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Groups, Geometry, and Dynamics

Groups with infinitely many ends are not fraction groups

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Abstract. We show that any finitely generated group F with infinitely many ends is not a group of fractions of any finitely generated proper subsemigroup P, that is F cannot be expressed as a product PP^{-1}. In particular this solves a conjecture of Navas in the positive. As a corollary we obtain a new proof of the fact that finitely generated free groups do not admit isolated left-invariant orderings.

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1. Introduction

The existence of a left-invariant order on a group G is equivalent to the existence of a positive cone $P \subset G$, that is a subsemigroup such that G can be written as a disjoint union $G = \{1\} \sqcup P \sqcup P^{-1}$. In fact there is a one-to-one correspondence between left-invariant orderings and such positive cones.

In this note we prove that whenever a finitely generated group F with infinitely many ends can be written as $F = PP^{-1}$, where P is a finitely generated subsemigroup of F, then $P = F$. Our result answers a question of Navas, who conjectured that finitely generated free groups are not groups of fractions of finitely generated subsemigroups P with $P \cap P^{-1} = \emptyset$.

As an application we obtain a new proof of the fact that the space of left-invariant orderings of a finitely generated free group (endowed with the Chabauty topology) does not have isolated points. This result follows from the work of McCleary [2], but appears in this form for the first time in the work of Navas [3]. It is worth noting that our proof is the first geometric one.

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We also deduce that the left-orderings of finitely generated groups with infinitely many ends do not have finitely generated positive cones. This was already known for free products of left-orderable groups by the work of Rivas [4].

Our theorem complements a folklore result stating that whenever $\mathcal{S}$ is a finite generating set for a group G, and G does not contain a free subsemigroup, then G is a group of fractions of P, the semigroup generated by $\mathcal{S}$.

We should note here that finitely generated groups with infinitely many ends have been classified by Stallings [5, 6]. They are precisely those fundamental groups of non-trivial graphs of groups with exactly one edge and a finite edge group, which are finitely generated and not virtually cyclic.

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2. The result

In the following we will use X to denote the (right) Cayley graph of a finitely generated group F with respect to some finite generating set. We will identify F with vertices of X, and use d to denote the standard metric on the Cayley graph X. The isometric left-action of F on X and its subsets will be denoted by left multiplication. The notation $B(x, \xi)$ will stand for the closed ball centred at x of radius ξ.

We will assume that F has infinitely many ends, and so there will exist a constant κ such that the ball $B = B(1, \kappa)$ disconnects X into a space with at least 3 infinite components. We will use S to denote the set of vertices of B.

Definition 2.1. We say that $A \subset X$ is a *shoot* if and only if there exists $w \in F$ such that A is a connected component of $X \smallsetminus wB$. We say that wB *bounds* the shoot.

Lemma 2.2. *Let A be an infinite shoot bounded by B. Then there exists $w \in F$ such that $w(X \smallsetminus A) \subseteq A$ and $w^{-1}(X \smallsetminus A) \subseteq A$.*

Proof. Note that the ball $B(1, 2\kappa)$ is finite, since X is locally finite. Since F has infinitely many ends and A is infinite, there exists λ such that

$$L = \{x \in F \mid d(1, x) = \lambda\} \cap A$$

has more than $|B(1, 2\kappa)|$ elements. Take $l \in L$. The cardinality of L guarantees that there exists $l' \in L$ such that $l'l^{-1} \notin B(1, 2\kappa)$.

Let $w = l'l^{-1}$. Observe that

$$d(w, l') = d(l'l^{-1}, l') = d(1, l) = \lambda$$

Consider a shortest path between w and l'. If it lies entirely in A, then in particular so does w. If not, then it must contain some point $b \in B$, since B bounds A. Now we have

$$\lambda = d(w, l') = d(w, b) + d(b, l') \geqslant d(w, b) + \lambda - \kappa$$

which implies that $d(w, b) \leqslant \kappa$, and hence that $w \in B(1, 2\kappa)$, which is a contradiction. We have thus established that $w \in A \setminus B(1, 2\kappa)$, and therefore that $wB \subset A$.

Note that $w^{-1} = ll'^{-1} \notin B(1, 2\kappa)$ enjoys the same properties as w, and so we immediately conclude that $w^{-1}B \subset A$, or equivalently that $B \subset wA$.

Since wB and B are disjoint, every shoot bounded by wB either contains B or is disjoint from it. Clearly, there is a unique shoot bounded by wB containing B, and we have already shown that it is wA. Each of the other shoots bounded by wB lies in a single shoot bounded by B, namely in the shoot bounded by B which contains wB. But we have already seen that this is A. We are left with the conclusion that $w(X \setminus A) \subset A$, and our proof is finished by making the analogous observations for w^{-1}. □

We are now ready for the main result.

Theorem 2.3. *Let P be a finitely generated subsemigroup of a finitely generated group F with infinitely many ends. If $PP^{-1} = F$, then $P = F$.*

Proof. For ease of notation we will refer to the elements of P as positive, and to the elements of P^{-1} as negative.

We first note that any finite generating set of P is a generating set for F. Let X be the Cayley graph of F with respect to some such generating set. Note that this allows us to view generators of P as positive edges of X, and hence any positive element $p \in P$ is realised by a positive path between 1 and the vertex p in X.

We will use the notation κ, B and S as defined above.

Step 1. We claim that $S(P^{-1} \cup \{1\}) = F$.

If P intersects each ball $B(x, \kappa) = xB$ then each $x \in F$ is a concatenation of an element of P (namely any positive path from 1 to xB) with an element in S

(connecting the end of the positive path to the centre of the ball). Thus we have $x \in PS$, and our claim follows by taking inverses.

Let us now suppose that there exists an $x \in F$ such that

$$P \cap xB = \emptyset$$

Let A_0 denote an infinite shoot bounded by xB such that $1 \notin A_0$.

Let $z \in F \smallsetminus S$ be any element, and let A be the shoot bounded by B containing z. We claim that there exists $y \in F$ such that $yA \subseteq A_0$.

There are two cases we need to consider. The first one occurs when

$$xA \subseteq A_0$$

in which case we take $y = x$. The other one (illustrated in Figure 1) occurs when $xA \nsubseteq A_0$, that is when xA is a shoot bounded by xB other than A_0. Lemma 2.2 applied to $x^{-1}A_0$ gives us an element $w \in F$ such that $w(X \smallsetminus x^{-1}A_0) \subseteq x^{-1}A_0$. So $y = xw$ satisfies

$$yA = xwA \subseteq xx^{-1}A_0 = A_0$$

and so we have proven the claim.

Now, since $yz \in F = PP^{-1}$, we can write $yz = pq$, where p is positive and q is negative. Since there are no positive elements in xB by assumption, we see that $p \notin A_0$, and therefore q is a negative path connecting a vertex $p \in X \smallsetminus A_0$ to $yz \in yA \subseteq A_0$. The shoot yA is bounded by yB and contained in A_0, hence any path from $X \smallsetminus A_0$ to yA has to cross yB. This is in particular true for q, so there is a negative path (a terminal subpath of q) from some vertex of yB to yz, and hence from a vertex of B to z (after translating by y^{-1}). In the group language we have thus shown that $z \in SP^{-1}$, and so

$$F \smallsetminus S \subseteq SP^{-1}$$

But clearly $S \subset S(P^{-1} \cup \{1\})$, and so we have proven the claim of Step 1.

STEP 2. We claim that $P = F$.

We have established above that $S(P^{-1} \cup \{1\}) = F$, with S being finite. Let Q be a minimal (with respect to cardinality) finite subset of F such that

$$Q(P^{-1} \cup \{1\}) = F.$$

Suppose that there exist distinct $q, q' \in Q$. Then $q^{-1}q' \in F = PP^{-1}$, and so $q^{-1}q' = ab^{-1}$ with $a, b \in P$. Hence

$$q, q' \in qaP^{-1}$$

and therefore we could replace Q by $(Q \cup \{qa\}) \setminus \{q, q'\}$ of smaller cardinality. This shows that $|Q| = 1$. Without loss of generality we can take $Q = \{1\}$, and thence get

$$P^{-1} \cup \{1\} = F$$

Now let $f \in F \setminus \{1\}$. We have $f, f^{-1} \in P^{-1}$, and since P^{-1} is a semigroup, also $1 = ff^{-1} \in P^{-1}$. So $P^{-1} = F$. Taking an inverse concludes the theorem. □

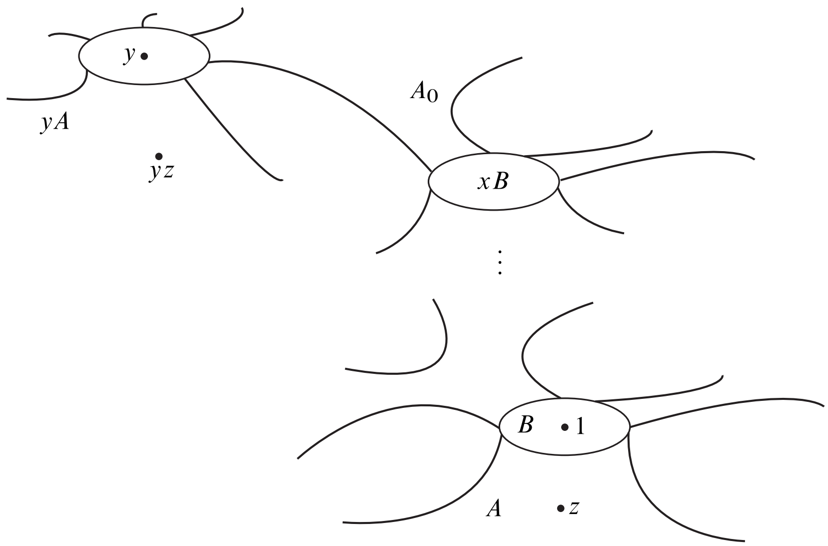

Figure 1. Step 1 of the theorem.

We now easily deduce the following.

Corollary 2.4. *Let F be a finitely generated group with infinitely many ends. Then F does not allow a left-invariant ordering with a finitely generated positive cone.*

Proof. Let P be the positive cone of a left-invariant ordering of F. Then

$$F = P \cup P^{-1} \cup \{1\}$$

and so in particular $F = PP^{-1}$. But also $P \cap P^{-1} = \emptyset$, and so $P \neq F$. Now the contrapositive of Theorem 2.3 tells us that P is not finitely generated. □

The statement above follows from the work of Rivas [4], since left-orderable groups are torsion-free, and so they have infinitely many ends only when they are free products.

We also get the following corollary.

Corollary 2.5. *The space of left-invariant orderings on any finitely generated free group has no isolated points.*

Proof. Let P be the positive cone of an isolated ordering of F, a finitely generated free group. By above, P is not finitely generated.

The order defined by P is isolated, and so there exists a finite set $S \subset F$ such that whenever we have another positive cone of an ordering P' such that $P \cap S = P' \cap S$, then $P = P'$. However the work of Smith and Clay [1, Theorem E] allows us to construct an order (in fact infinitely many such orders) whose positive cone P' satisfies $P \cap S = P' \cap S$, but such that $P \neq P'$. This is a contradiction. □

Added in proof. From the main theorem one can also easily deduce that groups with finite Garside structures have at most two ends.

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