\partial\tau}u_j(\tau,x)}{(t-\tau)^{\beta_k}}d\tau$. It is not hard to verify that $F_k(t,x)$ can be represented in the form $\mathcal{R}_k(t,D)\varphi(x)$ with a pseudo-differential operator $\mathcal{R}_k(t,D)$ whose symbol is given in (17). A unique solution to (26),(27) can be found applying the fractional Duhamel principle (see [35]):

$$ u_k(t,x) = S_k(t,D)u_{k-1}(t_{cr,k},x) + \int_{t_{cr,k}}^{t} S_k(t-(\tau-t_{cr,k}),D)\ {}_{t_{cr,k}}D_\tau^{1-\beta_k}F_k(\tau,x)d\tau, $$

for $k = 1,\dots,N$, $t_{cr,k} < t < t_{cr,k+1}$. Now taking into account that

$$ \begin{aligned} u_{k-1}(t_{cr,k},x) = &\left[\prod_{j=0}^{k-1} S_j(t_{cr,j+1}-t_{cr,j},D)\right]\varphi(x) \\ &+ \int_{t_{cr,k-1}}^{t_{cr,k}} S_{k-1}(t_{cr,k}-(\tau-t_{cr,k-1}),D)_{t_{cr,k-1}}D_\tau^{1-\beta_{k-1}}F_{k-1}(\tau,x)\,d\tau \end{aligned} $$

we arrive at (18). □

Remark 4.3. Assume in equation (11) $\beta(t) = \beta$, where β is a constant in $(0,1]$. Then the representation formula (18) is reduced to $u(t,x) = E_\beta(t^\beta\mathcal{A}(\mathcal{D}))\varphi(x)$, which coincides with the result of [11].

Applying the technique used in the paper [11] and the duality of the spaces $\Psi_{G,p}(\mathbb{R}^n)$ and $\Psi'_{-G,q}(\mathbb{R}^n)$ we can prove the following theorem.

Theorem 4.4. *Assume $\nu = 0$ and $\varphi \in \Psi'_{-G,q}(\mathbb{R}^n)$. Then the Cauchy problem* (11)-(13) (*with '-D' instead of 'D'*) *has a unique weak solution $u(t,x) \in C([0,T], \Psi'_{-G,q}(\mathbb{R}^n))$, $T < \infty$, which is represented in the form*

$$u(t,x) = \mathcal{I}'_0 S_0(t,-D)\varphi(x) + \sum_{k=1}^{N} \mathcal{I}'_k(t) \Big\{ M_k(t,-D)\varphi(x) + S_k(t,-D)$$
$$\times \int_{t_{cr,k-1}}^{t_{cr,k}} S_{k-1}(t_{cr,k}+t_{cr,k-1}-\tau,-D)\ {}_{t_{cr,k-1}}D_\tau^{1-\beta_k} \mathcal{R}_k(\tau,-D)\varphi(x)\, d\tau$$
$$+ \int_{t_{cr,k}}^{t} S_k(t+t_{cr,k}-\tau,-D)\ {}_{t_{cr,k}}D_\tau^{1-\beta_k}\, \mathcal{R}_k(\tau,-D)\varphi(x)\, d\tau \Big\},$$

Corollary 4.5. *If $\nu = 0$ then the fundamental solution of equation* (11) *with the continuity conditions in* (13) *is represented in the form*

$$U(t,x) = \mathcal{I}'_0(t) \frac{1}{(2\pi)^n} \int_{R^n} E_{\beta_0}(t^{\beta_0}\mathcal{A}(-\xi)) d\xi + \sum_{k=1}^{N} \mathcal{I}'_k(t) \frac{1}{(2\pi)^n}$$
$$\times \int_{R^n} \Big\{ E_{\beta_k}\big((t-t_k)^{\beta_k}\mathcal{A}(-\xi)\big) \prod_{j=0}^{k-1} E_{\beta_j}\big((t_{j+1}-t_j)^{\beta_j}\mathcal{A}(-\xi)\big)$$
$$+ E_{\beta_k}\big((t-t_k)^{\beta_k} A(-\xi)\big)$$
$$\times \int_{t_{k-1}}^{t_k} E_{\beta_{k-1}}\big((t_k-\tau)^{\beta_{k-1}} A(-\xi)\big)\ {}_{t_{k-1}}D_\tau^{1-\beta_{k-1}}\, \mathcal{R}_{k-1}(\tau,-\xi) d\tau$$
$$+ \int_{t_k}^{t} E_{\beta_k}\big((t-\tau)^{\beta_k} A(-\xi)\big)\ {}_{t_k}D_\tau^{1-\beta_k}\, \mathcal{R}_k(\tau,-\xi) d\tau \Big\} e^{ix\xi}\, d\xi,$$

where $t_j = t_{cr,j}$. Moreover, $U(t,x) \in \Psi'_{-G,q}(\mathbb{R}^n)$ for every fixed $t > 0$.

4.2. The case $-1 < \nu \le 1$. The solution $u(t,x) = \mathcal{S}(t,D)\varphi(x)$ obtained in Theorem 4.2 in the case $\nu = 0$ has the structure $u(t,x) = \Psi_1(t,D)\varphi(x) + \Psi_2(t,D)\varphi(x)$, where $\Psi_1(t,D)$ and $\Psi_2(t,D)$ are operators with symbols

$$\Psi_1(t,\xi) = \mathcal{I}'_0 S_0(t,\xi) + \sum_{k=1}^{N} \mathcal{I}'_k(t)\, M_k(t,\xi),$$

and

$$\Psi_2(t,\xi)$$
$$= \sum_{k=1}^{N} \mathcal{I}'_k(t) \Big\{ S_k(t,\xi) \int_{t_{cr,k-1}}^{t_{cr,k}} S_{k-1}(t_{cr,k}+t_{cr,k-1}-\tau,\xi)\ {}_{t_{cr,k-1}}D_\tau^{1-\beta_k}\, \mathcal{R}_k(\tau,\xi)\, d\tau$$
$$+ \int_{t_{cr,k}}^{t} S_k(t+t_{cr,k}-\tau,\xi)\ {}_{t_{cr,k}}D_\tau^{1-\beta_k}\, \mathcal{R}_k(\tau,\xi)\, d\tau \Big\}.$$

The term $v(t,x) = \Psi(t,D)\varphi(x)$ reflects an effect of diffusion modes, while the term $w(t,x) = \Psi_2(t,D)\varphi(x)$ is connected with a memory of the past. We note that this structure remains correct in the general case $\nu \in (0,1]$ also, however, the symbols of solution operators are further restructured depending on the intervals of mixture of (two or more) modes. The theorems formulated below concern some intervals free of mixed modes.

Theorem 4.6. *Assume $\mu \neq 0$, $\mu + \nu \neq 0$ and $\varphi \in \Psi_{G,p}(\mathbb{R}^n)$. Then there exists a number $T^* > 0$ and pseudo-differential operators $\mathcal{P}^*(D)$ and $\mathcal{R}^*(t,D)$ with continuous symbols, such that for $t > T^*$ the solution of the Cauchy problem* (11)–(13) *coincides with the solution of the Cauchy problem*

$$ {}_{T^*}D_*^{\beta_N} u(t,x) = \mathcal{A}(D)u(t,x) + f^*(t,x), \quad t > T^*,\ x \in \mathbb{R}^n \tag{28} $$

$$ u(T^*,x) = \varphi^*(x), \quad x \in \mathbb{R}^n. \tag{29} $$

where $f^*(t,x) = \mathcal{R}^*(t,D)\varphi(x)$ and $\varphi^*(x) = \mathcal{P}^*(D)\varphi(x)$.

Proof. Assume $\nu > 0$. Then as it follows from Theorem 3.3 that the actual mode changes occur at times $T_j^* = \frac{T_j}{\mu}$ and $t_j^* = \frac{T_j}{\mu+\nu}$, $j = 1,\dots,N$, if diffusion modes change at times $T_j, j = 1,\dots,N$. Obviously, $t_1^* < \dots < t_N^*$ and $T_1^* < \dots < T_N^*$ if $T_1 < \dots < T_N$. The order function $\beta(\mu t + \nu\tau)$ under the integral in $D_{*\{\mu,\nu\}}^{\beta(t)}$ takes the value β_N for all $t > T_N^*$ and $\tau > 0$. Hence, the variable order operator on the left side of (11) becomes $D_*^{\beta_N}$ if $t > T_N^*$. Analogously it follows from Theorem 3.4 that if $\nu < 0$, then $\beta(\mu t + \nu\tau)$ takes the value β_N for all $t > t_N^*$ and $\tau > 0$. Thus, if $\nu \neq 0$, then for all $t > T^* = \max\{T_N^*, t_N^*\}$ and $0 < \tau < t$ we have $\beta(\mu t + \nu\tau) = \beta_N$. Similar to the case $\nu = 0$, splitting the interval $(0,t)$, $t > T^*$, into subintervals, we can represent the equation (11) in the form (28). Further, from the continuity condition (13) we have $u(T^*,x) = \lim_{t\to T^*-0} v(t,x)$, where $v(t,x)$ is a solution to the Cauchy problem for fractional order pseudo-differential equations in subintervals of the interval $[0,T_N^*)$ constructed by continuation. Therefore there exists an operator $S^*(t,D)$, such that $v(t,x) = S^*(t,D)\varphi(x)$. Denote $\mathcal{P}^*(D) = S^*(T^*,D)$. Then $u(T^*,x) = \mathcal{P}^*(D)\varphi(x)$. This means that for $t > T^*$ solutions of problems (11)–(13) and (28),(29) coincide. If $\nu = 0$, then the statement follows from Theorem 4.2. □

Theorem 4.7. *Assume $\varphi \in \Psi_{G,p}(\mathbb{R}^n)$. Then there exists a number $t^* > 0$, such that for $0 < t < t^*$ the solution of the Cauchy problem* (11)–(13) *coincides with the solution of the Cauchy problem*

$$ D_*^{\beta_0} u(t,x) = \mathcal{A}(D)u(t,x), \quad t > 0,\ x \in \mathbb{R}^n \tag{30} $$

$$ u(0,x) = \varphi(x), \quad x \in \mathbb{R}^n. \tag{31} $$

Proof. It follows from Theorems 3.3 and 3.4 that the order function $\beta(\mu t + \nu\tau)$ under the integral in $D_{*\{\mu,\nu\}}^{\beta(t)}$ takes the value β_0 for all $t < t^* = \min\{t_1^*, T_1^*\}$

and $0 < \tau < t$. Hence, the variable order operator in (11) becomes $D_*^{\beta_0}$ if $0 < t < t^*$. The order β_1 (or diffusion mode $\{\beta_1, \mu, \nu\}$) has no influence in this interval. For $t > t^*$ two diffusion modes $\{\beta_0, \mu, \nu\}$ and $\{\beta_1, \mu, \nu\}$ are present. If $t < \min\{t_2^*, T_2^*\}$, then for all $\tau > 0$ we have $\mu t + \nu\tau < T_2$. That is, there is no influence of the mode $\{\beta_2, \mu, \nu\}$ if $t < \min\{t_2^*, T_2^*\}$. In the same manner the other values of β have no influence in the interval $0 < t < t^*$. This means that for $0 < t < t^*$ solutions of problems (11)–(13) and (30),(31) coincide. □

5. Some properties of the fundamental solution

In this section we study some basic properties of a solution $u(t, x)$ to the problem (11)–(13). Namely, we show that $u(t, x)$ is a density function under a rather general conditions on the pseudo-differential operator $\mathcal{A}(D)$. We also study the MSD(t) of the corresponding process near the initial time, which is important, in particular, in cell biology.

Theorem 5.1. *Assume $A(\xi)$ is a continuous symbol with negative values for $\xi \neq 0$, $A(-\xi) = A(\xi)$ and $A(0) = 0$. Then the Fourier transform $\hat{U}(t, \xi) = F[U](t, \xi)$ of the fundamental solution $U(t, x)$ to the problem* (11)–(13) *satisfies the following conditions:*

1. *$\hat{U}(t, \xi)$ is continuous in ξ for every fixed $t \geq 0$;*
2. *$\hat{U}(t, 0) = 1$ for all $t \geq 0$;*
3. *$\hat{U}(t, \xi)$ is positive definite for every fixed $t \geq 0$.*

Proof. First, let $\nu = 0$. Then Corollary 4.5 and the symmetry $A(-\xi) = A(\xi)$ imply that

$$\begin{aligned}
\hat{U}(t, \xi) = {} & \mathcal{I}_0'(t) E_{\beta_0}\big(t^{\beta_0} A(\xi)\big) \\
& + \sum_{k=1}^{N} \mathcal{I}_k'(t) \Bigg\{ E_{\beta_k}\big((t - t_k)^{\beta_k} A(\xi)\big) \prod_{j=0}^{k-1} E_{\beta_j}((t_{j+1} - t_j)^{\beta_j} A(\xi)) \\
& + E_{\beta_k}\big((t - t_k)^{\beta_k} A(\xi)\big) \\
& \times \int_{t_{k-1}}^{t_k} E_{\beta_{k-1}}\big((t_k - \tau)^{\beta_{k-1}} A(\xi)\big) \; {}_{t_{k-1}}D_\tau^{1-\beta_{k-1}} \mathcal{R}_{k-1}(\tau, \xi)\, d\tau \\
& + \int_{t_k}^{t} E_{\beta_k}\big((t - \tau)^{\beta_k} A(\xi)\big) \; {}_{t_k}D_\tau^{1-\beta_k} \mathcal{R}_k(\tau, \xi)\, d\tau \Bigg\},
\end{aligned} \tag{32}$$

where $E_\beta(z)$ is the Mittag–Leffler function and $\mathcal{R}_k(t, \xi)$ is defined in (17). Taking into account continuity of $E_\beta(z)$, we conclude that $\hat{U}(t, \xi)$ is continuous for every fixed $t > 0$. Further, it follows from the definition of $\mathcal{R}_k(t, \xi)$ that $\mathcal{R}(t, 0) = 0$. This implies $\hat{U}(t, 0) = \sum_{k=0}^{N} \mathcal{I}_k'(t)$, since $E_\beta(0) = 1$. Finally, as is known [21,29] since the function $E_\beta(-\lambda t^\beta)$, $0 < \beta \le 1$, is completely monotone for all $t > 0$ if $\lambda > 0$, we have $E_\beta(-\lambda t^\beta) > 0$ and $\frac{dE_\beta(-\lambda t^\beta)}{dt} < 0$ for all $t > 0$. The latter implies $\mathcal{R}_k(t,\xi) \ge 0$, $k = 1,\dots,N$. It follows from this fact together with $A(\xi) \le 0$ and positiveness of $E_\beta(t^\beta A(\xi))$ for all $t \ge 0$ that $\hat{U}(t,\xi) > 0$ for every fixed $t \ge 0$ and $\xi \in K \subset\subset R^n$, where K is an arbitrary compact. Now it is easy to verify positive definiteness of $\hat{U}(t,\xi)$ for each fixed $t > 0$. The idea of the proof in the general case $\nu \in (-1,1]$ is preserved, since the general structure of the representation formula for the fundamental solution remains unchanged. □

Corollary 5.2. *Under the assumption of Theorem* 5.1 *the fundamental solution* $U(t,x)$ *to the problem* (11)–(13) *is a probability density function for each fixed* $t \in (0,\infty)$.

The proof of this statement immediately follows from the Bochner–Khinchin theorem (see, e.g., [3]).

Thus there exists a stochastic process X_t with a density function $p_t(x) = U(t,x)$ for every fixed $t \ge 0$ with $p_0(x) = \delta(x)$. Denote by $\mu_t = E[X_t]$ the expectation of a random variable X_t (t is fixed) and $MSD(t) = E[|X_t - \mu_t|^2]$.

Now assume that the pseudo-differential operator on the right hand side of (11) is a negative definite second order homogeneous elliptic operator, that is the symbol of the operator $\mathcal{A}(D)$ has the form $A(\xi) = \frac{1}{2}\sum a_{ij}\xi_i\xi_j$. The matrix $\mathbf{A} = (a_{ij})$ is symmetric and negative definite: $V^T\mathbf{A}V \le -C|V|^2$, $C > 0$, where V is an n-dimensional vector, V^T is its transpose. By $Tr(\mathbf{A})$ we denote the trace of $\mathbf{A}$: $Tr(\mathbf{A}) = \sum a_{jj}$. It is not hard to verify that in this case $U(t,-x) = U(t,x)$ and $\mu_t = 0$. Hence,

$$MSD(U;t) = \int_{\mathbb{R}^n} |x|^2 U(t,x)dx. \tag{33}$$

Theorem 5.3. *Assume* $\mathcal{A}(D)$ *in* (11) *is a second order homogeneous negative definite elliptic operator. Then there exists* $t^* > 0$ *such that for* $t < t^*$ *the function* $MSD(U;t)$, *where* $U(t,x)$ *is the fundamental solution of the Cauchy problem* (11)–(13), *is represented in the form*

$$MSD(U;t) = \frac{Tr(\mathbf{A})}{\Gamma(\beta+1)} t^\beta, \quad 0 < t < t^*. \tag{34}$$

Proof. The proof is an implication of Theorem 4.7 and the fact that $MSD(U;t)$ for a solution of the Cauchy problem (30),(31) can be represented in the form $MSD(U;t) = (-\Delta_\xi)\hat{U}(t,\xi)|_{\xi=0}$. □

Theorem 5.4. *Under the condition of Theorem* 5.3 *for* $MSD(U;t)$, *where* U *is the fundamental solution of the Cauchy problem* (11)–(13), *the asymptotic relation* $MSD(U;t) = O(t^{\beta_N})$, $t \to \infty$, *holds.*

Proof. It follows from (32) for $t > T_N^*$ that $\hat{U(t,\xi)} = Q_1(t,\xi) + Q_2(t,\xi)$, where

$$Q_1(t,\xi) = E_{\beta_N}\big((t-t_N)^{\beta_N} A(\xi)\big) \left\{ \prod_{j=0}^{N-1} E_{\beta_j}\big((t_{j+1}-t_j)^{\beta_j} A(\xi)\big) \right.$$
$$\left. + \int_{t_{N-1}}^{t_N} E_{\beta_{N-1}}\big((t_N-\tau)^{\beta_{N-1}} A(\xi)\big)\ {}_{t_{N-1}}D_\tau^{1-\beta_{N-1}}\, \mathcal{R}_{N-1}(\tau,\xi)\, d\tau \right\},$$

and

$$Q_2(t,\xi) = \int_{t_N}^{t} E_{\beta_N}\big((t-\tau)^{\beta_N} A(\xi)\big)\ {}_{t_N}D_\tau^{1-\beta_N}\, \mathcal{R}_N(\tau,\xi)\, d\tau.$$

It is not hard to verify that $Q_1(t,\xi)$ dominates in terms of asymptotics for large t. We observe the same after applying $-\Delta_\xi$ as well. Elementary, but tedious calculations show that $(-\Delta_\xi)Q_1(t,\xi) = O(t^{\beta_N}),\ t \to \infty$. □

Corollary 5.5. *Let* $\beta(t) = \beta$, *where* β *is a constant in* $(0,1]$. *Then*

$$MSD(U;t) = \frac{Tr(\mathbf{A})}{\Gamma(\beta+1)} t^\beta, \quad t > 0.$$

Remark 5.6. A natural generalization of the model is to allow changing of random diffusion modes β_k at random times T_k with appropriate distributions, respectively. The questions on an asymptotic behaviour of $U(t,x)$ for large times, which shows how heavy is the tail of the distribution, as well as an asymptotic behaviour of $MSD(U;t)$, $t \to \infty$, which tells about the nature of the corresponding process, are important. We will discuss these challenging questions in a separate paper.

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References

[1] Andries, E., Umarov, S. and Steinberg, S., Monte Carlo random walk simulations based on distributed order differential equations with application to cell biology. *Frac. Calc. Appl. Anal.* 9 (2006)(4), 351 – 369.

[2] Applebaum, D., *Lévy Processes and Stochastic Calculus.* Cambridge: Cambridge Univ. Press 2004.

[3] Billingsley, P., *Probability and Measure.* New York: John Wiley & Sons 1995.

[4] Caputo, M., Linear models of dissipation whose Q is almost frequency independent. II. *Geophys. J. R. Astr. Soc.* 13 (1967), 529 – 539.

[5] Chechkin, A. V., Gorenflo, R. and Sokolov, I. M., Retarding subdiffusion and accelerating superdiffusion governed by distributed order fractional diffusion equation. *Phys. Rev.* E 66 (2002), 046129, 1 – 6.

[6] Chechkin, A. V., Gorenflo, R. and Sokolov, I. M., Fractional diffusion in inhomogeneous media. *J. Physics A: Math. Gen.* 38 (2005), L679 – L684.

[7] Djrbashian, M. M., *Harmonic Analysis and Boundary Value Problems in the Complex Domain.* Basel: Birkhäuser 1993.

[8] Dubinskiĭ, Yu. A., On a method of solving partial differential equations (in Russian). *Dokl. Akad. Nauk SSSR* 258 (1981), 780 – 784; transl. in: *Sov. Math. Dokl.* 23 (1981), 583 – 587.

[9] Edidin, M., Lipid microdomains in cell surface membranes. *Curr. Opin. Struct. Biol.* 7 (1997), 528 – 532.

[10] Ghosh, R. N. and Webb, W. W., Automated detection and tracking of individual and clustered cell surface low density lipoprotein receptor molecules. *Biophys. J.* 66 (1994), 1301 – 1318.

[11] Gorenflo, R., Luchko, Yu. and Umarov, S., On the Cauchy and multi-point problems for partial pseudo-differential equations of fractional order. *Fract. Calc. Appl. Anal.* 3 (2000)(3), 249 – 277.

[12] Gorenflo, R. and Mainardi, F., Simply and multiply scaled diffusion limits for continuous time random walk. *J. Phys. Conf. Ser.* 7 (2005), 1 – 16.

[13] Gorenflo, R., Mainardi, F., Moretti, D., Pagnini, G. and Paradisi, P., Discrete random walk models for space-time fractional diffusion. *Chemical Phys.* 284 (2002), 521 – 541.

[14] Hoh, W., Pseudo differential operators with negative definite symbols of variable order. *Rev. Mat. Iberoamericana* 16 (2000)(2), 219 – 241.

[15] Jacob, N. and Leopold H.-G., Pseudo-differential operators with variable order of differentiation generating Feller semigroups. *Integr. Equ. Oper. Theory* 17 (1993), 544 – 553.

[16] Liu, F., Shen, S., Anh, V. and Turner, I., Analysis of a discrete non-Markovian random walk approximation for the time fractional diffusion equation. *ANZIAM J.* 46 (2005), C488 – C504.

[17] Lorenzo, C. F. and Hartley, T. T., Variable order and distributed order fractional operators. *Nonlin. Dynam.* 29 (2002), 57 – 98.

[18] Meerschaert, M. and Scheffler, H.-P., Stochastic model for ultraslow diffusion. *Stochastic Process. Appl.* 116 (2006)(9), 1215 – 1235.

[19] Metzler, R. and Klafter, J., The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Reports* 339 (2000), 1 – 77.

[20] Metzler, R. and Klafter, J., The restaurant at the end of random walk: recent developments in the description of anomalous transport by fractional dynamics. *J. Physics. A: Math. Gen.* 37 (2004), R161 – R208.

[21] Pollard, H., The completely monotonic character of Mittag–Leffler function $E_\alpha(-x)$. *Bull. Amer. Math. Soc.* 54 (1948), 1115 – 1116.

[22] Ritchie, K., Shan, X. Y., Kondo, J., Iwasawa, K., Fujiwara, T. and Kusumi, A., Detection of non-Brownian diffusion in the cell membrane in single molecule tracking. *Biophys. J.* 88 (2005), 2266 – 2277.

[23] Saxton, M. J. and Jacobson, K., Single-particle tracking: applications to membrane dynamics. *Ann. Rev. Biophys. Biomol. Struct.* 26 (1997), 373 – 399.

[24] Saxton, M., Anomalous subdiffusion in fluorescence photobleaching recovery: a Monte Carlo study. *Biophys. J.* 81 (2001)4, 2226 – 2240.

[25] Samko, S. G., Kilbas, A. A. and Marichev, O. I., *Fractional Integrals and Derivatives: Theory and Applications*. New York: Gordon & Breach 1993.

[26] Samko, S. G. and Ross, B., Integration and differentiation to a variable fractional order. *Integr. Transforms Spec. Funct.* 1 (1993)(4), 277 – 300.

[27] Samko, S. G., Fractional integration and differentiation of variable order. *Anal. Math.* 21 (1995), 213 – 236.

[28] Slattery, J. P., Lateral mobility of FceRI on rat basophilic leukemia cells as measured by single particle tracking using a novel bright fluorescent probe. Ph.D. Thesis. Ithaca (NY): Cornell Univ. 1995.

[29] Schneider, W. R., Completely monotone generalized Mittag–Leffler functions. *Expositiones Mat.* 14 (1996), 3 – 16.

[30] Suzuki, K., Ritchie, K, Kajikawa, E., Fujiwara, T. and Kusumi, A., Rapid hop diffusion of a G-protein-coupled receptor in the plasma membrane as revealed by single-molecule techniques. *Biophys. J.* 88 (2005), 3659 – 3680.

[31] Umarov, S., Nonlocal boundary value problems for pseudo-differential and differential operator equations I. *Diff. Equ.* 33 (1997), 831 – 840.

[32] Umarov, S., Nonlocal boundary value problems for pseudo-differential and differential operator equations II. *Diff. Equ.* 34 (1998), 374 – 381.

[33] Umarov, S. and Gorenflo, R., On multi-dimensional random walk models approximating symmetric space-fractional diffusion processes. *Fract. Calc. Appl. Anal.* 8 (2005)(1), 73 – 88.

[34] Umarov, S. and Steinberg, S., Random walk models associated with distributed fractional order differential equations. In: *High Dimensional Probability*. IMS Lecture Notes Monogr. Ser. 51. Beachwood (OH): Inst. Math. Statist. 2006, pp. 117 – 127.

[35] Umarov, S. and Saydamatov, E., A fractional analog of the Duhamel principle. *Fract. Calc. Appl. Anal.* 9 (2006)(1), 57 – 70.

[36] Zaslavsky, G., Chaos, fractional kinetics, and anomalous transport. *Phys. Reports* 371 (2002), 461 – 580.

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