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Right-angled Artin groups and Out($\mathbb{F}_n$) I. Quasi-isometric embeddings

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Abstract. We construct quasi-isometric embeddings from right-angled Artin groups into the outer automorphism group of a free group. These homomorphisms are modeled on the homomorphisms into the mapping class group constructed by Clay, Leininger, and Mangahas. Toward this goal, we develop tools in the free group setting that mirror those for surface groups and discuss various analogs of subsurface projection.

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Contents

1. Introduction

For a finite simplicial graph Γ with vertex set Γ^0, the right-angled Artin group $A(\Gamma)$ is the group presented with generators $s_i \in \Gamma^0$ and relators $[s_i, s_j] = 1$ whenever s_i and s_j are joined by an edge in Γ. Although they are simple to define, right-angled Artin groups have been at the center of recent developments in geometric group theory and low-dimensional topology. This interest is, in part, because many geometrically significant groups contain right-angled Artin subgroups. For example, Wang constructed injective homomorphisms from certain right-angled Artin groups into $\mathrm{SL}_n(\mathbb{Z})$, for $n \geq 5$; see [28]. In [18], Kapovich proved that for any finite simplicial graph Γ and any symplectic manifold (M, ω), $A(\Gamma)$ embeds into the group of Hamiltonian symplectomorphisms of (M, ω). Turning our attention to the mapping class group of a surface, Koberda showed that under general conditions the subgroup generated by sufficiently high powers of finitely many mapping classes is a right-angled Artin subgroup of $\mathrm{Mod}(S)$; see [19]. In [7], Clay, Leininger, and Mangahas constructed quasi-isometric embeddings of right-angled Artin groups into mapping class groups using partial pseudo-Anosov mapping classes. Specifically, they prove the following:

Theorem 1.1 (Theorem 1.1 of [7]). *Suppose that $f_1, \dots, f_n \in \mathrm{Mod}(S)$ are fully supported on disjoint or overlapping non-annular subsurfaces. Then after raising to sufficiently high powers, the elements generate a quasi-isometrically embedded right-angled Artin subgroup of* $\mathrm{Mod}(S)$. *Furthermore, the orbit map to Teichmüller space is a quasi-isometric embedding.*

Corollary 1.2 (Corollary 1.2 of [7]). *Any right-angled Artin group admits a homomorphism to some mapping class group which is a quasi-isometric embedding, and for which the orbit map to Teichmüller space is a quasi-isometric embedding.*

In this paper, we develop the theory necessary to quasi-isometrically embed right-angled Artin groups into $\mathrm{Out}(\mathbb{F}_n)$. Here, we show the following (see Section 4 for definitions and a more general statement):

Theorem 1.3. *Suppose that $f_1, \dots, f_n \in \mathrm{Out}(\mathbb{F}_n)$ are fully supported on an admissible collection of free factors. Then after raising to sufficiently high powers, the elements generate a quasi-isometrically embedded right-angled Artin subgroup of* $\mathrm{Out}(\mathbb{F}_n)$.

The *admissible collection* condition on the set of free factors in Theorem 1.3 is meant to mimic the situation in Theorem 1.1, where the subsurfaces considered are either disjoint or overlapping. We note that if Γ is the "coincidence graph" for the involved free factors, then the right-angled Artin group generated in Theorem 1.3 is $A(\Gamma)$. This is made precise in Section 4. We also obtain

Corollary 1.4. *Any right-angled Artin group admits a homomorphism to* Out($\mathbb{F}_n$)*, for some n, which is a quasi-isometric embedding.*

Although much of the inspiration for this paper is drawn from [7], there are several significant points of departure. First, the methods of [7] rely heavily on subsurface projections for Mod(S), which were introduced by Masur and Minsky in [24]. When working in Out($\mathbb{F}_n$), however, there are different possible projections that one could employ. In [4], Bestvina and Feighn begin with free factors A and B of $\mathbb{F}_n$ that are in "general position," and they define the projection of A to the free splitting complex of B. These projections, though powerful in other settings, are not delicate enough for our application. In particular, the presence of commuting outer automorphisms in our construction precludes the free factors from satisfying the conditions for finite diameter Bestvina–Feighn projections. See [27] for recent work that extends the Bestvina–Feighn projections to a larger class of free factors. In [26], a different sort of projection is developed. Sabalka and Savchuk consider a topologically defined projection using sphere systems in M_n, the double of the handlebody of genus n. Although these projections are interesting in their own right, they do not always give free splittings of free factors and so they cannot be used in this paper. These difficulties are discussed in detail in Section 5.3. To resolve these issues, we develop our own projections which are tailored for the applications in this paper. In the process, we demonstrate the relationship between the projections of [4] and [26], answering a question that appears in both papers.

Second, the authors of [7] use the Masur–Minsky distance formulas for Mod(S) to verify that the homomorphisms they construct are quasi-isometric embeddings. For Out($\mathbb{F}_n$), however, there are no general distance formulas available. Instead, in Section 10 we address this issue by using the partial ordering on the syllables of $g \in A(\Gamma)$. This partial ordering allows us to control distance in Out(F_n) by using the projections that are defined in Section 3.2. This suffices for proving the lower bounds on Out($\mathbb{F}_n$)-distance that is needed in our main theorem.

Finally, we note that there is another method to construct quasi-isometrically embedded right-angled Artin subgroups of Out($\mathbb{F}_{2g}$). One could start with a once-punctured genus g surface $\dot{S}$ and use the methods of [7] to build a quasi-isometric embedding from $A(\Gamma)$ into Mod($\dot{S}$). In [11], the authors show that the injective homomorphism Mod($\dot{S}$) $\to$ Out($\mathbb{F}_{2g}$) induced by the action of Mod($\dot{S}$) on $\pi_1(\dot{S}) = \mathbb{F}_{2g}$, is itself a quasi-isometric embedding. Composing two such maps then gives a quasi-isometric embedding from $A(\Gamma)$ into $\mathrm{Out}(\mathbb{F}_{2g})$. These homomorphisms have the property that they factor through mapping class groups and, hence, fix the conjugacy class in $\mathbb{F}_{2g}$ corresponding to the puncture. In our approach, however, homomorphisms into $\mathrm{Out}(\mathbb{F}_n)$ do not factor through mapping class groups.

1.1. Outline of the paper and its sequel. The paper is organized as follows: Section 2 covers basic background material. Section 3 defines the subfactor projections that we use, gives their basic properties, and relates them to the projections of [4]. Section 4 defines the homomorphisms from right-angled Artin groups into $\mathrm{Out}(\mathbb{F}_n)$ that are of interest, gives a precise statement of our main theorem, and provides a few examples.

In order to control distance in $\mathrm{Out}(\mathbb{F}_n)$ for the proof of our main theorem, we require a version of Behrstock's inequality, which is an important tool for studying subsurface projections. To prove this, we work with a topological model of the projections that is developed in Section 5. This approach has the additional advantage that it can be used to relate the various notions of projection that are discussed above. In Section 6, we prove the version of Behrstock's inequality that is needed. This is followed by Sections 7 and 8 which give related partial orderings for both free factors and syllables of $g \in A(\Gamma)$. Section 9 closely follows the arguments of [7] and gives conditions when normal form words in $A(\Gamma)$ provide large projection distances.

Having arranged large projection distances, the last step is to argue that for "non-disjoint" free factors these distances independently contribute to distance in $\mathrm{Out}(\mathbb{F}_n)$; this is done in Section 10. This section can be thought of as making up for the lack of lower bounds coming from Masur–Minsky type formulas. The proof that our homomorphisms are quasi-isometric embeddings into $\mathrm{Out}(\mathbb{F}_n)$ is then concluded in Section 11.

In the sequel to this paper, we answer the following question. Fix a quasi-isometric embedding $\phi\colon A(\Gamma) \to \mathrm{Out}(\mathbb{F}_n)$, as constructed in this paper. What conditions on $g \in A(\Gamma)$ guarantee that $\phi(g)$ is a fully irreducible outer automorphism of $\mathrm{Out}(\mathbb{F}_n)$? To answer this question, we use the extension of the Bestvina–Feighn subfactor projections obtained in [27].

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2. Background

2.1. Quasi-isometries. Let (X, d_X) and (Y, d_Y) be metric spaces. $f : X \to Y$ is a (K, L)-*quasi-isometric embedding* if for all $x_1, x_2 \in X$

$$\frac{1}{K} d_X(x_1, x_2) - L \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + L.$$

If, in addition, every point of Y is within distance L from the image $f(X)$, then f is a *quasi-isometry* and X and Y are said to be *quasi-isometric*. In this paper, the metric spaces of interest arise from finite dimensional simplicial complexes. For a particular complex, the metric is induced by giving each simplex the structure of a standard Euclidean simplex. Recall that if K is a finite dimensional simplicial complex, then this piecewise Euclidean metric on K is quasi-isometric to K^1, the 1-skeleton of K, with its standard graph metric (see [5] for details). Since we are interested in the coarse geometry of such complexes, i.e. their metric structure up to quasi-isometry, this justifies our convention of when working with a complex K to consider only the graph metric on K^1. Here, and below, a *graph* is a 1-dimensional CW complex and a simply connected graph is a *tree*.

2.2. Out($\mathbb{F}$) basics. Fix $n \geq 2$ and let $\mathbb{F}_n$ denote the free group of rank n with outer automorphism group Out($\mathbb{F}_n$). When it is clear from context, the subscript n will be dropped from the notation. In this section, we recall some basic facts about Out($\mathbb{F}_n$) that we will need throughout the paper. First, a *splitting* of $\mathbb{F}$ is a minimal, simplicial action $\mathbb{F} \curvearrowright T$ on a non-trivial simplicial tree. The action is determined by a homomorphism $\psi : \mathbb{F} \to \mathrm{Aut}(T)$ into the simplicial automorphisms of T. An action on a tree is *minimal* if there is no proper invariant subtree. By a *free splitting*, we mean a splitting with trivial edge stabilizers and refer to a k-*edge splitting* as a free splitting with k natural edge orbits. Here, natural edges are the edges of the cell structure on T whose vertices all have valence ≥ 3. From Bass-Serre theory, k-edge splittings correspond to graph of groups decompositions of $\mathbb{F}$ with k edges, each edge with trivial edge group. Two actions $\mathbb{F} \curvearrowright T$ and $\mathbb{F} \curvearrowright T'$ are *conjugate* if there is a $\mathbb{F}$-equivariant homeomorphism $\chi : T \to T'$, and the conjugacy class of an action is denoted by $[\mathbb{F} \curvearrowright T]$. We will usually drop the action symbol from the notation and refer to the splitting by T. Finally, an equivariant surjection $c : T \to T'$ between $\mathbb{F}$-trees is a *collapse map* if all point preimages are connected. In this case, T is said to be a refinement of T'.

The *free splitting complex* $\mathcal{S}_n$ of the free group $\mathbb{F}_n$ is the simplicial complex defined as follows (see [14] for details). The vertex set $\mathcal{S}_n^0$ is the set of conjugacy classes of 1-edge splittings of $\mathbb{F}_n$, and $k+1$ vertices $[T_0], \ldots, [T_k]$ determine a k-simplex of $\mathcal{S}_n$ if there is a $(k+1)$-edge splitting T and collapse maps $c_i : T \to T_i$, for each $i = 0, \ldots, k$. That is, a collection of vertices span a simplex in $\mathcal{S}_n$ if they have a common refinement. We will mostly work with the barycentric subdivision of the free splitting complex, denoted by $\mathcal{S}'_n$. The vertices of $\mathcal{S}'_n$ are conjugacy classes of free splittings of $\mathbb{F}_n$ and two vertices are joined by an edge if, up to conjugacy, one refines the other. Higher dimensional simplicies are determined similarly.

For $n \geq 3$, the *free factor complex* $\mathcal{FF}_n$ of $\mathbb{F}_n$ is the simplicial complex defined as follows (see [17] or [3] for details). The vertices are conjugacy classes of free factors of $\mathbb{F}_n$ and $k+1$ conjugacy classes $[A_0], \ldots, [A_k]$ span a k-simplex if there are representative free factors in these conjugacy classes with $A_0 \subset A_1 \subset \cdots \subset A_k$. When $n = 2$, the definition is modified so that $\mathcal{FF}_2$ is the standard Farey graph. In this case, vertices of $\mathcal{FF}_2$ are conjugacy classes of rank 1 free factors and two vertices are jointed by an edge if there are representatives in these conjugacy classes that form a basis for $\mathbb{F}_2$.

$\mathrm{Out}(\mathbb{F}_n)$ acts simplicially on these complexes. For $\mathcal{FF}$, if $f \in \mathrm{Out}(\mathbb{F})$ is represented by an automorphism ϕ, we define $f[A] = [\phi A]$. It is clear that this is independent of choice of ϕ and that the action extends to a simplicial action on all of $\mathcal{FF}$. For $\mathcal{S}$ the action is defined as follows: with f and ϕ as above and $[T] \in \mathcal{S}^0$, suppose that the action on T is given by the homomorphism $\psi : \mathbb{F} \to \mathrm{Aut}(T)$. Then $f[T]$ is the conjugacy class of $\mathbb{F}$-tree determined by $\psi \circ \phi^{-1} : \mathbb{F} \to \mathrm{Aut}(T)$. That is, the underlying tree is unchanged and the action is precomposed with the inverse of a representative automorphism for f. Again, checking that this is a well-defined action that extends to all of $\mathcal{S}$ (or $\mathcal{S}'$) is an easy exercise. These definitions have the convenient property that if $[T]$ is a conjugacy class of free splitting with vertex stabilizers $[A_1], \ldots, [A_l]$, then $f[T]$ has vertex stabilizers $f[A_1], \ldots, f[A_l]$, for any $f \in \mathrm{Out}(\mathbb{F})$.

There is a natural, coarsely defined map $\pi : \mathcal{S}' \to \mathcal{FF}$. For $T \in (\mathcal{S}')^0$, we set $\pi(T)$ equal to the set of free factors that arise as a vertex group of a 1-edge collapse of T. That is, $A \in \pi(T)$ if and only if there is a tree $T_0 \in \mathcal{S}^0$, T refines T_0, and A is a vertex group of T_0. Letting $d_{\mathcal{FF}}$ denote distance in $\mathcal{FF}$ and setting $d_{\mathcal{FF}}(\pi(T), \pi(T')) = \mathrm{diam}_{\mathcal{FF}}(\pi(T) \cup \pi(T'))$, it is easily verified that π is coarsely 4-Lipschitz [3]. Note that here, and throughout the paper, the brackets that denote conjugacy classes of trees and free factors will often be suppressed when it should cause no confusion to do so.

Recent efforts to understand the free splitting and free factor complex have focused on their metric properties along with their similarity to the curve complex of a surface. In particular, both complexes are now known to be Gromov-hyperbolic. Hyperbolicity of the free factor complex was proven by Bestvina and Feighn in [3], and hyperbolicity of the free splitting complex was proven by Handel and Mosher in [14]. See [20] and [12] for alternative proofs and perspectives. Although these results represent significant progress in understanding the geometry of $\mathcal{FF}$ and $\mathcal{S}$, they are not directly used in this paper.

We remark that the action $\mathrm{Out}(\mathbb{F}_n) \curvearrowright \mathcal{S}_n$ is far from proper; all vertices have infinite stabilizers. There is, however, an invariant subcomplex of $\mathcal{S}_n'$ that is locally finite, and the inherited action is proper. This is the spine of Outer space and we refer the reader to [8] or [15] for details beyond what is discussed here. Also, see [10] or [1] for an alternative perspective.

The *spine of Outer space* $\mathcal{K}_n$ is the subcomplex of $\mathcal{S}_n'$ spanned by vertices that correspond to proper splittings of $\mathbb{F}_n$. Recall that a splitting T is *proper* if no element of $\mathbb{F}_n$ fixes a vertex in T. Hence, $T \in \mathcal{S}_n'$ is proper if and only if $T/\mathbb{F}_n$ is a graph with fundamental group isomorphic to $\mathbb{F}_n$. Observe that since $\mathrm{Out}(\mathbb{F}_n)$ preserves the vertices of $\mathcal{S}_n'$ corresponding to proper splittings there is an induced simplicial action $\mathrm{Out}(\mathbb{F}_n) \curvearrowright \mathcal{K}_n$.

It is well-known that $\mathcal{K}_n$ is a locally finite, connected complex and that the action $\mathrm{Out}(\mathbb{F}_n) \curvearrowright \mathcal{K}_n$ is proper and cocompact (see [8]). Hence, for any tree $T \in \mathcal{K}_n^0$, the orbit map $g \mapsto gT$ defines a quasi-isometry from $\mathrm{Out}(\mathbb{F}_n)$ to $\mathcal{K}_n$ by the Švarc-Milnor lemma [5]. As remarked above, the metric considered here is the standard graph metric on $\mathcal{K}_n^1$, the 1-skeleton of the spine of Outer space. This metric on $\mathcal{K}_n^1$ will serve as our geometric model for $\mathrm{Out}(\mathbb{F}_n)$.

2.3. The sphere complex. We recall the $\mathrm{Out}(\mathbb{F}_n)$-equivalent identification between the free splitting complex and the sphere complex. See [1] for details. Take $M_n = \#_n(S^1 \times S^2)$, or equivalently, the double of the handlebody of genus n. Let $M_{n,s}$ be M_n with s open 3-balls removed. Note that $\pi_1 M_n$ is isomorphic to $\mathbb{F}_n$ and, once and for all, fix such an isomorphism. A sphere S in $M_{n,s}$ is *essential* if it is not boundary parallel and does not bound a 3-ball. A collection of disjoint, essential, pairwise non-isotopic spheres in M_n is called a *sphere system*. By [21], spheres S_1 and S_2 are homotopic in M_n if and only if they are isotopic.

The *sphere complex* $\mathcal{S}(M_n)$ is the simplicial complex whose vertices are isotopy classes of essential spheres and vertices $[S_0], \dots, [S_k]$ span a k-simplex if there are representatives in these isotopy classes that are disjoint in M_n. It is a theorem of [21] that with $\mathrm{Mod}(M_n) = \pi_0(\mathrm{Diff} M_n)$ there is an exact sequence

$$1 \longrightarrow K \longrightarrow \mathrm{Mod}(M_n) \longrightarrow \mathrm{Out}(\mathbb{F}_n) \longrightarrow 1,$$

where K is a finite group generated by "Dehn twists" about essential spheres. Since elements of K act trivially on $\mathcal{S}(M_n)$, we have a well-defined action $\mathrm{Out}(\mathbb{F}_n) \curvearrowright \mathcal{S}(M_n)$. The following proposition of Aramayona and Souto identifies $\mathcal{S}_n$ and $\mathcal{S}(M_n)$. See Section 5.1 for how one constructs splittings from essential spheres.

Proposition 2.1 ([1]). *For $n \geq 2$, $\mathcal{S}_n$ and $\mathcal{S}(M_n)$ are* $\mathrm{Out}(\mathbb{F}_n)$*-equivariantly isomorphic.*

2.4. Translation length in $\mathcal{FF}_n$. An outer automorphism $f \in \mathrm{Out}(\mathbb{F}_n)$ is *fully irreducible* if no positive power of f fixes a conjugacy class of a free factor. That is, for any $A \in \mathcal{FF}_n^0$, $f^n(A) = A$ implies that $n = 0$. Recall that the (*stable*) *translation length* of an outer automorphism $f \in \mathrm{Out}(\mathbb{F}_n)$ on $\mathcal{FF}_n$ is defined as

$$\ell_{\mathcal{FF}}(f) = \lim_{k \to \infty} \frac{d_{\mathcal{FF}}(A, f^k A)}{k}$$

where $A \in \mathcal{FF}_n^0$. It is not difficult to verify that $\ell_{\mathcal{FF}}(f)$ is well-defined and independent of $A \in \mathcal{FF}_n^0$. It also satisfies the property $\ell_{\mathcal{FF}}(f^n) = n \cdot \ell_{\mathcal{FF}}(f)$ for $n \geq 0$. Further, $\ell_{\mathcal{FF}}(f) \geq c$ if and only if for all $A \in \mathcal{FF}_n^0$, $d_{\mathcal{FF}}(A, f^n A) \geq c|n|$. The following proposition of Bestvina and Feighn characterizes those outer automorphisms with positive translation length on $\mathcal{FF}_n$.

Proposition 2.2 ([3]). *Let $f \in$* $\mathrm{Out}(\mathbb{F}_n)$*, f is fully irreducible if and only if $\ell_{\mathcal{FF}}(f) > 0$.*

It appears to be an open question whether there is a uniform lower bound on translation length for fully irreducible outer automorphisms of $\mathrm{Out}(\mathbb{F}_n)$. In the mapping class group situation, this is indeed the case. That is, for a fixed surface S there is an $\eta > 0$ so that if $f \in \mathrm{Mod}(S)$ is pseudo-Anosov then the curve complex translation length of f is greater than or equal to η [23]. It is worthwhile to note that when $n = 2$, Proposition 2.2 reduces to the following statement: if an outer automorphism is infinite order and does not fix a conjugacy class of a primitive element in $\mathbb{F}_2$, then it acts with positive translation length on $\mathcal{FF}_2$, which as noted above is the Farey graph.

3. Projections to free factor complexes

For a finitely generated subgroup $H \leq \mathbb{F}$, let $\mathcal{S}(H)$ and $\mathcal{F}(H)$ denote the free splitting complex and free factor complex of H, respectively. A subgroup H is *self-normalizing* if $N(H) = H$, where $N(H)$ is the normalizer of H in $\mathbb{F}$. When H is self-normalizing the complexes $\mathcal{S}(H)$ and $\mathcal{F}(H)$ depend only on the conjugacy class of H in $\mathbb{F}$. More precisely, if $H' = gHg^{-1}$ for $g \in \mathbb{F}$, then g induces an isomorphism between $\mathcal{S}(H)$ and $\mathcal{S}(H')$ (and between $\mathcal{F}(H)$ and $\mathcal{F}(H')$) via conjugation. For any other $x \in \mathbb{F}$ with $H' = xHx^{-1}$ we see that $x^{-1}g$ normalizes H and so $x^{-1}g \in H$. In this case, $gH = xH$ and it is easily verified that g and x induce identical isomorphisms between $\mathcal{S}(H)$ and $\mathcal{S}(H')$. Hence, when H is self-normalizing we obtain a canonical identification between the free splitting complex of H and the free splitting complex of each of its conjugates. The same holds for the free factor complex of H. This allows us to unambiguously refer to the free splitting complex or free factor complex for the conjugacy class $[H]$. Finally, recall that a subgroup $C \leq \mathbb{F}$ is *malnormal* if $xCx^{-1} \cap C \neq \{1\}$ implies that $x \in C$. For example, free factors of $\mathbb{F}$ are malnormal and malnormal subgroups are self-normalizing.

3.1. Projecting trees. Given a free splitting $T \in \mathcal{S}'$ and a finitely generated subgroup $H \leq \mathbb{F}$ denote by T^H the *minimal H-subtree* of T. This is the unique minimal H-invariant subtree of the restricted action $H \curvearrowright T$. For any such H, T^H is either trivial, in which case H fixes a unique vertex in T, or T^H is the union of axes of elements in H that act hyperbolically on T. When T^H is not trivial, we define the *projection* of T to the free splitting complex of H as $\pi_{\mathcal{S}(H)}(T) = [H \curvearrowright T^H]$, where the brackets denote conjugacy of H-trees. Note that this projection is a well-defined vertex of $\mathcal{S}'(H)$ and it depends only on the conjugacy class of T. To see this, note that any conjugacy between $\mathbb{F}$-trees will induce a conjugacy between their minimal H-subtrees. Further define the projection to the free factor complex of H to be the composition $\pi_H(T) = \pi(\pi_{\mathcal{S}(H)}(T))$, where $\pi\colon \mathcal{S}(H) \to \mathcal{F}(H)$ is the 4-Lipschitz map defined in Section 2.2. Hence, $\pi_H(T) \subset \mathcal{F}(H)$ is the collection of free factors of H that arise as a vertex group of a one-edge collapse of the splitting $H \curvearrowright T^H$. When H is also self-normalizing, e.g. a free factor, these projections are independent of the choice of H within its conjugacy class. The following lemma verifies that such projections are coarsely Lipschitz.

Lemma 3.1. *Let $\mathbb{F}_n \curvearrowright T$ be a free splitting and $H \le \mathbb{F}_n$ a finitely generated subgroup with T^H non-trivial. Let T_0 be a refinement of T with equivariant collapse map $c\colon T_0 \to T$. Then there is an induced collapse map $c_H\colon T_0^H \to T^H$. Hence, T_0^H is a refinement of T^H.*

Proof. Since $c(T_0^H) \subset T$ is an invariant H-tree, it contains T^H. Also, the axis in T_0 of any hyperbolic $h \in H$ is mapped by c to either h's axis in T or a singe vertex stabilized by h; each of which is contained in T^H. Since T_0^H is the union of such axes, we see that $c(T_0^H) = T^H$. Hence the map c_H described in the lemma is given by restriction. It remains to show that c_H is a collapse map. This is the case since for any $p \in T^H$,

$$c_H^{-1}(p) = T_0^H \cap c^{-1}(p)$$

is the intersection of two subtrees of T_0 and is, therefore, connected. □

For a free factor A of $\mathbb{F}$ we use the symbol d_A to denote distance in $\mathcal{F}(A)$ and for $\mathbb{F}_n$-trees T_1, T_2 we use the shorthand

$$d_A(T_1, T_2) := d_A(\pi_A(T_1), \pi_A(T_2)) = \mathrm{diam}_A(\pi_A(T_1) \cup \pi_A(T_2))$$

when both projections are defined. The following proposition follows immediately from the definitions in this section and Lemma 3.1.

Proposition 3.2 (Basic properties I). *Let T_1, T_2 be adjacent vertices in $\mathcal{K}_n$, $A \in \mathcal{FF}_n$, and H a finitely generated and self-normalizing subgroup of $\mathbb{F}_n$ containing A, up to conjugacy. Then we have the following:*

(1) $\mathrm{diam}_{\mathcal{F}(A)}(\pi_A(T)) \le 4$*;*

(2) $d_A(T_1, T_2) \le 4$*;*

(3) $\pi_A(T_1) = \pi_A(\pi_{\mathcal{S}(H)}(T_1))$ *and so* $d_A(T_1, T_2) = d_A(\pi_{\mathcal{S}(H)}(T_1), \pi_{\mathcal{S}(H)}(T_2))$.

3.2. Projecting factors. Let A and B be rank ≥ 2 free factors of $\mathbb{F}_n$. Define A and B to be *disjoint* if they are nonconjugate vertex groups of a free splitting of $\mathbb{F}_n$. Disjoint free factors are those that will support commuting outer automorphisms in our construction. Define A and B to *meet* if there exist representatives in their conjugacy classes whose intersection is nontrivial and proper in each factor. In this section, we show that this intersection provides a well-defined projection of $[B]$ to $\mathcal{F}(A)$, the free factor complex of A. Note that if A and B meet, then $d_{\mathcal{FF}}([A], [B]) = 2$.

Fix free factors A and B in $\mathbb{F}_n$. Define the projection of B into $\mathcal{F}(A)$ to be

$$\pi_A(B) = \{[A \cap gBg^{-1}] \colon g \in \mathbb{F}_n\} \setminus \{[1], [A]\},$$

where conjugacy is taken in A. Observe that A and B meet exactly when $\pi_A(B) \neq \emptyset \neq \pi_B(A)$. We show that members of $\pi_A(B)$ are vertex groups of a single (non-unique) free splitting of A and so $\pi_A(B)$ has diameter less than or equal to 4 in $\mathcal{F}(A)$. Since the projection is independent of the conjugacy class of B, this provides the desired projection from $[B]$ to the free factor complex of A.

Lemma 3.3. *Suppose the free factors A and B meet. Then $\mathrm{diam}_{\mathcal{F}(A)}\pi_A(B) \leq 4$.*

Proof. First, observe that g uniquely determines the class $[A \cap gBg^{-1}] \in \pi_A(B)$ up to double coset in $\mathbb{F}$. Precisely, $[A \cap gBg^{-1}] = [A \cap hBh^{-1}] \neq 1$ if and only if $AgB = AhB$; this follows from the fact that free factors are malnormal. Now choose any marked graph G which contains a subgraph G^B whose fundamental group represents B up to conjugacy. Let $p_A \colon \widetilde{G}_A \to G$ be the cover of G corresponding to free factor A and let G_A denote the core of $\widetilde{G}_A$. By covering space theory, the components of $p^{-1}(G^B)$ are in bijective correspondence with the double cosets $\{AgB \colon g \in \mathbb{F}\}$. Also, the fundamental group of the component corresponding to AgB is $A \cap gBg^{-1}$. Since the core carries the fundamental group of $\widetilde{G}_A$, all nontrivial subgroups $A \cap gBg^{-1}$ correspond to double cosets representing components of $p^{-1}(G^B)$ in the core G_A. Hence, G_A is a marked A-graph that contains disjoint subgraphs whose fundamental groups (up to conjugacy in A) are the subgroups of $\pi_A(B)$. This completes the proof. □

If $A \in \mathcal{FF}_n^0$ and $f \in \mathrm{Out}(\mathbb{F}_n)$ stabilizes A, then f induces an outer automorphism of A, denoted $f|_A \in \mathrm{Out}(A)$. In this case, let $\ell_A(f)$ represent the translation length of $f|_A$ on $\mathcal{F}(A)$. By Proposition 2.2, if $f|_A$ is fully irreducible in $\mathrm{Out}(A)$, then $\ell_A(f) > 0$. The following proposition provides the additional properties of the projections that will be needed throughout the paper. Its proof is a straightforward exercise in working through the definitions of this section.

Proposition 3.4 (Basic properties II). *Let $A, B, C \in \mathcal{FF}_n^0$ so that A and B meet and A and C are disjoint. Let $c \in \mathrm{Out}(\mathbb{F})$ stabilize the free factors A and C with $c|_A = 1$ in $\mathrm{Out}(A)$. Finally, let $T \in \mathcal{K}^0$ and $f \in \mathrm{Out}(\mathbb{F})$ be arbitrary. Then f induces an isomorphism*

$$f \colon \mathcal{F}(A) \longrightarrow \mathcal{F}(fA)$$

and we have the following:

(1) *$f(A)$ and $f(B)$ meet and $\pi_{fA}(fB) = f(\pi_A(B)) \subset \mathcal{F}(fA)$;*

(2) *$\pi_{fA}(fT) = f(\pi_A(T)) \subset \mathcal{F}(fA)$;*

(3) *$\pi_A(cB) = \pi_A(B) \subset \mathcal{F}(A)$;*

(4) *$\pi_A(cT) = \pi_A(T) \subset \mathcal{F}(A)$.*

For the applications in this paper, a slightly stronger condition than meeting is necessary on the free factors A and B. In particular, we need their meeting representatives to generate the "correct" subgroup of $\mathbb{F}$. More precisely, say that two free factors A and B of $\mathbb{F}$ *overlap* if there are representatives in their conjugacy classes, still denoted A and B, so that $A \cap B = x \neq \{1\}$ is proper in both A and B and the subgroup generated by these representatives $\langle A, B\rangle \leq \mathbb{F}$ is isomorphic to $A *_x B$. Note that the first condition here is exactly that A and B meet.

Example 1. Here is an example of free factors that meet but do not overlap. Let $\mathbb{F}_6 = \langle a, b, c, d, e, f\rangle$ and consider the free factors $A = \langle a, b, c, d, f\rangle$ and $B = \langle aec, bed, f\rangle$. It is quickly verified that $A \cap B = \langle f\rangle$, so A and B meet. However, $\langle A, B\rangle = \mathbb{F}_6$ is not isomorphic to $A *_{\langle f\rangle} B = \langle a, b, c, d\rangle * B$, which has rank 7.

Remark 3.5. Suppose the free factors $[A], [B] \in \mathcal{FF}$ overlap and select representatives in their conjugacy classes so that $A \cap B = x$ is nontrivial and proper in both A and B. Note that as in Lemma 3.3 the free factor x is not necessarily unique up to conjugacy, but once the conjugacy class of x is fixed the subgroup $H = \langle A, B\rangle$ generated by these conjugacy class representatives is itself determined up to conjugacy in $\mathbb{F}$. Since A and B overlap, x can be chosen so that $H \cong A *_x B$ and it is not difficult to verify that H is finitely generated and self-normalizing. So, for example, if $T \in S'$, then $\pi_A(T) = \pi_A(\pi_{\mathcal{S}(H)})(T)$ by Lemma 3.2. Projections of meetings factors, however, may slightly change. In particular, A and B are free factors of H that overlap but, now as subgroups of H, x is their unique intersection up to conjugacy. In general, we use the notation $\pi_A(B \leq H)$ to denote the projection of B to the free factor complex of A when B is considered as a free factor of H. Note that in this case $\pi_A(B \leq H) = \{[x]\} \subset \pi_A(B) \subset \mathcal{F}(A)$ and so although the choice of x and, hence, H is not uniquely determined by the overlapping free factors A and B, this ambiguity is not significant when considering projections.

3.3. The Bestvina–Feighn Projections. In [4], the authors show that there is a finite coloring of the vertices of the free factor complex $\mathcal{FF}_n$ so that if A and B are free factors of $\mathbb{F}_n$ with either

(1) A and B have the same color, or

(2) $d_{\mathcal{FF}}(A, B) > 4$,

then there is a well-defined projection $\pi^{\mathrm{BF}}_{\mathcal{S}(A)}(B) \subset \mathcal{S}(A)$ with uniformly bounded diameter. Moreover, these projections have properties similar to those of subsurface projections. The Bestvina–Feighn projection is defined as follows: first choose $T \in \mathcal{K}^0_n$ with the property that the marked graph $T/\mathbb{F}_n$ contains an embedded subgraph whose fundamental group represents B, then define $\pi^{\mathrm{BF}}_{\mathcal{S}(A)}(B) = \pi_{\mathcal{S}(A)}(T) \subset \mathcal{S}(A)$. It is shown that when A and B satisfy the stated conditions, this projection is coarsely independent of the choice of T. See [4] for details.

Free factors that meet, however, do *not* satisfy the conditions stated above, and it is easy to construct examples where A and B meet but the projection $\pi^{\mathrm{BF}}_{\mathcal{S}(A)}(B)$ does not have finite diameter in $\mathcal{S}(A)$ (as the choice of T is varied). Despite this, Lemma 3.3 shows that if we further project to the free factor complex of A we obtain a set with finite diameter. This shows that when the free factors A and B meet, the projection $\pi_A(B)$ defined in this paper agrees coarsely with the projection $\pi(\pi^{\mathrm{BF}}_{\mathcal{S}(A)}(B)) \subset \mathcal{F}(A)$. See [27] for further discussion. In Section 5.3, we relate the projections discussed here with those of [26].

4. The homomorphisms $A(\Gamma) \to \mathrm{Out}(\mathbb{F}_n)$

In this section, we present the most general version of our theorem. Technical conditions are unavoidable since, unlike the surface case, free factors do not uniquely determine splittings. Also, some care must be taken when defining the support of an outer automorphism. After presenting the general conditions, we also give a specific construction for applying the main theorem. The idea is to replace the surface in the mapping class group situation with a graph of groups decomposition of $\mathbb{F}$.

4.1. Admissible systems. Let $\mathcal{A} = \{A_1, \dots, A_n\} \subset \mathcal{FF}^0$ be a collection of (conjugacy classes of) rank ≥ 2 free factors of $\mathbb{F}$ such that for $i \neq j$ either

(1) A_i and A_j are *disjoint*, that is they are vertex groups of a common splitting, or

(2) A_i and A_j *overlap*, so in particular $\pi_{A_i}(A_j) \neq \emptyset \neq \pi_{A_j}(A_i)$.

Then we say that $\mathcal{A}$ is an *admissible collection* of free factors of $\mathbb{F}$. Let $\Gamma = \Gamma_{\mathcal{A}}$ be the coincidence graph for $\mathcal{A}$. This is the graph with a vertex v_i for each A_i and an edge connecting v_i and v_j whenever the free factors A_i and A_j are disjoint.

An outer automorphism $f_i \in \mathrm{Out}(\mathbb{F})$ is said to be *supported* on the factor A_i if $f_i(A_j) = A_j$ for each v_j in the star of $v_i \in \Gamma^0$ and $f_i|_{A_j} = 1 \in \mathrm{Out}(A_j)$ for each v_j in the link of $v_i \in \Gamma^0$. Informally, f_i is required to stabilize and act trivially on each free factor in $\mathcal{A}$ that is disjoint from A_i as well as stabilize A_i itself. We say that f_i is *fully supported* on A_i if, in addition, $f_i|_{A_i} \in \mathrm{Out}(A_i)$ is fully irreducible. Finally, we call the pair $\mathcal{S} = (\mathcal{A}, \{f_i\})$ an *admissible system* if the f_i are fully supported on the collection of free factors $\mathcal{A}$ and for each v_i, v_j joined by an edge in Γ, f_i and f_j commute in $\mathrm{Out}(\mathbb{F})$ (this condition is made unnecessary in the construction of the next section).

Given an admissible system $\mathcal{S} = (\mathcal{A}, \{f_i\})$, we have the induced homomorphism

$$\phi = \phi_{\mathcal{S}} \colon A(\Gamma) \longrightarrow \mathrm{Out}(\mathbb{F}_n)$$

defined by mapping $v_i \mapsto f_i$. Our main theorem is the following:

Theorem 4.1. *Given an admissible collection $\mathcal{A}$ of free factors for $\mathbb{F}$ with coincidence graph Γ there is a $C \geq 0$ so that if outer automorphisms $\{f_i\}$ are chosen to make $\mathcal{S} = (\mathcal{A}, \{f_i\})$ an admissible system with $\ell_{A_i}(f_i) \geq C$ then the induced homomorphism $\phi = \phi_{\mathcal{S}} \colon A(\Gamma) \to \mathrm{Out}(\mathbb{F})$ is a quasi-isometric embedding.*

It is worth noting that since right-angled Artin groups are torsion-free, homomorphisms from $A(\Gamma)$ that are quasi-isometric embeddings are injective.

4.2. Splitting construction. Here we present a particular type of graph of groups decomposition of $\mathbb{F}$ that allows for easy applications of Theorem 4.1. Let $\mathcal{G}$ be a free splitting of $\mathbb{F}$ along with a family of collapse maps

$$p_i \colon \mathcal{G} \longrightarrow \mathcal{G}_i$$

to splittings $\mathcal{G}_i$, satisfying the following conditions:

(1) Each splitting $\mathcal{G}_i$ has a preferred vertex $v_i \in \mathcal{G}_i$ so that all edges of $\mathcal{G}_i$ are incident to v_i.

(2) Setting $G_i = p^{-1}(v_i) \subset \mathcal{G}$ we require that for $i \neq j$ one of the two following conditions hold: either (i) G_i and G_j are *disjoint*, meaning that $G_i \cap G_j = \emptyset$, or (ii) $G_i \cap G_j$ is a subgraph whose induced subgroup is nontrivial and proper in each of the subgroups induced by G_i and G_j. In the latter case, we say the subgraphs *overlap*.

We call the splitting $\mathcal{G}$ satisfying these conditions a *support graph*, and we note that the above data is determined by the collection of subgraphs G_i. For such a splitting of $\mathbb{F}$, we set $A_i = \pi_1(G_i) = (\mathcal{G}_i)_{v_i} \in \mathcal{FF}$. This is the vertex groups of the vertex v_i in $\mathcal{G}_i$. It is clear from the above conditions that such a collection of free factors forms an admissible collection $\mathcal{A}(\mathcal{G})$ and that $\Gamma_{\mathcal{A}(\mathcal{G})}$ is precisely the coincidence graph of the G_i in $\mathcal{G}$.

Next, we consider the outer automorphisms that will generate the image of our homomorphism. For each i, chose an $f_i \in \mathrm{Out}(\mathbb{F}_n)$ which preserves the splitting $\mathcal{G}_i$, induces the identity automorphism on the underlying graph of $\mathcal{G}_i$, and restricts to the identity on the complement of v_i in $\mathcal{G}_i$. In this case, we say that f_i is *supported* on G_i (or v_i), and if the restriction of f_i to the free factor A_i is fully irreducible, we say that f_i is *fully supported* on G_i (or v_i). With these choices, the pair $\mathcal{S}(\mathcal{G}) = (\mathcal{A}(\mathcal{G}), \{f_i\})$ is an admissible system. Indeed, the only condition to check is that if v_i and v_j represent disjoint free factors, then the outer automorphisms f_i and f_j commute. Observe that since G_i and G_j are disjoint subgraphs of $\mathcal{G}$ we may collapse each to a vertex to obtain a common refinement $\mathcal{G}_{ij}$ of $\mathcal{G}_i$ and $\mathcal{G}_j$, which has vertices with associated groups $(\mathcal{G}_i)_{v_i}$ and $(\mathcal{G}_j)_{v_j}$. Label these vertices of $\mathcal{G}_{ij}$ v_i and v_j corresponding to the subgraphs G_i and G_j of $\mathcal{G}$. From the fact that f_i and f_j are supported on G_i and G_j, respectively, it follows that they both stabilize the common refinement $\mathcal{G}_{ij}$ and are each supported on distinct vertices, namely v_i and v_j. This implies that f_i and f_j commute in $\mathrm{Out}(\mathbb{F})$. Hence, $\mathcal{S}(\mathcal{G}) = (\mathcal{A}(\mathcal{G}), \{f_i\})$ is an admissible system inducing a homomorphism

$$\phi_{\mathcal{S}(\mathcal{G})} \colon A(\Gamma_{\mathcal{G}}) \longrightarrow \mathrm{Out}(\mathbb{F})$$

given by

$$v_i \longmapsto f_i$$

as before. With this setup, our main result can be restated as follows.

Corollary 4.2. *Suppose $\mathcal{G}$ is a free splitting of $\mathbb{F}$ that is a support graph with subgraphs G_i for $1 \le i \le k$. Let Γ be the coincidence graph for these subgraphs. There is a $C \ge 0$ so that if for each i, $f_i \in \mathrm{Out}(\mathbb{F})$ is fully supported on G_i with $\ell_{A_i}(f_i) \ge C$, then the induced homomorphism*

$$\phi_{\mathcal{S}(\mathcal{G})} \colon A(\Gamma) \longrightarrow \mathrm{Out}(\mathbb{F})$$

is a quasi-isometric embedding.

We remark that once a support graph $\mathcal{G}$ is constructed with $\pi_1\mathcal{G} = \mathbb{F}_n$, there is no obstruction to finding f_i fully supported on G_i with large translation length on $\mathcal{F}(A_i)$. Corollary 4.2 then implies that there exist maps $\phi_{\mathcal{S}(\mathcal{G})} \colon A(\Gamma) \to \mathrm{Out}(\mathbb{F}_n)$ which are quasi-isometric embeddings.

4.3. Constructions and applications. We use our main theorem to construct quasi-isometric homomorphisms into $\mathrm{Out}(\mathbb{F}_n)$ beginning with an arbitrary right-angled Artin group $A(\Gamma)$. We provide a bound on n given a measurement of complexity of Γ.

First, it is easy to use the splitting construction of Section 4.2 to start with a graph Γ and find a quasi-isometric embedding $A(\Gamma) \to \mathrm{Out}(\mathbb{F}_n)$, with n depending on Γ. We illustrate this with an example and then give a general procedure. Note that although using the splitting construction is simple, it will always require that n is rather large compared to Γ. As demonstrated in Example 3, more creative choices of admissible systems can be used to reduce n.

Example 2. Let $\Gamma = \Gamma_5$ be the pentagon graph with vertices labeled counterclockwise v_0, v_2, v_4, v_1, v_3 as in Figure 1, and let Γ^c be the same graph with vertices labeled cyclically $v_0, \dots, v_4$. Take $\mathcal{G}$ to be the graph of groups with underlying graph $(\Gamma^c)'$, the barycentric subdivision of Γ^c, with trivial vertex group labels on the vertices of Γ^c and infinite cyclic group labels on the subdivision vertices. Note that $\pi_1\mathcal{G} = \mathbb{F}_6$. Set G_i $(1 \le i \le 4)$ equal to the subgraph of $\mathcal{G}$ consisting of the vertex labeled v_i, its two adjacent subdivision vertices, and the edges joining these vertices to v_i. Observe that G_i and G_j have empty intersection if and only if v_i and v_j are joined by an edge in Γ. Also, if G_i and G_j intersect then their intersection is a vertex with nontrivial vertex group. Hence, $\mathcal{G}$ is a support graph with subgraphs G_i whose coincidence graph is Γ. By Corollary 4.2 there is a constant C such that choosing any collection of outer automorphisms f_i fully supported on the collection G_i with $\ell_{A_i}(f_i) \ge C$ determines a homomorphism $A(\Gamma_5) \to \mathrm{Out}(\mathbb{F}_6)$ that is a quasi-isometric embedding. In Example 3, we improve this construction by modifying $\mathcal{G}$.

Now fix any simplicial graph Γ with n vertices labeled $v_1, \dots, v_n$. We give a general procedure for producing a support graph $\mathcal{G}$ with subgraphs G_i whose coincidence graph is Γ. By Corollary 4.2, this provides examples of homomorphisms $A(\Gamma) \to \mathrm{Out}(\pi_1(\mathcal{G}))$ that are quasi-isometric embeddings. First, assume that the *complement graph* Γ^c is connected. Recall that Γ^c is the subgraph of the complete graph on Γ^0 whose edge set is the complement of the edge set of Γ. Let $(\Gamma^c)'$ be the barycentric subdivision of Γ^c. We reserve labels v_i for the vertices of $(\Gamma^c)'$ that are vertices of Γ^c and label the vertex of $(\Gamma^c)'$ corresponding to the edge (v_i, v_j) of Γ^c by v_{ij}. Hence, in $(\Gamma^c)'$ the vertex v_{ij} is valence two and is connected by an edge to both v_i and v_j. Set G_i equal to the star of the vertex v_i in $(\Gamma^c)'$, i.e. G_i is the union of edges incident to v_i together with their vertices. Now take $\mathcal{G}$ to be the graph of groups with underlying graph $(\Gamma^c)'$ and infinite cyclic vertex group labels for each vertex v_{ij}, $i \neq j$. For vertices v_i there are two cases for vertex groups. If v_i has valence one in $\mathcal{G}$ then we label it with an infinite cyclic vertex group and otherwise we give it a trivial vertex group.

With these vertex groups, $\mathcal{G}$ becomes of graph of groups decomposition for $\mathbb{F}_n$. Moreover, $\mathcal{G}$ is a support graph for the collection of subgraphs G_i with coincidence graph Γ. Indeed, G_i and G_j have nonempty intersection in $\mathcal{G}$ if and only if v_i and v_j are joined by an edge in Γ^c. When this is the case, their intersection is a single vertex with infinite cyclic vertex group and this vertex group is proper in each of the groups induced by G_i and G_j. We can also calculate the rank of $\pi_1\mathcal{G}$. By construction, the rank of $\pi_1\mathcal{G}$ is equal to the rank of the fundamental group of the underlying graph plus the number of nontrivial vertex groups on $\mathcal{G}$. Since there is a nontrivial vertex group for each edge of Γ^c and each vertex of Γ^c of valence one, the rank of $\pi_1\mathcal{G}$ equals

$$1 + 2|\mathrm{E}(\Gamma^c)| - |\mathrm{V}(\Gamma^c)| + |\text{valence 1 vertices of } \Gamma^c|.$$

Translating this into a function of Γ, we see that the rank of $\pi_1\mathcal{G}$ is

$$1 + |\mathrm{V}(\Gamma)| \cdot (|\mathrm{V}(\Gamma)| - 2) - |\mathrm{E}(\Gamma)| + |\text{valence } n-2 \text{ vertices of } \Gamma|,$$

and we refer to this quantity as the complexity of Γ, denoted $c(\Gamma)$.

When Γ^c is not connected it decomposes into components $\Gamma^c = \sqcup_{i=1}^{l} \Delta_i$ and it is not difficult to show that $A(\Gamma) = A(\Delta_1^c) \times \cdots \times A(\Delta_l^c)$. In this case, we set $c(\Gamma) = \sum_i c(\Delta_i^c)$ and the corresponding supported graph is constructed as follows. Let $\mathcal{G}(\Delta_i^c)$ be the support graph constructed as above for the graph Δ_i^c. Let $\mathcal{G}$ to be the support graph built by taking the wedge of l intervals (at one endpoint of each) and attaching the other endpoint of the ith interval to an arbitrary vertex of $\mathcal{G}(\Delta_i^c)$. The graph of groups structure on $\mathcal{G}$ is induced by that of $\mathcal{G}(\Delta_i^c)$ along with a trivial group label at the wedge vertex. Then $\mathcal{G}$ is a support graph with coincidence graph Γ and complexity $c(\Gamma)$. As noted above, the existence of a support graph with coincidence graph Γ implies the following:

Corollary 4.3. *For any simplicial graph Γ, $A(\Gamma)$ admits a homomorphism into* Out($\mathbb{F}_n$)*, with $n \leq c(\Gamma)$, which is a quasi-isometric embedding.*

The next example shows how Theorem 11.1 can be used to give quasi-isometric embeddings into Out($\mathbb{F}_n$) for smaller n than by using support graphs.

Example 3. Again, let $\Gamma = \Gamma_5$ be the pentagon graph with vertices labeled counter-clockwise v_0, v_2, v_4, v_1, v_3 as in Figure 1. Take $\mathcal{G}$ as in Figure 1. This is a graph of groups decomposition for $\mathbb{F}_5$; the central vertex has trivial vertex group

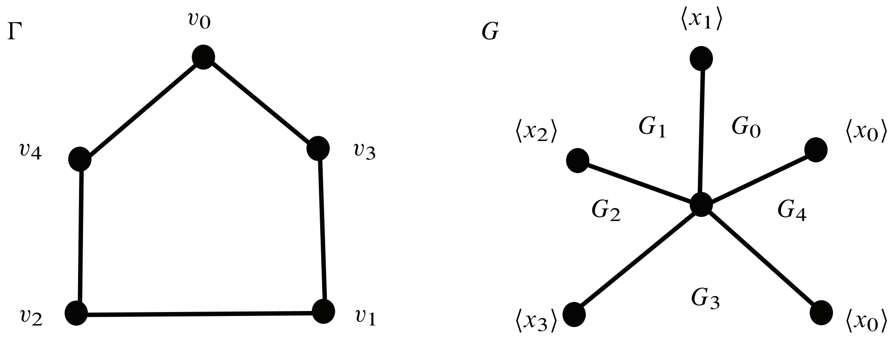

Figure 1. $\mathbb{F}_5 = \pi_1(\mathcal{G})$.

and the 5 valence one vertices joined to the central vertex each have infinite cyclic vertex group, with generators labeled $x_0 \ldots, x_4$. $\mathcal{G}$ can be thought of as a "folded" version of the support graph that appears in Example 2. For $0 \le i \le 4$, let G_i be the smallest connected subgraph containing the vertices labeled x_i and x_{i+1}, with indices taken mod 5. Note that $\mathcal{G}$ together with the subgraphs G_i is *not* a support graph; for example G_0 and G_2 intersect in a vertex with trivial vertex group. Despite this, for $i = 0, \ldots, 4$, $A_i = \pi_1 G_i = \langle x_i, x_{i+1}\rangle$ does form an admissible collection of free factors with coincidence graph Γ_5. Hence, by Theorem 4.1, there exists a $C \ge 0$ so that if there are outer automorphisms $f_i \in \mathrm{Out}(\mathbb{F}_5)$ making $(\{A_i\}, \{f_i\})$ an admissible system with $\ell_{A_i}(f_i) \ge C$ then the induced homomorphism $\phi\colon A(\Gamma_5) \to \mathrm{Out}(\mathbb{F}_5)$ is a quasi-isometric embedding. Choosing such a collection in this case is straightforward. Specifically, let $B_i = \langle x_{i+2}, x_{i+3}, x_{i+4}\rangle$ and choose $f_i \in \mathrm{Out}(\mathbb{F}_5)$ for $i = 0, \ldots, 4$ so that

(1) $f_i(A_i) = A_i$ and $f_i(B_i) = B_i$,

(2) the restriction $f_i|A_i \in \mathrm{Out}(A_i)$ is fully irreducible with $\ell_{A_i}(f_i) \ge C$, and

(3) the restriction $f_i|B_i = 1 \in \mathrm{Aut}(B_i)$.

With these choices, it is clear that each f_i is fully supported on A_i and that f_i and f_j commute if and only if v_i and v_j are joined by an edge of Γ. This makes $\mathcal{S} = (\{A_i\}, \{f_i\})$ into an admissible system with $\ell_{A_i}(f_i) \ge C$ and so the induced homomorphism

$$\phi_{\mathcal{S}}\colon A(\Gamma_5) \longrightarrow \mathrm{Out}(\mathbb{F}_5)$$

is a quasi-isometric embedding. In fact, as we shall see in the proof of the main theorem, the required translation length is simple to determine. Further, as each of free factors A_i in the admissible system is rank 2, the free factor complex $\mathcal{F}(A_i)$ is the Farey graph where translation lengths can be computed.

For an application, recall that $A(\Gamma_5)$ contains quasi-isometrically embedded copies of $\pi_1(\Sigma_2)$, the fundamental group of the closed genus 2 surface (see [9]). Restricting the homomorphism constructed above to such a subgroup, we obtain quasi-isometric embeddings

$$\pi_1(\Sigma_2) \longrightarrow \mathrm{Out}(\mathbb{F}_5).$$

5. Splittings and submanifolds

We need a topological interpretation of our projections in order to prove the version of Behrstock's inequality that appears in the next section. We first review some facts about embedded surfaces in 3-manifolds and the splittings they induce.

5.1. Surfaces and splittings. It is well-known that codimension 1 submanifolds induce splittings of the ambient manifold group [25]. We review some details here, focusing on the case when then inclusion map is not necessarily π_1-injective.

For our application, begin with an orientable, connected 3-manifold X possibly with boundary and a property embedded, orientable surface F. We do not require that F is connected or that each component of F is π_1-injective. Working, for example, in the smooth setting, choose a tubular neighborhood $N \cong F \times I$ of F in X whose restriction $N \cap \partial X$ is a tubular neighborhood of the boundary of F in ∂X. Let G denote the graph *dual* to F in X. This is the graph with a vertex for each component of $X \setminus \mathrm{int}(N)$ and an edge e_f, for each component $f \subset F$, that joins the vertices corresponding to the (not necessarily distinct) components on either side of f. We may consider G as embedded in X and, after choosing an appropriate embedding, G is easily seen to be a retract of X. The retraction is obtained by collapsing each complementary component of N to its corresponding vertex and projecting $f \times I$ to I for each component f of F. Here, I is the closed interval $[-1, 1]$ and $f \times \{0\}$ corresponds under the identification $N \cong F \times I$ to $f \subset N$.

Let $\widetilde{X}$ denote the universal cover of X and let $\widetilde{N}$ and $\widetilde{F}$ denote the complete preimage of N and F, respectively. Let T_F denote the graph dual to $\widetilde{F}$ in $\widetilde{X}$. Since T_F is a retract of the connected, simply connected space $\widetilde{X}$, T_F is a tree. We call T_F the *dual tree* to the surface F in X. As $\widetilde{F}$ and $\widetilde{X} \setminus \widetilde{N}$ are permuted by the action of $\pi_1(X)$, we obtain a simplicial action $\pi_1(X) \curvearrowright T_F$, up to the usual ambiguity of choosing basepoints. The following is an exercise in covering space theory; it appears in [25].

Proposition 5.1. *With the above notation, let v be a vertex of T_F corresponding to a lift of a component $C \subset X \setminus N$ and e an edge of T_F corresponding to a lift of a component $f \subset F$. Then*

(1) $stab(v) = \mathrm{im}(\pi_1 C \to \pi_1 X)$

(2) $stab(e) = \mathrm{im}(\pi_1 f \to \pi_1 X)$

where both equalities are up to conjugation in $\pi_1 X$.

The action $\pi_1 X \curvearrowright T_F$ provides a splitting of $\pi_1 X$ via Basse-Serre theory. The corresponding graph of groups decomposition of $\pi_1 X$ has underlying graph $G = T_F / \pi_1 X$ with vertex and edge groups as given in Proposition 5.1. A subgraph $G' \subset G$ *carries* a subgroup $H \leq \pi_1 X$ if the subgroup induced by G' contains H, up to conjugacy.

We now specialize to the situation where the action $\pi_1 X \curvearrowright T_F$ has trivial edge stabilizers. The following proposition determines when the dual tree to a surface is minimal. First, say that a connected component $f \subset F$ is *superfluous* if f separates X and to one side bounds a relatively simply connected submanifold, i.e. $X \setminus f = X_1 \sqcup X_2$ and $\mathrm{im}(\pi_1(X_1) \to \pi_1(X)) = 1$. A component of F that is not superfluous is said to *split* X. Also, use the notation $T^{\min}$ to denote the unique minimal subtree associated to an action on the tree T, see Section 2.2.

Proposition 5.2. *Let F be an orientable, properly embedded surface in the orientable 3-manifold X with $\mathrm{im}(\pi_1 f \to \pi_1 X) = 1$ for each component f of F. Then the edge $e_f \subset T$ corresponding to a lift of the component $f \subset F$ is contained in the minimal subtree $T_F^{\min}$ if and only if f splits X.*

Proof. First suppose that the edge e_f whose orbit corresponds to the lifts of f is not in the minimal subtree $T^{\min}$. Setting $G = T_F / \pi_1 X$ and $G^{\min} = T_F^{\min} / \pi_1 X$, the image of e_f in G does not lie in $G^{\min}$. Since $G^{\min}$ carries the fundamental group of X, the image of e_f in G must separate and the component of its complement not containing $G^{\min}$ has all trivial vertex groups. In X, this implies that the component $f \subset F$ separates X and to one side bounds a component whose fundamental group, when included into $\pi_1 X$, is trivial. Hence, f is superfluous.

Now suppose that f is a component of F that is superfluous. Then f corresponds to a separating edge e in $G = T_F / \pi_1 X$, with lift $e_f \subset T_F$, whose complement in G contains a component with trivial induced subgroup. Hence, this component of $G \setminus e$ is a tree with trivial vertex groups. Set G' equal to the other component of the complement of e in G. Then G' carries all of $\pi_1 X$ and so its complete preimage in T_F is connected, $\pi_1 X$ invariant, and does not contain the edge e_f. Hence, e_f is not in $T^{\min}$. □

We will use the above proposition in the following manner: If $f \subset F$ splits X, then T_f is a 1-edge collapse of T_F corresponding to a 1-edge splitting of $\pi_1 X$.

5.2. Topological projections. The purpose of this section is to give a topological description of the projection $\pi_A(T)$ in terms of submanifolds of the manifold M_n. As discussed below, these are similar to the submanifold projections of [26], and this section serves to explain the connection between these projections and the projections of [4]. To verify that our description is accurate, we rely on Hatcher's normal position for spheres in $M = M_n$ and its generalization in [16]. Let $\widetilde{M}$ denote the universal covers of M. We say that essential sphere systems S_1 and S_2 in M are in *normal position* if for $\widetilde{S_1}$ and $\widetilde{S}_2$, the complete preimage of S_1 and S_2 in $\widetilde{M}$, any spheres $s_1 \in \widetilde{S}_1$ and $s_2 \in \widetilde{S}_2$ satisfy each of the following:

(1) s_1 and s_2 intersect in at most one component and

(2) no component of $s_1 \setminus s_2$ is a disk that is isotopic relative its boundary to a disk in s_2.

This definition is easily seen to be equivalent to Hatcher's original notion of normal position in the case where one of the sphere systems is maximal [10]. In particular, the authors of [16] use Hatcher's original proof of existence and uniqueness of normal position to show the following:

Lemma 5.3. *Any two essential sphere systems S_0 and S can be isotoped to be in normal position. Also, normal position is unique in the following sense: Let S_0 be a sphere system of M_n, and let S, S' be two isotopic spheres in M_n which are in normal position with respect to S_0. Then there is a homotopy between S and S' which restricts to an isotopy on S_0.*

Fix sphere systems S and S_A and a preferred component $C_A \subset M \setminus S_A$. In what follows we assume that $S_A = \partial C_A$. When this is the case, we say C_A is a *splitting component* and observe that C_A is homeomorphic to $M_{k,s}$, as defined in Section 2.3. Let A be the (conjugacy class of) free factor $\pi_1(C_A)$ and let $T = T_S$ be the free splitting of $\mathbb{F}$ determined by the sphere system S. Since we are interested the projection of the splitting $\mathbb{F} \curvearrowright T$ to the free splitting complex of A, our aim is a topological interpretation of the projection $\pi_A(T) = [A \curvearrowright T^A]$.

Put S and S_A in normal position and consider the collection of connected components of the surface $F = S \cap C_A$. This family of surfaces is well-defined up to homotopy in C_A that restricts to isotopy on S_A, by Lemma 5.3. Consider the graph of spaces decomposition of C_A given by F with dual tree T_F, see Section 5.1. Recall that a connected component $f \subset F$ is *superfluous* if f separates C_A and to one side bounds a relatively simply connected submanifold, that is

$C_A \setminus f = C_1 \sqcup C_2$ and $\mathrm{im}(\pi_1(C_1) \to \pi_1(C_A)) = 1$. A component of F that is not superfluous is said to *split* C_A. Set $\bar{F}$ equal to F minus its superfluous components and let $T_{\bar{F}}$ be its dual tree (that is, the tree dual to the complete preimage of $\bar{F}$ in the universal cover of C_A).

We claim the following about the associated splitting of $\pi_1 C_A = A$:

(1) there is an A-equivariant simplicial embedding $\chi\colon T_F \to T$ whose image contains T^A,

(2) an edge e of T_F maps to an edge in T^A if and only if e corresponds to the lift of a component of $f \subset F$ that splits C_A, and

(3) the projection $\pi_A(T) = A \curvearrowright T^A$ is conjugate to the A-tree $T_{\bar{F}}$.

To prove the above claim we refer to Figure 2, where as above the free splitting $\mathbb{F} \curvearrowright T$ corresponds to the sphere system $S \subset M$. Let $\pi\colon \widetilde{M} \to M$ be the universal cover of M and let $\widetilde{S}$ be the complete preimage of S in $\widetilde{M}$. The map labeled $\tilde{p}$ is the equivariant map from $\widetilde{M}$ to the tree T obtained by retracting $\widetilde{M}$ to the tree dual to $\widetilde{S} \subset \widetilde{M}$, as explained in Section 5.1. Hence, if we let m denote the set of midpoints of edges of T then $\widetilde{S} = \tilde{p}^{-1}(m)$. Setting $F = S \cap C_A$ as above, we note that if $\widetilde{C}_A$ is a fixed component of the preimage of C_A in $\widetilde{M}$ then $\pi|_{\widetilde{C}_A}\colon \widetilde{C}_A \to C_A$ is the universal cover and $\widetilde{F} = (\pi|_{\widetilde{C}_A})^{-1}(F) = \widetilde{S} \cap \widetilde{C}_A$. Hence, by definition of the dual tree to F in C_A, T_F is precisely the tree dual to $\widetilde{S} \cap \widetilde{C}_A$ in $\widetilde{C}_A$.

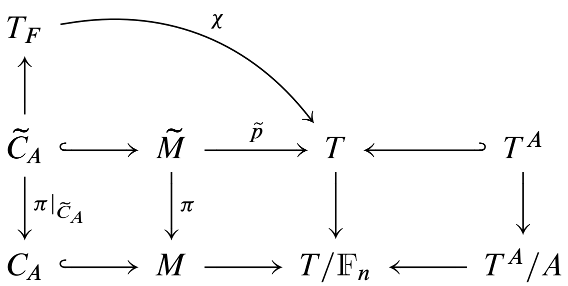

Figure 2. Defining the map $\chi\colon T_F \to T$.

Because $\tilde{p}$ is $\mathbb{F}$-equivariant, $T' = \tilde{p}(\widetilde{C}_A)$ is an A-invariant subtree of T and so it contains T^A, the minimal A-subtree of T. Note that by carefully choosing the projection $\tilde{p}$, we may assume that T' is a subcomplex of T. We first show that the A-tree T' is conjugate to the A-tree T_F. Since T is dual to $\widetilde{S}$ in $\widetilde{M}$ and T_F is dual to $\widetilde{F} = \widetilde{S} \cap \widetilde{C}_A$ in $\widetilde{C}_A \subset \widetilde{M}$ each complementary component of $\widetilde{F}$ in $\widetilde{C}_A$ corresponds to a complementary component of $\widetilde{S}$ in $\widetilde{M}$. This induces a map from the vertices of T_F to those of T. As components of $\widetilde{F}$ are contained in components of $\widetilde{S}$, this map extends to a simplicial map of A-trees $\chi\colon T_F \to T$ with image T'. We show that this map does not fold edges and is, therefore, an immersion. This suffices to prove that $\chi\colon T_F \to T'$ is an A-conjugacy.

To see that χ does not fold edge, suppose to the contrary that two edges e_1 and e_2 with common initial vertex v are identified by χ (Figure 3). Then the edge e_i is dual to a component $f_i \subset \widetilde{F}$ in $\widetilde{C}_A$ and these components are disjoint. Since e_1 and e_2 are folded by χ, their common image e in T corresponds to a sphere $s \subset \widetilde{S}$ which must contain f_1 and f_2 as subsurfaces. Let τ be an arc in s that connects the interiors of f_1 and f_2 and intersects only the components of $s \cap \partial\widetilde{C}_A$ that separate f_1 and f_2. Since each component of $\partial\widetilde{C}_A$ separates $\widetilde{M}$, as do all essential spheres in $\widetilde{M}$, the first and last components of $\partial\widetilde{C}_A$ intersected by τ must be the same. This implies that f_1 and f_2 each have a boundary component on the same component of $\partial\widetilde{C}_A$. Hence, the sphere s intersects the same component of $\partial\widetilde{C}_A$ in at least 2 circles. This, however, contradicts normal position of the sphere systems S and ∂C_A. We conclude that the A-trees T_F and T' are simplicially conjugate. This proves claim (1) and justifies identifying T_F and T' through χ. Observe that since T' contains T^A, we get an induced A-conjugacy $\chi\colon T_F^A \to T^A$ on minimal subtrees.

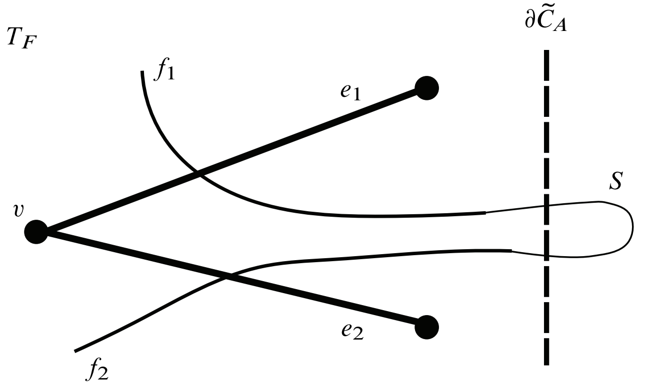

Figure 3. Folding edges.

It remains to show that $T_{\bar{F}} = T_F^A$, as this identifies the edges of T_F that correspond to components of F that split C_A with those contained in T^A. Since the components of $\bar{F}$ are precisely those that split C_A, Proposition 5.2 implies that the minimal A-subtree of T_F is $T_{\bar{F}}$ and so $T_{\bar{F}} = T_F^A$, as required. This completes the proofs of claims (2) and (3).

To summarize the above discussion:

Proposition 5.4. *Let $T \in \mathcal{S}'_n$ be a free splitting of $\mathbb{F}_n$ corresponding to the sphere system $S \subset M_n$. Fix a submanifold C_A, as above, with $\pi_1(C_A) = A$ and ∂C_A and S in normal position. If $\bar{F}$ is the surface obtained from $F = S \cap C_A$ by removing the components that separate and bound relatively simply connected components, then $\bar{F}$ is nonempty if and only if T^A is nontrivial. When this is the case, the resulting splitting $T_{\bar{F}}$ is conjugate as an A-tree to $\pi_{\mathcal{S}(A)}(T)$.*

5.3. Relations between the various projections. In [26], Sabalka and Savchuk define projections from the sphere complex $\mathcal{S}(M_n)$ to the sphere and disk complex of certain submanifolds of M_n. Their projections can be interpreted within the framework developed in this section, providing a simple relationship to $\pi_{\mathcal{S}(A)}(T)$. This answers a question asked in [26, 4]. However, it is important to note that, as demonstrated below, it is possible for each of the projections to be defined in situations when the other is not. Also, it is not clear whether the distances in the target complexes of the two projections are comparable. This section is not necessary for the rest of the paper.

Let $X \subset M_n$ denote a component of the complement of some sphere system. In [26], such X are referred to as *submanifolds*. Note that X is homeomorphic to $M_{k,s}$ for some $k < n$ and $s > 0$. The disk and sphere complex of X, denoted $\mathcal{DS}(X)$, is defined to be the simplicial complex whose vertices are isotopy classes of essential spheres and essential properly embedded disks in X with $k+1$ vertices spanning a k-simplex whenever the disks and spheres representing these vertices can be realized disjointly in X. Sabalka and Savchuk define their projections as follows. Let S be an essential sphere system in M_n. Put S and ∂X in normal position and set $F = S \cap X$. The projection $\pi_X^{\mathrm{SS}}([S]) \subset \mathcal{DS}(X)$ is then defined to be the components of F which are either spheres or disks. If there are no such components of F, then the projection is left undefined.

Fix a submanifold X with $A = \pi_1 X$ a rank ≥ 2 free factor of $\mathbb{F}_n = \pi_1 M_n$. There is a *partially* defined map $\Phi\colon \mathcal{DS}^0(X) \to \mathcal{S}^0(A)$ given by taking $D \in \mathcal{DS}^0(X)$ and mapping it to the A-tree T_D if D splits X. If D does not split X, then $\Phi(D)$ is left undefined. Recall that as in Section 5.1, T_D is the dual tree to $D \subset X$. Note that this map will be defined on all vertices of $\mathcal{DS}(X)$ *only* when X is homeomorphic to $M_{k,1}$. When D and D' are adjacent in $\mathcal{DS}(X)$ and both $\Phi(D)$ and $\Phi(D')$ are defined, then it is clear that $d_{\mathcal{S}(A)}(\Phi(D), \Phi(D')) \leq 1$. With this setup, we can show the following:

Proposition 5.5. *Let T be a free splitting of $\mathbb{F}_n$ and S its corresponding sphere system in M_n. Let X be a submanifold of M_n with $\pi_1 X = A \neq 1$. If the composition $\Phi \circ \pi_X^{SS}(S)$ is defined, then it is a free splitting of A that has $\pi_{\mathcal{S}(A)}(T)$ as a refinement.*

Proof. By Proposition 5.4, if S and ∂X are in normal position and $F = S \cap X$ then T^A is conjugate to $T_{\bar{F}}$ where $\bar{F}$ is the union of connected components of F that split X. By definition, $\Phi \circ \pi_X^{SS}(S)$ is the tree dual to the collection of disks and spheres $D \subset F$ that split X, which is nonempty by assumption. Since $D \subset \bar{F}$, the induced map $T_{\bar{F}} \to T_D$ is a collapse map. Hence, $T_{\bar{F}}$ refines $\Phi \circ \pi_X^{SS}(S)$. □

This proposition also gives the connection between the projections of [26] and those of [4]. Recall that the projection $\pi_{\mathcal{S}(A)}^{BF}(B)$ is well-defined, i.e. has bounded diameter image, when either (1) A and B have the same color in a specific finite coloring of the vertices of $\mathcal{FF}_n$ or (2) $d_{\mathcal{FF}}(A, B) > 4$. See [4] for the definition of the coloring and further details.

Corollary 5.6. *Let A, B be free factors of $\mathbb{F}_n$ satisfying one of the above conditions so that the projection $\pi_{\mathcal{S}(A)}^{BF}(B)$ is well-defined. Let X be a submanifold of M with $\pi_1 X = A$ and let S be any sphere system that contains a sphere system $S' \subset S$ whose dual tree $T_{S'}$ has B as a vertex stabilizer. If the composition $\Phi \circ \pi_X^{SS}(S)$ is defined, then it has bounded distance from $\pi_{\mathcal{S}(A)}^{BF}(B)$ in $\mathcal{S}(A)$, where the bound depends only on n.*

It is important to note that whether $\Phi \circ \pi_X^{SS}(S)$ is defined is highly dependent on the choice of X and S that represent the free factors A and B in Corollary 5.6. This is demonstrated in the examples below.

Proof. By definition, we may take $\pi_{\mathcal{S}(A)}^{BF}(B) = \pi_{\mathcal{S}(A)}(T_{S'})$. By Lemma 3.1, this is refined by the projection $\pi_{\mathcal{S}(A)}(T_S)$, and by Proposition 5.5, $\pi_{\mathcal{S}(A)}(T_S)$ also refines $\Phi \circ \pi_X^{SS}(S)$. This completes the proof since we may take as our bound the diameter of the Bestvina–Feighn projection plus 2. □

We end this section with some examples that illustrate cases when one of the projections is defined and the other is not. The general idea is that while the Bestvina–Feighn projections are robust, i.e. they do not depend on how a factor is complemented, the Sabalka and Savchuk projections are highly sensitive to the submanifold that is chosen to represent a free factor.

Example 4. Take $M = M_4$ and $S = S_1 \cup S_2$ to be a union of two essential spheres so that $X = M \setminus S$ connected with $\pi_1 X = A$. Let $f \in \mathrm{Out}(\mathbb{F}_n)$ with $f(A) = A$ but f has no power that fixes S in $\mathcal{S}(M)$. Then $\pi_{\mathcal{S}(A)}(f^n T_S) = \pi_{\mathcal{S}(A)}(T_S)$ is undefined, as A fixes a vertex of T_S, but $\pi_X^{\mathrm{SS}}(f^n S)$ is defined for all $n \geq 1$ by construction. Hence, it must be the case that each disk of $\pi_X^{\mathrm{SS}}(f^n S)$ is superfluous in X. Informally, each disk of $\pi_X^{\mathrm{SS}}(f^n S)$ $(n \geq 1)$ simply encloses some boundary components of X without splitting $\pi_1 X$.

Example 5. Take M, X, A as above and refer to Figure 4 where M is drawn as a handlebody and spheres are drawn as properly embedded disks; doubling the picture gives an illustration of what is described. Let S_3 be any sphere that separates M into two components, one of which contains $S = \partial X$ and the other, denoted Y, has $\pi_1 Y = A$. Let R be the essential sphere shown in Figure 4 with dual tree T_R; R is in normal position with S_3. Note that R splits Y with non-trivial projection $\pi_{\mathcal{S}(A)}(T_R)$. However, $Y \cap R$ has no disks of intersection and so $\pi_Y^{\mathrm{SS}}(R)$ is undefined. If instead we use the submanifold X to represent the free factor A, we see that $\pi_X^{\mathrm{SS}}(R)$ is the sphere $R \subset X$ and $\Phi \circ \pi_X^{\mathrm{SS}}(R) = \pi_{\mathcal{S}(A)}(T_R)$.

Even if we only use the submanifold X, which *exhausts* M in the terminology of [26], to represent the free factor A, the question of whether the composition $\Phi \circ \pi_X^{\mathrm{SS}}$ is defined still depends on the choice of sphere that is projected. This is because the existence of a disk in $\pi_X^{\mathrm{SS}}(R)$ that splits X is highly depended on R itself. In fact, it is not difficult to show the following: for any nonseperating sphere $R \subset X$ there is a $f \in \mathrm{Out}(\mathbb{F}_4)$ with $f(A) = A$ and $f|_A = 1 \in \mathrm{Out}(A)$, so in particular $\pi_A(f T_R) = \pi_A(T_R) = \Phi \circ \pi_X^{\mathrm{SS}}(R)$, but $\Phi \circ \pi_X^{\mathrm{SS}}(fR)$ is undefined. This implies that all disks of $\pi_X^{\mathrm{SS}}(fR)$ are superfluous even though $\pi_A(f T_R) = \pi_A(T_R)$.

6. Behrstock's Inequality

We now introduce an analog of Behrstock's inequality for projections to the free factor complex of a free factor. For the original statement and proof in the case of subsurface projections from the curve complex, see [2]. The proof of the free group version given in Proposition 6.1 is similar in spirit to the proof of the original version of Behrstock's inequality that is recorded in [22], where it is attributed to Chris Leininger. Both proofs investigate intersections of submanifolds and give explicit bounds on the distances of the projections that are considered.

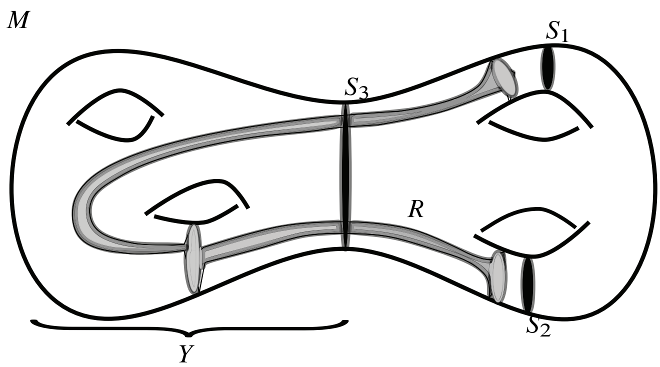

Figure 4. Projecting R to Y.

Proposition 6.1. *There is an $M \geq 0$ so that if A and B are free factors of $\mathbb{F}$ of rank ≥ 2 that overlap, then for any $T \in \mathcal{S}'$ with $\pi_A(T) \neq \emptyset \neq \pi_B(T)$ we have*

$$\min\{d_A(B,T), d_B(A,T)\} \leq M.$$

Proof. Fix $T \in \mathcal{S}'$ that has nontrivial projection to both the free factors complex of A and the free factor complex of B. Since A and B overlap we may, as in Section 3.2, choose conjugates (still denoted A and B) so that $A \cap B = x$, where $x \neq \{1\}$ is a proper free factor of A and B. Write $A = A' * x$ and $B = B' * x$ so that

$$H = \langle A, B \rangle \cong A *_x B \cong A' * x * B'.$$

Since $\pi_A(B \leq H) = \{[x]\} \subset \pi_A(B)$ in $\mathcal{F}(A)$ and $\pi_B(A \leq H) = \{[x]\} \subset \pi_B(A)$ in $\mathcal{F}(B)$ and by Lemma 3.4, $\pi_A(\pi_{\mathcal{S}(H)}(T)) = \pi_A(T)$ and $\pi_B(\pi_{\mathcal{S}(H)}(T)) = \pi_B(T)$, we have

$$\begin{aligned} d_A(B,T) &\leq d_A(\pi_A(B \leq H), \pi_{\mathcal{S}(H)}(T)) + \mathrm{diam}_A(\pi_A(B)) \\ &\leq d_A(x, \pi_{\mathcal{S}(H)}(T)) + 4 \end{aligned}$$

and similarly

$$\begin{aligned} d_B(A,T) &\leq d_B(\pi_B(A \leq H), \pi_{\mathcal{S}(H)}(T)) + \mathrm{diam}_B(\pi_B(A)) \\ &\leq d_B(x, \pi_{\mathcal{S}(H)}(T)) + 4. \end{aligned}$$

Hence, it suffices to show that for $T \in \mathcal{S}'$ with $\pi_A(T) \neq \emptyset \neq \pi_B(T)$

$$\min\{d_A(x, \pi_{\mathcal{S}(H)}(T)), d_B(x, \pi_{\mathcal{S}(H)}(T))\} \leq M - 4,$$

where H is fixed as above.

To transition to the topological picture, suppose that $\mathrm{rank}(H) = k$ and set $M = M_k$ with a fixed identification $\pi_1 M = H$. Let S_A, S_B be two disjoint spheres in M that correspond to the splitting $H = A' * x * B'$ via Proposition 2.1. Take C_A to be the submanifold with boundary S_A and $\pi_1 C_A = A$ and take C_B to be the submanifold with boundary S_B and $\pi_1 C_B = B$. By construction $S_A \subset C_B$ and $S_B \subset C_A$ and so, in particular, ∂C_A induces a splitting of $B = \pi_1 C_B$ whose projection to $\mathcal{F}(B)$ contains $\pi_B(A \le H) = \{[x]\}$. Similarly, ∂C_B induces a splitting of $A = \pi_1 C_A$ whose projection to $\mathcal{F}(A)$ contains $\pi_A(B \le H) = \{[x]\}$.

Now choose any tree $T \in \mathcal{S}'(H)$ with nontrivial projections to $\mathcal{F}(A)$ and $\mathcal{F}(B)$ and let S be the corresponding sphere system in M. Put S and $\partial C_A \cup \partial C_B$ in normal position and recall that by Proposition 5.4, $\pi_{\mathcal{S}(A)}(T)$ is given by the collection of components of $C_A \cap S$ that split C_A. With this set-up, we show that

$$\min\{d_A(\partial C_B, S), d_B(\partial C_A, S)\} \le 12$$

where for any sphere system R in M, $\pi_A(R)$ denotes $\pi_A(T_R)$.

Suppose, toward a contraction, that $d_B(\partial C_A, S)$ and $d_A(\partial C_B, S)$ are greater than 12 and consider the forest G on S that is dual to the circles of intersection $\partial C_A \cap S$ and $\partial C_B \cap S$. We label the edges of G dual to circles of $\partial C_A \cap S$ with "a" and those dual to $\partial C_B \cap S$ with "b". Label the vertices of G that represent components $S \setminus (\partial C_A \cup \partial C_B)$ contained in $C_A \cap C_B$ with "AB", those in C_A but not C_B with "A", and those in C_B but not C_A with "B".

Call a subtree of G *terminal* if it has a unique vertex that separates it from its complement in G. We say a subtree is an a- tree (or b- tree) if all of its edges are labeled a (or b).

Claim 1. *No AB-vertex which is the boundary of both an a-edge and a b-edge is a vertex for either a terminal a-tree or a terminal b-tree.*

Proof of claim 1. We prove the claim for terminal a-trees. The proof for b-trees is obtained by switching the symbols a and b.

Suppose that there is an AB-vertex v of G which bounds both an a-edge and a b-edge and is the vertex for a terminal a-tree. Observe that the component S' of $S \cap C_B$ that corresponds to the union of b-edges at v (as in Figure 5) splits C_B and so it can be used for the projection $\pi_B(S)$ (see the remark following Proposition 5.2). To see that S' splits C_B, recall that if this were not the case then $C_B \setminus S' = C_1 \cup C_2$, where C_1 is relatively simply connected in M. As S' contains a disk of intersection with either $C_B \cap C_A$ or $C_B \setminus C_A$ coming from a valence one vertex of the terminal a-tree, this disk cobounds a region R contained in C_1 with a disk of ∂C_A. This shows that R is relatively simply connected with sphere boundary and basic combinatorial topology implies that R is simply connected in M. This implies that R is a 3-ball and so it can be used to reduce the number of intersections of S and ∂C_A, contradicting normal position. Hence, S' splits C_B.

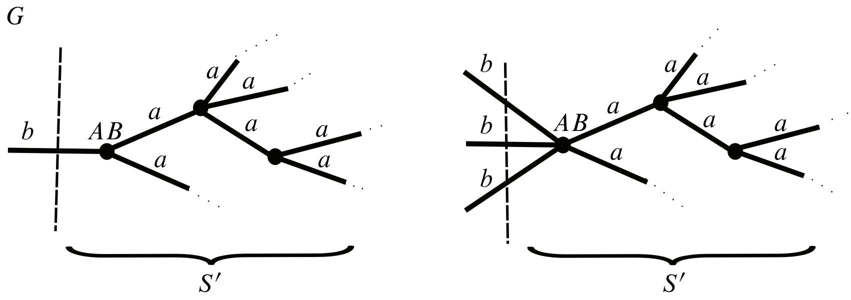

Figure 5. Two cases for S'.

Now there are two cases (see Figure 5). Suppose first that v is an endpoint for at least two b-edges. This implies that S' has at least two boundary components; each of which is contained in ∂C_B. Since these edges share the endpoint v, these boundary components co-bound the same component of $S' \cap C_A$. Let d_1, d_2 be two boundary components of S' which are not separated by another such boundary component of S' in ∂C_B. Let α be an arc between d_1 and d_2 in ∂C_B which intersects no other boundary component of S' and let β be an arc in S' joining d_1 and d_2 with $\partial\beta = \partial\alpha$ that does not intersect ∂C_A. Since S and ∂C_B are in normal position, β is not homotopic relative endpoints into ∂C_B and so $\gamma = \alpha * \beta$ is an essential loop in $C_A \cap C_B$ which is disjoint from ∂C_A and can be homotoped not to intersect S'. Hence, if $[\gamma]$ denotes the conjugacy class of the smallest free factor containing $\langle\gamma\rangle$ then

$$\begin{aligned} d_B(\partial C_A, S) &\leq \operatorname{diam}_{\mathcal{F}(B)}(S) + d_B(\partial C_A, S') \\ &\leq \operatorname{diam}_{\mathcal{F}(B)}(S) + d_B(\partial C_A, [\gamma]) + d_B([\gamma], S') \\ &\leq 4 + 4 + 4 = 12, \end{aligned}$$

a contradiction.

If v is the endpoint of only one b-edge, this argument does not work. In this case, S' is a disk and we argue as follows: first, any disk component of $S' \cap C_A$ splits C_A and is disjoint from ∂C_B providing the bound $d_A(\partial C_B, S) \leq 4$, a contradiction. So assume that each components of $S' \cap C_A$ has at least two boundary components on ∂C_A, except possibly the unique component with a boundary component on ∂C_B. Among all components of $S' \cap C_A$ choose the component S'' which has two boundary components d_1, d_2 on ∂C_A which are least separated by other components of $S' \cap \partial C_A$. Let α be an arc in ∂C_A between d_1 and d_2 that intersects only the circles of $S' \cap \partial C_A$ that separate d_1 from d_2 in ∂C_A. Note that by our choice of S'' each circle of $S' \cap \partial C_A$ that is crossed by α bounds a distinct component of $S' \cap C_A$. Let β be an arc in S'' joining these boundary components with $\partial\beta = \partial\alpha$. As before, $\gamma = \alpha * \beta$ is an essential loop in $C_A \cap C_B$ and we obtain a similar contradiction as above if γ intersects S' at most once; so suppose that this is not the case. Since intersections between S' and γ must occur along α we conclude that there is a component C of $S' \cap C_A$ which does not have boundary on ∂C_B and intersects γ exactly once. This implies that C is nonseparating in C_A. Hence, C splits C_A and is disjoint from ∂C_B. This provides the upper bound on distance

$$\begin{aligned} d_A(S, \partial C_B) &\leq \operatorname{diam}_A(S) + d_A(C, \partial C_B) \\ &\leq 4 + 4 = 8, \end{aligned}$$

a contradiction. □

Claim 2. *There exists an AB-vertex of G that has both an a-edge and a b-edge.*

Proof of claim 2. Assume to the contrary; that is assume that no component of $S \cap C_A \cap C_B$ has its boundary on both ∂C_A and ∂C_B. Let $s \in S$ be a sphere of S that splits C_B, this sphere exists by assumption. If s intersection ∂C_B then it does not meet ∂C_A and so $d_B(s, \partial C_A) \leq 4$, a contraction. Hence, $s \subset C_B$. If s also splits C_A, i.e. if some component of $s \cap C_A$ splits C_A, then we conclude $d_A(s, \partial C_B) \leq 4$; so it must be the case that every component of $s \cap C_A$ is superfluous, that is, it separates C_A and bounds to one side a component that is relatively simply connected. Note this implies in particular that no component of $s \cap C_A$ is a disk. We show that this also leads to a contraction. The argument is similar to that of the second part of Claim 1.

Among all components of $s \cap C_A$ choose the one with boundary components on ∂C_A that are least separated by circles of $s \cap \partial C_A$, call this component s''. As in the poof of Claim 2, let α be an arc in ∂C_A between these boundary components of s'' that intersects only the circles of $s \cap \partial C_A$ that separate these boundary components. Note that by our choice of s'' each circle of $s \cap \partial C_A$ that is crossed by α bounds a distinct component of $s \cap C_A$. Let β be an arc in s'' joining these boundary components, with the same endpoints as α. By normal position, $\gamma = \alpha * \beta$ is an essential loop in $C_A \cap C_B$ that can be homotoped to miss s'' and we obtain a similar contradiction as above if γ does not intersect any other components of $s \cap C_A$; so assume that this is not the case. Since additional intersections with s must occur along α we conclude that there is a component C of $s \cap C_A$ that intersects γ exactly once. This implies that C is nonseparating in C_A and contradicts the statement that all components of $s \cap C_A$ are superfluous. □

To conclude the proof of the proposition, first locate an AB-vertex v that has both an a-edge and a b-edge. The existence of v is guaranteed by Claim 2. By the Claim 1, the b-edges at v are not contained in a terminal b-tree. Hence, there is an a-edge adjacent to this b-tree in the complement of the initial vertex; the adjacency necessarily occurring at an AB-vertex. At this new vertex, Claim 1 now implies that the a-edges are not contained in a terminal a-tree. Hence we may repeat the process and find a new AB-vertex to which we may again apply Claim 1. Since G is a forest, these AB vertices are distinct and we conclude that G is infinite. This contradicts that fact that edges of G correspond to components of the intersection of transverse sphere systems S and $S_A \cup S_B$ in M_k and must, therefore, be finite.
□

7. Order on overlapping factors

For trees $T, T' \in \mathcal{K}^0$ and $K \geq 2M + 1$, define $\Omega(K, T, T')$ to be the set of (conjugacy classes of) free factors with the property that $A \in \Omega(K, T, T')$ if and only if $d_A(T, T') \geq K$. This definition is analogous to [7], where the authors put a partial ordering on the set of subsurfaces with large projection distance between two fixed markings. See [24] and [6] for details on this partial ordering on subsurfaces. Defining a partial ordering on $\Omega(K, T, T')$, however, requires a more general notion of projection than is available in our situation. We resolve this issue by defining a relation that is not necessarily transitive. Lemma 10.1 will then compensate for this lack of transitivity.

For $A, B \in \Omega(K, T, T')$ that *overlap* we define

$$A \prec B$$

to mean that

$$d_A(T, B) \geq M + 1,$$

where M is as in Proposition 6.1. As noted above, this does not define a partial order. In particular, if $A \prec B$ and $B \prec C$ there is no reason to expect that A and C will meet as free factors. We do, however, have the following version of Proposition 3.6 from [7].

Proposition 7.1. *Let $K \geq 2M + 1$ and choose $A, B \in \Omega(K, T, T')$ that overlap. Then A and B are ordered and the following are equivalent:*

(1) $A \prec B$;

(2) $d_A(T, B) \geq M + 1$;

(3) $d_B(T, A) \leq M$;

(4) $d_B(T', A) \geq M + 1$;

(5) $d_A(T', B) \leq M$.

Proof. (1) implies (2) is by definition, (2) implies (3) is Proposition 6.1, (3) implies (4) is the observation that

$$d_B(T', A) \geq d_B(T, T') - d_B(T, A) \geq 2M + 1 - M = M + 1,$$

and the proofs of the remaining implications are similar. To show that $A, B \in \Omega(K, T, T')$ which overlap are ordered, note that by the equivalence of the above conditions if $A \not\prec B$ then $d_A(T, B) \leq M$ and if $B \not\prec A$, switching the roles of A and B, $d_A(T', B) \leq M$ so that

$$d_A(T, T') \leq d_A(T, B) + d_A(B, T') \leq 2M \leq K,$$

a contradiction. □

8. Normal forms in $A(\Gamma)$

Let Γ be a simplicial graph with vertex set $V(\Gamma) = \{s_1, \ldots, s_n\}$ and edge set $E(\Gamma) \subset V(\Gamma) \times V(\Gamma)$. The right-angled Artin group, $A(\Gamma)$, associated to Γ is the group presented by

$$\langle s_i \in V(\Gamma) \colon [s_i, s_j] = 1 \iff (s_i, s_j) \in E(\Gamma) \rangle.$$

We refer to $s_1, \ldots, s_n$ as the *standard generators* of $A(\Gamma)$.

8.1. The Clay–Leininger–Mangahas partial order. In this section, we briefly recall a normal form for elements of a right-angled Artin group. For details see Section 4 of [7] and the references provided there. Fix a word $w = x_1^{e_1} \ldots x_k^{e_k}$ in the vertex generators of $A(\Gamma)$, with $x_i \in \{s_1, \ldots, s_n\}$ for each $i = 1, \ldots, k$. Each $x_i^{e_i}$ together with its index, which serves to distinguish between duplicate occurrences of the same generator, is a *syllable* of the word w. Let $\mathrm{syl}(w)$ denote the set of syllables for the word w. We consider the following 3 moves that can be applied to w without altering the element in $A(\Gamma)$ it represents:

(1) If $e_i = 0$, then remove the syllable $x_i^{e_i}$.

(2) If $x_i = x_{i+1}$ as vertex generators, then replace $x_i^{e_i} x_{i+1}^{e_{i+1}}$ with $x_i^{e_i + e_{i+1}}$.

(3) If the vertex generators x_i and x_{i+1} commute, then replace $x_i^{e_i} x_{i+1}^{e_{i+1}}$ with $x_{i+1}^{e_{i+1}} x_i^{e_i}$.

For $g \in A(\Gamma)$, set Min(g) equal to the set of words in the standard generators of $A(\Gamma)$ that have the fewest syllables among words representing g. We refers to words in Min(g) as the *normal form* representatives of g. Hermiller and Meier showed in [13] that any word representing g can be brought to any word in Min(g) by applications of the three moves above. Since these moves do not increase the word (or syllable) length, we see that words in Min(g) are also minimal length with respect to the standard generators and that any two words in Min(g) differ by repeated application of move (3) only. It is verified in [7] that for any $g \in A(\Gamma)$ and $w, w' \in \text{Min}(g)$ there is a natural bijection between syl(w) and syl(w'). Because of this, for $g \in A(\Gamma)$ we can define syl(g) = syl(w) for $w \in \text{Min}(g)$. For each $g \in A(\Gamma)$, this permits a strict partial order $\prec$ on the set syl(g) by setting $x_i^{e_i} \prec x_j^{e_j}$ if and only if for every $w \in \text{Min}(g)$ the syllable $x_i^{e_i}$ precedes $x_j^{e_j}$ in the spelling of w.

8.2. Order on meeting syllables. By analogy with the weaker notion of order on free factors, for $g \in A(\Gamma)$ let $\overset{m}{\prec}$ be the relation on syl(g) defined as follows: $x_i^{e_i} \overset{m}{\prec} x_j^{e_j}$ if and only if $x_i^{e_i} \prec x_j^{e_j}$ and there is a normal form $w \in \text{Min}(g)$ where $x_i^{e_i}$ and $x_j^{e_j}$ are adjacent. The following observation will be important in proving the lower bound on distance in our main theorem.

Lemma 8.1. *The strict partial ordering $\prec$ on* syl(g) *is the transitive closure of the relation $\overset{m}{\prec}$.*

Proof. From the definition of $\overset{m}{\prec}$ it suffices to show that if $x_i^{e_i} \prec x_j^{e_j}$ in syl(w) then $x_i^{e_i}$ and $x_j^{e_j}$ cobound a chain of syllables where adjacent terms are ordered by $\overset{m}{\prec}$. To this end, let

$$x_i^{e_i} = a_1 \prec a_2 \prec \cdots \prec a_n = x_j^{e_j}$$

be a chain of maximal length joining $x_i^{e_i}$ and $x_j^{e_j}$ in syl(g). We show that each pair of consecutive terms in the chain is ordered by $\overset{m}{\prec}$. Take $1 \leq i \leq n$ and consider the $w \in \text{Min}(g)$ for which a_i and a_{i+1} are separated by the least number of syllables in w. If a_i and a_{i+1} are adjacent in w we are done, otherwise write

$$w = w_1 \cdot a_i \cdot s \cdot w_2 \cdot a_{i+1} \cdot w_3$$

where w_1, w_2, w_3 are possibly empty subwords of w and s is a syllable of w. By our choice of w, $a_i \prec s$, for otherwise we could commute s past a_i resulting in a normal form for g with fewer syllables separating a_i and a_{i+1}. Then either $s \prec a_{i+1}$, which contradicts the assumption that the chain is maximal, or s can be commuted past a_{i+1} resulting in a normal form $w' \in \mathrm{Min}(g)$ with

$$w' = w_1 \cdot a_i \cdot w_2' \cdot a_{i+1} \cdot s \cdot w_3'$$

where w_2' is a subword of w_2. This contradicts our choice of w. Hence, a_i and a_j must occur consecutively in w and so $a_i \overset{m}{\prec} a_{i+1}$ as required. □

9. Large projection distance

Fix an admissible system $\mathcal{S} = (\mathcal{A}, \{f_i\})$ for $\mathbb{F}$ with coincidence graph Γ. This determines a homomorphism $\phi = \phi_{\mathcal{S}} : A(\Gamma) \to \mathrm{Out}(\mathbb{F})$ by mapping the vertex generator s_i to the outer automorphism f_i.

For $g \in A(\Gamma)$ with $w = x_1^{e_1} \dots x_k^{e_k} \in \mathrm{Min}(g)$, let $J : \{1, \dots, k\} \to \{1, \dots, n\}$ be defined so that $x_i = s_{J(i)}$, as generators of $A(\Gamma)$. Hence, $\phi(x_i) = f_{J(i)}$ is supported on $A_{J(i)}$. Write

$$A^w(x_i^{e_i}) = \phi(x_1^{e_1} \dots x_{i-1}^{e_{i-1}})(A_{J(i)})$$

for $i = 2, \dots, k$ and $A^w(x_1^{e_1}) = A_{J(1)}$. This defines a map

$$A^w : \mathrm{syl}(w) \to \mathcal{FF}^0.$$

It is verified in [7] that this map is well-defined for $g \in A(\Gamma)$, independent of the choice of normal form. Then, set $A^g = A^w$ for $w \in \mathrm{Min}(g)$ and set $\textsc{fact}(g)$ equal to the image of the map $A^g : \mathrm{syl}(g) \to \mathcal{FF}^0$. We refer to the free factors in $\textsc{fact}(g)$ as the *active free factors* for $g \in A(\Gamma)$. For notional convenience, set $B_i = A_{J(i)}$ and $g_i = \phi(x_i^{e_i}) = f_{J(i)}^{e_i}$. Note that this notation is for a fixed $w \in \mathrm{Min}(g)$.

Having developed the necessary tools in the free group setting, the proof of the first part of the following theorem is a verification that the arguments of [7] extend to this situation, even with a weaker form of Proposition 6.1 . We repeat their argument here for completeness. Let M be the constant determined in Proposition 6.1 and let $L = 4$ be the Lipschitz constant for the projection $\pi_A : \mathcal{K} \to \mathcal{F}(A)$, $A \in \mathcal{FF}$.

Theorem 9.1. *Given an admissible collection $\mathcal{A}$ of free factors for $\mathbb{F}$ with coincidence graph Γ and $T \in \mathcal{K}^0$, there is a $K \geq 5M + 3L$ so that if outer automorphisms $\{f_i\}$ are chosen to make $(\mathcal{A}, \{f_i\})$ an admissible system with $\ell_{A_i}(f_i) \geq 2K$ then the induced homomorphism $\phi\colon A(\Gamma) \to \mathrm{Out}(\mathbb{F})$ satisfies the following: for any $g \in A(\Gamma)$ with normal form $w = x_1^{e_1} \dots x_k^{e_k} \in \mathrm{Min}(g)$,*

(1) $d_{A^g(x_i^{e_i})}(T, \phi(g)T) \geq K|e_i|$ *for* $1 \leq i \leq k$. *In particular,* $\textsc{fact}(g) \subset \Omega(K, T, \phi(g)T)$.

(2) *If* $x_i^{e_i} \overset{m}{\prec} x_j^{e_j}$, *then* $A^g(x_i^{e_i})$ *and* $A^g(x_j^{e_j})$ *overlap and*

$$A^g(x_i^{e_i}) \prec A^g(x_j^{e_j}).$$

Proof. Set $K = 5M + 3L + 2 \cdot \max\{d_{A_i}(T, A_j)\}$ and observe that this choice of K has the property that if A_i and A_j overlap then $d_{A_i}(T, A_j) \leq K/2 - M$. The proof of (1) is by induction on the syllable length of $w \in \mathrm{Min}(g)$. If w has only one syllable then

$$d_{A_{J(1)}}(T, f_{J(1)}^{e_1} T) \geq \ell_{A_{J(1)}}(f_{J(1)}^{e_1}) \geq 2K|e_1|.$$

Now suppose that (1) has been proven for all elements in $A(\Gamma)$ that have representative with less than or equal to $k - 1$ syllables. Take $g \in A(\Gamma)$ with $w = x_1^{e_1} \dots x_k^{e_k}$ a k-syllable normal form representative for g. Using the notation at the beginning of this section, write $\phi(w)$ as $g_1 \dots g_k$ so that for $1 \leq i \leq k$ we must show

$$d_{g_1 \dots g_{i-1} B_i}(T, g_1 \dots g_k T) \geq K|e_i|.$$

With $x_i^{e_i} \in \mathrm{syl}(g)$ fixed and $g_i = \phi(x_i^{e_i})$, we write $\phi(g)$ as $abg_i c$ by choosing a normal form $w \in \mathrm{Min}(g)$ so that

(1) $c = g_{i+1} \dots g_k$ and g_i and g_{i+1} do not commute,

(2) $a = g_1 \dots g_l$ with l the largest index among $w \in \mathrm{Min}(g)$ so that g_l and g_i do not commute, and

(3) $b = g_{l+1} \dots g_{i-1}$, all of which commute with g_i.

Note that we allow a, b or c to be empty.

Using this notation, we show that $d_{abB_i}(T, abg_i cT) \geq K|e_i|$. By Lemma 3.4 and the triangle inequality,

$$\begin{aligned} d_{abB_i}(T, abg_i cT) &= d_{B_i}(b^{-1}a^{-1}T, g_i cT) \\ &\geq d_{B_i}(T, g_i T) - d_{B_i}(b^{-1}a^{-1}T, T) - d_{B_i}(g_i cT, g_i T). \end{aligned}$$

Since b is written in terms of generators that restrict to the identity outer automorphism on B_i and g_i restricts to an isometry of the free factor complex of B_i, Lemma 3.4 implies

$$d_{B_i}(b^{-1}a^{-1}T, T) = d_{B_i}(a^{-1}T, T)$$

and

$$d_{B_i}(g_i cT, g_i T) = d_{B_i}(cT, T).$$

This, along with our hypothesis on translation length, allows us to write

$$d_{abB_i}(T, abg_i cT) \geq 2K|e_i| - d_{B_i}(a^{-1}T, T) - d_{B_i}(cT, T). \tag{1}$$

We use the induction hypotheses to show that both terms subtracted in (3) are $\leq K/2$. This will complete the proof of (1). First, observe that each of $a^{-1} = g_l^{-1} \dots g_1^{-1}$ and $c = g_{i+1} \dots g_k$ is either trivial or is the image of a normal form subword of w with strictly fewer than k syllables and begins with a syllable not commuting with $x_i^{e_i}$. This is all that is needed for the remainder of the proof. We show the inequality $d_{B_i}(a^{-1}T, T) \leq K/2$, the other appears in [7] where the proof follows through without change.

By the induction hypothesis applied to a^{-1},

$$d_{B_l}(T, a^{-1}T) = d_{B_l}(T, g_l^{-1} \dots g_1^{-1}T) \geq K|e_l|,$$

and so since $d_{B_l}(T, B_i) \leq K/2 - M$ by our choice of K, we have $d_{B_l}(B_i, a^{-1}T) \geq K - (K/2 - M) \geq M + 1$. Since B_i and B_l overlap, Proposition 6.1 implies that $d_{B_i}(B_l, a^{-1}T) \leq M$, so by another application of $d_{B_i}(T, B_l) \leq K/2 - M$,

$$d_{B_i}(a^{-1}T, T) \leq M + (K/2 - M) \leq K/2,$$

as required. This completes the proof of the first part of the theorem.

The second part of the theorem is also proven by induction on syllable length. If $g \in A(\Gamma)$ has syllable length equal to 1, then there is nothing to prove. Suppose that the ordering statement holds for all g with a minimal syllable representative with less then or equal to $k - 1$ syllables. As in the first part of the proof, take $g \in A(\Gamma)$ with $w = x_1^{e_1} \dots x_k^{e_k}$ a k-syllable normal form representative for g. Write $\phi(w)$ as $g_1 \dots g_k$ and suppose that $x_i^{e_i} \overset{m}{\prec} x_j^{e_j}$ as syllables of g. If $j \leq k - 1$ then we may apply the induction hypothesis to a prefix of w and conclude $A^w(x_i^{e_i}) \prec A^w(x_j^{e_j})$. More precisely, let w' be the word formed by the first $k - 1$ syllables of w; this is a normal form word for some $g' \in A(\Gamma)$. By the induction hypothesis $A^{w'}(x_i^{e_i})$ and $A^{w'}(x_j^{e_j})$ overlap and $A^{w'}(x_i^{e_i}) \prec A^{w'}(x_j^{e_j})$. This suffices since for $l \leq k-1$ we have $A^w(x_l^{e_l}) = A^{w'}(x_l^{e_l})$, using the obvious identification of the syllables of w' with those of w.

Otherwise, $j = k$ and by definition of $\overset{m}{\prec}$ we may choose $w \in \mathrm{Min}(g)$ so that $w = a x_i^{e_i} x_k^{e_k}$ and so $\phi(w) = \phi(a) g_i g_k$. Since $x_i^{e_i} \overset{m}{\prec} x_k^{e_k}$, B_i and B_k overlap and so $\phi(a) g_i B_i = \phi(a) B_i$ and $\phi(a) g_i B_k$ also overlap. We have

$$\begin{aligned} d_{A^g(x_k^{e_k})}(A^g(x_i^{e_i}), \phi(g)T) &= d_{\phi(a) g_i B_k}(\phi(a) B_i, \phi(a) g_i g_k T) \\ &= d_{B_k}(B_i, g_k T) \\ &\geq d_{B_k}(T, g_k T) - d_{B_k}(B_i, T) \\ &\geq d_{A_{J(k)}}(T, f_{J(k)}^{e_k} T) - d_{A_{J(k)}}(A_{J(i)}, T) \\ &\geq 2K - K \\ &\geq M + 1, \end{aligned}$$

and so since $A^g(x_i^{e_i}), A^g(x_k^{e_k}) \in \Omega(K, T, \phi(g)T)$, by Proposition 7.1

$$A^w(x_i^{e_i}) \prec A^w(x_k^{e_k}). \qquad \square$$

10. The lower bound on distance for admissible systems

Let $\mathcal{A} = (\{A_i\}, \{f_i\})$ be an admissible system satisfying the hypotheses of Theorem 9.1 for $T \in \mathcal{K}^0$ and let $K \geq 5M + 3L$ be as in Theorem 9.1. For $g \in A(\Gamma)$ and $w \in \mathrm{Min}(g)$ write in normal form

$$w = x_1^{e_1} \ldots x_k^{e_k}.$$

We make use of the notation introduced at the beginning of the previous section.

Set $T' = \phi(g)T$ and choose a geodesic $T = T_0, T_1, \ldots, T_N = T'$ in the 1-skeleton of $\mathcal{K}_n$. Similar to [24], we define the subinterval $I_A = [a_A, b_A] \subset [0, N]$ associated to the free factor $A \in \Omega(K, T, T')$ as follows. Set

$$a_A = \max\{k \in \{0, \ldots, N\} \colon d_A(T, T_k) \leq 2M + L\}$$

and

$$b_A = \min\{k \in \{a_A, \ldots, N\} \colon d_A(T_k, T') \leq 2M + L\}.$$

Since $A \in \Omega(K, T, T')$, $d_A(T, T') \geq K \geq 5M + 3L$ and so both a_A and b_A are well-defined and not equal. Hence, the interval I_A is nonempty and for all $k \in I_A$,

$$d_A(T_k, T) \geq 2M + 1 \quad \text{and} \quad d_A(T_k, T') \geq 2M + 1.$$

This uses that fact that the projection from $\mathcal{K}^0$ to $\mathcal{F}(A)$ is L-Lipschitz. The next lemma shows that if syllables are ordered, then distance in their associated free factors cannot be made simultaneously.

Lemma 10.1. *With notation fixed as above, if* $x_i^{e_i}, x_j^{e_j} \in \mathrm{syl}(w)$ *and* $x_i^{e_i} \prec x_j^{e_j}$ *then*

$$I_{A^w(x_i^{e_i})} < I_{A^w(x_j^{e_j})}.$$

That is, the intervals are disjoint and correctly ordered in $[0, N]$.

Proof. We first prove the proposition when $x_i^{e_i} \overset{m}{\prec} x_j^{e_j}$. Recall that since $x_i^{e_i} \overset{m}{\prec} x_j^{e_j}$, Theorem 9.1 implies that the free factors $A^w(x_i^{e_i})$ and $A^w(x_j^{e_j})$ overlap and are ordered, $A^w(x_i^{e_i}) \prec A^w(x_j^{e_j})$. If $k \in I_{A^w(x_i^{e_i})}$, then $d_{A^w(x_i^{e_i})}(T_k, T') \geq 2M + 1$ and since $A^w(x_i^{e_i}) \prec A^w(x_j^{e_j})$ we have $d_{A^w(x_i^{e_i})}(A^w(x_j^{e_j}), T') \leq M$. The triangle inequality then implies that

$$d_{A^w(x_i^{e_i})}(T_k, A^w(x_j^{e_j})) \geq M + 1.$$

As the free factors $A^w(x_i^{e_i})$ and $A^w(x_j^{e_j})$ overlap, by Proposition 6.1 we have

$$d_{A^w(x_j^{e_j})}(T_k, A^w(x_i^{e_i})) \leq M.$$

Combining this with the inequality $d_{A^w(x_j^{e_j})}(A^w(x_i^{e_i}), T) \leq M$, again coming from the ordering, provides

$$d_{A^w(x_j^{e_j})}(T, T_k) \leq 2M.$$

Since this is true for each $k \in I_{A^w(x_i^{e_i})}$ it follows from the definition of $I_{A^w(x_j^{e_j})}$ that $I_{A^w(x_i^{e_i})} \cap I_{A^w(x_j^{e_j})} = \emptyset$. So if there were an index $k \in I_{A^w(x_i^{e_i})}$ with $k > a_{A^w(x_j^{e_j})}$ then by disjointness of the intervals $a_{A^w(x_i^{e_i})} > a_{A^w(x_j^{e_j})}$. This contradict the choice of $a_{A^w(x_j^{e_j})}$ as the largest index k with $d_{A^w(x_j^{e_j})}(T, T_k) \leq 2M + 1$ and shows that the intervals of interest are disjoint and ordered as $I_{A^w(x_i^{e_i})} < I_{A^w(x_j^{e_j})}$.

Now, if more generally we have that $x_i^{e_i} \prec x_j^{e_j}$, then by Lemma 8.1, $x_i^{e_i}$ and $x_j^{e_j}$ can be joined by a chain of syllables

$$x_i^{e_i} = a_0 \overset{m}{\prec} a_1 \overset{m}{\prec} \cdots \overset{m}{\prec} a_l = x_j^{e_j}.$$

Hence, we conclude

$$I_{A^w(x_i^{e_i})} < I_{A^w(a_1)} < \cdots < I_{A^w(x_j^{e_j})},$$

as required. ∎

Let $s = s(\Gamma)$ be the size of the largest complete subgraph of Γ. This is also the maximal rank of a free abelian subgroup of $A(\Gamma)$. Note by the definition of an admissible system, s is bounded above by a constant depending only on the rank of $\mathbb{F}$. To simplify notations, associated to the free factor $A^g(x_i^{e_i})$ we set $a_i = a_{A^g(x_i^{e_i})}$ and $b_i = b_{A^g(x_i^{e_i})}$.

Lemma 10.2 (Lower bound on distance). *With notation fixed as above, K as in Theorem 9.1 and $w \in \mathrm{Min}(g)$ in normal form*

$$\sum_{1 \le i \le k} d_{A^g(x_i^{e_i})}(T, \phi(g)T) \le 5sL \cdot d_{\mathcal{K}}(T, \phi(g)T).$$

Proof. Since $A^g(x_i^{e_i}) \in \Omega(K, T, \phi(g)T)$ for all $x_i^{e_i} \in \mathrm{syl}(g)$ by Theorem 9.1, we have the collection of nonempty subintervals

$$\{I_{A^g(x_i^{e_i})} : 1 \le i \le k\}$$

of $\{0, 1, \ldots, N\}$. If, for $i \le j$, it is the case that $x_i^{e_i} \prec x_j^{e_j}$ then by Lemma 10.1, $I_{A^g(x_i^{e_i})}$ and $I_{A^g(x_j^{e_j})}$ are ordered and, hence, disjoint. Further, any collection of syllables pairwise unordered by $\prec$ has size bounded above by s. This is clear since such a collection of syllables can be commuted to be consecutive in w using move (3) and so correspond to distinct pairwise commuting standard generators. We conclude that for any integer $j \in [0, N]$, j is contained in at most s of the intervals $I_{A^g(x_i^{e_i})}$. Hence,

$$\sum_{1 \le i \le k} |b_i - a_i| \le s \cdot d_{\mathcal{K}}(T, \phi(w)T).$$

Using the Lipschitz condition on the projections and the triangle inequality,

$$\begin{aligned} d_{A^g(x_i^{e_i})}(T, \phi(g)T) &\le d_{A^g(x_i^{e_i})}(T_{a_i}, T_{b_i}) + 4M + 2L \\ &\le L|b_i - a_i| + 4M + 2L. \end{aligned}$$

Since for each $A \in \Omega(K, T, \phi(g)T)$, $d_A(T, \phi(g)T) \ge K \ge 5M + 3L$ we have $|b_A - a_A| \ge \frac{M+L}{L}$. This implies that $d_A(T, \phi(g)T) \le 5L \cdot |b_A - a_A|$ and so putting this with the inequality above

$$\sum_{1 \le i \le k} d_{A^g(i)}(T, \phi(g)T) \le 5sL \cdot d_{\mathcal{K}}(T, \phi(g)T),$$

as required. □

11. The quasi-isometric embedding

We can now prove Theorem 11.1.

Theorem 11.1. *Given an admissible collection $\mathcal{A}$ of free factors for $\mathbb{F}_n$ with coincidence graph Γ there is a $C \geq 0$ so that if outer automorphism $\{f_i\}$ are chosen making $\mathcal{S} = (\mathcal{A}, \{f_i\})$ an admissible system with $\ell_{A_i}(f_i) \geq C$ then the induced homomorphism $\phi = \phi_{\mathcal{S}} \colon A(\Gamma) \to \mathrm{Out}(\mathbb{F}_n)$ is a quasi-isometric embedding.*

Proof. Suppose $\mathcal{A}$ is an admissible collection of free factors and $T \in \mathcal{K}^0$. Take $C = 2K$, for K as in Theorem 9.1. We show that the orbit map $A(\Gamma) \to \mathcal{K}_n^1$

$$g \mapsto \phi(g)T$$

is a quasi-isometric embedding, where $A(\Gamma)$ is given the word metric in its standard generators. Since $\mathrm{Out}(\mathbb{F}_n)$ is quasi-isometric to $\mathcal{K}_n^1$, this suffices to prove the theorem. First, recall that the orbit map is Lipschitz, as is any orbit map induced by an isometric action of a finitely generated group on a metric space. Specifically, $d_{\mathcal{K}}(T, \phi(g)T) \leq A \cdot |g|$, where $A = \max\{d_{\mathcal{K}}(T, \phi(s_i)T \colon 1 \leq i \leq n\}$ and $s_1, \ldots, s_n$ are the standard generators.

Let $g \in A(\Gamma)$. By Theorem 9.1, we know that if $w = x_1^{e_1} \ldots x_k^{e_k} \in \mathrm{Min}(g)$, then

$$d_{A^g(x_i^{e_i})}(T, \phi(g)T) \geq K|e_i|$$

for $1 \leq i \leq k$. Hence, by Lemma 10.2

$$\begin{aligned} |g| &= \sum_{1 \leq i \leq k} |e_i| \\ &\leq \frac{1}{K} \sum_{1 \leq i \leq k} d_{A^g(x_i^{e_i})}(T, \phi(g)T) \\ &\leq \frac{5sL}{K} \cdot d_{\mathcal{K}}(T, \phi(g)T). \end{aligned}$$

We conclude that for any $g, h \in A(\Gamma)$

$$\frac{1}{A} d_{\mathcal{K}}(\phi(g)T, \phi(h)T) = \frac{1}{A} d_{\mathcal{K}}(T, \phi(g^{-1}h)T) \leq |g^{-1}h| = d_{A(\Gamma)}(g, h)$$

and

$$d_{A(\Gamma)}(g, h)) \leq \frac{5sL}{K} \cdot d_{\mathcal{K}}(T, \phi(g^{-1}h))T) = \frac{5sL}{K} \cdot d_{\mathcal{K}}(\phi(g)T, \phi(h))T),$$

as required. □

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