

## Some applications of $\ell_p$ -cohomology to boundaries of Gromov hyperbolic spaces

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**Abstract.** We study quasi-isometry invariants of Gromov hyperbolic spaces, focusing on the  $\ell_p$ -cohomology and closely related invariants such as the conformal dimension, combinatorial modulus, and the Combinatorial Loewner Property. We give new constructions of continuous  $\ell_p$ -cohomology, thereby obtaining information about the  $\ell_p$ -equivalence relation, as well as critical exponents associated with  $\ell_p$ -cohomology. As an application, we provide a flexible construction of hyperbolic groups which do not have the Combinatorial Loewner Property, extending [8] and complementing the examples from [10]. Another consequence is the existence of hyperbolic groups with Sierpinski carpet boundary which have conformal dimension arbitrarily close to 1. In particular, we answer questions of Mario Bonk, Juha Heinonen and John Mackay.

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## 1. Introduction

**Background.** In this paper we will be interested in quasi-isometry invariant structure in Gromov hyperbolic spaces, primarily structure which is reflected in the boundary. For some hyperbolic groups  $\Gamma$ , the topological structure of the boundary  $\partial\Gamma$  alone contains substantial information: witness the JSJ decomposition encoded in the local cut point structure of the boundary [14], and many situations where one can detect boundaries of certain subgroups  $H \subset \Gamma$  by means of topological criteria. However, in many cases, for instance for generic hyperbolic groups, the topology reveals little of the structure of the group and is completely inadequate for addressing rigidity questions, since the homeomorphism group of the boundary is highly transitive. In these cases it is necessary to use the finer quasi-Möbius structure of the boundary and analytical invariants attached to it, such as modulus (Pansu, metric-measure, or combinatorial),  $\ell_p$ -cohomology, and closely related quantities such as the conformal dimension. The seminal work of Heinonen-Koskela [24], followed by [16, 25, 38, 29] indicates that when  $\partial\Gamma$  is quasi-Möbius homeomorphic to a Loewner space (an Ahlfors  $Q$ -regular  $Q$ -Loewner space in the sense of [24]), there should be a rigidity theory resembling that of lattices in rank 1 Lie groups. This possibility is illustrated by [11, 39].

It remains unclear which hyperbolic groups  $\Gamma$  have Loewner boundary in the above sense. Conjecturally,  $\partial\Gamma$  is Loewner if and only if it satisfies the Combinatorial Loewner Property [30]. To provide some evidence of the abundance of such groups (modulo the conjecture), in [10] we gave a variety of examples with the Combinatorial Loewner Property. On the other hand, it had already been shown in [13, 8] that there are groups  $\Gamma$  whose boundary is not Loewner, which can still be effectively studied using  $\ell_p$ -cohomology and its cousins. Our main purpose in this paper is to advance the understanding of this complementary situation by providing new constructions of  $\ell_p$ -cohomology classes, and giving a number of applications.

**Setup.** We will restrict our attention (at least in the introduction) to proper Gromov hyperbolic spaces which satisfy the following two additional conditions:

- (Bounded geometry) For every  $0 < r \leq R$ , there is a  $N(r, R) \in \mathbb{N}$  such that every  $R$ -ball can be covered by at most  $N = N(r, R)$  balls of radius  $r$ .
- (Nondegeneracy) There is a  $C \in [0, \infty)$  such that every point  $x$  lies within distance at most  $C$  from all three sides of some ideal geodesic triangle  $\Delta_x$ .

The visual boundary  $\partial X$  of such a space  $X$  is a compact, doubling, uniformly perfect metric space, which is determined up to quasi-Möbius homeomorphism by the quasi-isometry class of  $X$ . Conversely, every compact, doubling, uniformly perfect metric space is the visual boundary of a unique hyperbolic metric space as above, up to quasi-isometry (see Section 2).

To simplify the discussion of homological properties, we will impose (without loss of generality) the additional standing assumption that  $X$  is a simply connected metric simplicial complex with links of uniformly bounded complexity, and with all simplices isometric to regular Euclidean simplices with unit length edges.

**Quasi-isometry invariant function spaces.** Let  $X$  be a Gromov hyperbolic simplicial complex as above, with boundary  $\partial X$ .

We recall (see Section 3 and [35, 20, 18, 8]) that for  $p \in (1, \infty)$ , the continuous (first)  $\ell_p$ -cohomology  $\ell_p H_{\text{cont}}^1(X)$  is canonically isomorphic to the space  $A_p(\partial X)/\mathbb{R}$ , where  $A_p(\partial X)$  is the space of continuous functions  $u : \partial X \rightarrow \mathbb{R}$  that have a continuous extension  $f : X^{(0)} \cup \partial X \rightarrow \mathbb{R}$  with  $p$ -summable coboundary:

$$\|df\|_{\ell_p}^p = \sum_{[vw] \in X^{(1)}} |f(v) - f(w)|^p < \infty,$$

and where  $\mathbb{R}$  denotes the subspace of constant functions. Associated with the continuous  $\ell_p$ -cohomology are several other quasi-isometry invariants.

- (1) The  $\ell_p$ -equivalence relation  $\sim_p$  on  $\partial X$ , where  $z_1 \sim_p z_2$  iff  $u(z_1) = u(z_2)$  for every  $u \in A_p(\partial X)$ .
- (2) The infimal  $p$  such that  $\ell_p H_{\text{cont}}^1(X) \simeq A_p(\partial X)/\mathbb{R}$  is nontrivial. We will denote this by  $p_{\neq 0}(X)$ . Equivalently  $p_{\neq 0}(X)$  is the infimal  $p$  such that  $\sim_p$  has more than one coset.
- (3) The infimum  $p_{\text{sep}}(X)$  of the  $p$  such that  $A_p(\partial X)$  separates points in  $\partial X$ , or equivalently,  $p_{\text{sep}}(X)$  is the infimal  $p$  such that all cosets of  $\sim_p$  are points.

These invariants were exploited in [13, 8, 10] due to their connection with conformal dimension and the Combinatorial Loewner Property. Specifically, when  $\partial X$  is approximately self-similar (e.g. if  $\partial X$  is the visual boundary of a hyperbolic group) then  $p_{\text{sep}}(X)$  coincides with the Ahlfors regular conformal dimension of  $\partial X$ ; and if  $\partial X$  has the Combinatorial Loewner Property then the two critical exponents  $p_{\neq 0}(X)$  and  $p_{\text{sep}}(X)$  coincide, i.e. for every  $p \in (1, \infty)$ , the function space  $A_p(\partial X)$  separates points iff it is nontrivial (we refer the reader to Sections 2 and 3 for the precise statements and the relevant terminology).

**Construction of nontrivial continuous  $\ell_p$ -cohomology.** The key results in this paper are constructions of nontrivial elements in the  $\ell_p$ -cohomology. The general approach for the construction is inspired by [8], and may be described as follows. Inside the Gromov hyperbolic complex  $X$ , we identify a subcomplex  $Y$  such that the relative cohomology of the pair  $(X, X \setminus Y)$  reduces – essentially by excision – to the cohomology of  $Y$  relative to its frontier in  $X$ . Then we prove that the latter contains an abundance of nontrivial classes. This yields nontrivial classes in  $\ell_p H_{\text{cont}}^1(X)$  with additional control, allowing us to make deductions about the cosets of the  $\ell_p$ -equivalence relation.

Let  $Y$  be a Gromov hyperbolic space satisfying our standing assumptions. We recall [18] that if  $Y$  has Ahlfors  $Q$ -regular visual boundary  $\partial Y$ , then  $A_p(\partial Y)$  contains the Lipschitz functions on  $\partial Y$  for any  $p > Q$ , and in particular it separates points. Our first result says that if  $W \subset Y$  is a subcomplex with well-separated connected components, then for  $p$  slightly larger than  $Q$ , the relative cohomology  $\ell_p H_{\text{cont}}^1(Y, W)$  is highly nontrivial. In other words, at the price of increasing the exponent slightly, one can arrange for the representing functions  $f : Y^{(0)} \rightarrow \mathbb{R}$  to be constant on the (0-skeleton of the) connected components of  $W$ , provided the components are far apart.

**Theorem 1.1** (Corollary 4.6). *Let  $Y$  be hyperbolic simply connected metric simplicial complex satisfying the assumptions above, and let  $\text{Confdim}(\partial Y)$  denote the Ahlfors regular conformal dimension of  $\partial Y$ .*

*For every  $\alpha \in (0, 1)$ ,  $C \geq 0$ , there is a  $D \geq 1$  with the following properties. Suppose  $W \subset Y$  is a subcomplex of  $Y$  such that*

- (1) *every connected component of  $W$  is  $C$ -quasiconvex in  $Y$ ;*
- (2) *for every  $y \in W$  there is a complete geodesic  $\gamma \subset W$  lying in the same connected component of  $W$ , such that  $\text{dist}(y, \gamma) \leq C$ ;*
- (3) *the distance between distinct components of  $W$  is at least  $D$ .*

*Then for  $p > \frac{1}{\alpha} \text{Confdim}(\partial Y)$ , the relative cohomology  $\ell_p H_{\text{cont}}^1(Y, W)$  separates every pair of distinct points  $z_1, z_2 \in \partial Y$ , such that  $\{z_1, z_2\} \not\subseteq \partial H$  for every connected component  $H$  of  $W$ .*

We prove Theorem 1.1 by translating it to an existence theorem for functions on  $\partial Y$ . It then reduces to the following theorem about Hölder functions, which is of independent interest.

**Theorem 1.2** (Theorem 4.1). *For every  $\alpha \in (0, 1)$ , there is a  $D \geq 1$  with the following property. Suppose  $Z$  is a bounded metric space, and  $\mathcal{C}$  is a countable collection of closed positive diameter subsets of  $Z$  such that the pairwise relative distance satisfies*

$$\Delta(C_1, C_2) = \frac{d(C_1, C_2)}{\min(\text{diam}(C_1), \text{diam}(C_2))} \geq D$$

for all  $C_1, C_2 \in \mathcal{C}$ ,  $C_1 \neq C_2$ . Then for every pair of distinct points  $z_1, z_2 \in Z$ , either  $\{z_1, z_2\} \subseteq C$  for some  $C \in \mathcal{C}$ , or there is a Hölder function  $u \in C^\alpha(Z)$ , such that

- (1)  $u|_C$  is constant for every  $C \in \mathcal{C}$ ;
- (2) if  $C_1, C_2 \in \mathcal{C}$  and  $u(C_1) = u(C_2)$ , then  $C_1 = C_2$ ;
- (3)  $u(z_1) \neq u(z_2)$ .

We remark that one can construct an example  $Z$ ,  $\mathcal{C}$  as in the theorem, where  $Z = S^k$  and  $\bigcup_{C \in \mathcal{C}} C$  has full Lebesgue measure. In this case, if  $u: Z \rightarrow \mathbb{R}$  is a Lipschitz function that is constant on every  $C \in \mathcal{C}$ , then almost every point  $z \in Z$  will be a point of differentiability of  $u$  and a point of density of some element  $C \in \mathcal{C}$ . Hence  $Du = 0$  almost everywhere, and  $u$  is constant. This shows that it is necessary to take  $\alpha < 1$ .

Our second construction of  $\ell_p$ -cohomology classes pertains to a special class of 2-complexes.

**Definition 1.3.** An *elementary polygonal complex* is a connected, simply connected, 2-dimensional cell complex  $Y$  whose edges are colored black or white, that enjoys the following properties:

- $Y$  is a union of 2-cells that intersect pairwise in at most a single vertex or edge;
- every 2-cell is combinatorially isomorphic to a polygon with even perimeter at least 6;
- the edges on the boundary of every 2-cell are alternately black and white;
- every white edge has thickness one, and every black edge has thickness at least 2. Here the thickness of an edge is the number of 2-cells containing it.

The union of the white edges in  $Y$  is the *frontier* of  $Y$ , and is denoted  $\mathcal{F}$ ; its connected components are trees.

A simple example of an elementary polygonal complex is the orbit  $\Gamma(P)$  of a right-angled hexagon  $P$  in the hyperbolic plane, under the group  $\Gamma$  generated by reflections in 3 alternate sides  $e_1, e_2, e_3 \subset \partial P$ .

Elementary polygonal complexes are relevant for us because they turn up naturally as embedded subcomplexes  $Y \subset X$ , where  $X$  is a generic polygonal 2-complex, such that the pair  $(Y, \mathcal{T})$  is the result of applying excision to the pair  $(X, X \setminus Y)$ .

For every elementary polygonal 2-complex  $Y$ , we define two invariants:

- $p_{\neq 0}(Y, \mathcal{T})$  is the infimum of the set of  $p \in [1, \infty)$  such that the relative cohomology  $\ell_p H_{\text{cont}}^1(Y, \mathcal{T})$  is nontrivial;
- $p_{\text{sep}}(Y, \mathcal{T})$  is the infimum of the set of  $p \in [1, \infty)$ , such that for every pair of distinct points  $z_1, z_2 \in \partial Y$ , either  $\{z_1, z_2\} \subseteq \partial T$  for some connected component  $T$  of  $\mathcal{T}$ , or the relative cohomology  $\ell_p H_{\text{cont}}^1(Y, \mathcal{T})$  separates  $\{z_1, z_2\}$ .

Our main result about elementary polygonal complexes is the following pair of estimates on these invariants.

**Theorem 1.4** (Corollary 6.6). *Let  $Y$  be an elementary polygonal complex. Assume that the perimeter of every 2-cell of  $Y$  lies in  $[2m_1, 2m_2]$  and that the thickness of every black edge lies in  $[k_1, k_2]$ , with  $m_1 \geq 3$  and  $k_1 \geq 2$ . Then*

$$1 + \frac{\log(k_1 - 1)}{\log(m_2 - 1)} \leq p_{\neq 0}(Y, \mathcal{T}) \leq p_{\text{sep}}(Y, \mathcal{T}) \leq 1 + \frac{\log(k_2 - 1)}{\log(m_1 - 1)}.$$

Note that the estimate becomes sharp when the thickness and perimeter are constant.

**Applications to spaces and groups.** We now discuss some of applications given in the paper.

We first apply Theorem 1.1 to amalgams, generalizing the construction of [8].

**Theorem 1.5** (Corollary 5.3). *Suppose  $A, B$  are hyperbolic groups, and we are given malnormal quasiconvex embeddings  $C \hookrightarrow A, C \hookrightarrow B$ . Suppose that there is a decreasing sequence  $\{A_n\}_{n \in \mathbb{N}}$  of finite index subgroups of  $A$  such that  $\bigcap_{n \in \mathbb{N}} A_n = C$ , and set*

$$\Gamma_n := A_n \star_C B.$$

- (1) *If  $p_{\text{sep}}(A) < p_{\text{sep}}(B)$  then, for all  $p \in (p_{\text{sep}}(A), p_{\text{sep}}(B)]$  and every  $n$  large enough, the  $\ell_p$ -equivalence relation on  $\partial\Gamma_n$  possesses a coset different from a point and the whole  $\partial\Gamma$ . In particular for  $n$  large enough,  $\partial\Gamma_n$  does not have the CLP.*
- (2) *If  $p_{\text{sep}}(A) < p_{\neq 0}(B)$  then, for  $p \in (p_{\text{sep}}(A), p_{\neq 0}(B))$  and every  $n$  large enough, the cosets of the  $\ell_p$ -equivalence relation on  $\partial\Gamma_n$  are single points and the boundaries of cosets  $gB$ , for  $g \in \Gamma_n$ . In particular, for large  $n$ , any quasi-isometry of  $\Gamma_n$  permutes the cosets  $gB$ , for  $g \in \Gamma_n$ .*

We now sketch the proof of the theorem.

Let  $K_A, K_B$ , and  $K_C$  be finite 2-complexes with respective fundamental groups  $A, B$ , and  $C$ , such that there are simplicial embeddings  $K_C \hookrightarrow K_A, K_C \hookrightarrow K_B$  inducing the given embeddings of fundamental groups. Let  $K_A^n \rightarrow K_A$ , for every  $n$ , be the finite covering corresponding to the inclusion  $A_n \subset A$ , and fix a lift  $K_C \hookrightarrow K_A^n$  of the embedding  $K_C \hookrightarrow K_A$ . We let  $K^n$  be the result of gluing  $K_A^n$  to  $K_B$  along the copies of  $K_C$ , so  $\pi_1(K^n) \simeq A_n \star_C B = \Gamma_n$ . The universal cover  $\tilde{K}^n$  is a union of copies of the universal covers  $\tilde{K}_A^n = \tilde{K}_A$  and  $\tilde{K}_B$ , whose incidence graph is the Bass-Serre tree of the decomposition  $\Gamma_n = A_n \star_C B$ . If we choose a copy  $Y \subset \tilde{K}^n$  of  $\tilde{K}_A$ , then the frontier  $W_n$  of  $Y$  in  $\tilde{K}^n$  breaks up into connected components which are stabilized by conjugates of  $C$ , where the minimal pairwise separation between distinct components of  $W_n$  tends to infinity as  $n \rightarrow \infty$ . Theorem 1.1 then applies, yielding nontrivial functions in  $\ell_p H_{\text{cont}}^1(Y, W_n)$  for every  $p > \text{Confdim}(A)$  and every  $n$  sufficiently large; by excision these give functions in  $\ell_p H_{\text{cont}}^1(\tilde{K}^n) \simeq A_p(\partial\Gamma_n)/\mathbb{R}$  which are constant on  $\partial Z \subset \partial\Gamma_n$  for every copy  $Z \subset \tilde{K}^n$  of  $\tilde{K}_B$  in  $\tilde{K}^n$ . These functions provide enough information about the  $\ell_p$ -equivalence relation  $\sim_p$  to deduce (1) and (2).

As a corollary, we obtain examples of hyperbolic groups with Sierpinski carpet boundary which do not have the Combinatorial Loewner Property, and which are quasi-isometrically rigid. See Example 5.5. These examples answer a question of Mario Bonk.

Using our estimates for the critical exponents for elementary polygonal complexes, we obtain upper bounds for the Ahlfors regular conformal dimension of boundaries of a large class of 2-complexes.

**Theorem 1.6** (Proposition 7.1). *Let  $X$  be a simply connected hyperbolic 2-complex whose boundary is connected and approximately self-similar. Assume that  $X$  is a union of 2-cells, where 2-cells intersect pairwise in at most a vertex or edge.*

- (1) *If the perimeter of every 2-cell is at least  $n \geq 5$ , the thickness of every edge lies is at most  $k \geq 2$ , and the link of every vertex contains no circuit of length 3, then*

$$\text{Confdim}(\partial X) \leq 1 + \frac{\log(k-1)}{\log(n-3)}.$$

- (2) *If the perimeter of every 2-cell is at least  $n \geq 7$  and the thickness of every edge is at most  $k \geq 2$ , then*

$$\text{Confdim}(\partial X) \leq 1 + \frac{\log(k-1)}{\log(n-5)}.$$

To prove Theorem 1.6(1), by a straightforward consequence of [28, 15], it suffices to show that the function space  $A_p(\partial X)$  separates points in  $\partial X$  for  $p > 1 + \frac{\log(k-1)}{\log(n-3)}$ . To do this, we find that (a subdivision of)  $X$  contains many elementary polygonal complexes  $Y$  with thickness at most  $k$  and perimeter at least  $2(n-2)$ , such that excision applied to  $(X, X \setminus Y)$  yields the pair  $(Y, \mathcal{T})$ . Theorem 1.4 then produces enough elements in  $A_p(\partial X)$  to separate points.

As an application of Theorem 1.6 we obtain, for every  $\epsilon > 0$ , a hyperbolic group  $\Gamma$  with Sierpinski carpet boundary with the Combinatorial Loewner Property, such that  $\text{Confdim}(\partial \Gamma) < 1 + \epsilon$ . See Example 7.2. This answers a question of Juha Heinonen and John Mackay.

**Organization of the paper.** Section 2 is a brief presentation of some useful topics in geometric analysis. Section 3 discusses first  $\ell_p$ -cohomology and related invariants. We present our new construction of continuous  $\ell_p$ -cohomology, and discuss several connections between  $\ell_p$ -cohomology and the geometry of the boundary. In Section 4 we establish Theorems 1.1 and 1.2. Applications to amalgamated products are given in Section 5. Section 6 focuses on elementary polygonal complexes. Applications to polygonal complexes and Coxeter groups are given in Sections 7 and 8.

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## 2. Preliminaries

This section is a brief presentation of some topics in geometric analysis that will be useful in the sequel. This includes the Ahlfors regular conformal dimension,



the Combinatorial Loewner Property, approximately self-similar spaces, and a few aspects of Gromov hyperbolic spaces.

**Ahlfors regular conformal dimension.** We refer to [34] for a detailed treatment of the conformal dimension and related subjects.

Recall that a metric space  $Z$  is called a *doubling metric space* if there is a constant  $n \in \mathbb{N}$  such that every ball  $B(z, r) \subset Z$  can be covered by at most  $n$  balls of radius  $\frac{r}{2}$ .

A space  $Z$  is *uniformly perfect*, if there exists a constant  $0 < \lambda < 1$  such that for every ball  $B(z, r)$  in  $Z$  with  $0 < r \leq \text{diam } Z$  one has  $B(z, r) \setminus B(z, \lambda r) \neq \emptyset$ .

A space  $Z$  is *Ahlfors  $Q$ -regular* (for some  $Q \in (0, +\infty)$ ) if there is a measure  $\mu$  on  $Z$  such that for every ball  $B \subset Z$  of radius  $0 < r \leq \text{diam}(Z)$  one has  $\mu(B) \asymp r^Q$ .

Every compact, doubling, uniformly perfect metric space is quasi-Möbius homeomorphic to a Ahlfors regular metric space (see [23]). This justifies the following definition.

**Definition 2.1.** Let  $Z$  be a compact, doubling, uniformly perfect metric space. The *Ahlfors regular conformal dimension* of  $Z$  is the infimum of the Hausdorff dimensions of the Ahlfors regular metric spaces which are quasi-Möbius homeomorphic to  $Z$ . We shall denote it by  $\text{Confdim}(Z)$ .

**The Combinatorial Loewner Property (CLP).** The Combinatorial Loewner Property was introduced in [30, 10]. We start with some basic related notions.

Let  $Z$  be a compact metric space, let  $k \in \mathbb{N}$ , and let  $\kappa \geq 1$ . A finite graph  $G_k$  is called a  $\kappa$ -*approximation of  $Z$  on scale  $k$* , if it is the incidence graph of a covering of  $Z$ , such that for every  $v \in G_k^0$  there exists  $z_v \in Z$  with

$$B(z_v, \kappa^{-1}2^{-k}) \subset v \subset B(z_v, \kappa 2^{-k}),$$

and, for  $v, w \in G_k^0$  with  $v \neq w$ ,

$$B(z_v, \kappa^{-1}2^{-k}) \cap B(z_w, \kappa^{-1}2^{-k}) = \emptyset.$$

Note that we identify every vertex  $v$  of  $G_k$  with the corresponding subset in  $Z$ . A collection of graphs  $\{G_k\}_{k \in \mathbb{N}}$  is called a  $\kappa$ -*approximation* of  $Z$ , if for each  $k \in \mathbb{N}$  the graph  $G_k$  is a  $\kappa$ -approximation of  $Z$  on scale  $k$ .

Let  $\gamma \subset Z$  be a curve and let  $\rho: G_k^0 \rightarrow \mathbb{R}_+$  be any function. The  $\rho$ -*length* of  $\gamma$  is

$$L_\rho(\gamma) = \sum_{v \cap \gamma \neq \emptyset} \rho(v).$$

For  $p \geq 1$  the  $p$ -mass of  $\rho$  is

$$M_p(\rho) = \sum_{v \in G_k^0} \rho(v)^p.$$

Let  $\mathcal{F}$  be a non-void family of curves in  $Z$ . We define the  $G_k$ -combinatorial  $p$ -modulus by

$$\text{Mod}_p(\mathcal{F}, G_k) = \inf_{\rho} M_p(\rho),$$

where the infimum is over all  $\mathcal{F}$ -admissible functions, i.e. functions  $\rho: G_k^0 \rightarrow \mathbb{R}_+$  which satisfy  $L_{\rho}(\gamma) \geq 1$  for every  $\gamma \in \mathcal{F}$ .

We denote by  $\mathcal{F}(A, B)$  the family of curves joining two subsets  $A$  and  $B$  of  $Z$  and by  $\text{Mod}_p(A, B, G_k)$  its  $G_k$ -combinatorial  $p$ -modulus.

Suppose now that  $Z$  is a compact arcwise connected doubling metric space. Let  $\{G_k\}_{k \in \mathbb{N}}$  be a  $\kappa$ -approximation of  $Z$ . In the following statement  $\Delta(A, B)$  denotes the *relative distance* between two disjoint non degenerate continua  $A, B \subset Z$  i.e.

$$\Delta(A, B) = \frac{\text{dist}(A, B)}{\min\{\text{diam } A, \text{diam } B\}}. \tag{2.2}$$

**Definition 2.3.** Suppose  $p > 1$ . Then  $Z$  satisfies the *Combinatorial  $p$ -Loewner Property* if there exist two positive increasing functions  $\phi, \psi$  on  $(0, +\infty)$  with  $\lim_{t \rightarrow 0} \psi(t) = 0$ , such that for all disjoint non-degenerate continua  $A, B \subset Z$  and for all  $k$  with  $2^{-k} \leq \min\{\text{diam } A, \text{diam } B\}$  one has

$$\phi(\Delta(A, B)^{-1}) \leq \text{Mod}_p(A, B, G_k) \leq \psi(\Delta(A, B)^{-1}).$$

We say that  $Z$  satisfies the *combinatorial Loewner property* if it satisfies the Combinatorial  $p$ -Loewner Property for some  $p > 1$ .

The CLP is invariant under quasi-Möbius homeomorphisms. A compact Ahlfors  $p$ -regular,  $p$ -Loewner metric space satisfies the Combinatorial  $p$ -Loewner Property (see [10] Th.2.6 for these results). It is conjectured in [30] that if  $Z$  satisfies the CLP and is approximately self-similar (see below for the definition), then  $Z$  is quasi-Möbius homeomorphic to a regular Loewner space.

**Approximately self-similar spaces.** The following definition appears in [30] and [10].

**Definition 2.4.** A compact metric space  $(Z, d)$  is called *approximately self-similar* if there is a constant  $L_0 \geq 1$  such that if  $B(z, r) \subset Z$  is a ball of radius  $0 < r \leq \text{diam}(Z)$ , then there is an open subset  $U \subset Z$  which is  $L_0$ -bi-Lipschitz homeomorphic to the rescaled ball  $(B(z, r), \frac{1}{r}d)$ .

Observe that approximately self-similar metric spaces are doubling and uniformly perfect. Examples include some classical fractal spaces like the square Sierpinski carpet and the cubical Menger sponge. Other examples are the visual boundaries of the hyperbolic spaces which admit an isometric properly discontinuous and cocompact group action [10]. A further source of examples comes from expanding Thurston maps, [2], [22].

The following result is due to S. Keith and the second (named) author [28]. A proof is written in [15].

**Theorem 2.5.** *Suppose  $Z$  is an arcwise connected, approximately self-similar metric space. Let  $\{G_k\}_{k \in \mathbb{N}}$  be a  $\kappa$ -approximation of  $Z$ . Pick a positive constant  $d_0$  that is small compared to the diameter of  $Z$  and to the constant  $L_0$  of Definition 2.4. Denote by  $\mathcal{F}_0$  the family of curves  $\gamma \subset Z$  with  $\text{diam}(\gamma) \geq d_0$ . Then*

$$\text{Confdim}(Z) = \inf\{p \in [1, +\infty) \mid \lim_{k \rightarrow +\infty} \text{Mod}_p(\mathcal{F}_0, G_k) = 0\}.$$

**Hyperbolic spaces.** Let  $X$  be a hyperbolic proper geodesic metric space. We denote the distance between any pair of points  $x, x' \in X$  by  $|x - x'|$ . Let  $\partial X$  be the boundary at infinity of  $X$ . It carries a *visual metric*, i.e. a metric  $d$  for which there is a constant  $a > 1$  such that for all  $z, z' \in \partial X$ , one has

$$d(z, z') \asymp a^{-L}, \tag{2.6}$$

where  $L$  denotes the distance from  $x_0$  (an origin in  $X$ ) to a geodesic  $(z, z') \subset X$ . Moreover  $X \cup \partial X$  is naturally a metric compactification of  $X$ . There is a metric  $d$  on  $X \cup \partial X$  that enjoys the following property: for all  $x, x' \in X \cup \partial X$ , one has

$$d(x, x') \asymp a^{-L} \min\{1, |x - x'|\}, \tag{2.7}$$

where  $a$  is the exponential parameter in (2.6), and where  $L$  denotes the distance in  $X$  from the origin  $x_0$  to a geodesic whose endpoints are  $x$  and  $x'$ . See e.g. [19, 3] for more details. For a subset  $E \subset X$  we denote by  $\partial E$  its *limit set* in  $\partial X$ , i.e.  $\partial E = \bar{E}^{X \cup \partial X} \cap \partial X$ .

If  $X$  satisfies the bounded geometry and nondegeneracy conditions of the introduction, then  $\partial X$  is a doubling uniformly perfect metric space. Conversely, every compact, doubling, uniformly perfect metric space is the visual boundary of a unique hyperbolic metric space as above, up to quasi-isometry; this follows e.g. from a construction of G. Elek [18] (see also [12] for more details).

We also notice that – thanks to Rips’ construction (see [19, 5, 27]) – every proper bounded geometry hyperbolic space is quasi-isometric to a contractible simplicial metric complex, with links of uniformly bounded complexity, and all simplices isometric to regular Euclidean simplices with unit length edges.

### 3. $\ell_p$ -cohomology

This section presents aspects of the  $\ell_p$ -cohomology that will be useful in the sequel. Only the first  $\ell_p$ -cohomology will play a role. Therefore, instead of considering contractible simplicial complexes, we will content ourselves with simply connected ones.

We consider in this section a connected, simply connected, metric simplicial complex  $X$ , with links of uniformly bounded complexity, and all simplices isometric to regular Euclidean simplices with unit length edges. We suppose that it is a hyperbolic metric space, and we denote its visual boundary by  $\partial X$ .

**First  $\ell_p$ -cohomology.** For  $k \in \mathbb{N}$  denote by  $X^{(k)}$  the set of the  $k$ -simplices of  $X$ . For a countable set  $E$  and for  $p \in [1, \infty)$ , let  $\ell_p(E)$  be the Banach space of  $p$ -summable real functions on  $E$ . The  $k$ -th space of  $\ell_p$ -cochains is

$$C_p^{(k)}(X) := \ell_p(X^{(k)}).$$

The standard coboundary operator

$$d^{(k)} : C_p^{(k)}(X) \longrightarrow C_p^{(k+1)}(X)$$

is bounded because of the bounded geometry assumption on  $X$ . When  $k = 0$ , the operator  $d^{(0)}$  is simply the restriction to  $\ell_p(X^{(0)})$  of the differential operator  $d$  defined for every  $f : X^{(0)} \rightarrow \mathbb{R}$  by

$$df(a) = f(a_+) - f(a_-) \quad \text{for all } a = (a_-, a_+) \in X^{(1)}.$$

The  $k$ -th  $\ell_p$ -cohomology group of  $X$  is

$$\ell_p H^k(X) = \ker d^{(k)} / \text{Im } d^{(k-1)}.$$

Since  $X$  is simply connected, every 1-cocycle on  $X$  is the differential of a unique function  $f : X^{(0)} \rightarrow \mathbb{R}$  up to an additive constant. Therefore in degree 1 we get a canonical isomorphism

$$\ell_p H^1(X) \simeq \{f : X^{(0)} \rightarrow \mathbb{R} \mid df \in \ell_p(X^{(1)})\} / \ell_p(X^{(0)}) + \mathbb{R},$$

where  $\mathbb{R}$  denotes the set of constant functions on  $X^{(0)}$ . In the sequel we shall always represent  $\ell_p H^1(X)$  via this isomorphism.

Equipped with the semi-norm induced by the  $\ell_p$ -norm of  $df$  the topological vector space  $\ell_p H^1(X)$  is a quasi-isometric invariant of  $X$ . Moreover if  $X$  satisfies a linear isoperimetric inequality, then  $\ell_p H^1(X)$  is a Banach space, and  $\ell_p H^1(X)$  injects in  $\ell_q H^1(X)$  for  $1 < p \leq q < +\infty$ . See [35], [20], [9] for a proof of these results.

The *continuous first  $\ell_p$ -cohomology group* of  $X$  is

$$\ell_p H^1_{\text{cont}}(X) := \{[f] \in \ell_p H^1(X) \mid f \text{ extends continuously to } X^{(0)} \cup \partial X\},$$

where  $X^{(0)} \cup \partial X$  is the metric compactification of  $X^{(0)}$  (see Section 2 for the definition).

Following P. Pansu [35] we introduce the quasi-isometric numerical invariant of  $X$

$$p_{\neq 0}(X) = \inf\{p \geq 1 \mid \ell_p H^1_{\text{cont}}(X) \neq 0\}.$$

**$\ell_p$ -Equivalence relations.** For  $[f] \in \ell_p H^1_{\text{cont}}(X)$  denote by  $f_\infty: \partial X \rightarrow \mathbb{R}$  its boundary extension. Following M. Gromov ([20, p.259], see also [18], [8]) we set

$$A_p(\partial X) := \{u: \partial X \rightarrow \mathbb{R} \mid u = f_\infty \text{ with } [f] \in \ell_p H^1_{\text{cont}}(X)\},$$

and we define the  *$\ell_p$ -equivalence relation* on  $\partial X$  by

$$z_1 \sim_p z_2 \iff u(z_1) = u(z_2) \text{ for all } u \in A_p(\partial X).$$

This is a closed equivalence relation on  $\partial X$  which is invariant by the boundary extensions of the quasi-isometries of  $X$ . Its cosets are called the  *$\ell_p$ -cosets*. We define

$$p_{\text{sep}}(X) = \inf\{p \geq 1 \mid A_p(\partial X) \text{ separates points in } \partial X\}.$$

Equivalently  $p_{\text{sep}}(X)$  is the infimal  $p$  such that all  $\ell_p$ -cosets are points.

Recall (from the introduction) that  $X$  is *nondegenerate*, if every  $x \in X$  lies within uniformly bounded distance from all three sides of some ideal geodesic triangle. For nondegenerate spaces  $X$ , every class  $[f] \in \ell_p H^1_{\text{cont}}(X)$  is fully determined by its boundary value  $f_\infty$  ([37], see also [35], [13] Theorem 3.1). More precisely  $[f] = 0$  if and only if  $f_\infty$  is constant. In particular we get the following result.

**Proposition 3.1.** *Suppose  $X$  is nondegenerate. Then*

$$p_{\neq 0}(X) = \inf\{p \geq 1 \mid (\partial X / \sim_p) \text{ is not a singleton}\}.$$

We also notice that for nondegenerate spaces  $X$ , the  $\ell_p$ -cosets are always connected in  $\partial X$  ([10] Proposition 10.1).

**Relative  $\ell_p$ -cohomology.** Let  $Y$  be a subcomplex of  $X$ . The  $k$ -th space of relative  $\ell_p$ -cochains of  $(X, Y)$  is

$$C_p^{(k)}(X, Y) := \{\omega \in C_p^{(k)}(X) \mid \omega|_{Y^{(k)}} = 0\}.$$

The  $k$ -th relative  $\ell_p$ -cohomology group of  $(X, Y)$  is

$$\ell_p H^k(X, Y) = \frac{\ker(d^{(k)}: C_p^{(k)}(X, Y) \rightarrow C_p^{(k+1)}(X, Y))}{\text{Im}(d^{(k-1)}: C_p^{(k-1)}(X, Y) \rightarrow C_p^{(k)}(X, Y))}.$$

A straightforward property is the following excision principle.

**Proposition 3.2.** *Suppose  $U \subset Y$  is a subset such that  $Y \setminus U$  is a subcomplex of  $Y$ . Then for every  $k \in \mathbb{N}$  the restriction map induces a canonical isomorphism*

$$\ell_p H^k(X, Y) \simeq \ell_p H^k(X \setminus U, Y \setminus U).$$

Since  $X$  is simply connected, by integrating every relative 1-cocycle, we obtain the canonical isomorphism

$$\ell_p H^1(X, Y) \simeq \{f: X^{(0)} \rightarrow \mathbb{R} \mid df \in \ell_p(X^{(1)}) \text{ and } f|_{E^{(0)}} \text{ is constant in every connected component } E \text{ of } Y\} / \sim,$$

where  $f \sim g$  if and only if  $f - g$  belongs to  $\ell_p(X^{(0)}) + \mathbb{R}$ . We will always represent  $\ell_p H^1(X, Y)$  via this isomorphism. Note that  $\ell_p H^1(X, Y)$  injects canonically in  $\ell_p H^1(X)$ .

We denote by  $\ell_p H_{\text{cont}}^1(X, Y)$  the subspace of  $\ell_p H^1(X, Y)$  consisting of the classes  $[f]$  such that  $f$  extends continuously to  $X^{(0)} \cup \partial X$ . We introduce two numerical invariants:

- $p_{\neq 0}(X, Y)$  is the infimum of the  $p \in [1, +\infty)$  such that the relative cohomology  $\ell_p H_{\text{cont}}^1(X, Y)$  is nontrivial;
- $p_{\text{sep}}(X, Y)$  is the infimum of the  $p \in [1, +\infty)$  such that for every pair of distinct points  $z_1, z_2 \in \partial X$ , either  $\{z_1, z_2\} \subseteq \partial E$  for some connected component  $E$  of  $Y$ , or the relative cohomology  $\ell_p H_{\text{cont}}^1(X, Y)$  separates  $\{z_1, z_2\}$ .

In case there is no such  $p$ , we just declare the corresponding invariant to be equal to  $+\infty$ . We emphasize that the property:  $\partial E \cap \partial E' = \emptyset$  for every pair of distinct connected components  $E, E'$  of  $Y$ , is a necessary condition for the finiteness of  $p_{\text{sep}}(X, Y)$ .

Obviously one has  $p_{\neq 0}(X) \leq p_{\neq 0}(X, Y)$ . When  $X$  is nondegenerate, every element  $[f] \in \ell_p H_{\text{cont}}^1(X, Y)$  is determined by its boundary extension  $f_\infty$ , and we get  $p_{\neq 0}(X, Y) \leq p_{\text{sep}}(X, Y)$ .

**A construction of cohomology classes.** One of the goals of the paper is to construct  $\ell_p$ -cohomology classes of  $X$ . We present here a construction that uses the relative cohomology of special subcomplexes of  $X$ .

**Definition 3.3.** A subcomplex  $Y \subset X$  *decomposes*  $X$ , if it is connected, simply connected and quasi-convex, and if its frontier  $W := Y \setminus \text{int}(Y)$  enjoys the following properties:

- (1) for every pair  $H_1, H_2$  of distinct connected components of  $W$  one has

$$\partial H_1 \cap \partial H_2 = \emptyset;$$

- (2) every sequence  $\{H_i\}_{i \in \mathbb{N}}$  of distinct connected components of  $W$  subconverges in  $X \cup \partial X$  to a singleton of  $\partial X$ .

A collection  $\{Y_j\}_{j \in J}$  of subcomplexes of  $X$  *fully decomposes*  $X$ , if it satisfies the following properties:

- (3) every subcomplex  $Y \in \{Y_j\}_{j \in J}$  decomposes  $X$ ;
- (4) for every pair of distinct points  $z_1, z_2 \in \partial X$ , there is a  $Y \in \{Y_j\}_{j \in J}$ , such that for every connected component  $E$  of  $\overline{X \setminus Y}$  one has  $\{z_1, z_2\} \not\subseteq \partial E$ .

The origin of Definition 3.3 lies in group amalgams, as illustrated by the following example.

**Example 3.4.** Let  $A, B, C$  be three hyperbolic groups, suppose that  $A$  and  $B$  are non-elementary and that  $C$  is a proper quasi-convex malnormal subgroup of  $A$  and  $B$ . Then the amalgamated product  $\Gamma := A \star_C B$  is a hyperbolic group [26]. Let  $K_A, K_B$ , and  $K_C$  be finite 2-complexes with respective fundamental groups  $A, B$ , and  $C$ , such that there are simplicial embeddings  $K_C \hookrightarrow K_A, K_C \hookrightarrow K_B$  inducing the given embeddings of fundamental groups. We let  $K$  be the result of gluing  $K_A$  to  $K_B$  along the copies of  $K_C$ , so  $\pi_1(K) \simeq A \star_C B = \Gamma$ . The universal cover  $\tilde{K}$  is a union of copies of the universal covers  $\tilde{K}_A$  and  $\tilde{K}_B$ , whose incidence graph is the Bass-Serre tree of the decomposition  $\Gamma = A \star_C B$ . If we choose a copy  $Y \subset \tilde{K}$  of  $\tilde{K}_A$ , then  $Y$  decomposes  $\tilde{K}$ . The frontier of  $Y$  in  $\tilde{K}$  breaks up into connected components which are stabilized by conjugates of  $C$ .

The following proposition and corollary will serve repeatedly in the sequel.

**Proposition 3.5.** *Suppose that a subcomplex  $Y$  decomposes  $X$  and let*

$$W := Y \setminus \text{int}(Y)$$

*be its frontier in  $X$ . Then*

(1) *The restriction map*

$$\ell_p H_{\text{cont}}^1(X, \overline{X \setminus Y}) \longrightarrow \ell_p H_{\text{cont}}^1(Y, W),$$

*defined by*

$$[f] \longmapsto [f|_{Y^{(0)}}],$$

*is an isomorphism.*

(2)  $p_{\neq 0}(X, \overline{X \setminus Y}) = p_{\neq 0}(Y, W)$  and  $p_{\text{sep}}(X, \overline{X \setminus Y}) = p_{\text{sep}}(Y, W)$ .

**Corollary 3.6.** (1) *Suppose that a subcomplex  $Y$  decomposes  $X$ , and let  $W$  be its frontier. Then one has*

$$p_{\neq 0}(X) \leq p_{\neq 0}(Y, W).$$

(2) *Suppose that a subcomplex collection  $\{Y_j\}_{j \in J}$  fully decomposes  $X$ , and let  $W_j$  be the frontier of  $Y_j$ . Then one has*

$$p_{\text{sep}}(X) \leq \sup_{j \in J} p_{\text{sep}}(Y_j, W_j).$$

The proof of the proposition will use the

**Lemma 3.7.** *Let  $Y \subset X$  be a connected subcomplex, let  $W$  be its frontier in  $X$ , and let  $\mathcal{E}$  be the set of the connected components of  $\overline{X \setminus Y}$ . Then*

- (1) *the connected components of  $W$  are precisely the subsets of the form  $E \cap Y$ , with  $E \in \mathcal{E}$ ;*
- (2) *their limit sets in  $\partial X$  satisfy  $\partial(E \cap Y) = \partial E \cap \partial Y$ ;*
- (3) *if  $E \in \mathcal{E}$  is such that  $\partial E \cap \partial Y = \emptyset$ , then  $\partial E$  is open (and closed) in  $\partial X$ .*

*Proof of Lemma 3.7.* (1) Note that  $W$  can be expressed as  $W = \overline{(X \setminus Y)} \cap Y$ . Hence it is enough to show that every subset  $E \cap Y$  (with  $E \in \mathcal{E}$ ) is contained in a connected component of  $W$ . Consider the following exact sequence associated to the ordinary simplicial homology of the couple  $(X, \overline{X \setminus Y})$ :

$$H_1(X) \rightarrow H_1(X, \overline{X \setminus Y}) \xrightarrow{\bar{\partial}} H_0(\overline{X \setminus Y}).$$



Since  $X$  is simply connected one has  $H_1(X) = 0$ . By excising the open subset  $X \setminus Y$  from  $X$  and  $\overline{X \setminus Y}$ , we get a canonical isomorphism

$$H_1(Y, W) \longrightarrow H_1(X, \overline{X \setminus Y}).$$

Therefore the boundary map  $\bar{\partial}: H_1(Y, W) \rightarrow H_0(\overline{X \setminus Y})$  is injective. Let  $w_1, w_2$  be two distinct points in  $E \cap Y^{(0)}$ . Since  $Y$  is connected there is path  $\gamma \subset Y^{(1)}$  joining  $w_1$  to  $w_2$ . It defines a  $(Y, W)$ -relative 1-cycle, such that  $\bar{\partial}([\gamma]) = [w_2 - w_1] = 0$  in  $H_0(\overline{X \setminus Y})$ . Since  $\bar{\partial}$  is injective,  $[\gamma]$  is trivial in  $H_1(Y, W)$ . Therefore there is a 2-chain  $\sigma$  in  $Y$ , such that  $\partial\sigma - \gamma$  is a 1-chain in  $W$  whose boundary is  $w_1 - w_2$ . Hence  $w_1$  and  $w_2$  lie in the same connected component of  $W$ .

(2) It is enough to prove that

$$(\partial E \cap \partial Y) \subset \partial(E \cap Y).$$

Let  $z \in \partial E \cap \partial Y$ , and pick points  $x_n \in E, y_n \in Y$  that both converge to  $z$  as  $n \rightarrow \infty$ . Consider the geodesic segment  $[x_n, y_n]$ ; it must meet  $E \cap Y$ , let  $w_n$  be an intersection point. Then  $w_n$  converges to  $z$ ; thus  $z \in \partial(E \cap Y)$ .

(3) Let  $E \in \mathcal{E}$  with  $\partial E \cap \partial Y = \emptyset$ . We claim that  $\{\partial E, \partial(X \setminus E)\}$  is a partition of  $\partial X$ . Clearly, one has  $\partial X = \partial E \cup \partial(X \setminus E)$ . Suppose by contradiction that  $\partial E \cap \partial(X \setminus E)$  is nonempty. Let  $z \in \partial E \cap \partial(X \setminus E)$ , and pick  $x_n \in E$  and  $x'_n \in X \setminus E$  that both converge to  $z$ . Consider the geodesic segment  $[x_n, x'_n]$ ; it must meet  $Y$ , let  $y_n$  be an intersection point. We get  $z = \lim y_n \in \partial Y$ , which is a contradiction. The claim follows.  $\square$

We can now give the

*Proof of Proposition 3.5.* (1) We apply the excision property (Proposition 3.2) to the complexes  $X, \overline{X \setminus Y}$  and to the subset  $U = X \setminus Y$ . Since  $(\overline{X \setminus Y}) \setminus U = (\overline{X \setminus Y}) \cap Y = W$ , we obtain that  $\ell_p H^1(X, \overline{X \setminus Y})$  is isomorphic to  $\ell_p H^1(Y, W)$ . The complexes  $X$  and  $Y$  are simply connected, thus, via the canonical representation, the isomorphism can simply be written as  $[f] \mapsto [f|_{Y^{(0)}}]$ .

We now suppose that  $f|_{Y^{(0)}}$  extends continuously to  $Y^{(0)} \cup \partial Y$ , and we prove that  $f$  extends continuously to  $X^{(0)} \cup \partial X$ . Let  $\mathcal{E}$  be the set of connected components of  $\overline{X \setminus Y}$ . Consider a sequence  $\{x_n\}_{n \in \mathbb{N}} \subset X^{(0)}$  that converges in  $X^{(0)} \cup \partial X$  to a point  $x_\infty \in \partial X$ . We wish to prove that  $\{f(x_n)\}_{n \in \mathbb{N}}$  is a convergent sequence. We distinguish several cases.

If  $x_\infty$  belongs to  $\partial X \setminus \partial Y$ , then for  $n$  and  $m$  large enough the geodesic segments  $[x_n, x_m]$  do not intersect  $Y$ . Therefore they are all contained in the same component  $E \in \mathcal{E}$ . Thus  $f(x_n)$  is constant for  $n$  large enough.

Suppose now that  $x_\infty \in \partial Y$ . Denote by  $(f|_{Y^{(0)}})_\infty$  the boundary extension of  $f|_{Y^{(0)}}$  to  $\partial Y$ . We will prove that  $f(x_n) \rightarrow (f|_{Y^{(0)}})_\infty(x_\infty)$ . By taking a subsequence if necessary, it is enough to consider the following special cases:

- (A) for every  $n \in \mathbb{N}$ ,  $x_n$  belongs to  $Y$ ;
- (B) there is an  $E \in \mathcal{E}$  such that for every  $n \in \mathbb{N}$ ,  $x_n$  belongs to  $E$ ;
- (C) there is a sequence  $\{E_n\}_{n \in \mathbb{N}}$  of distinct elements of  $\mathcal{E}$  such that  $x_n \in E_n$ .

Case (A) is obvious. In case (B), let  $y_n \in Y^{(0)}$  which converges to  $x_\infty$ . When one travels from  $x_n$  to  $y_n$  along a geodesic path  $\gamma_n \subset X^{(1)}$ , one meets for the first time  $Y^{(0)}$  at a point  $y'_n \in (E \cap Y^{(0)})$ . Since  $y'_n \in \gamma_n$ , it satisfies  $y'_n \rightarrow x_\infty$  when  $n \rightarrow \infty$ . Therefore we obtain

$$f(x_n) = f(y'_n) = f|_{Y^{(0)}}(y'_n) \rightarrow (f|_{Y^{(0)}})_\infty(x_\infty).$$

In case (C), let  $y$  be an origin in  $Y$ . Every geodesic segment of  $X$  joining  $y$  to  $E_n$  meets  $E_n \cap Y$ . It follows from property (2) in Definition 3.3 and from Lemma 3.7(1), that the sequence  $\{E_n\}_{n \in \mathbb{N}}$  converges in  $X \cup \partial X$  to the singleton  $\{x_\infty\}$ . For  $n \in \mathbb{N}$ , pick  $y_n \in E_n \cap Y^{(0)}$ . The sequence  $\{y_n\}_{n \in \mathbb{N}}$  tends to  $x_\infty$ ; this leads to

$$f(x_n) = f|_{Y^{(0)}}(y_n) \rightarrow (f|_{Y^{(0)}})_\infty(x_\infty).$$

(2) The equality  $p_{\neq 0}(X, \overline{X \setminus Y}) = p_{\neq 0}(Y, W)$  follows trivially from part (1). We now establish the equality  $p_{\text{sep}}(X, \overline{X \setminus Y}) = p_{\text{sep}}(Y, W)$ .

The equality  $p_{\text{sep}}(Y, W) \leq p_{\text{sep}}(X, \overline{X \setminus Y})$  is a straightforward consequence of Lemma 3.7(2) and part (1) above.

To establish the converse inequality, suppose that  $\ell_p H^1_{\text{cont}}(Y, W)$  separates every pair of distinct points  $z'_1, z'_2 \in \partial Y$  such that  $\{z'_1, z'_2\} \not\subseteq \partial H$  for every component  $H$  of  $W$ . Given a pair of distinct points  $z_1, z_2 \in \partial X$ , with  $\{z_1, z_2\} \not\subseteq \partial E$  for every component  $E \in \mathcal{E}$ , we are looking for an element  $[f] \in \ell_p H^1_{\text{cont}}(X, \overline{X \setminus Y})$  such that  $f_\infty(z_1) \neq f_\infty(z_2)$ .

First, suppose there is an  $E \in \mathcal{E}$  such that  $z_1 \in \partial E$  and  $\partial E \cap \partial Y = \emptyset$ . Then, by Lemma 3.7(3), the subset  $\partial E$  is open and closed in  $\partial X$ . As an easy consequence the characteristic function  $u$  of  $\partial E$  writes  $u = f_\infty$ , with  $[f] \in \ell_q H^1_{\text{cont}}(X, \overline{X \setminus Y})$ , for every  $q \geq 1$ .

Suppose on the contrary, that neither  $z_1$  nor  $z_2$  belong to a limit set  $\partial E$  with  $\partial E \cap \partial Y = \emptyset$ . For  $i = 1, 2$ , define a point  $z'_i \in \partial Y$  as follows:

- if  $z_i \in \partial Y$  set  $z'_i := z_i$ ;
- if  $z_i \notin \partial Y$ , then there is an  $E_i \in \mathcal{E}$  with  $z_i \in \partial E_i$ ; pick  $z'_i \in \partial E_i \cap \partial Y$ .

From Lemma 3.7(2) and from the assumption (1) in Definition 3.3, we have  $\partial E_1 \cap \partial E_2 \cap Y = \emptyset$  for every distinct components  $E_1, E_2 \in \mathcal{E}$ . It follows that  $z'_1 \neq z'_2$ , and that  $\{z'_1, z'_2\} \not\subseteq \partial H$  for every component  $H$  of  $W$ . Our separability assumption on  $\ell_p H_{\text{cont}}^1(Y, W)$ , in combination with part (1) isomorphism, yields a desired element  $[f] \in \ell_p H_{\text{cont}}^1(X, \overline{X \setminus Y})$ .  $\square$

*Proof of Corollary 3.6.* (1) From a standard inequality and Proposition 3.5(2), one has

$$p_{\neq 0}(X) \leq p_{\neq 0}(X, \overline{X \setminus Y}) = p_{\neq 0}(Y, W).$$

(2) From Definition 3.3 and Proposition 3.5(2), one has

$$p_{\text{sep}}(X) \leq \sup_{j \in J} p_{\text{sep}}(X, \overline{X \setminus Y_j}) = \sup_{j \in J} p_{\text{sep}}(Y_j, W_j). \quad \square$$

**Cohomology and the geometry of the boundary.** The following result relates the  $\ell_p$ -cohomology of  $X$  with the structure of the boundary  $\partial X$ , more precisely with the Ahlfors regular conformal dimension and the Combinatorial Loewner Property (see Section 2 for the definitions). It will serve as a main tool in the paper.

**Theorem 3.8.** *Assume that  $X$  is non-degenerate and that  $\partial X$  is connected and approximately self-similar, let  $p \geq 1$ . Then*

(1)  $p > \text{Confdim}(\partial X)$  if and only if

$$(\partial X / \sim_p) = \partial X;$$

in particular

$$p_{\text{sep}}(X) = \text{Confdim}(\partial X).$$

(2) If  $\partial X$  satisfies the CLP, then for  $1 \leq p \leq \text{Confdim}(\partial X)$  the quotient  $\partial X / \sim_p$  is a singleton; in particular

$$p_{\neq 0}(X) = p_{\text{sep}}(X) = \text{Confdim}(\partial X).$$

*Proof.* It follows immediately from Theorem 2.5 in combination with Proposition 3.1 and [10], Corollary 10.5.  $\square$

#### 4. A qualitative bound for $p_{\text{sep}}(Y, W)$

In this section we make a qualitative connection between the relative invariant  $p_{\text{sep}}(Y, W)$  defined in Section 3 and certain geometric properties of the pair  $(Y, W)$

(see Corollary 4.6). This relies on the following result which is also of independent interest. In the statement  $\Delta(\cdot, \cdot)$  denotes the relative distance, see (2.2) for the definition.

**Theorem 4.1.** *For every  $\alpha \in (0, 1)$  there is a  $D \geq 1$  with the following property. Suppose  $Z$  is a bounded metric space, and  $\mathcal{C}$  is a countable collection of closed positive diameter subsets of  $Z$  where  $\Delta(C_1, C_2) \geq D$  for all  $C_1, C_2 \in \mathcal{C}, C_1 \neq C_2$ . Then, for every pair of distinct points  $z_1, z_2 \in Z$ , either  $\{z_1, z_2\} \subseteq C$  for some  $C \in \mathcal{C}$ , or there is a Hölder function  $u \in C^\alpha(Z)$ , such that*

- (1)  $u|_C$  is constant for every  $C \in \mathcal{C}$ ;
- (2) if  $C_1, C_2 \in \mathcal{C}$  and  $u(C_1) = u(C_2)$ , then  $C_1 = C_2$ ;
- (3)  $u(z_1) \neq u(z_2)$ .

**Remarks.** 1) The countability of  $\mathcal{C}$  is only used to obtain (2); one gets plenty of functions without countability.

2) The argument given in the proof of Prop.1.3 of [8] shows that there is a function  $\alpha: (0, \infty) \rightarrow (0, 1)$  with the following property. Let  $Z$  be bounded metric space, and let  $\mathcal{C}$  be a countable collection of closed positive diameter subsets of  $Z$  with  $\Delta(C_1, C_2) \geq D > 0$  for all  $C_1, C_2 \in \mathcal{C}, C_1 \neq C_2$ . Then for  $\alpha = \alpha(D)$  there is a Hölder function  $u \in C^\alpha(Z)$  which satisfies conditions (1),(2),(3) above. Theorem 4.1 above asserts that we can choose the function  $\alpha$  so that  $\alpha(D) \rightarrow 1$  when  $D \rightarrow \infty$ .

To prove Theorem 4.1, we consider a bounded metric space  $Z$ , and a countable collection  $\mathcal{C}$  of closed positive diameter subsets of  $Z$ , such that  $\Delta(C_1, C_2) \geq D$  for all  $C_1, C_2 \in \mathcal{C}, C_1 \neq C_2$ ; here  $D$  is a constant that is subject to several lower bounds during the course of the proof. We can assume that  $\text{diam}(Z) \leq 1$  by rescaling the metric of  $Z$  if necessary. We will assume that  $D > 8$ , and pick  $\Lambda \in [4, \frac{D}{2})$  (we will need  $\Lambda \geq 4$  in the proof of Sublemma 4.5). Let  $r_k = 2^{-k}$ , and let  $\mathcal{C}_k = \{C \in \mathcal{C} \mid \text{diam}(C) \in (r_{k+1}, r_k]\}$ . Since  $\text{diam}(Z) \leq 1$  we have  $\mathcal{C} = \bigcup_{k \geq 0} \mathcal{C}_k$ . Given a Lipschitz function  $v_{-1}: Z \rightarrow \mathbb{R}$ , we will construct the Hölder function  $u$  as a convergent series  $\sum_{j=-1}^\infty v_j$  where, for every  $k \geq 0$ ,

- $v_k$  is Lipschitz;
- $v_k$  is supported in  $\bigcup_{C \in \mathcal{C}_k} N_{\Lambda r_k}(C)$ ;
- for every  $0 \leq j \leq k$  and  $C \in \mathcal{C}_j$ , the partial sum  $u_k = \sum_{j=-1}^k v_j$  is constant on  $C$ .

Thus one may think of  $v_k$  as a “correction” which adjusts  $u_{k-1}$  so that it becomes constant on elements of  $\mathcal{C}_k$ .

**Lemma 4.2.** *Suppose  $0 \leq i, j < k, i \neq j$ , and there are  $C_1 \in \mathcal{C}_i, C_2 \in \mathcal{C}_j$  such that*

$$\text{dist}(N_{\Lambda r_i}(C_1), N_{\Lambda r_j}(C_2)) \leq r_k.$$

*Then  $|i - j| \geq \log_2(\frac{D}{6\Lambda})$ .*

*Proof.* We may assume  $i < j$ . We have

$$D \leq \Delta(C_1, C_2) \leq \frac{\text{dist}(C_1, C_2)}{\binom{r_j}{2}} \implies \text{dist}(C_1, C_2) \geq \frac{Dr_j}{2}$$

and

$$\text{dist}(C_1, C_2) \leq \Lambda r_i + \Lambda r_j + r_k \leq 3\Lambda r_i$$

so

$$\frac{r_i}{r_j} \geq \frac{D}{6\Lambda} \implies j - i \geq \log_2\left(\frac{D}{6\Lambda}\right). \quad \square$$

Let  $n$  be the integer part of  $\log_2(\frac{D}{6\Lambda})$ ; we will assume that  $n \geq 1$ .

Let  $\{L_j\}_{j \in \mathbb{Z}} \subset [0, \infty)$  be an increasing sequence such that  $L_j = 0$  for all  $j \leq -2$ , and let  $\hat{L}_k = \sum_{j=1}^{\infty} L_{k-jn}$  (this is a finite sum since  $L_j = 0$  for  $j \leq -2$ ).

**Definition 4.3.** The sequence  $\{L_j\}$  is *feasible* if  $L_k \geq \hat{L}_k$  for all  $k \geq 0$ .

As an example let  $L_j = e^{\lambda j}$  for  $j \geq -1$ , and  $L_j = 0$  for  $j \leq -2$ . Then  $\{L_j\}$  is feasible if  $e^{-\lambda n} < \frac{1}{2}$ . In particular, we may take  $\lambda$  small when  $n$  is large.

**Lemma 4.4.** *Suppose  $\{L_j\}$  is feasible, and  $v_{-1}: Z \rightarrow \mathbb{R}$  is  $L_{-1}$ -Lipschitz. Then there is a sequence  $\{v_k\}$ , where for every  $k \geq 0$ ,*

- (1)  $v_k$  is  $L_k$ -Lipschitz;
- (2)  $\text{Spt}(v_k) \subset N_{\Lambda r_k}(\bigcup_{C \in \mathcal{C}_k} C)$ ;
- (3)  $\|v_k\|_{C^0} \leq 2L_k r_k$ ;
- (4) for every  $0 \leq j \leq k$ , and every  $C \in \mathcal{C}_j$ , the partial sum  $u_k = \sum_{j=-1}^k v_j$  is constant on  $C$ .

*Proof.* Assume inductively that for some  $k \geq 0$ , there exist functions  $v_{-1}, \dots, v_{k-1}$  satisfying the conditions of the lemma. For every  $C \in \mathcal{C}_k$ , we would like to specify the constant value of the function  $u_k$ . To that end, choose a point  $p_C \in C$  and some  $u_C \in [u_{k-1}(p_C) - \widehat{L}_k r_k, u_{k-1}(p_C) + \widehat{L}_k r_k]$ . Let

$$W_k = \left( Z \setminus \left( \bigcup_{C \in \mathcal{C}_k} N_{\Delta r_k}(C) \right) \right) \sqcup \left( \bigcup_{C \in \mathcal{C}_k} C \right),$$

and define

$$\bar{v}_k: W_k \longrightarrow \mathbb{R}$$

by

$$\bar{v}_k(x) = \begin{cases} 0 & x \in Z \setminus \bigcup_{C \in \mathcal{C}_k} N_{\Delta r_k}(C), \\ u_C - u_{k-1}(x) & x \in C \in \mathcal{C}_k. \end{cases}$$

Since  $\Lambda < \frac{D}{2}$ , the subset  $W_k$  contains every  $C \in \mathcal{C}_i$  with  $i < k$ . Thus  $\bar{v}_k(x) = 0$  for  $x \in C \in \mathcal{C}_i$  and for  $i < k$ .

**Sublemma 4.5.** *One has*

$$\text{Lip}(\bar{v}_k) \leq \sum_{j=1}^{\infty} L_{k-jn} = \widehat{L}_k.$$

*Proof of the sublemma.* Pick  $x, y \in W_k$ .

CASE 1. SUPPOSE  $x, y \in C$  FOR SOME  $C \in \mathcal{C}_k$ . Then

$$\begin{aligned} & \frac{|\bar{v}_k(x) - \bar{v}_k(y)|}{d(x, y)} \\ &= \frac{|u_{k-1}(x) - u_{k-1}(y)|}{d(x, y)} \\ &\leq \sum \{ \text{Lip}(v_j) \mid j < k, \text{Spt}(v_j) \cap C \neq \emptyset \} \\ &\leq \sum \{ L_j \mid j < k, \text{Spt}(v_j) \cap C \neq \emptyset \} \\ &\leq \sum \{ L_j \mid j < k, \text{there exists } C_1 \in \mathcal{C}_j \text{ such that } N_{\Delta r_j}(C_1) \cap C \neq \emptyset \} \\ &\leq \sum_{j \geq 1} L_{k-jn} = \widehat{L}_k, \end{aligned}$$

since the sequence  $L_j$  is increasing, and consecutive elements of the set

$$\{j \mid j < k \text{ and there exists } C_1 \in \mathcal{C}_j \text{ such that } N_{\Lambda r_j}(C_1) \cap C \neq \emptyset\}$$

differ by at least  $n$ , by Lemma 4.2.

CASE 2. THERE IS A  $C \in \mathcal{C}_k$  SUCH THAT  $x \in C$ , AND  $y \notin C$ . Then  $d(x, y) \geq \Lambda r_k$ . Reasoning as in Case 1, we have  $|u_{k-1}(pC) - u_{k-1}(x)| \leq \widehat{L}_k r_k$ . Hence

$$|\bar{v}_k(x)| = |u_C - u_{k-1}(x)| \leq |u_{k-1}(pC) - u_{k-1}(x)| + \widehat{L}_k r_k \leq 2\widehat{L}_k r_k.$$

Therefore

$$|\bar{v}_k(x) - \bar{v}_k(y)| \leq |\bar{v}_k(x)| + |\bar{v}_k(y)| \leq 4\widehat{L}_k r_k,$$

so, since  $\Lambda \geq 4$ ,

$$\frac{|\bar{v}_k(x) - \bar{v}_k(y)|}{d(x, y)} \leq \frac{4}{\Lambda} \widehat{L}_k \leq \widehat{L}_k.$$

Thus the sublemma holds. □

By McShane's extension lemma (see [23]), there is an  $\widehat{L}_k$ -Lipschitz extension  $v_k : Z \rightarrow \mathbb{R}$  of  $\bar{v}_k$ , where  $\|v_k\|_{C^0} \leq \|\bar{v}_k\|_{C^0} \leq 2\widehat{L}_k r_k$ . Since  $\widehat{L}_k \leq L_k$  by the feasibility assumption,  $v_k$  is  $L_k$ -Lipschitz and  $\|v_k\|_{C^0} \leq 2L_k r_k$ . Moreover, by construction,  $u_k$  is constant equal to  $u_C$  on every  $C \in \mathcal{C}_k$ , and is equal to  $u_{k-1}$  on every  $C \in \mathcal{C}_i$  with  $i < k$ . Therefore condition (4) is satisfied and the lemma holds by induction. □

*Proof of Theorem 4.1.* To get the Hölder bound, we use the feasible sequence  $L_j = e^{\lambda j}$ , and keep in mind that we may take  $\lambda$  close to 0 provided  $D$  is large. Suppose  $x, y \in Z$  and  $d(x, y) \in [r_{k+1}, r_k]$ . Then

$$\begin{aligned} |u_{k-1}(x) - u_{k-1}(y)| &\leq r_k \sum \{L_j \mid j < k, \text{Spt}(v_j) \cap \{x, y\} \neq \emptyset\} \\ &\leq r_k \widehat{L}_k \leq L_k r_k. \end{aligned}$$

Also,

$$\begin{aligned} \left| \sum_{j \geq k} v_j(x) - \sum_{j \geq k} v_j(y) \right| &\leq 2 \sum_{j \geq k} \|v_j\|_{C^0} \leq 4 \sum_{j \geq k} L_j r_j \\ &= 4 \sum_{j \geq k} e^{\lambda j} 2^{-j} \leq 9L_k r_k \end{aligned}$$

when  $\lambda$  is small. So

$$\frac{|u(x) - u(y)|}{d(x, y)^\alpha} \leq \frac{10L_k r_k}{\left(\frac{r_k}{2}\right)^\alpha} = 10 \cdot 2^\alpha \cdot (e^\lambda \cdot 2^{\alpha-1})^k$$

which is bounded independent of  $k$  when  $\lambda$  is small.

It is clear from the construction that if  $\mathcal{C}$  is countable, then we may arrange that  $u$  takes different values on different  $C$ 's. Similarly, for a given pair of distinct points  $z_1, z_2 \in Z$  such that  $\{z_1, z_2\} \notin C$  for every  $C \in \mathcal{C}$ , we can perform the above construction so that  $u(z_1) \neq u(z_2)$ .  $\square$

We now give an application to hyperbolic spaces. Again we consider  $Y$  a hyperbolic non-degenerate simply connected metric simplicial complex, with links of uniformly bounded complexity, and all simplices isometric to regular Euclidean simplices with unit length edges.

**Corollary 4.6.** *Let  $Y$  be as above. For every  $\alpha \in (0, 1)$ ,  $C \geq 0$ , there is a  $D \geq 1$  with the following properties. Suppose  $W \subset Y$  is a subcomplex of  $Y$  such that*

- (1) *every connected component of  $W$  is  $C$ -quasiconvex in  $Y$ ;*
- (2) *for every  $y \in W$  there is a complete geodesic  $\gamma \subset W$  lying in the same connected component of  $W$ , such that  $\text{dist}(y, \gamma) \leq C$ ;*
- (3) *the distance between distinct components of  $W$  is at least  $D$ .*

*Then  $p_{\text{sep}}(Y, W) \leq \frac{1}{\alpha} \text{Confdim}(\partial Y)$ .*

The proof relies on Theorem 4.1 and on the following elementary lemma.

**Lemma 4.7.** *Let  $Y$  be a hyperbolic space and let  $d$  be a visual metric on  $\partial Y$ . For every  $C \geq 0$  there are constants  $A > 0$ ,  $B \geq 0$  such that for every  $C$ -quasiconvex subsets  $H_1, H_2 \subset Y$  with non empty limit sets, one has*

$$\Delta(\partial H_1, \partial H_2) \geq A \cdot a^{\frac{1}{2} \text{dist}(H_1, H_2)} - B,$$

*where  $a$  is the exponential parameter of  $d$  (see (2.6)). Moreover  $A, B$  depend only on  $C$ , the hyperbolicity constant of  $Y$ , and the constants of the visual metric  $d$ .*

*Proof of the lemma.* We denote the distance in  $Y$  by  $|y_1 - y_2|$ . Pick  $h_1 \in H_1$ ,  $h_2 \in H_2$  with  $|h_1 - h_2| \leq \text{dist}(H_1, H_2) + 1$ . If  $\text{dist}(H_1, H_2)$  is large enough – compared with  $C$  and the hyperbolicity constant of  $Y$  – then for every  $y_1 \in H_1$ ,  $y_2 \in H_2$  the subset  $[y_1, h_1] \cup [h_1, h_2] \cup [h_2, y_2]$  is a quasi-geodesic segment with controlled constants. In particular the segment  $[h_1, h_2]$  lies in a metric neighborhood  $N_r([y_1, y_2])$  where  $r$  is controlled. Therefore  $H_1 \cup [h_1, h_2] \cup H_2$  is a



$C'$ -quasiconvex subset, where  $C'$  depends only on  $C$  and the hyperbolicity constant of  $Y$ .

Pick an origin  $y_0 \in Y$  and let  $p$  be a nearest point projection of  $y_0$  on  $H_1 \cup [h_1, h_2] \cup H_2$ . We distinguish two cases.

CASE 1:  $p \in [h_1, h_2]$ . Set  $L = \max_{i=1,2} \text{dist}(p, H_i)$ ; one has, with relations (2.6),

$$\Delta(\partial H_1, \partial H_2) \gtrsim \frac{a^{-|y_0-p|}}{a^{-|y_0-p|-L}} = a^L \geq a^{\frac{1}{2} \text{dist}(H_1, H_2)}.$$

CASE 2:  $p \in H_1 \cup H_2$ . Suppose for example that  $p$  belongs to  $H_1$ . Since every geodesic joining  $H_1$  to  $H_2$  passes close by  $h_1$  one has

$$\text{dist}(\partial H_1, \partial H_2) \gtrsim a^{-|y_0-h_1|}.$$

In addition every geodesic joining  $y_0$  to  $H_2$  passes close by  $p$  and thus close by  $h_1$ . Therefore,

$$\text{diam}(\partial H_2) \lesssim a^{-|y_0-h_1|-\text{dist}(H_1, H_2)},$$

and so  $\Delta(\partial H_1, \partial H_2) \gtrsim a^{\text{dist}(H_1, H_2)}$ . □

*Proof of Corollary 4.6.* Let  $\alpha \in (0, 1)$  and let  $d$  be a visual metric on  $\partial Y$ . From the definition of the Ahlfors regular conformal dimension, and because  $\frac{\alpha+1}{2\alpha} > 1$ , there is an Ahlfors  $Q$ -regular metric  $\delta$  on  $\partial Y$ , with  $Q \leq \frac{\alpha+1}{2\alpha} \text{Confdim}(\partial Y)$ , that is *quasi-Möbius equivalent* to  $d$  i.e. the identity map  $(\partial Y, d) \rightarrow (\partial Y, \delta)$  is a quasi-Möbius homeomorphism. This last property implies that the relative distances associated to  $d$  and  $\delta$  are quantitatively related (see [4] Lemma 3.2). Thus, thanks to Theorem 4.1 and Lemma 4.7, there is a constant  $D \geq 1$  such that if  $W \subset Y$  satisfies the conditions (1), (2), (3) of the statement, and if  $\mathcal{H}$  denotes the family of its connected components, then, for every pair of distinct points  $z_1, z_2 \in \partial Y$  with  $\{z_1, z_2\} \not\subseteq \partial H$  for every  $H \in \mathcal{H}$ , there is a function  $u \in C^{\frac{\alpha+1}{2}}(\partial Y, \delta)$  with the following properties

- $u|_{\partial H}$  is constant for every  $H \in \mathcal{H}$ .
- $u(z_1) \neq u(z_2)$ .

We wish to extend  $u$  to a continuous function  $f : Y^{(0)} \cup \partial Y \rightarrow \mathbb{R}$  that is constant on every  $H \in \mathcal{H}$ . To do so pick an origin  $y_0 \in Y$  and observe that the nondegeneracy property of  $Y$  yields the existence of a constant  $R \geq 0$  such that for every  $y \in Y$  there is a  $z \in \partial Y$  with  $\text{dist}(y, [y_0, z]) \leq R$ .

Let  $y \in Y^{(0)}$ . If  $y$  belongs to a  $H \in \mathcal{H}$ , set  $f(y) = u(\partial H)$ . If not, pick a  $z \in \partial Y$  such that  $\text{dist}(y, [y_0, z]) \leq R$  and define  $f(y) = u(z)$ .

We claim that  $f$  is a continuous function of  $Y^{(0)} \cup \partial Y$ . Let  $a > 1$  be the exponential parameter of the visual metric  $d$ . Our hypothesis (2) implies that there is a constant  $R' \geq 0$  so that for every  $y \in Y^{(0)}$  there is a  $z \in \partial Y$  with  $\text{dist}(y, [y_0, z]) \leq R'$  and  $f(y) = u(z)$ . Let  $y_1, y_2 \in Y^{(0)}$  with  $y_1 \neq y_2$ . By relation (2.7), their mutual distance in  $Y^{(0)} \cup \partial Y$  is comparable to  $a^{-L}$ , where  $L = \text{dist}(y_0, [y_1, y_2])$ . The metrics  $d$  and  $\delta$  being quasi-Möbius equivalent they are Hölder equivalent (see [23], Corollary 11.5). Thus  $u \in C^\beta(\partial Y, d)$  for some  $\beta > 0$ , and we obtain

$$|f(y_1) - f(y_2)| = |u(z_1) - u(z_2)| \lesssim a^{-\beta \text{dist}(y_0, (z_1, z_2))} \lesssim a^{-\beta L}. \tag{4.8}$$

To establish the last inequality one notices that  $\text{dist}(y_0, (z_1, z_2)) - L$  is bounded by below just in terms of  $R'$  and the hyperbolicity constant of  $Y$ . The claim follows.

We now claim that  $df \in \ell_p(Y^{(1)})$  for  $p > \frac{2Q}{\alpha+1}$ . To see it, observe that  $u$  is a Lipschitz function of  $(\partial Y, \delta^{\frac{\alpha+1}{2}})$  and  $\delta^{\frac{\alpha+1}{2}}$  is an Ahlfors  $\frac{2Q}{\alpha+1}$ -regular metric on  $\partial Y$  which is quasi-Möbius equivalent to a visual metric. Therefore the claim follows from Elek’s extension process [18] (see [9], Proposition 3.2, for more details).

Hence the function  $f$  defines an element of  $\ell_p H^1_{\text{cont}}(Y, W)$  for  $p > \frac{2Q}{\alpha+1}$ , which separates  $z_1$  and  $z_2$ . Thus

$$p_{\text{sep}}(Y, W) \leq \frac{2Q}{\alpha + 1} \leq \frac{1}{\alpha} \text{Confdim}(\partial Y). \tag{□}$$

### 5. Applications to amalgamated products

This section uses Section 3 and Corollary 4.6 to construct examples of group amalgams whose  $\ell_p$ -cohomology has specified behavior (see Corollary 5.3 especially).

Let  $A$  be a hyperbolic group and let  $C \leq A$  be a finitely presentable subgroup. Let  $K_A$  and  $K_C$  be finite 2-complexes with respective fundamental groups  $A$  and  $C$ , such that there is a simplicial embedding  $K_A \hookrightarrow K_C$  inducing the given embedding of fundamental groups. Denote by  $\tilde{K}_A$  the universal cover of  $K_A$  and pick a copy  $\tilde{K}_C \subset \tilde{K}_A$  of the universal cover of  $K_C$ . The spaces and invariants  $\ell_p H^1_{\text{cont}}(Y, W)$ ,  $p_{\neq 0}(Y)$ ,  $p_{\text{sep}}(Y)$ ,  $p_{\neq 0}(Y, W)$ ,  $p_{\text{sep}}(Y, W)$ , associated to the pair

$$(Y, W) = \left( \tilde{K}_A, \bigcup_{a \in A} a \tilde{K}_C \right),$$

will be denoted simply by  $\ell_p H^1_{\text{cont}}(A, C)$ ,  $p_{\neq 0}(A)$ ,  $p_{\text{sep}}(A)$ ,  $p_{\neq 0}(A, C)$  and  $p_{\text{sep}}(A, C)$ .

We notice that  $p_{\text{sep}}(A, C)$  is finite when  $C$  is a quasi-convex malnormal subgroup (see Remark 2 after Theorem 4.1 and [8] for more details).

Recall from [26] that, given  $A, B$  two hyperbolic groups and  $C$  a quasi-convex malnormal subgroup of  $A$  and  $B$ , the amalgamated product  $A *_C B$  is hyperbolic.

**Proposition 5.1.** *Let  $A, B, C$  three hyperbolic groups, suppose that  $A$  and  $B$  are non elementary and that  $C$  is a proper quasi-convex malnormal subgroup of  $A$  and  $B$ . Let  $\Gamma$  be the amalgamated product  $A *_C B$ . We have*

- (1) *If  $p_{\neq 0}(A, C) < p_{\text{sep}}(B)$  then for all  $p \in (p_{\neq 0}(A, C), p_{\text{sep}}(B)]$  the  $\ell_p$ -equivalence relation on  $\partial\Gamma$  possesses a coset different from a point and the whole boundary  $\partial\Gamma$ .*
- (2) *If  $p_{\text{sep}}(A, C) < p_{\neq 0}(B)$  then for all  $p \in (p_{\text{sep}}(A, C), p_{\neq 0}(B))$  the  $\ell_p$ -equivalence relation on  $\partial\Gamma$  is of the form*

$$z_1 \sim_p z_2 \iff z_1 = z_2 \text{ or there exists } g \in \Gamma \text{ such that } \{z_1, z_2\} \subset g(\partial B).$$

*Proof of Proposition 5.1.* Part (2) is established in [8], Theorem 0.1. We provide here a more enlightening proof.

(1) Let  $p \in (p_{\neq 0}(A, C), p_{\text{sep}}(B)]$  be as in the statement. From Example 3.4 and Corollary 3.6(1) we have  $p > p_{\neq 0}(\Gamma)$ . Hence, by Proposition 3.1, the  $\ell_p$ -cosets are different from  $\partial\Gamma$ . On the other hand we have the obvious inequality  $p \leq p_{\text{sep}}(B) \leq p_{\text{sep}}(\Gamma)$ . Thus Theorem 3.8(1) shows that  $\partial\Gamma$  admits a  $\ell_p$ -coset which is different from a singleton.

(2) Let  $p \in (p_{\text{sep}}(A, C), p_{\neq 0}(B))$  be as in the statement. Let  $u \in A_p(\partial\Gamma)$ . Its restriction to every  $g(\partial B)$  (with  $g \in \Gamma$ ) is constant since  $p < p_{\neq 0}(B)$ . Thus every  $g(\partial B)$  is contained in a  $\ell_p$ -coset.

Conversely to establish that

$$z_1 \sim_p z_2 \implies z_1 = z_2 \text{ or there exists } g \in \Gamma \text{ such that } \{z_1, z_2\} \subset g(\partial B),$$

we shall see that  $A_p(\partial\Gamma)$  separates the limit sets  $g(\partial B)$ . Consider finite simplicial complexes  $K_A, K_B, K_C, K$ , with respective fundamental group  $A, B, C, \Gamma$ , as described in Example 3.4. Let  $\tilde{K}$  be the universal cover of  $K$  and choose copies  $\tilde{K}_A, \tilde{K}_B \subset \tilde{K}$  of the universal covers of  $K_A, K_B$ . We know from Example 3.4 that  $\tilde{K}_A$  decomposes  $\tilde{K}$ .

Pick two distinct subcomplexes  $g\tilde{K}_B, g'\tilde{K}_B \subset \tilde{K}$  and a geodesic  $\gamma \subset \tilde{K}$  joining them. It passes through a subcomplex  $g''\tilde{K}_A$ . Applying  $(g'')^{-1}$  if necessary, we may assume that  $g'' = 1$ . Therefore  $g\tilde{K}_B$  and  $g'\tilde{K}_B$  lie in different components of  $\tilde{K} \setminus \tilde{K}_A$ . By Proposition 3.5(2), we have  $p_{\text{sep}}(A, C) = p_{\text{sep}}(\tilde{K}, \tilde{K} \setminus \tilde{K}_A)$ . Since  $p > p_{\text{sep}}(A, C)$ , we obtain that  $A_p(\partial\Gamma)$  separates  $g(\partial B)$  from  $g'(\partial B)$ .  $\square$

From Theorem 3.8(2), and since  $\sim_p$  is invariant under the boundary extensions of the quasi-isometries of  $\Gamma$ , we get the following result.

**Corollary 5.2.** *Let  $A, B, C, \Gamma$  be as in Proposition 5.1.*

- (1) *If  $p_{\neq 0}(A, C) < p_{\text{sep}}(B)$  then  $\partial\Gamma$  does not admit the CLP.*
- (2) *If  $p_{\text{sep}}(A, C) < p_{\neq 0}(B)$  then any quasi-isometry of  $\Gamma$  permutes the cosets  $gB$ , for  $g \in \Gamma$ . More precisely, the image of a coset  $gB$  by a quasi-isometry of  $\Gamma$  lies within bounded distance (quantitatively) from a unique coset  $g'B$ .*

In combination with Corollary 4.6 this leads to the following result.

**Corollary 5.3.** *Let  $A, B, C$  be as in Proposition 5.1 and suppose that there is a decreasing sequence  $\{A_n\}_{n \in \mathbb{N}}$  of finite index subgroups of  $A$  such that  $\bigcap_{n \in \mathbb{N}} A_n = C$ . Set  $\Gamma_n := A_n *_C B$ .*

- (1) *If  $p_{\text{sep}}(A) < p_{\text{sep}}(B)$  then, for all  $p \in (p_{\text{sep}}(A), p_{\text{sep}}(B)]$  and every  $n$  large enough, the  $\ell_p$ -equivalence relation on  $\partial\Gamma_n$  possesses a coset different from a point and the whole  $\partial\Gamma$ . In particular for  $n$  large enough,  $\partial\Gamma_n$  does not admit the CLP.*
- (2) *If  $p_{\text{sep}}(A) < p_{\neq 0}(B)$  then, for  $p \in (p_{\text{sep}}(A), p_{\neq 0}(B))$  and every  $n$  large enough, the cosets of the  $\ell_p$ -equivalence relation on  $\partial\Gamma_n$  are single points and the boundaries of cosets  $gB$ , for  $g \in \Gamma_n$ . In particular, for large  $n$ , any quasi-isometry of  $\Gamma_n$  permutes the cosets  $gB$ , for  $g \in \Gamma_n$ .*

*Proof.* According to Proposition 5.1 and Corollary 5.2 it is enough to prove that

$$\limsup_n p_{\text{sep}}(A_n, C) \leq p_{\text{sep}}(A). \quad (5.4)$$

We consider the simplicial complexes  $\tilde{K}_A, \tilde{K}_C$  introduced in the beginning of the section, and we set, for  $n \in \mathbb{N}$ ,

$$(Y, W_n) := \left( \tilde{K}_A, \bigcup_{a \in A_n} a\tilde{K}_C \right).$$

Then just by definition we have  $p_{\text{sep}}(A_n, C) = p_{\text{sep}}(Y, W_n)$ . On the other hand, since  $CA_nC = A_n$ , one has in the group  $A$

$$\inf\{\text{dist}(C, aC) \mid aC \in A_n/C, aC \neq C\} = \text{dist}(1, A_n \setminus C).$$

The group  $A$  is quasi-isometric to the space  $\tilde{K}_A$ . Therefore the last equality, in combination with the hypothesis  $\bigcap_{n \in \mathbb{N}} A_n = C$ , implies that the minimal pairwise separation between distinct components of  $W_n$  tends infinity as  $n \rightarrow \infty$ . It follows from Corollary 4.6 that

$$\limsup_n p_{\text{sep}}(Y, W_n) \leq \text{Confdim}(\partial A).$$

With Theorem 3.8(1) we get the desired inequality (5.4). □

**Example 5.5.** It is now possible to answer M. Bonk’s question about examples of approximately self-similar Sierpinski carpets without the CLP. Pick two hyperbolic groups  $A$  and  $B$  whose boundaries are homeomorphic to the Sierpinski carpet, and which admit an isomorphic peripheral subgroup  $C$ . Assume that  $\text{Confdim}(\partial A) < \text{Confdim}(\partial B)$  and that there is a sequence  $\{A_n\}_{n \in \mathbb{N}}$  of finite index subgroups of  $A$  such that  $\bigcap_{n \in \mathbb{N}} A_n = C$ . Such examples can be found among hyperbolic Coxeter groups, because quasi-convex subgroups of Coxeter groups are separable. See Example 8.3 and [21]. Then for  $n$  large enough the boundary of  $A_n *_C B$  is homeomorphic to the Sierpinski carpet. Moreover, according to Corollary 5.3, it doesn’t admit the CLP for  $n$  large enough.

**Example 5.6.** Let  $M, M'$  and  $N$  be closed hyperbolic (i.e. constant curvature  $-1$ ) manifolds with  $1 \leq \dim(N) < \dim(M) < \dim(M')$ . Suppose that  $M$  and  $M'$  contain as a submanifold an isometric totally geodesic copy of  $N$ . The group  $\pi_1(N)$  is separable in  $\pi_1(M)$  (see [1]). For the standard hyperbolic space  $\mathbb{H}^k$  of dimension  $k \geq 2$ , by [35], one has  $\text{Confdim}(\partial \mathbb{H}^k) = p_{\neq 0}(\mathbb{H}^k) = k - 1$ . Therefore the assumptions in items (1) and (2) of Corollary 5.3 are satisfied with  $A = \pi_1(M)$ ,  $B = \pi_1(M')$  and  $C = \pi_1(N)$ . It follows that the manifold  $M$  admits a finite cover  $M_n$  containing an isometric totally geodesic copy of  $N$ , such that the space

$$K := M_n \sqcup_N M'$$

possesses the following properties.

- For  $p \in (\dim M - 1, \dim M' - 1)$ , by letting  $\tilde{K}$  be the universal cover of  $K$ , the cosets of the  $\ell_p$ -equivalence relation on  $\partial \tilde{K}$  are points and the boundaries of lifts of  $M'$ . In particular  $\partial \tilde{K}$  doesn’t satisfy the CLP.
- Every quasi-isometry of  $\tilde{K}$  permutes the lifts of  $M' \subset K$ .

The second property may also be proven using the topology of the boundary, or using coarse topology. By combining topological arguments with the zooming method of R. Schwartz [36], one can deduce from the second property that every quasi-isometry of  $\tilde{K}$  lies within bounded distance from an isometry.

## 6. Elementary polygonal complexes

In this section we compute the invariants  $p_{\neq 0}(Y, W)$  and  $p_{\text{sep}}(Y, W)$  in some very special cases. They will serve in the next sections to obtain upper bounds for the conformal dimension and for the invariant  $p_{\neq 0}(X)$ .

**Definition 6.1.** A *polygonal complex* is a connected simply connected 2-cell complex  $X$  of the following form:

- every 2-cell is isomorphic to a polygon with at least 3 sides;
- every pair of 2-cells shares at most a vertex or an edge.

The number of sides of a 2-cell is called its *perimeter*, the number of 2-cells containing an edge is called its *thickness*.

**Definition 6.2.** An *elementary polygonal complex* is a polygonal complex  $Y$  whose edges are colored black or white, that enjoys the following properties:

- every 2-cell has even perimeter at least 6;
- the edges on the boundary of every 2-cell are alternately black and white;
- every white edge has thickness 1, and every black edge has thickness at least 2.

We will equip every elementary polygonal complex  $Y$  with a length metric of negative curvature, by identifying every 2-cell with a constant negative curvature right angled regular polygon of unit length edges; in particular  $Y$  is a Gromov hyperbolic metric space (quasi-isometric to a tree). The union of the white edges of  $Y$  is called its *frontier*. The frontier is a locally convex subcomplex of  $Y$ , and hence every connected component is a  $CAT(0)$  space, i.e. subtree of  $Y$ . We call such components *frontier trees* of  $Y$ .

The (unique) elementary polygonal complex, whose 2-cells have constant perimeter  $2m$  and whose black edges have constant thickness  $k$  ( $m \geq 3, k \geq 2$ ), will be denoted by  $Y_{m,k}$ .

We denote by  $\mathcal{T}$  the frontier of  $Y$ , and we consider the associated invariants  $p_{\neq 0}(Y, \mathcal{T})$ ,  $p_{\text{sep}}(Y, \mathcal{T})$ . When  $Y = Y_{m,k}$  we write  $\mathcal{T}_{m,k}$ ,  $p_{m,k}$ ,  $q_{m,k}$  for brevity.

**Theorem 6.3.** *For  $m \geq 3$  and  $k \geq 2$  one has*

$$p_{m,k} = q_{m,k} = 1 + \frac{\log(k-1)}{\log(m-1)}.$$

*Proof.* We first establish the upper bound

$$q_{m,k} \leq 1 + \frac{\log(k-1)}{\log(m-1)}. \tag{6.4}$$

For this purpose we will construct some elements in  $\ell_p H_{\text{cont}}^1(Y_{m,k}, \mathcal{T}_{m,k})$ . Set  $Y_m := Y_{m,2}$  for simplicity. Observe that  $Y_m$  is a planar polygonal complex.

For any choice of 2-cells  $c \subset Y_{m,k}$ ,  $d \subset Y_m$  there is an obvious continuous polygonal map

$$r_{c,d} : Y_{m,k} \longrightarrow Y_m$$

sending  $c$  to  $d$  and such that for every  $y_0 \in c$  and every  $y \in Y_{m,k}$  one has

$$|r_{c,d}(y_0) - r_{c,d}(y)| = |y_0 - y|.$$

Observe that the frontier of  $Y_m$  is an union of disjoint geodesics. We will define a continuous map  $\varphi : Y_m \cup \partial Y_m \rightarrow \mathbb{H}^2 \cup \mathbb{S}^1$  which maps every frontier geodesic of  $Y_m$  to an ideal point in  $\mathbb{S}^1$ . To do so, fix a 2-cell  $d \subset Y_m$ . At first we define  $\varphi|_d$  so that its image is a regular ideal  $m$ -gon. In other words  $\varphi$  collapses every white edge of  $d$  to an ideal point in  $\mathbb{S}^1$ , and these ideal points are  $m$  regularly distributed points in  $\mathbb{S}^1$ . For  $n \in \mathbb{N}$ , let  $P_n$  be the union of the 2-cells of  $Y_m$  whose combinatorial distance to  $d$  is less than or equal to  $n$ . By induction we define  $\varphi$  on the subcomplex  $P_n$ , so that

- (i) the images of the 2-cells of  $Y_m$  form a tessellation of  $\mathbb{H}^2$  by ideal  $m$ -gons;
- (ii) the images of the frontier geodesics of  $Y_m$ , passing through the subcomplex  $P_n$ , are  $m(m-1)^{n-1}$  ideal points equally spaced on  $\mathbb{S}^1$ .

Let  $u : \mathbb{S}^1 \rightarrow \mathbb{R}$  be a Lipschitz function. Since  $\varphi(Y_m^{(0)}) \subset \mathbb{S}^1$ , the composition  $u \circ \varphi \circ r_{c,d}$  is well defined on  $Y_{m,k}^{(0)}$ . Let  $f : Y_{m,k}^{(0)} \rightarrow \mathbb{R}$  be this function. By construction its restriction to every frontier tree  $T \subset \mathcal{T}_{m,k}$  is constant. Moreover  $f$  extends continuously to  $Y_{m,k}^{(0)} \cup \partial Y_{m,k}$  since  $r_{c,d}$  does. It remains to estimate the  $p$ -norm of  $df$ . Let  $C$  be the Lipschitz constant of  $u$ . There are  $m(m-1)^n(k-1)^n$  black edges at the frontier of the subcomplex  $r_{c,d}^{-1}(P_n)$ . If  $a$  is such an edge, one has from property (ii) above

$$|df(a)| \leq \frac{2\pi C}{m(m-1)^{n-1}}.$$

Therefore

$$\|df\|_p^p \lesssim \sum_{n \in \mathbb{N}} \frac{(m-1)^n (k-1)^n}{(m-1)^{pn}} = \sum_{n \in \mathbb{N}} ((m-1)^{1-p} (k-1))^n.$$

Thus  $[f]$  belongs to  $\ell_p H_{\text{cont}}^1(Y_{m,k}, \mathcal{T}_{m,k})$  for  $(m-1)^{1-p} (k-1) < 1$  i.e. for

$$p > 1 + \frac{\log(k-1)}{\log(m-1)}.$$

In addition, by varying the 2-cell  $c$  and the fonction  $u$ , one obtains functions  $f$  that separate any given pair of distinct points  $z_1, z_2 \in \partial Y_{m,k}$  such that  $\{z_1, z_2\} \not\subseteq \partial T$  for every  $T \in \mathcal{T}_{m,k}$ . Inequality (6.4) now follows.

To establish the theorem it remains to prove

$$p_{m,k} \geq 1 + \frac{\log(k-1)}{\log(m-1)}. \tag{6.5}$$

To do so we consider, for  $m, k, \ell \geq 3$ , the polygonal complex  $\Delta_{m,k,\ell}$  defined by the following properties:

- every 2-cell has perimeter  $2m$ ;
- every edge is colored black or white, and the edge colors on the boundary of each 2-cell are alternating;
- the thickness of every black edge is  $k$ , while the thickness of white edges is  $\ell$ ;
- the link of every vertex is the full bipartite graph with  $k + \ell$  vertices.

As with elementary polygonal complexes, we metrize  $\Delta_{m,k,\ell}$  so that each cell is a regular right-angled hyperbolic polygon, and hence  $\Delta_{m,k,\ell}$  is a right-angled Fuchsian building. The union of its white edges is a disjoint union of totally geodesic trees in  $\Delta_{m,k,\ell}$ . By cutting  $\Delta_{m,k,\ell}$  along these trees, one divides  $\Delta_{m,k,\ell}$  into subcomplexes, each isometric to  $Y_{m,k}$ . Let  $Y \subset \Delta_{m,k,\ell}$  be one such subcomplex. It decomposes  $\Delta_{m,k,\ell}$ . Denote by  $\mathcal{T}$  its frontier. By Corollary 3.6 one gets that  $p_{\neq 0}(\Delta_{m,k,\ell}) \leq p_{\neq 0}(Y, \mathcal{T}) = p_{m,k}$ . On the other hand  $\partial \Delta_{m,k,\ell}$  is known to admit the CLP (see [11, 10]). Hence, with Theorem 3.8(2) we obtain

$$p_{m,k} \geq p_{\neq 0}(\Delta_{m,k,\ell}) = \text{Confdim}(\partial \Delta_{m,k,\ell}).$$

From [7] formula (0.2), one has  $\text{Confdim}(\partial \Delta_{m,k,\ell}) = 1 + \frac{1}{x}$  where  $x$  is the unique positive number which satisfies

$$\frac{(k-1)^x + (\ell-1)^x}{(1+(k-1)^x)(1+(\ell-1)^x)} = \frac{1}{m}.$$



By an easy computation we get that

$$\lim_{\ell \rightarrow +\infty} \text{Confdim}(\partial\Delta_{m,k,\ell}) = 1 + \frac{\log(k-1)}{\log(m-1)}.$$

Inequality 6.5 follows. □

**Corollary 6.6.** *Let  $Y$  be an elementary polygonal complex. Assume that the perimeter of every 2-cell of  $Y$  lies in  $[2m_1, 2m_2]$  and that the thickness of every black edge lies in  $[k_1, k_2]$ , with  $m_1 \geq 3$  and  $k_1 \geq 2$ . Then*

$$1 + \frac{\log(k_1 - 1)}{\log(m_2 - 1)} \leq p_{\neq 0}(Y, \mathcal{T}) \leq p_{\text{sep}}(Y, \mathcal{T}) \leq 1 + \frac{\log(k_2 - 1)}{\log(m_1 - 1)}.$$

*Proof.* We first establish the last inequality. By adding 2-cells to  $Y$  if necessary, we obtain an elementary polygonal complex  $Y'$ , whose black edges are of constant thickness  $k_2$ , and whose 2-cell perimeters are larger than or equal to  $2m_1$ . Let  $\mathcal{T}'$  be its frontier. The cellular embedding  $(Y, \mathcal{T}) \rightarrow (Y', \mathcal{T}')$  induces a restriction map  $\ell_p H_{\text{cont}}^1(Y', \mathcal{T}') \rightarrow \ell_p H_{\text{cont}}^1(Y, \mathcal{T})$ , which leads to  $p_{\text{sep}}(Y, \mathcal{T}) \leq p_{\text{sep}}(Y', \mathcal{T}')$ .

We now compare  $p_{\text{sep}}(Y', \mathcal{T}')$  with  $q_{m_1, k_2}$ . Observe that any polygon  $P$  of perimeter larger than  $2m_1$  contains a  $2m_1$ -gon whose black edges are contained in black edges of  $P$ , and whose white edges are contained in the interior of  $P$ . With this observation one can construct embeddings  $\varphi: Y_{m_1, k_2} \rightarrow Y'$  in such a way that  $\varphi(Y_{m_1, k_2})$  decomposes  $Y'$  and Proposition 3.5 yields a monomorphism

$$\ell_p H_{\text{cont}}^1(Y_{m_1, k_2}, \mathcal{T}_{m_1, k_2}) \hookrightarrow \ell_p H_{\text{cont}}^1(Y', \mathcal{T}').$$

For  $p > q_{m_1, k_2}$ , by varying the embedding  $\varphi$ , we see that  $\ell_p H_{\text{cont}}^1(Y', \mathcal{T}')$  separates any given pair of distinct points  $z_1, z_2 \in \partial Y'$  such that  $\{z_1, z_2\} \not\subseteq \partial T'$  for every  $T' \in \mathcal{T}'$ . Therefore  $p_{\text{sep}}(Y', \mathcal{T}') \leq q_{m_1, k_2}$ , and the last inequality of Corollary 6.6 follows from Theorem 6.3.

The first inequality can be proved in a similar way. □

### 7. Applications to polygonal complexes

Building on earlier Sections 3 and 6, this section derives some results on the conformal dimension of polygonal complex boundaries (Proposition 7.1).

Let  $X$  be a polygonal complex. If the perimeter of every 2-cell is larger than or equal to 7, then we identify every 2-cell with a constant negative curvature regular polygon of unit length edges and of angles  $\frac{2\pi}{3}$ . The resulting length metric on  $X$  is of negative curvature, in particular  $X$  is hyperbolic.

If every 2-cell has perimeter at least 5, and the link of every vertex contains no circuit of length 3, then we identify every 2-cell with a constant negative curvature regular polygon of unit length edges and of angles  $\frac{\pi}{2}$ . Again the resulting length metric on  $X$  is of negative curvature.

The following result gives an upper bound for the conformal dimension of the boundary at infinity of such polygonal complexes. In some cases a lower bound can be obtained by M. Gromov's method of "round trees", see [20] p. 207, [6], and [33].

**Proposition 7.1.** *Let  $X$  be a polygonal complex of negative curvature as above; suppose that  $\partial X$  is connected and approximately self-similar.*

- (1) *If the perimeter of every 2-cell is at least  $n \geq 5$ , the thickness of every edge is at most  $k \geq 2$ , and the link of every vertex contains no circuit of length 3, then*

$$\text{Confdim}(\partial X) \leq 1 + \frac{\log(k-1)}{\log(n-3)}.$$

- (2) *If the perimeter of every 2-cell is at least  $n \geq 7$  and the thickness of every edge is at most  $k \geq 2$ , then*

$$\text{Confdim}(\partial X) \leq 1 + \frac{\log(k-1)}{\log(n-5)}.$$

*Proof.* We shall construct a family of embedded elementary polygonal complexes that fully decomposes  $X$ . The statement will then follow from Corollaries 3.6(2), 6.6 and Theorem 3.8(1).

(1) First we associate to every edge  $e \subset X$  a *rooted directed tree* defined by the following process:

- (A) join by a directed segment the middle of  $e$  to the center  $O$  of every 2-cell  $c$  containing  $e$ ;
- (B) connect by a directed segment the center  $O$  to the middle of every edge  $e'$  of  $c$  that is not adjacent nor equal to  $e$ ;
- (C) restart the process with  $e'$  and every 2-cell distinct from  $c$  that contains  $e'$ .

Let  $T_e$  be the resulting graph in  $X$ . We claim that  $T_e$  is a bi-Lipschitz embedded tree in  $X$ . To do so, we first notice that from its definition every edge of  $T_e$  admits a direction. A path in  $T_e$  will be called a *directed path* if its edges are all directed in the same way. Pick a directed path  $\gamma \subset T_e$ . Since  $X$  is right angled and non-positively curved, item (B) in combination with angle considerations implies that the union of the 2-cells met by  $\gamma$  is a convex subset of  $X$ . Therefore all directed

paths are uniformly bilipschitz embedded in  $X$ . In addition, two different directed paths in  $T_e$  issuing from the same point have distinct endpoints. Thus  $T_e$  is an embedded tree in  $X$ . Moreover every backtrack free path in  $T_e$  is the union of at most two directed paths making an angle bounded away from 0. Since  $X$  is non-positively curved, one obtains that all backtrack free paths in  $T_e$  are uniformly bilipschitz embedded. The claim follows now easily.

Pick an  $0 < \epsilon \leq \frac{1}{4}$  and consider the  $\epsilon$ -neighborhood  $Y_e$  of  $T_e$  in  $X$ . This is a bi-Lipschitz embedded elementary polygonal complex. The perimeter of every of its 2-cell is larger than or equal to  $2(n-2)$ , and the thickness of its edges is smaller than or equal to  $k$ . The family  $\{Y_e\}_{e \in X^{(1)}}$  fully decomposes  $X$ . Indeed, for a given pair of distinct points  $z_1, z_2 \in \partial X$ , let  $c \in X^{(2)}$  be a 2-cell such that the geodesic  $(z_1, z_2)$  connects two non-adjacent sides  $e_1, e_2$  of  $c$  (possibly  $(z_1, z_2)$  intersects  $c$  in a side  $e_3$  of  $c$  which joins  $e_1$  to  $e_2$ ). Let  $e$  be a side of  $c$  which is different from  $e_1$  and  $e_2$ . Then  $Y_e$  separates  $z_1$  and  $z_2$ . Therefore according to Theorem 3.8(1), Corollaries 3.6(2) and 6.6 we get

$$\text{Confdim}(\partial X) = p_{\text{sep}}(X) \leq \sup_{e \in X^{(1)}} p_{\text{sep}}(Y_e, \mathcal{T}_e) \leq 1 + \frac{\log(k-1)}{\log(n-3)},$$

where  $\mathcal{T}_e$  denotes the frontier of  $Y_e$ .

(2) The method is the same apart from a slight modification in the construction process of the rooted directed trees. Item (B) becomes

(B') Connect  $O$  to the middle of every edge  $e'$  of  $c$  whose combinatorial distance to  $e$  is at least 2.

We claim that  $T_e$  is again a bi-Lipschitz embedded tree in  $X$ . To see this, consider again a directed path  $\gamma \subset T_e$  and the 2-cells  $c_1, \dots, c_n \subset X$  successively met by  $\gamma$ . Their union is not convex in  $X$  but a slight modification is. Indeed pick  $i \in \{1, \dots, n-1\}$ , let  $x$  be one of the vertices of the segment  $c_i \cap c_{i+1}$ , and let  $y \in c_i \setminus c_{i+1}, z \in c_{i+1} \setminus c_i$  be the vertices adjacent to  $x$ . Denote by  $\sigma_x$  the convex hull of the subset  $\{x, y, z\} \subset X$ . By considering the link at  $x$ , one sees that it either degenerates to the union  $[yx] \cup [xz]$ , or it is a simplex contained in the unique 2-cell which contains  $x, y, z$ . Since  $X$  is a non positively curved polygonal complex with unit length edges and  $\frac{2\pi}{3}$  angles, the angles of  $\sigma_x$  at  $y$  and  $z$  are smaller than  $\frac{\pi}{6}$ . Moreover item (B') implies that  $y \notin c_{i-1}$  and  $z \notin c_{i+2}$ . It follows from angle considerations that the union

$$\left(\bigcup_i c_i\right) \cup \left(\bigcup_x \sigma_x\right)$$

defines a locally convex isometric immersion of an abstract CAT(0) space into  $X$ , and it is therefore a global embedding with convex image.

The rest of the proof is similar to case (1). Here the 2-cells of the associated elementary polygonal complex  $Y_e$  have perimeter larger than or equal to  $2(n - 4)$  because of (B').  $\square$

**Example 7.2.** Assume that the link of every vertex is the 1-skeleton of the  $k$ -dimensional cube and that the perimeter of every 2-cell is equal to  $n \geq 5$ . Then Proposition 7.1(1) in combination with the lower bound established in [6] p.140 gives

$$1 + \frac{\log(k - 1)}{\log(n - 3) + \log 15} \leq \text{Confdim}(\partial X) \leq 1 + \frac{\log(k - 1)}{\log(n - 3)}.$$

When the link is the 1-skeleton of the 3-dimensional cube  $\partial X$  is homeomorphic to the Sierpinski carpet. For even  $n$ , examples of such complexes are provided by Davis complexes of Coxeter groups; their boundaries admit the CLP (see [10], 9.4). Therefore one obtains examples of Sierpinski carpet boundaries satisfying the CLP, whose conformal dimension is arbitrarily close to 1. This answers a question of J. Heinonen and J. Mackay.

**Example 7.3.** Assume now that the link of every vertex is the full bipartite graph with  $k + k$  vertices and that the perimeter of every 2-cell is equal to  $n \geq 5$ . From [7] one has

$$\text{Confdim}(\partial X) = 1 + \frac{\log(k - 1)}{\log\left(\frac{n}{2} - 1 + \sqrt{\left(\frac{n}{2} - 1\right)^2 - 1}\right)},$$

which is quite close to the upper bound obtained in Proposition 7.1. We emphasize that the Hausdorff dimensions of the visual metrics on  $\partial X$  do not give in general such precise upper bounds. For example it follows from [31] that if every 2-cell of  $X$  is isometric to the same right angled polygon  $P \subset \mathbb{H}^2$ , then the Hausdorff dimension of the associated visual metric is larger than

$$1 + \frac{\text{length}(\partial P)}{\text{area}(P)} \log(k - 1).$$

## 8. Applications to Coxeter groups

This section applies earlier results for the invariants  $\text{Confdim}(\partial\Gamma)$  and  $p_{\neq 0}(\Gamma)$  to Coxeter groups  $\Gamma$  (Corollary 8.1 and Proposition 8.2)

Recall that a group  $\Gamma$  is a *Coxeter group* if it admits a presentation of the form

$$\Gamma = \langle s, s \in S \mid s^2 = 1, (st)^{m_{st}} = 1 \text{ for } s \neq t \rangle,$$

with  $|S| < +\infty$ , and  $m_{st} \in \{2, 3, \dots, +\infty\}$ . To such a presentation one associates a finite simplicial graph  $L$  whose vertices are the elements  $s \in S$  and whose edges join the pairs  $(s, t)$  such that  $s \neq t$  and  $m_{st} \neq +\infty$ . We label every edge of  $L$  with the corresponding integer  $m_{st}$ . The labelled graph  $L$  is called the *defining graph* of  $\Gamma$ . The valence of a vertex  $s \in L^{(0)}$  will be denoted by  $\text{val}(s)$ .

Using Proposition 7.1 we will deduce the following result.

**Corollary 8.1.** *Suppose  $\Gamma$  is a Coxeter group with defining graph  $L$ .*

- (1) *If for every  $(s, t) \in L^{(1)}$  one has  $\text{val}(s) \leq k$ ,  $m_{st} \geq m \geq 3$  and  $L$  contains no circuit of length 3, then  $\Gamma$  is hyperbolic and*

$$\text{Confdim}(\partial\Gamma) \leq 1 + \frac{\log(k - 1)}{\log(2m - 3)}.$$

- (2) *If for every  $(s, t) \in L^{(1)}$  one has  $\text{val}(s) \leq k$ ,  $m_{st} \geq m \geq 4$ , then  $\Gamma$  is hyperbolic and*

$$\text{Confdim}(\partial\Gamma) \leq 1 + \frac{\log(k - 1)}{\log(2m - 5)}.$$

The above corollary shows that global bounds for the valence and the integers  $m_{st}$  yield upper bounds for the conformal dimension. In contrast, the following result asserts that local bounds are enough to obtain upper bounds for  $p_{\neq 0}(\Gamma)$ .

**Proposition 8.2.** *Suppose  $\Gamma$  is a hyperbolic Coxeter group with defining graph  $L$ . For  $s \in L^{(0)}$  set  $m_s := \inf_{(s,t) \in L^{(1)}} m_{st}$ .*

- (1) *Suppose that there is an  $s \in L^{(0)}$  with  $\text{val}(s) \geq 2$ ,  $m_s \geq 3$  which does not belong to any length 3 circuit of  $L$ . Then*

$$p_{\neq 0}(\Gamma) \leq 1 + \frac{\log(\text{val}(s) - 1)}{\log(m_s - 1)}.$$

- (2) *Suppose that there is an  $s \in L^{(0)}$  with  $\text{val}(s) \geq 2$ ,  $m_s \geq 5$ . Then*

$$p_{\neq 0}(\Gamma) \leq 1 + \frac{\log(\text{val}(s) - 1)}{\log(m_s - 3)}.$$

**Remarks.** The definition of the invariant  $p_{\neq 0}(\Gamma)$  given in Section 3 requires  $\Gamma$  to be hyperbolic. However the above proposition extends to non hyperbolic Coxeter groups as well. In this case the conclusion is simply that  $\ell_p H^1(\Gamma) \neq 0$  for  $p$  larger than the right hand side.

The proofs of Corollary 8.1 and Proposition 8.2 rely on the fact that every Coxeter group  $\Gamma$  has a properly discontinuous, cocompact, isometric action on a  $CAT(0)$  cellular complex  $X$  called the Davis complex of  $\Gamma$ . We list below some of its properties (see [17] Chapters 7 and 12 for more details).

For  $I \subset S$ , denote by  $\Gamma_I$  the subgroup of  $\Gamma$  generated by  $I$ . It is again a Coxeter group; its defining graph is the maximal subgraph of  $L$  whose vertex set is  $I$ . By attaching a simplex to every subset  $I \subset S$  such that  $\Gamma_I$  is finite, one obtains a simplicial complex  $\Sigma L$  whose 1-skeleton is the graph  $L$ . The Davis complex  $X$  enjoys the following properties.

- The 1-skeleton of  $X$  identifies naturally with the Cayley graph of  $(\Gamma, S)$ ; in particular every edge of  $X$  is labelled by a generator  $s \in S$ .
- Every  $k$ -cell is isometric to a  $k$ -dimensional Euclidean polytope.
- The link of every vertex is isomorphic to the simplicial complex  $\Sigma L$ .
- For  $(s, t) \in L^{(1)}$  the corresponding 2-cells of  $X$  are regular Euclidean polygons of perimeter  $2m_{st}$  and unit length edges alternately labelled  $s$  and  $t$ . More generally if  $I \subset S$  spans a  $k$ -simplex in  $\Sigma L$ , then the corresponding  $k$ -cells of  $X$  are isometric to the Euclidean polytope which is the Davis complex of the finite Coxeter group  $\Gamma_I$ .
- For every  $I \subset S$ , the Davis complex of  $\Gamma_I$  has a canonical isometric embedding in  $X$ .

*Proof of Corollary 8.1.* Given a free product  $G = A \star B$  of two hyperbolic groups  $A, B$ , it is well known that

$$\text{Confdim}(\partial G) = \max\{\text{Confdim}(\partial A), \text{Confdim}(\partial B)\}.$$

Thus, by decomposing  $\Gamma$  as a free product, where each factor is a Coxeter group associated to a connected component of  $L$ , we can and will restrict ourself to the case  $L$  is connected.

For  $I \subset S$ , the hypotheses in (1) or (2) imply that the subgroup  $\Gamma_I$  is finite if and only if  $|I| = 1$  or  $2$ . Therefore one has  $\Sigma L = L$  and the Davis' complex  $X$  is a polygonal complex. It clearly satisfies the hypothesis (1) or (2) in Proposition 7.1 with  $n = 2m$ . The corollary follows.  $\square$

*Proof of Proposition 8.2.* (1) Pick  $s \in L^{(0)}$  as in the statement, set  $J := \{t \in S \mid (s, t) \in L^{(1)}\}$  and consider the Coxeter subgroup  $A \leq \Gamma$  generated by  $\{s\} \cup J$ . Its Davis complex, denoted by  $X_A$ , isometrically embeds in  $X$ .

Since there is no length 3 circuit containing  $s$ , the defining graph of  $A$  consists only of segments joining  $s$  to its neighbour vertices  $t \in J$ . Therefore  $X_A$  is an elementary polygonal complex; its black edges are those labelled by  $s$  and the white ones are labelled by an element of  $J$ . The perimeter of its 2-cells is at least  $2m_s \geq 6$  and the thickness of its black edges is equal to  $\text{val}(s)$ . Denote by  $\mathcal{T}_A$  its frontier.

The absence of a length 3 circuit containing  $s$  implies that for every  $t \in J$  and  $K \subset S$  with  $\{s, t\} \not\subseteq K$ , the subgroup  $\Gamma_K$  is infinite. From the previous description of the Davis complex  $X$  we obtain that  $X_A \setminus \mathcal{T}_A$  is an open subset of  $X$ . Therefore  $X_A$  decomposes  $X$ , since  $X$  is simply connected. With Corollaries 3.6(1) and 6.6 we get

$$p_{\neq 0}(X) \leq p_{\neq 0}(X_A, \mathcal{T}_A) \leq 1 + \frac{\log(\text{val}(s) - 1)}{\log(m_s - 1)}.$$

(2) Let  $s$  and  $J$  be as in case (1). Consider the union of the 2-cells of  $X$  that correspond to the family of edges  $(s, t) \in L^{(1)}$  with  $t \in J$ . Let  $\Xi$  be one of its connected components. It is a 2-cell complex whose universal cover is an elementary polygonal complex. We color its edges black and white: the black edges are those labelled by  $s$  and the white ones those labelled by an element of  $J$ . The perimeter of its 2-cells is larger than or equal to  $2m_s \geq 10$  and the thickness of its black edges is equal to  $\text{val}(s)$ .

Since  $m_s \geq 5$  we see that for every  $t \in J$  and  $K \subset S$  with  $\{s, t\} \not\subseteq K$ , the subgroup  $\Gamma_K$  is infinite. Thus  $\Xi$  minus the union of its white edges is an open subset of  $X$ . We will construct an elementary polygonal complex  $Y \subset \Xi$  that decomposes  $X$ . The statement will then follow from Corollaries 3.6(1) and 6.6 again.

The construction of  $Y$  uses a variant of the method presented in the proof of Proposition 7.1. We associate to every black edge  $e \subset \Xi$  a *rooted directed tree* defined by the following process:

- (A) join by a directed segment the middle of  $e$  to the center  $O$  of every 2-cell  $c$  containing  $e$ ;
- (B) connect by a directed segment the center  $O$  to the middle of every black edge  $e'$  of  $c$  that is distinct from  $e$  and from the two nearest ones;
- (C) restart the process with  $e'$  and every 2-cell distinct from  $c$  that contains  $e'$ .

Let  $T_e$  be the resulting graph in  $\Xi$ . We claim that  $T_e$  is a bi-Lipschitz embedded tree in  $X$ . To see this we first modify the constant curvature of every 2-cell of  $\Xi$  so that their angles become equal to  $\frac{3\pi}{4}$ . Since the original angles were at least equal to  $\pi - \frac{\pi}{m_s} \geq \frac{3\pi}{4}$  and since  $\frac{3\pi}{4} + \frac{3\pi}{4} + \frac{\pi}{2} = 2\pi$ , the complex  $X$  remains nonpositively curved.

Consider a directed path  $\gamma \subset T_e$  as defined in the proof of Proposition 7.1, and let  $c_1, \dots, c_n \subset \Xi$  be the 2-cells successively met by  $\gamma$ . Their union is not convex in  $X$ , but a slight modification is. Indeed, pick  $i \in \{1, \dots, n - 1\}$ , let  $x$  be one of the vertices of the segment  $c_i \cap c_{i+1}$ , and let  $y \in c_i \setminus c_{i+1}$ ,  $z \in c_{i+1} \setminus c_i$  be the vertices adjacent to  $x$ . Consider the geodesic simplex  $\sigma_x \subset X$  whose vertices are  $x, y, z$ . Since  $X$  is non positively curved with unit length edges and  $\frac{3\pi}{4}$  angles, the angles of  $\sigma_x$  at  $y$  and  $z$  are smaller than  $\frac{\pi}{4}$ . Moreover item (B) implies that for two consecutive segments  $c_{i-1} \cap c_i$  and  $c_i \cap c_{i+1}$  the associated simplices are disjoint (they are separated by a black edge of  $c_i$  at least). It follows from angle considerations that

$$\left(\bigcup_i c_i\right) \cup \left(\bigcup_x \sigma_x\right)$$

is a convex subset of  $X$ . The rest of the argument and of the construction is similar to the proof of Proposition 7.1. Here the 2-cells of the associated elementary polygonal complex  $Y$  have perimeter larger than or equal to  $2(m_s - 2)$  because of the second item of the construction process.  $\square$

**Example 8.3.** When a hyperbolic group boundary  $\partial\Gamma$  satisfies the CLP one knows from Theorem 3.8(2) that  $p_{\neq 0}(\Gamma) = \text{Confdim}(\partial\Gamma)$ . So Proposition 8.2 yields an upper bound for the conformal dimension of CLP Coxeter boundaries. For example consider the following hyperbolic Coxeter group

$$\Gamma = \langle s_1, \dots, s_4 \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1, \text{ for } i \neq j \rangle,$$

where the order  $m_{ij}$  is finite for all  $i \neq j$  and  $\sum_{i \neq j} \frac{1}{m_{ij}} < 1$  for all  $j \in \{1, \dots, 4\}$ . The associated graph is the 1-skeleton of the tetrahedron. The visual boundary is homeomorphic to the Sierpinski carpet, so its conformal dimension is larger than 1, see [32]. Moreover it admits the CLP [10]. Define

$$m = \max_{1 \leq i \leq 4} (\min_{j \neq i} m_{ij}).$$

If  $m \geq 5$  then from Proposition 8.2(2) we get that

$$\text{Confdim}(\partial\Gamma) \leq 1 + \frac{\log 2}{\log(m - 3)}.$$

In particular if we choose an  $i \in \{1, \dots, 4\}$ , fix the orders  $\{m_{jk}\}_{j \neq i, k \neq i}$ , and let the  $\{m_{ij}\}_{j \neq i}$  go to  $+\infty$ , the conformal dimension tends to 1. We obtain in such a way a family of Coxeter groups with Sierpinski carpet boundaries of different conformal dimensions, which all contain an isomorphic peripheral subgroup, namely the subgroup generated by  $\{s_j\}_{j \neq i}$ . Existence of groups with these properties was evoked in Example 5.5.



**Example 8.4.** The previous examples are quite special because they satisfy the CLP independently of the choice of the coefficients  $m_{st}$ . In general the CLP for Coxeter group boundaries is very sensitive to the coefficients  $m_{st}$ . The reason is the following: suppose that the graph of a hyperbolic Coxeter group  $\Gamma$  decomposes as  $L = L_1 \cup L_2$  where  $L_i$  ( $i = 1, 2$ ) is a *flag* subgraph of  $L$  (i.e. every edge of  $L$  with both endpoints in  $L_i$  belongs to  $L_i$ ). Denote by  $\Gamma_i$  the Coxeter group with defining graph  $L_i$ , and assume that

- there is a vertex  $s \in L_1$  such that no edges issuing from  $s$  belong to  $L_2$  and
- the Coxeter group  $\Gamma_2$  satisfies  $\text{Confdim}(\partial\Gamma_2) > 1$ .

Then if the coefficients of the edges issuing from  $s$  are large enough compared to  $\text{Confdim}(\partial\Gamma_2)$ , we get from Proposition 8.2(2) and Theorem 3.8(1)

$$p_{\neq 0}(\Gamma) < \text{Confdim}(\partial\Gamma_2) \leq \text{Confdim}(\partial\Gamma),$$

and so, according to Theorem 3.8(2), the CLP fails for  $\partial\Gamma$ .

Concrete examples are provided, for instance, by Coxeter groups whose defining graphs are complete bipartite. Let  $L(k, \ell)$  be the full bipartite graph with  $k$  black vertices and  $\ell$  white vertices ( $k \geq 3, \ell \geq 3$ ). When all the coefficients  $m_{st}$  are equal to the same integer  $m \geq 3$  then it is known that the corresponding Coxeter group boundary admits the CLP [11, 10]. Now suppose that the number  $k$  of black vertices is at least equal to 4. Pick one black vertex and decompose  $L(k, \ell)$  accordingly:

$$L(k, \ell) = L(1, \ell) \cup L(k - 1, \ell).$$

Next, choose the coefficients  $m_{st} \geq 3$  arbitrarily on the edges of the second factor graph and let the coefficients of the first one go to  $+\infty$ . Since we have  $k - 1 \geq 3$ , the boundary of the second factor group is homeomorphic to the Menger sponge; thus its conformal dimension is larger than 1 [32]. Therefore the above discussion applies and the CLP fails.

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