

Dynamics on the $\mathrm{PSL}(2, \mathbb{C})$ -character variety of a twisted I -bundle

Michelle Lee¹

Abstract. Let M be a twisted interval bundle over a closed nonorientable hyperbolizable surface. Let $\mathcal{X}(M)$ be the $\mathrm{PSL}(2, \mathbb{C})$ -character variety of $\pi_1(M)$. We examine the dynamics of the action of $\mathrm{Out}(\pi_1(M))$ on $\mathcal{X}(M)$, and in particular, we find an open set on which the action is properly discontinuous that is strictly larger than the interior of the deformation space of marked hyperbolic 3-manifolds homotopy equivalent to M . Furthermore, we identify which discrete and faithful representations can lie in a domain of discontinuity for the action of $\mathrm{Out}(\pi_1(M))$ on $\mathcal{X}(M)$.

Mathematics Subject Classification (2010). Primary: 57M50; Secondary: 57M60.

Keywords. Twisted interval bundle, hyperbolic 3-manifold, character variety, outer automorphism group.

1. Introduction

In this paper we use the deformation theory of hyperbolic 3-manifolds to study the dynamics of $\mathrm{Out}(\pi_1(M))$ on the $\mathrm{PSL}(2, \mathbb{C})$ -character variety of $\pi_1(M)$ when M is a hyperbolizable twisted I -bundle. The deformation space $\mathrm{AH}(M)$ is the space of conjugacy classes of discrete and faithful representations of $\pi_1(M)$ into $\mathrm{PSL}(2, \mathbb{C})$. It can also be thought of as the space of marked hyperbolic 3-manifolds homotopy equivalent to a given compact 3-manifold with boundary M . It sits inside the $\mathrm{PSL}(2, \mathbb{C})$ -character variety of $\pi_1(M)$

$$\mathcal{X}(M) = \mathrm{Hom}(\pi_1(M), \mathrm{PSL}(2, \mathbb{C})) // \mathrm{PSL}(2, \mathbb{C})$$

the quotient of $\mathrm{Hom}(\pi_1(M), \mathrm{PSL}(2, \mathbb{C}))$ from geometric invariant theory. The outer automorphism group $\mathrm{Out}(\pi_1(M))$ acts on $\mathcal{X}(M)$ and on $\mathrm{AH}(M)$ in the fol-

¹ Partially supported by NSF RTG grant DMS 0602191.

lowing way: an outer automorphism $[f]$ maps a representation $[\rho]$ to $[\rho \circ f^{-1}]$. Using the parametrization of the interior of $\text{AH}(M)$ (see [6] for more details on this parametrization), it is well known that this action is properly discontinuous on the interior of $\text{AH}(M)$. In this paper we find a larger domain of discontinuity for the action of $\text{Out}(\pi_1(M))$ in the case where M is a twisted I -bundle over a closed nonorientable hyperbolic surface. Namely, we prove the following.

Theorem 1. *If M is a hyperbolizable twisted I -bundle over a closed nonorientable surface, then there exists an open, $\text{Out}(\pi_1(M))$ -invariant subset $\mathcal{PS}(M)$, called the set of primitive-stable representations, in $\mathcal{X}(M)$ containing the interior of $\text{AH}(M)$ as well as points on $\partial \text{AH}(M)$ such that the action of $\text{Out}(\pi_1(M))$ is properly discontinuous on $\mathcal{PS}(M)$.*

It is well known that in the case where M is a trivial I -bundle over an orientable hyperbolic surface no point on the boundary of $\text{AH}(M)$ can lie in a domain of discontinuity for the action of $\text{Out}(\pi_1(M))$ on $\mathcal{X}(M)$ (see Section 3). Theorem 1 shows that, surprisingly, the dynamics of the outer automorphisms of a nonorientable surface group differ from those of the outer automorphisms of an orientable surface group.

Naturally, we would like to know whether $\mathcal{PS}(M)$ is a maximal domain of discontinuity. Toward this end, we prove Theorem 2 that characterizes which points in $\text{AH}(M)$ lie in $\mathcal{PS}(M)$ and also characterizes which points in $\text{AH}(M)$ can lie in a domain of discontinuity for the action of $\text{Out}(\pi_1(M))$. So far, the only other case in which the points of $\text{AH}(M)$ that can lie in a domain of discontinuity have been characterized is the quasi-Fuchsian case.

Theorem 2. *Let M be a hyperbolizable twisted I -bundle and let $[\rho]$ be an element of $\text{AH}(M)$. Then $[\rho]$ lies in the complement of $\mathcal{PS}(M)$ if and only if there exists a primitive element g of $\pi_1(M)$ such that $\rho(g)$ is parabolic. Moreover, if ρ lies in $\text{AH}(M) - \mathcal{PS}(M)$, then ρ does not lie in any domain of discontinuity for the action of $\text{Out}(\pi_1(M))$ on $\mathcal{X}(M)$.*

A primitive element of $\pi_1(M)$ is one which is associated to a simple closed curve on the base surface. In particular, all purely hyperbolic points in $\text{AH}(M)$ lie in $\mathcal{PS}(M)$.

The notion of primitive-stable representations was first introduced by Minsky in [15] where he proved a result analogous to Theorem 1 for handlebodies; namely, if H_g is a genus g hyperbolizable handlebody, then the set of primitive-stable representations, denoted $\mathcal{PS}(H_g)$, is a domain of discontinuity for the action of

$\mathrm{Out}(F_g)$ strictly larger than the interior of $\mathrm{AH}(H_g)$, where F_g is the free group on g generators. In an upcoming paper ([11]), we extend the notion of primitive-stability to compression bodies.

Canary and Storm ([7]) showed that when M is a compact, hyperbolizable 3-manifold with incompressible boundary and no toroidal boundary components that is not an I -bundle, then there exists an $\mathrm{Out}(\pi_1(M))$ -invariant open set $W(M)$ containing the interior of $\mathrm{AH}(M)$ and containing points on $\partial \mathrm{AH}(M)$ on which the action is properly discontinuous. Canary and Storm use a different method of constructing this set that involves the characteristic submanifold of M but the set they construct $W(M)$ also contains all purely hyperbolic points. Combining their results with Theorem 1 and the quasi-Fuchsian case (see Section 3) we can conclude the following.

Corollary 3. *Let M be a compact, orientable, hyperbolizable 3-manifold with nonempty incompressible boundary and no toroidal boundary components. Then there exists an open, $\mathrm{Out}(\pi_1(M))$ -invariant set, containing the interior of $\mathrm{AH}(M)$ and points on the boundary of $\mathrm{AH}(M)$, on which $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously if and only if M is not a trivial I -bundle over an orientable hyperbolic surface. Moreover, in the case that M is not a trivial I -bundle, then this set contains all purely hyperbolic points in $\mathrm{AH}(M)$.*

Corollary 3 suggests that when M is a trivial I -bundle over an orientable, hyperbolic surface the dynamics of $\mathrm{Out}(\pi_1(M))$ on $\mathcal{X}(M)$ is an anomalous case. In this case, it is an open question whether the interior of deformation space is the largest domain of discontinuity for the action of $\mathrm{Out}(\pi_1(M))$ on $\mathcal{X}(M)$.

We conclude this introduction with a brief outline of the paper. In Section 2 we define the notion of primitive-stable representations and show that the collection of primitive-stable representations $\mathcal{PS}(M)$ forms an open $\mathrm{Out}(\pi_1(M))$ -invariant subset of $\mathcal{X}(M)$ on which $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously. In Section 3 we break up the proof of Theorem 2 into two parts. In Proposition 15 we show that an element $[\rho]$ in $\mathrm{AH}(M)$ lies in $\mathcal{PS}(M)$ if and only if $\rho(g)$ is not parabolic for any primitive element g of $\pi_1(M)$. Minsky observed that if a representation $[\sigma]$ maps the core curve of an essential annulus to a parabolic element, then $[\sigma]$ cannot lie in any domain of discontinuity. Using this result and Proposition 15 it follows that any point in $\mathrm{AH}(M)$ outside of $\mathcal{PS}(M)$ does not lie in a domain of discontinuity, proving the second assertion of Theorem 2. We conclude with the observation that Minsky's result can also be applied to show that no element in the boundary of quasi-Fuchsian space can lie in a domain of discontinuity.

Acknowledgements. I would like to thank my advisor Dick Canary for his constant support and guidance. I would also like to thank Juan Souto and Nina White for several useful conversations.

2. Primitive stability

The goal of this section is to show that the set $\mathcal{PS}(M)$ of primitive-stable representations is an open, $\text{Out}(\pi_1(M))$ -invariant set on which the action is properly discontinuous. We follow the argument given by Minsky in [15] for the case when M is a handlebody. Let B be a closed nonorientable hyperbolic surface and let \tilde{B} be its orientable double cover. Let M be a twisted I -bundle over B , namely

$$M = \tilde{B} \times I / (x, t) \sim (\theta(x), 1 - t)$$

where θ is an orientation reversing, fixed-point free involution of \tilde{B} such that $\tilde{B}/\langle\theta\rangle$ is homeomorphic to B . Let G be the fundamental group of M .

Definition 4. We say an element g in G is *primitive* if it can be represented by a simple closed curve on the base surface B .

As G is torsion free, every element of G acts with North-South dynamics on $C_S(G)$ the Cayley graph of G with finite symmetric generating set S . If g_- and g_+ denote the repelling and attracting fixed points of g on $\partial C_S(G)$, let $L(g)$ denote the set of geodesics connecting g_- and g_+ . Let \mathcal{P} denote the set of geodesics l such that l is contained in $L(g)$ for some primitive element g .

Given a representation $\rho: G \rightarrow \text{PSL}_2(\mathbb{C})$ and a basepoint x in \mathbb{H}^3 , there exists a unique ρ -equivariant map $\tau_{\rho,x}: C_S(G) \rightarrow \mathbb{H}^3$ taking the identity to x and edges to geodesic segments.

Definition 5. A representation $\rho: G \rightarrow \text{PSL}_2(\mathbb{C})$ is called (K, A) -*primitive-stable* if there exists a basepoint x in \mathbb{H}^3 such that $\tau_{\rho,x}$ takes all geodesics of \mathcal{P} to (K, A) -quasi-geodesics. A representation ρ is *primitive-stable* if it is (K, A) -primitive-stable for some (K, A) .

A (K, A) -quasi-geodesic is a (K, A) -quasi-isometric image of \mathbb{R} . Here we take the parametrization to be $\tau_{\rho,x}$ composed with the arclength parametrization in $C_S(G)$.

Primitive-stability is independent of the choice of basepoint in \mathbb{H}^3 and the choice of generators S , although the constants (K, A) will change (see [12], Lemma III.2).

Remark 6. (K, A) -primitive-stability is invariant under conjugation by elements of $\mathrm{PSL}(2, \mathbb{C})$. Two irreducible representations in $\mathrm{Hom}(G, \mathrm{PSL}(2, \mathbb{C}))$ lie in the same fiber of $\pi : \mathrm{Hom}(\pi_1(M), \mathrm{PSL}(2, \mathbb{C})) \rightarrow \mathcal{X}(M)$ if and only if they differ by conjugation (see [9]). As reducible representations are never primitive-stable, primitive-stability is well defined on $\mathcal{X}(M)$.

In light of Remark 6, let $\mathcal{PS}(M) \subset \mathcal{X}(M)$ denote the set of primitive-stable representations in $\mathcal{X}(M)$.

We will begin by showing that $\mathcal{PS}(M)$ is open and $\mathrm{Out}(\pi_1(M))$ -invariant. Openness will follow from the fact that quasi-geodesics remain quasi-geodesics under small perturbations. The $\mathrm{Out}(\pi_1(M))$ -invariance will follow from the fact that automorphisms of G preserve the set of primitive elements.

Lemma 7. *$\mathcal{PS}(M)$ is an open, $\mathrm{Out}(\pi_1(M))$ -invariant subset of $\mathcal{X}(M)$. Moreover given any $[\rho_0]$ in \mathcal{PS} there exists constants (K_0, A_0) and a neighborhood $U_{[\rho_0]}$ such that any element $[\sigma]$ in $U_{[\rho_0]}$ is (K_0, A_0) -primitive-stable.*

Proof. We start by showing that $\mathcal{PS}(M)$ is open. The second statement will follow as a consequence of the proof of openness. Let L be a geodesic in $C_S(G)$, L' the image of L under $\tau_{\rho,x}$, $\{v_i\}$ the image of the vertex sequence of L , and $P_{j,i}$ the plane that perpendicularly bisects the geodesic segment $[v_{ji}, v_{(j+1)i}]$. We will need the following characterization of quasi-geodesics due to Minsky (for proof see [15] or [12]).

Lemma 8 (Minsky). *Given (K, A) , there exists $c > 0$ and $i \in \mathbb{N}$ such that if $L' = \tau_{\rho,x}(L)$ is a (K, A) -quasi-geodesic, then $P_{j,i}$ separates $P_{(j+1),i}$ and $P_{(j-1),i}$ and $d(P_{j,i}, P_{(j+1),i}) > c$. Conversely, given $c > 0$ and $i \in \mathbb{N}$ there exists (K', A') such that if $L' = \tau_{\rho,x}(L)$ has the property that $P_{j,i}$ separates $P_{(j+1),i}$ and $P_{(j-1),i}$ and $d(P_{j,i}, P_{(j+1),i}) > c$, then L' is a (K', A') -quasi-geodesic.*

To show that $\mathcal{PS}(M)$ is open, it suffices to show the following:

Lemma 9. *For any primitive-stable representation $[\rho_0]$ in $\mathcal{X}(M)$, there exists $U_{[\rho_0]}$ a neighborhood of $[\rho_0]$ and constants $c' > 0$, $i' \in \mathbb{N}$ such that for any $[\sigma]$ in $U_{[\rho_0]}$ and any geodesic l in \mathcal{P} , the planes, $P_{j,i'}$ corresponding to $\tau_{\sigma,x}(l)$ have the property that $P_{j,i'}$ separates $P_{(j+1),i'}$ and $P_{(j-1),i'}$ and $d(P_{j,i'}, P_{(j+1),i'}) > c'$.*

We outline the proof of Lemma 9 whose details can be found in [12]. Suppose that $[\rho_0]$ is (K_0, A_0) -primitive-stable. Let i_0 and c_0 be the constants obtained from

Lemma 8. Let $i' = i_0$ and choose $c' < c_0$. Since the map

$$\begin{aligned} \phi_g : \text{Hom}(G, \text{PSL}(2, \mathbb{C})) &\longrightarrow \mathbb{H}^3, \\ \rho &\longmapsto \rho(g) \cdot x, \end{aligned}$$

is continuous and geodesics segments vary continuously with respect to their endpoints, for any pair of vertices v_{ji_0} and $v_{(j+1)i_0}$ the point in the unit tangent bundle determining P_{j,i_0} will vary continuously over $\text{Hom}(G, \text{PSL}(2, \mathbb{C}))$. As G acts transitively and isometrically on $C_S(G)$ it suffices to check that the separation and distance properties of the planes P_{j,i_0} hold for subpaths of elements of \mathcal{P} beginning at the identity of length $3i_0$. In particular we are only concerned with what happens to a finite number of group elements. So if we take a lift ρ_0 of $[\rho_0]$ in $\text{Hom}(G, \text{PSL}(2, \mathbb{C}))$, then there exists an open neighborhood U_{ρ_0} of ρ_0 such that $P_{j,i'}$ separates $P_{(j+1),i'}$ and $P_{(j-1),i'}$ and $d(P_{j,i'}, P_{(j+1),i'}) > c'$ whose image in $\mathcal{X}(M)$ is the desired neighborhood of $[\rho_0]$. This completes the proof of Lemma 9 and hence openness.

To see that $\mathcal{PS}(M)$ is $\text{Out}(\pi_1(M))$ -invariant first observe that since homotopy equivalences of closed surfaces are homotopic to homeomorphisms, any automorphism f of $\pi_1(M) \cong \pi_1(B)$ preserves the set of primitive elements. The image of $\tau_{\rho \circ f^{-1}, x} : C_{f(S)}(G) \rightarrow \mathbb{H}^3$ coincides with the image of $\tau_{\rho, x} : C_S(G) \rightarrow \mathbb{H}^3$. Since primitive-stability is independent of the choice of generators of G , $\rho \circ f^{-1}$ is also primitive-stable. □

We finish this section by showing that the action of $\text{Out}(\pi_1(M))$ on $\mathcal{PS}(M)$ is properly discontinuous. The idea is that primitive-stability will imply that translation length of a primitive element in the Cayley graph is coarsely the same as translation length of the corresponding isometry in \mathbb{H}^3 (Lemma 10). To show proper discontinuity of the $\text{Out}(\pi_1(M))$ -action it will suffice to show that only finitely many automorphisms, up to conjugation, can change the translation length of primitive elements in the Cayley graph by a bounded amount (Lemma 13).

Lemma 10. *Let C be a compact subset of $\mathcal{PS}(M)$. Let $l_\rho(g)$ denote the translation length of $\rho(g)$ in \mathbb{H}^3 and let $\|g\|$ denote the translation length of g in $C_S(G)$. Then, there exist constants r and R such that*

$$r \leq \frac{l_\rho(g)}{\|g\|} \leq R$$

for all primitive elements g and all representations $[\rho]$ in C .

Proof. We will start by finding the upper bound. Let R be the maximum of $\{d(x, \rho(s)x)\}$ over s in S and $[\rho]$ in C ; such a R exists as S is finite and the map $\mathcal{X}(M) \rightarrow \mathbb{R}$ that maps $[\rho]$ to $d(x, \rho(s)x)$ for a fixed s and x is continuous. Then, $\tau_{\rho,x}$ is R -Lipschitz for all $[\rho]$ in C and so

$$l_\rho(g) \leq R\|g\|.$$

To see the lower bound we can first assume by Lemmata 8 and 9 that there exists (K, A) such that every element $[\rho]$ in C is (K, A) -primitive-stable. We would like to compare translation length of g along the geodesics in $L(g)$ with $l_\rho(g)$. Unfortunately, g need not act by translation on the elements of $L(g)$. Instead, there exists uniform quasi-axes for g , meaning there exists a (not necessarily unique) (K', A') -quasi-geodesic l' invariant under the action of g such that $d(z, gz) = \|g\|$ for any z in l' (see [12] Lemma II.6).

It will suffice to consider the image of these quasi-axes by the following lemma (see [12], Lemma II.9).

Lemma 11. *Given (K, A) and (K', A') , there exists $K'' = K''(K, K', A, A')$ and $A'' = A''(K, K', A, A')$ such that the following holds. If γ is a geodesic in $C_S(G)$ such that $\tau_{\rho,x}(\gamma)$ is a (K, A) -quasi-geodesic and γ' is a (K', A') -quasi-geodesic in $C_S(G)$ with the same endpoints at infinity as γ , then $\tau_{\rho,x}(\gamma')$ is a (K'', A'') -quasi-geodesic.*

By Lemma 11, if l' is a (K', A') -quasi-axis for g , then $\tau_{\rho,x}(l')$ is a (K'', A'') -quasi-geodesic in \mathbb{H}^3 . In particular, $\tau_{\rho,x}(l')$ lies in a $R'' = R''(K'', A'')$ neighborhood of the axis for $\rho(g)$. Hence if we take y to lie on the image of a quasi-axis for g , we have

$$l_\rho(g) \geq d(y, \rho(g) \cdot y) - 2R'' \geq \frac{\|g\|}{K''} - A'' - 2R''.$$

So,

$$\frac{l_\rho(g)}{\|g\|} \geq \frac{1}{K''} - \frac{A''}{\|g\|} - \frac{2R''}{\|g\|}.$$

For $\|g\|$ larger than $2(A'' + 2R'')K''$,

$$\frac{1}{K''} - \frac{A''}{\|g\|} - \frac{2R''}{\|g\|} > \frac{1}{2K''} > 0.$$

Let $[g_1], \dots, [g_m]$ be the conjugacy classes of elements of G such that $\|g_i\|$ is less than $2(A'' + 2R'')K''$. Since $\frac{l_\rho(g_i)}{\|g_i\|}$ varies continuously over $\mathcal{X}(M)$ for each g_i there exists a minimum value r_i for $\frac{l_\rho(g_i)}{\|g_i\|}$ over C . Take r to be the minimum of $\{r_1, \dots, r_m, \frac{1}{2K''}\}$. □

Proposition 12. $\text{Out}(\pi_1(M))$ acts properly discontinuously on $\mathcal{PS}(M)$.

Proof. Let C be a compact set in $\mathcal{PS}(M)$. Suppose that $[f]$ is an element in $\text{Out}(G)$ such that $f \cdot C \cap C \neq \emptyset$. By Lemma 10, for any $[\rho]$ in C and any primitive word w ,

$$\|f^{-1}(w)\| \leq \frac{1}{r} l_\rho(f^{-1}(w)) = \frac{1}{r} l_{\rho \circ f^{-1}}(w) \leq \frac{R}{r} \|w\|.$$

Let x_1, \dots, x_n be a set of generators for G such that each x_i is primitive and $x_i x_j$ where $i \neq j$ is primitive; the standard generators for $\pi_1(B)$ will do. Let $\mathcal{W} = \{x_i, x_i x_j \mid i \neq j\}$. The following lemma shows that there are only finitely many outer automorphisms $[f]$ satisfying the inequality above.

Lemma 13. For any $N > 0$, the set

$$\mathcal{A} = \{[f] \in \text{Out}(G) \mid \|f(w)\| \leq N \|w\| \text{ for all } w \in \mathcal{W}\}$$

is finite.

Proof. Suppose that $\{[f_k]\}$ is a sequence of infinitely many distinct elements in \mathcal{A} . As G acts cocompactly on \mathbb{H}^2 , there is a G -equivariant quasi-isometry

$$\tau': C_S(G) \longrightarrow \mathbb{H}^2.$$

Fix the following notation: let \bar{g} denote the isometry of \mathbb{H}^2 induced by the action of g , $l(\bar{g})$ its translation length and $\text{Ax}(\bar{g})$ its invariant geodesic axis. As G acts cocompactly on \mathbb{H}^2 , there exists $r > 0$ such that $l(\overline{f_k(x_i)}) \geq r$ on \mathbb{H}^2 for all i and k . Since $\|f_k(x_i x_j)\| \leq 2N$, there exists R such that $l(\overline{f_k(x_i x_j)}) \leq R$ all k and all pairs i, j such that $i \neq j$.

This implies that there exists an upper bound D on the distance between $\text{Ax}(\overline{f_k(x_i)})$ and $\text{Ax}(\overline{f_k(x_j)})$ for all k and all pairs i, j . If, on the contrary, $\{d(\text{Ax}(\overline{f_k(x_i)}), \text{Ax}(\overline{f_k(x_j)}))\}$ was unbounded, since $l(\overline{f_k(x_i)})$ and $l(\overline{f_k(x_j)})$ are bounded from below by r , $\{l(\overline{f_k(x_i) f_k(x_j)})\}$ would also be unbounded, a contradiction.

Then, there also exists an upper bound D' on the distance between $\text{Ax}(f_k(x_i))$, a quasi-axis of $f_k(x_i)$, and $\text{Ax}(f_k(x_j))$, a quasi-axis of $f_k(x_j)$, for all i, j , and k . Up to conjugation, we can assume that $\text{Ax}(f_k(x_i))$ is a uniformly bounded distance D'' from the identity e for all k and i . If y is a point on $\text{Ax}(f_k(x_i))$ closest to e , then we can bound the distance between the identity and $f_k(x_i)$ in the Cayley

graph as follows:

$$\begin{aligned} d(e, f_k(x_i)) &\leq D'' + d(y, f_k(x_i)y) + D'' \\ &\leq 2D'' + \|f_k(x_i)\| \\ &\leq 2D'' + N. \end{aligned}$$

This implies that up to conjugation, there are only finitely many possibilities for $f_k(x_i)$. Hence \mathcal{A} must be finite. \square

Since there are only finitely many outer automorphisms $[f]$ such that $f \cdot C \cap C \neq \emptyset$, the action is properly discontinuous on $\mathcal{PS}(M)$. \square

To complete the proof of Theorem 1 it remains to show that $\mathcal{PS}(M)$ is strictly larger than the interior of $\mathrm{AH}(M)$. This will follow immediately from Theorem 2, which will be proven in Section 3.

3. Primitive-stable points on the boundary of $\mathrm{AH}(M)$

In this section we prove Theorem 2. We break up the proof into Propositions 15 and 17. We start by characterizing which points in $\mathrm{AH}(M)$ lie in $\mathcal{PS}(M)$. The interior of $\mathrm{AH}(M)$ consists of convex cocompact representations (see [18]), namely those representations whose associated hyperbolic manifold has a compact convex core. We will show that $[\rho]$ in $\mathrm{AH}(M)$ lies in $\mathcal{PS}(M)$ if and only if $\rho(g)$ is hyperbolic for all primitive elements g of G . In particular, $\mathcal{PS}(M)$ will contain the interior of $\mathrm{AH}(M)$ as well as all purely hyperbolic points on the boundary of $\mathrm{AH}(M)$. This will complete the proof of Theorem 1. Then, to complete the proof of Theorem 2 we will use an observation by Minsky that if $[\sigma]$ in $\mathrm{AH}(M')$ maps a core curve of an essential annulus to a parabolic element of $\mathrm{PSL}(2, \mathbb{C})$ then $[\sigma]$ cannot lie in a domain of discontinuity of the action of $\mathrm{Out}(\pi_1(M'))$ on $\mathcal{X}(M')$. Finally we conclude with the result that no point on the boundary of quasi-Fuchsian space can lie in a domain of discontinuity.

We start this section by reviewing some basic facts from hyperbolic geometry that we will need. There exists a constant $\mu_3 > 0$ such that for any hyperbolic 3-manifold $N \cong \mathbb{H}^3/\Gamma$, where Γ a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$ and any $\epsilon < \mu_3$ each component of $N_{\mathrm{thin}(\epsilon)} = \{x \in N \mid \mathrm{inj}_N(x) < \epsilon\}$ is either a metric neighborhood of a closed geodesic, called a Margulis tube, or a parabolic cusp homeomorphic to either $S^1 \times \mathbb{R} \times (0, \infty)$ or $T \times (0, \infty)$ where T is a torus (see [1], Chapter D). Let N_ϵ^0 denote the complement of the non-compact portions of $N_{\mathrm{thin}(\epsilon)}$ in N . The convex core $C(N)$ of N is the smallest convex submanifold

of N such that the inclusion of $C(N)$ into N is a homotopy equivalence. A hyperbolic 3-manifold N is called convex cocompact if $C(N)$ is compact. In general, when $\pi_1(N)$ is finitely generated, there exists a compact submanifold C , called the compact core, whose inclusion induces a homotopy equivalence with N (see [17]). Moreover, C can be chosen such that C intersects each parabolic cusp of the non-compact portions of $N_{\text{thin}(\epsilon)}$ in a single incompressible annulus if the cusp is homeomorphic to $S^1 \times \mathbb{R} \times (0, \infty)$ or a torus if the cusp is homeomorphic to $T \times (0, \infty)$ (see [13]). A compact core of the latter type is called a relative compact core.

The goal of this section is to prove the following.

Theorem 2. *Let $[\rho]$ be an element of $\text{AH}(M)$. Then, $[\rho]$ lies in the complement of $\mathcal{PS}(M)$ if and only if there exists a primitive element g of $\pi_1(M)$ such that $\rho(g)$ is parabolic. Moreover, if ρ lies in $\text{AH}(M) - \mathcal{PS}(M)$, then ρ does not lie in any domain of discontinuity for the action of $\text{Out}(\pi_1(M))$ on $\mathcal{X}(M)$.*

We will use the following characterization of which discrete and faithful representations mapping all primitive elements to hyperbolic elements of $\text{PSL}(2, \mathbb{C})$ are primitive-stable.

Lemma 14. *Let ρ be a discrete and faithful representation of $\pi_1(M)$ into $\text{PSL}(2, \mathbb{C})$ such that $\rho(g)$ is hyperbolic for any primitive element g . Then ρ is primitive-stable if and only if there exists a compact subset Ω of $N_\rho = \mathbb{H}^3 / \rho(\pi_1(M))$ such that the set of geodesics corresponding to primitive elements of $\pi_1(M)$ is contained in Ω .*

Proof. Suppose there exists a compact set Ω such that all primitive geodesics of N_ρ are contained in Ω . Without loss of generality we can assume that Ω is a compact core C of N_ρ containing the image of $C_S(G) / \rho(G)$ in N_ρ . This implies, in particular, that $\tilde{\Omega}$, the preimage of Ω in \mathbb{H}^3 is connected. For some (K, A) , the orbit map $\tau_{\rho,x} : C_S(G) \rightarrow \tilde{\Omega} \subset \mathbb{H}^3$ is a (K, A) -quasi-isometry from $C_S(G)$ to $\tilde{\Omega}$ with the intrinsic metric. Any geodesic l in \mathcal{P} connecting g_- and g_+ , the fixed points of g , maps to a (K, A) -quasi-geodesic in $\tilde{\Omega}$, with the intrinsic metric. In particular, $\tau_{\rho,x}(l)$ lies in a $R = R_\Omega(K, A)$ -neighborhood of $\text{Ax}(g)$, a lift of the geodesic representing g in N_ρ . Then $\tau_{\rho,x}(l)$ lies in a R neighborhood of $\text{Ax}(g)$ with the extrinsic metric on $\tilde{\Omega}$. If x, y lie on $\tau_{\rho,x}(l)$ and if π denotes the closest point projection onto $\text{Ax}(g)$ in $\tilde{\Omega}$, then

$$d_{\tilde{\Omega}}(x, y) \leq d_{\tilde{\Omega}}(\pi(x), \pi(y)) + 2R = d_{\mathbb{H}^3}(\pi(x), \pi(y)) + 2R \leq d_{\mathbb{H}^3}(x, y) + 4R$$

This implies that $\tau_{\rho,x}(l)$ is a $(K, A + 4R)$ -quasi-geodesic in $\tilde{\Omega}$ with the extrinsic metric. Hence ρ is $(K, A + 4R)$ -primitive-stable.

Conversely, if ρ is (K, A) -primitive-stable then elements of \mathcal{P} stay within a bounded neighborhood of their corresponding geodesic axes in \mathbb{H}^3 . In particular, geodesics representing primitive elements will stay in a bounded neighborhood of the image of the Cayley graph in N_ρ , which is a compact set. \square

We will start by proving the first assertion of Theorem 2.

Proposition 15. *Let $[\rho]$ be an element of $\mathrm{AH}(M)$. Then $[\rho]$ does not lie in $\mathcal{PS}(M)$ if and only if there exists a primitive element g of $\pi_1(M)$ such that $\rho(g)$ is parabolic.*

Proof. The backwards direction is easy for if $\rho(g)$ is parabolic, then for any geodesic l connecting the fixed points g_+ and g_- on $\partial C_S(G)$, $\tau_{\rho,x}(l)$ is not quasi-geodesic.

For the forward direction, if $\rho(g)$ is hyperbolic for every primitive g , then by Lemma 14 it suffices to check that closed geodesics corresponding to primitive elements remain in a compact set. Let γ_g denote the unique geodesic representative of $\rho(g)$ in $N = \mathbb{H}^3/\rho(\pi_1(M))$. The representation ρ induces a homotopy equivalence $h_\rho: M \rightarrow N$. Precompose with the inclusion $B \rightarrow M$, to obtain an incompressible map $h'_\rho: B \rightarrow N$. Let α_g be a simple closed curve on B such that $h'_\rho(\alpha_g)$ is freely homotopic to γ_g . Fix a point x_0 on α_g . Extend α_g and x_0 to a one vertex triangulation of B , meaning a collection of mutually nonisotopic arcs k_i with all endpoints at x_0 and disjoint interiors such that $B - [(\cup k_i) \cup \alpha_g]$ is a collection of triangles.

The map h'_ρ is homotopic to a map h_g that maps α_g to γ_g , each arc k_i to a geodesic arc and each triangle to a totally geodesic triangle. If we endow B with the pull-back metric, then B has a hyperbolic metric with one cone singularity at x_0 . By construction, the sum of the angles of the sectors around x_0 is at least 2π . In particular, the area of B is bounded above by $-2\pi\chi(B)$. We have constructed a so-called simplicial hyperbolic surface (for more details see [2] §1.2).

Given $\epsilon \leq \mu_3$, we claim that given any such simplicial hyperbolic surface, $h: B \rightarrow N$, there is a uniform upper bound on how far its image can venture out of N_ϵ^0 . Observe first that there exists a constant A depending only on the Euler characteristic of B such that for any point x in B there exists a homotopically nontrivial simple curve through x of length less than A . To produce such a curve take a ball centered at x and blow it up until it intersects itself. Since the area of B is bounded, there is a uniform upper bound on the area and radius of such a ball.

For any $L > 2A$, if $h(B) \cap (N - \mathcal{N}_L(N_\epsilon^0)) \neq \emptyset$, where $\mathcal{N}_L(N_\epsilon^0)$ is the L -neighborhood of N_ϵ^0 , then there exists a homotopically non-trivial simple curve entirely contained within a noncompact component of $N_{\mathrm{thin}(\epsilon)}$. This implies that the curve

represents a parabolic element, a contradiction. So, there exists ϵ_0 such that simplicial hyperbolic surfaces realizing primitive geodesics are contained in $N_{\epsilon_0}^0$. Furthermore, we can choose ϵ_0 small enough such that the distance between any two Margulis tubes is bounded away from zero, i.e. there exists $\delta > 0$ such that for any two Margulis tubes T_1 and T_2 , the distance between them $d(T_1, T_2)$ is at least δ (see [14] Lemma 6.1).

Now suppose to the contrary that $\{\gamma_i\}$ is a sequence of primitive geodesics not contained in any compact set. Let $h_i : B \rightarrow N$ denote a simplicial hyperbolic surface containing γ_i . We can lift h_i to a map $\tilde{h}_i : S \rightarrow \tilde{N}$ where $S = \partial M$ and \tilde{N} is the double cover of N associated to the subgroup $\pi_1(S)$. By construction, \tilde{h}_i satisfies $\tilde{\theta} \circ \tilde{h}_i = \tilde{h}_i \circ \theta$, where $\tilde{\theta}$ is the nontrivial covering transformation of \tilde{N} . Moreover, the map \tilde{h}_i is a simplicial hyperbolic surface containing γ_i and $\tilde{\theta}(\gamma_i)$ where the associated triangulation is the preimage of the triangulation on B .

Fix C a compact core for N . The preimage \tilde{C} of C in \tilde{N} is a compact core for \tilde{N} . As \tilde{C} is homotopy equivalent to a fiber surface S which separates $\tilde{N} \cong S \times \mathbb{R}$, its complement $\tilde{N} - \tilde{C}$ has two components. Since \tilde{C} covers C and C has only one boundary component, $\tilde{\theta}$ must exchange the two boundary components of \tilde{C} , and hence $\tilde{\theta}$ must exchange the two components of $\tilde{N} - \tilde{C}$.

Let C_i be the following set

$$\{x \in N_{\epsilon_0}^0 \mid \text{there exists a path } c \text{ in } N_{\epsilon_0}^0 \text{ from } x \text{ to } C \text{ with } l(c \cap N_{\text{thick}(\epsilon_0)}) \leq i\}.$$

Since we chose ϵ_0 small enough so that the Margulis tubes are a definite distance apart, for each i there exists R_i such that C_i is contained in $\mathcal{N}_{R_i}(C)$, the R_i neighborhood of C . In particular, each C_i is compact. As $\{\gamma_i\}$ is not contained in any compact set, we can assume, up to subsequence, that there exists a point x_i on γ_i such that x_i lies outside the compact set C_i .

As $\tilde{\theta}$ interchanges the two components of $\tilde{N} - \tilde{C}$, the two lifts $\tilde{x}_i, \tilde{x}'_i$ of x_i lie in different components of $\tilde{N} - \tilde{C}$, but by the equivariance property of \tilde{h}_i , they both lie on $\tilde{h}_i(S)$. Any path c on $\tilde{h}_i(S)$ connecting \tilde{x}_i and \tilde{x}'_i satisfies the condition $d(c \cap \tilde{N}_{\text{thick}(\epsilon_0)}) \geq 2i$, for if not, in N there would be a path c' connecting x_i to C with $d(c' \cap N_{\text{thick}(\epsilon_0)}) < i$. For i large enough, this contradicts Bonahon's bounded diameter lemma ([2] Lemma 1.11) that states that the diameter of any incompressible simplicial hyperbolic surface modulo the ϵ_0 -thin part is bounded above where the bound depends only on ϵ_0 and the topology of the surface. \square

This completes the proof of the first assertion of Theorem 2. To see the second assertion we will need the following observation due to Minsky.

Lemma 16 (Minsky). *Let M be a compact hyperbolizable manifold with no toroidal boundary components. Let γ be the core curve of an essential annulus in M . Suppose that $\rho: \pi_1(M) \rightarrow \mathrm{PSL}_2(\mathbb{C})$ is a discrete and faithful representation such that $\rho(\gamma)$ is parabolic. Then, any neighborhood of $[\rho_0]$ contains points with infinite stabilizers. In particular, $[\rho]$ cannot lie in a domain of discontinuity for the action of $\mathrm{Out}(\pi_1(M))$ on $\mathcal{X}(M)$.*

Proof. Consider the map

$$\mathrm{tr}_\gamma^2: \mathcal{X}(M) \longrightarrow \mathbb{C}$$

where $[\rho] \mapsto \mathrm{tr}([\rho(\gamma)])^2$. As a neighborhood of $\mathrm{AH}(M)$ is a smooth complex manifold (see [10] Chapter 4) on which tr_γ^2 is a holomorphic map, the map tr^2 is either constant or open on that neighborhood. As the interior of $\mathrm{AH}(M)$ consists of convex cocompact representations that are dense in $\mathrm{AH}(M)$ (see [4] and [16]), the image of tr_γ^2 cannot be constant on all of $\mathrm{AH}(M)$ and hence it must be an open map on that neighborhood of $\mathrm{AH}(M)$. Since isometries of \mathbb{H}^3 are determined, up to conjugacy, by their trace and there are finite order elliptic isometries with trace arbitrarily close to 2 or -2 , there exist representations ρ_i approaching ρ such that $\rho_i(\gamma)$ corresponds to a finite order elliptic isometry. Let n_i denote the order of $\rho_i(\gamma)$. Then $D_\gamma^{n_i}$ the Dehn twist of order n_i about the annulus whose core curve is γ is an element in $\mathrm{Out}(\pi_1(M))$ that fixes $[\rho_i]$. Hence, elements arbitrarily close to $[\rho]$ have infinite stabilizers. \square

Proposition 17. *If ρ lies in $\mathrm{AH}(M) - \mathcal{PS}(M)$, then ρ does not lie in any domain of discontinuity for the action of $\mathrm{Out}(\pi_1(M))$ on $\mathcal{X}(M)$.*

Proof. If $[\rho]$ lies in the complement of $\mathcal{PS}(M)$ in $\mathrm{AH}(M)$, then there exists a primitive element g such that $\rho(g)$ is parabolic. Then g is either the core curve of an essential annulus or the core curve of an essential Mobius band. By Lemma 16 it suffices to show that the latter is impossible. Suppose that γ is a closed essential curve in M mapping to a parabolic element in $N_\rho = \mathbb{H}^3/\rho(G)$. Then we claim that γ must be homotopic into ∂M . If C is a relative compact core for N_ρ , consider the map $\phi: S = \partial M \rightarrow C$ in the homotopy class of $\rho|_{\pi_1(\partial M)}$. If \tilde{C} is the cover of C associated to the subgroup $\phi_*(\pi_1(S))$ it is a compact manifold with $\pi_1(\tilde{C}) \cong \pi_1(S)$. This implies that \tilde{C} must be a trivial I -bundle over S ([8] Theorem 10.6). Then the lift $S \rightarrow \tilde{C}$ can be homotoped into $\partial\tilde{C}$. Hence the map $S \rightarrow C$ is also homotopic into the boundary of C . Then, we can homotope the map $M \rightarrow C$ to a map that sends ∂M into ∂C . This map is either homotopic to a homeomorphism or is homotopic to a map $M \rightarrow \partial C$ ([8] Theorem 13.6). The latter cannot happen as this would imply that ∂C is a nonorientable closed surface. Hence any curve

in M mapping to a parabolic element in N_ρ is homotopic into ∂M . As the core curve of an essential Möbius band cannot be homotoped into ∂M , it cannot be mapped to a parabolic element in N_ρ . \square

This completes the proof of Theorem 2. We end this section with an application of Minsky's observation in the quasi-Fuchsian case.

Proposition 18. *Let F be an orientable hyperbolic surface. Then no point on the boundary of $\text{AH}(F \times I)$ can lie in a domain of discontinuity for the action of $\text{Out}(\pi_1(F \times I))$ on $\mathcal{X}(F \times I)$.*

Proof. Since geometrically finite points are dense on the boundary of $\text{AH}(F \times I)$ (see [3] and [5]) and since any simple closed curve on F is the core curve of an essential annulus, the result follows from Lemma 16. \square

References

- [1] R. Benedetti and C. Petronio, *Lectures on hyperbolic geometry*. Universitext, Springer, Berlin, 1992. [Zbl 0768.51018](#) [MR 1219310](#)
- [2] F. Bonahon, Bouts des variétés hyperboliques de dimension 3. *Ann. of Math. (2)* **124** (1986), no. 1, 71–158. [Zbl 0671.57008](#) [MR 0847953](#)
- [3] J. F. Brock, R. D. Canary, and Y. N. Minsky, The classification of Kleinian surface groups, II: The ending lamination conjecture. *Ann. of Math. (2)* **171** (2010), no. 1, 1–107. [Zbl 1193.30063](#) [MR 2630036](#)
- [4] J. Brock, R. Canary, and Y. Minsky, The classification of finitely-generated Kleinian groups. In preparation.
- [5] R. D. Canary and S. Hersensky, Ubiquity of geometric finiteness in boundaries of deformation spaces of hyperbolic 3-manifolds. *Amer. J. Math.* **126** (2004), no. 6, 1193–1220. [Zbl 1062.57021](#) [MR 2102392](#)
- [6] R. D. Canary and D. McCullough, Homotopy equivalences of 3-manifolds and deformation theory of Kleinian groups. *Mem. Amer. Math. Soc.* **172** (2004), no. 812. [Zbl 1062.57027](#) [MR 2096234](#)
- [7] R. D. Canary and P. Storm, Moduli spaces of hyperbolic 3-manifolds and dynamics on character varieties. *Comment. Math. Helv.* **88** (2013), no. 1, 221–251. [Zbl 1267.57019](#) [MR 3008919](#)
- [8] J. Hempel, *3-manifolds*. Annals of Mathematics Studies, 86. Princeton University Press, Princeton, N.J., and University of Tokyo Press, Tokyo, 1976. [Zbl 0345.57001](#) [MR 0415619](#)
- [9] M. Heusener and J. Porti, The variety of characters in $\text{PSL}_2(\mathbb{C})$. *Bol. Soc. Mat. Mexicana* (3) **10** (2004), Special Issue, 221–237. [Zbl 1100.57014](#) [MR 2199350](#)

- [10] M. Kapovich, *Hyperbolic manifolds and discrete groups*. Reprint of the 2001 edition. Modern Birkhäuser Classics, Birkhäuser Boston, Boston, MA, 2009. [Zbl 1180.57001](#) [MR 2553578](#)
- [11] M. Lee, Dynamics on the $\mathrm{PSL}(2, \mathbb{C})$ -character variety of a compression body. *Algebr. Geom. Topol.* **14** (2014), no. 4, 269–299. [Zbl 06339367](#) [MR 3252055](#)
- [12] M. Lee, *Dynamics on $\mathrm{PSL}(2, \mathbb{C})$ -character varieties of certain hyperbolic 3-manifolds*. Ph.D. Thesis, University of Michigan, Ann Arbor, MI 2012. [MR 3094006](#) <http://hdl.handle.net/2027.42/94050>
- [13] D. McCullough, Compact submanifolds of 3-manifolds with boundary. *Quart. J. Math. Oxford Ser. (2)* **37** (1986), no. 147, 299–307. [Zbl 0628.57008](#) [MR 0854628](#)
- [14] Y. N. Minsky, The classification of punctured-torus groups. *Ann. of Math. (2)* **149** (1999), no. 2, 559–626. [Zbl 0939.30034](#) [MR 1689341](#)
- [15] Y. Minsky, On dynamics of $\mathrm{Out}(F_n)$ on $\mathrm{PSL}_2\mathbb{C}$ characters. *Israel J. Math.* **193** (2013), no. 1, 47–70. [Zbl 1282.57023](#) [MR 3038545](#)
- [16] H. Namazi and J. Souto, Non-realizability and ending laminations: proof of the density conjecture. *Acta Math.* **209** (2012), no. 2, 323–395. [Zbl 1258.57010](#) [MR 3001608](#)
- [17] G. P. Scott, Compact submanifolds of 3-manifolds. *J. London Math. Soc. (2)* **7** (1973), 246–250. [Zbl 0266.57001](#) [MR 0326737](#)
- [18] D. Sullivan, Quasiconformal homeomorphisms and dynamics. II. Structural stability implies hyperbolicity for Kleinian groups. *Acta Math.* **155** (1985), no. 3-4, 243–260. [Zbl 0606.30044](#) [MR 0806415](#)

Received September 17, 2012

Michelle Lee, University of Michigan, Department of Mathematics, 2074 East Hall,
530 Church Street, Ann Arbor, MI 48109-1043, U.S.A.

e-mail: michelle.dongeeun.lee@gmail.com