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Groups, Geometry, and Dynamics

Thompson's group F is not SCY

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Abstract. In this note we prove that Thompson's group F cannot be the fundamental group of a symplectic 4-manifold with canonical class $K = 0 \in H^2(M)$ by showing that its Hausmann–Weinberger invariant $q(F)$ is strictly positive.

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Symplectic 4-manifolds with trivial canonical class, oftentimes referred to as *symplectic Calabi–Yau* manifolds are, conjecturally, a fairly restricted class of manifolds, see [13, 5]. Part of this restriction is reflected in known constraints for their fundamental groups, that we will refer to as *SCY groups*. In the case of $b_1 > 0$, these results, for which we refer to [1, 14, 8], corroborate the expectation that such groups are (virtually) poly-$\mathbb{Z}$.

We are interested here in the following constraints, that apply to the fundamental group $G = \pi_1(M)$ of a symplectic Calabi–Yau 4-manifold M with $b_1(M) = b_1(G) > 0$:

(1) $2 \leq b_1(G) \leq vb_1(G) \leq 4$, where $vb_1(G) = \sup\{b_1(G_i) | G_i \leq_{f.i.} G\}$ denotes the supremum of the first Betti number of all finite index subgroups of G;

(2) if the first L^2-Betti number $b_1^{(2)}(G)$ vanishes, then $q(G) = 0$, where the Hausmann–Weinberger invariant $q(G) = \inf\{\chi(X) | \pi_1(X) = G\}$ is defined as the infimum of the Euler characteristic among all 4-manifolds whose fundamental group is G ([11]).

(In [8, Proposition 2.2] the vanishing of $q(G)$ is stated under the assumption that G is residually finite, but in fact only the condition $b_1^{(2)}(G) = 0$ is used in the proof.)

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The purpose of this note is to apply these constraints to the case of Thompson's group F. The group F (about which we refer to [4, 10] for some preliminary facts) is a group that admits the finite presentation

$$F = \langle x_0, x_1 \mid [x_0 x_1^{-1}, x_0^{-1} x_1 x_0] = 1, [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] = 1 \rangle. \tag{1}$$

This group has a number of peculiar features, that make it a natural testing ground for conjectures and speculations. We should mention that S. Bauer asked (cf. [1, Question 1.5]) if another of Thompson's groups, T (which is a finitely presented simple group) is SCY, in this case with $b_1 = 0$: that question partly motivated the present note.

From a geometer's viewpoint, Thompson's group F has already been knocked out from the royalty of groups, i.e. Kähler groups, by the work of [17] (whose authors will hopefully condone us for the slight plagiarism in our title). However, as any finitely presented group, it keeps a footing as fundamental group of a symplectic 4-manifold, by [9] and, pushing the dimension up by 2, of symplectic 6-manifolds with trivial canonical class by [7]. In spite of that, we will show that the constraints discussed above are sufficient to show that F is not SCY. The main difficulty lies in the fact that the constraint on the first virtual Betti number, that is often very effective, is inconclusive:

Proposition. *Thompson's group F satisfies $b_1(F) = vb_1(F) = 2$.*

Proof. This is a consequence of the fact that ([4, Theorems 4.5]) the commutator subgroup $[F, F]$ is simple. Indeed, let $N \trianglelefteq_f F$ be a finite index normal subgroup. Then $[N, N]$ is a normal subgroup of $[F, F]$. Since $[F, F]$ is simple (and, as F is not virtually abelian, N is not abelian) it follows that $[N, N] = [F, F]$. We therefore see that $H_1(N) = N/[N, N] = N/[F, F]$ is a subgroup of $F/[F, F] \cong \mathbb{Z}^2$. Now, as the Betti number is non decreasing on finite index subgroups, $b_1(N) \geq b_1(F)$. This entails that $H_1(N)$ is a finite index subgroup of $H_1(F)$, hence a copy of $\mathbb{Z}^2$ itself. □

As the constraint on the virtual Betti number is inconclusive, we must resort to the Hausmann–Weinberger invariant $q(F)$ (whose calculation, to the authors' knowledge, has not appeared in the literature). While we are not able to calculate it exactly, we will show that it is strictly positive, whence F is not a SCY group.

Theorem. *The Hausmann–Weinberger invariant of Thompson's group F satisfies $0 < q(F) \leq 2$.*

Proof. As is well known (see e.g. [6]) the Hausmann–Weinberger invariant satisfies the basic inequalities $2 - 2b_1(F) \leq q(F) \leq 2 - 2\mathrm{def}(F)$, where $\mathrm{def}(F)$ denotes the deficiency of F. The upper bound is easily obtained then from the fact that the presentation in (1) has deficiency 0. To prove the lower bound, we will argue by contradiction. To start, we will compute the first L^2-Betti number. If F were residually finite, the proposition, together with the Lück Approximation Theorem [15], would immediately imply its vanishing, but as this isn't the case one must argue differently. There is more than one way to proceed to this calculation (see [16, Theorem 7.10] for the original calculation, or [2, Theorem 1.8]). For the reader's benefit, we present the following, which is fairly explicit. Start with a well-known infinite presentation of the group F:

$$F = \langle x_0, x_1, \ldots \mid x_n x_i = x_i x_{n+1}, \text{ for all } 0 \leq i < n \rangle,$$

that reduces to that in (1) putting $x_n = x_0^{1-n} x_1 x_0^{n-1}$ for all $n \geq 2$. Defining the shift monomorphism $\phi\colon F \to F$ as $\phi(x_i) = x_{i+1}$ for all $i \geq 0$, the images $F(m) = \phi^m(F)$ are isomorphic to F itself, and F is the properly ascending HNN-extension with base $F(1)$ itself, bonding subgroups $F(1)$ and $F(2)$ and stable letter x_0, i.e.

$$F = \langle F(1), x_0 \mid x_0^{-1} F(1) x_0 = \phi(F(1)) \rangle.$$

As F (hence $F(1)$, $F(2)$) admits a finite presentation, the L^2-Betti number $b_1^{(2)}(F)$ vanishes by [12, Lemma 2.1]. As F is an infinite group, $b_0^{(2)}(F)$ vanishes as well. Let M be a 4-manifold with fundamental group F. By standard facts of L^2-invariants (see e.g. [16]) we have

$$\chi(M) = 2b_0^{(2)}(F) - 2b_1^{(2)}(F) + b_2^{(2)}(M) = b_2^{(2)}(M) \geq 0,$$

whence $q(F) \geq 0$. Assume then, by contradiction, that equality holds for some manifold M; by [6, Theorem 6] the only obstruction for M to be an Eilenberg–Mac Lane space $K(F, 1)$ is $H^2(F, \mathbb{Z}[F])$. Now for Thompson's group F all cohomology groups $H^*(F, \mathbb{Z}[F])$ vanish ([10, Theorem 13.11.1]), so the obstruction vanishes; but in that case F would be a Poincaré duality group of dimension 4, hence satisfy $H^4(F, \mathbb{Z}[F]) = \mathbb{Z}$, that is false by the above. □

We observe that the result above entails that the deficiency of F is actually equal to zero. However, as the homology of F is known (see e.g. [3]), this follows also from Morse inequality $\mathrm{def}(F) \leq b_1(F) - b_2(F) = 0$ and the existence of the presentation of (1) of deficiency 0.

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