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Quasi-total actions and translation numbers

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Abstract. We show that a group admits a non-zero homogeneous quasimorphism if and only if it admits a certain type of action on a poset. Our proof is based on a construction of quasimorphisms which generalizes the construction of the classical translation number quasimorphism. We then develop a correspondence between quasimorphisms and actions on posets, which allows us to translate properties of orders into properties of quasimorphisms and vice versa. Concerning examples we obtain new realizations of the Rademacher quasimorphism, certain Brooks type quasimorphisms, the Dehornoy floor quasimorphism as well as Guichardet–Wigner quasimorphisms on simple Hermitian Lie groups of tube type. The latter we relate to Kaneyuki causal structures on Shilov boundaries, following an idea by Clerc and Koufany. As applications we characterize those quasimorphisms which arise from circle actions, and subgroups of Hermitian Lie groups with vanishing Guichardet–Wigner quasimorphisms.

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Contents

1. Introduction

1.1. Background and motivation. Let G be a group. A function $f: G \to \mathbb{R}$ is called a *quasimorphism* if its defect

$$D(f) := \sup_{g,h \in G} |f(gh) - f(g) - f(h)|$$

is finite. Every bounded perturbation of a homomorphism is a quasimorphism, but there exist also many quasimorphisms which are not of this form. In geometric group theory such non-trivial quasimorphisms arise from group actions on non-positively curved spaces with rank-one isometries (see [3] and the references therein and also [21, 2] for some recent developments); in Lie theory they arise from certain classical decompositions of Hermitian Lie groups [18, 7]. There is also a connection between quasimorphisms to dynamical systems; for example Ghys has observed [17] that the classical Rademacher quasimorphism on $\mathrm{PSL}_2(\mathbb{Z})$ is related to the Lorentz attractor. More exotic examples have been constructed during the last decade on infinite-dimensional groups of diffeomorphisms arising in contact and symplectic topology (see [12] for a recent survey). The interrelations between these different classes of quasimorphisms are not yet fully understood.

The goal of this article is to develop the foundations for a unified approach to quasimorphisms, which allows one to study all the examples mentioned above in a uniform way. This new approach, which was foreshadowed e.g. in [9, 6], is based on group actions on posets. It generalizes various classical constructions from the theory of left-orderable groups. At the heart of our approach lies a very general procedure (introduced in [11]) which associates to a certain type of action of a group G on a poset a numerical function $G \to \mathbb{R}$.

Definition 1.1. Let $(X, \preceq)$ be a poset and G be a group acting on X (but not necessarily preserving $\preceq$). An element $g \in G$ is called *dominant* with respect to the action if for all $h \in G$ there exists $n \in \mathbb{N}_0$ such that

$$kg^n.x \succeq kh.x \quad \text{for all } k \in G, x \in X. \tag{1}$$

Given a dominant $g \in G$ the associated *growth function* $\gamma_g: G \to \mathbb{R}$ by the formula

$$\gamma_g(h) := \lim_{n\to\infty} \frac{1}{n} \cdot \inf\{p \in \mathbb{Z} \mid kg^p.x \succeq kh^n.x \text{ for all } k \in G, x \in X\}.$$

It is easy to see that the limit in the definition of a growth function always exists (see Subsection 2.1). Let us record for later use that growth functions are *homogeneous* in the sense that for all $h \in G$, $n \in \mathbb{N}$ and any dominant g,

$$\gamma_g(h^n) = n \cdot \gamma_g(h).$$

Note also that the definitions of a dominant and its growth function simplify considerably if the action is assumed to be order-preserving. In this case, the notion of a growth function is closely related to the notion of *relative growth* as introduced in [11] (see Subsection 2.1 for a detailed discussion).

We will now relate growth functions to homogeneous quasimorphisms. Our starting point is the following classical example, which will be our guide throughout the article (see [27, 28] for the original or [16] for a modern treatment). Let $H^+_{\mathbb{Z}}(\mathbb{R})$ denote the group of those monotone homeomorphisms of the real line which commute with integral translations. On the one hand, a homogeneous quasimorphism $T_{\mathbb{R}}\colon H^+_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ can be defined by the formula

$$T_{\mathbb{R}}(g) := \lim_{n\to\infty} \frac{g^n.x - x}{n},$$

where $x \in \mathbb{R}$ is an arbitrary basepoint; this quasimorphism $T_{\mathbb{R}}$ is known as *Poincaré's translation number*. On the other hand, the group $H^+_{\mathbb{Z}}(\mathbb{R})$ acts order-preservingly on the poset $(\mathbb{R}, \leq)$. These two facts are related by the following observation (see e.g. [5, Cor. 2.19]).

Proposition 1.2. *The action of $H^+_{\mathbb{Z}}(\mathbb{R})$ on $(\mathbb{R}, \leq)$ admits dominants, and for every dominant $g \in G$ the associated growth function γ_g is a positive multiple of Poincaré's translation number.*

The first goal of this article consists of defining a class of group actions on posets, which generalizes the $H^+_{\mathbb{Z}}(\mathbb{R})$-action on $(\mathbb{R}, \leq)$, and to establish the analog of Proposition 1.2 for these actions. More precisely, we are going to describe in Definition 1.7 below a class of actions on posets called *quasi-total actions*, such that

(a) growth functions of quasi-total actions are homogeneous quasimorphism;
(b) conversely, every quasimorphism arises (up to a multiplicative constant) as the growth function of some quasi-total action;
(c) special classes of quasi-total actions give rise to special classes of quasimorphisms.

Properties (a) and (b) of quasi-total orders will be established in Theorem 1.8 below. The statement of (c) is intentionally kept vague at this point; we will encounter explicit incarnations of this principle in Theorem 1.10 and Theorem 1.13 below. Note that Part (a) is essentially a generalization of Proposition 1.2; its proof mimics closely the classical proof. Part (b) is based on a tautological construction, which implicitly appeared already in our previous work [6]. While the existence of such a tautological realization is very satisfactory from a theoretical point of view, the use of these realizations in applications is limited. It is therefore important to remark that the same homogeneous quasimorphism can appear as the growth function of very different orders. In fact, we observe heuristically that many quasimorphisms of particular interest happen to admit explicit realizations as growth functions of orders with very peculiar properties. Once such a realization has been identified, some version of (c) can be used to obtain further properties of the quasimorphisms in question. We will see various examples of this general strategy below.

1.2. Quasi-total triples. The notion of a quasi-total action arises by abstracting basic properties of the action of $H^+_{\mathbb{Z}}(\mathbb{R})$ on $(\mathbb{R}, \leq)$. The most notable property of the poset $(\mathbb{R}, \leq)$ is that it is totally ordered. It also admits an automorphism T given by $T(x) := x + 1$ such that for any $x, y \in \mathbb{R}$ we have $T^n x \geq y$ for all sufficiently large n. Finally, the group $H^+_{\mathbb{Z}}(\mathbb{R})$ centralizes the translation T. It turns out that these are the only three properties of the $H^+_{\mathbb{Z}}(\mathbb{R})$-action on $(\mathbb{R}, \leq)$ that enter into the proof of Proposition 1.2. In fact, the totality assumption on $\leq$ can be weakened. This leads to the following definition.

Definition 1.3. (i) Let $(X, \preceq)$ be a poset and $\mathrm{Aut}(X, \preceq)$ the group of order-preserving permutations of X. An element $T \in \mathrm{Aut}(X, \preceq)$ is called *dominant* if for all $a, b \in X$ there exists $n \in \mathbb{N}$ such that

$$T^n a \succ b.$$

(ii) Let $(X, \preceq)$ be a poset and $T \in \mathrm{Aut}(X, \preceq)$ a dominant. The triple $(X, \preceq, T)$ is called a *quasi-total triple* if there exists $N(X) \in \mathbb{N}$ such that, for all $a, b \in X$, there exists k, $0 \leq k \leq N(X)$, such that

$$a \preceq T^k b \quad \text{or} \quad b \preceq T^k a \tag{2}$$

and *complete* if, for all $a \in X$, $a \preceq Ta$. We denote by $\mathrm{Aut}_T(X, \preceq)$ the centralizer of T in $\mathrm{Aut}(X, \preceq)$.

Then the following result can be proved along the same lines as Proposition 1.2.

Theorem 1.4. *Let $(X, \preceq, T)$ be a complete quasi-total triple and $G < \mathrm{Aut}_T(X, \preceq)$ containing a dominant. Then all growth functions associated with the G action on $(X, \preceq)$ are (mutually proportional) nonzero homogeneous quasimorphisms.*

The proof of Theorem 1.4 falls into two separate steps. In a first step, we use the automorphism T in order to construct a homogeneous quasimorphism on $\mathrm{Aut}_T(X, \preceq)$; this quasimorphism coincides with the translation number in the classical case and should thus be thought of as a *generalized translation number*. In a second step one then has to show that the growth function associated with the $\mathrm{Aut}_T(X, \preceq)$-action on $(X, \preceq)$ are proportional to this generalized translation number. This second part of the proof relies heavily on results from [6]. We will carry out both steps under slightly weaker assumptions on the G-action; see Subsection 3.1 and Subsection 3.2 respectively. As mentioned earlier there is also a partial converse to Theorem 1.4.

Proposition 1.5. *Let G be a group and $f : G \to \mathbb{R}$ a homogeneous quasimorphism. Then there exists a complete quasi-total triple $(X, \preceq, T)$ and an embedding of G into $\mathrm{Aut}_T(X, \preceq)$ such that the image of G contains a dominant and all growth functions associated with the G-action on $(X, \preceq)$ are positive multiples of f.*

Proposition 1.5 will be established in Subsection 3.4. The proof is constructive, but the resulting G-action is rather tautological and not of much interest. As an almost immediate consequence of Theorem 1.4 and Proposition 1.5 we have

Corollary 1.6. *A group G admits a non-zero homogeneous quasimorphism if and only if it admits an action on a quasi-total triple $(X, \preceq, T)$ containing a dominant.*

Indeed, the previous results show that G admits a non-zero homogeneous quasimorphism if and only if it admits an *effective* action containing a dominant on a *complete* quasi-total triple $(X, \preceq, T)$. Both the effectiveness and the completeness assumption can actually be dropped, see Subsection 4.3.

1.3. Beyond quasi-total triples. In view of Theorem 1.4 and Proposition 1.5 one might be tempted to define a quasi-total action as an embedding

$$G \hookrightarrow \mathrm{Aut}_T(X, \preceq),$$

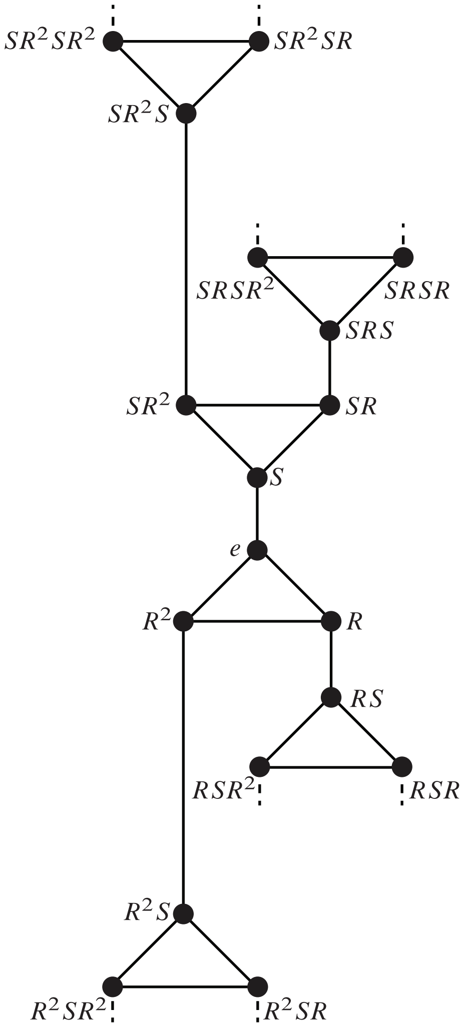

Figure 1. A Cayley graph embedding of $\mathrm{PSL}_2(\mathbb{Z})$ corresponding to the Rademacher quasi-morphism.

where $(X, \preceq, T)$ is a quasi-total triple. By the results of the last section this class of orders would actually satisfy Properties (a) and (b) as defined above. However, it is too narrow to host various examples, which appear naturally in applications. Consider for instance the group

$$G := \mathrm{PSL}_2(\mathbb{Z}) = \langle S, R \mid S^2, R^3 \rangle.$$

Continuing the pattern given in Figure 1 one obtains a map from the Cayley graph Γ of G (with respect to the generating set $\{S, R, R^2\}$) into $\mathbb{R}^2$. This map will not be injective, since edges will start to intersect, but we can deform it into an embedding of Γ into $\mathbb{R}^3$ without changing the (x, y)-coordinates. Once such an embedding is fixed, any partial order on $\mathbb{R}^3$ induces a partial order on Γ. Let us choose the order on $\mathbb{R}^3$ in such a way that going right in the picture increases the order and denote by $\preceq$ the induced order on Γ. Since the element R acts by rotations on the triangles in Γ, the order $\preceq$ will not be G-invariant. In particular, we cannot even embed G into $\mathrm{Aut}(\Gamma, \preceq)$, hence also not into $\mathrm{Aut}_T(\Gamma, \preceq)$ for any automorphism T. Nevertheless one can show that the growth functions of the G-action on $(\Gamma, \preceq)$ are homogeneous quasimorphisms (and actually proportional to the Rademacher quasimorphism on G). This indicates the need for a broader definition of quasi-total action. A closer investigation of the above example shows that while the order $\preceq$ is not G-invariant, it is in fact invariant *up to a bounded error*. The following definition of quasi-total actions is flexible enough to allow for this kind of behavior.

Definition 1.7. Let X be a set. (i) A family of subsets $\{H_n\}_{n\in\mathbb{Z}}$ of X is called a *halfspace filtration* of X if

(H1) $H_{n+1} \subsetneq H_n$, $n \in \mathbb{Z}$.

(H2) $\bigcap H_n = \emptyset, \bigcup H_n = X$.

Given a halfspace filtration $\{H_n\}_{n\in\mathbb{Z}}$ of X and elements $a, b \in X$ we denote by

$$h(a) := \sup\{n \in \mathbb{Z} \mid a \in H_n\}$$

the *height* of a and by

$$h(a, b) := h(a) - h(b)$$

the *relative height* of a over b with respect to $\{H_n\}$.

(ii) A partial order $\preceq$ on X is called a *halfspace order* if there exists a halfspace filtration $\{H_n\}$ of X and a constant $w := w(X, \preceq, \{H_n\})$ such that, for all $a, b \in X$,

$$h(a,b) \geq w \implies a \succeq b.$$

We then say that $\{H_n\}$ is *compatible* with the order $\preceq$ and that the triple $(X, \preceq, \{H_n\}_{n\in\mathbb{Z}})$ has *width* bounded by w.

(iii) Let $(X, \preceq)$ be a halfspace order and G be a group. An action of G on X is called *quasi-total* if it is effective, admits a dominant and there exists a compatible halfspace filtration $\{H_n\}$ of X and a constant $d > 0$ such that the associated height function satisfies

$$|h(ga, gb) - h(a,b)| \leq d \quad \text{for all } g \in G, a, b \in X. \tag{3}$$

It is easy to see that in the example above the G-action on the embedded Cayley graph $(\Gamma, \preceq)$ is quasi-total. Also, one can show that given any complete quasi-total triple $(X, \preceq, T)$ the action of $\mathrm{Aut}_T(X, \preceq)$ on $(X, \preceq)$ is quasi-total (see Corollary 3.9). We can now formulate our first main theorem.

Theorem 1.8. (i) *The growth functions of any quasi-total action are* (*mutually proportional*) *nonzero homogeneous quasimorphisms.*

(iii) *Up to a positive multiple, every nonzero homogeneous quasimorphism arises as the growth function of some quasi-total action of the underlying group.*

Note that Part (ii) of the theorem is just a special case of Proposition 1.5; on the other hand, Part (i) of Theorem 1.8 is more general than Theorem 1.4. Fortunately, the same two step approach used in the proof of Theorem 1.4 applies also in the present case (see again Subsection 3.1 and Subsection 3.2).

As a special case of Theorem 1.8 we see that the growth functions of the $\mathrm{PSL}_2(\mathbb{R})$-action on the above Cayley graph embedding into $\mathbb{R}^3$ are homogeneous quasimorphisms. We refer the reader to Example 4.5 for a proof of the fact that they are proportional to the Rademacher quasimorphism. This provides a new construction method for the Rademacher quasimorphism via an embedding of the Cayley graph of the underlying group into $\mathbb{R}^3$. This method generalizes to other quasimorphisms on finitely generated groups; see e.g. Example 4.6 for the case of Brooks quasimorphisms on free groups.

1.4. Subgroups with vanishing quasimorphism. By definition, every quasi-total action admits a dominant; in terms of a compatible halfspace filtration $\{H_n\}$

and associated height function h this means that the function $g \mapsto h(g.x)$ is unbounded for some (hence any) basepoint $x \in X$. This unboundedness assumption ensures that the corresponding quasimorphism is nonzero, hence unbounded. On the other hand, unboundedness of the function $g \mapsto h(g.x)$ is clearly necessary to obtain an unbounded growth function. This simple observation yields immediately the following characterization result.

Corollary 1.9. *Let G be a group acting quasi-totally on $(X, \preceq)$, $x_0 \in X$ and $f = \gamma_{x_0}$ the associated quasimorphism. Then for a subgroup $H < G$ the following are equivalent*:

(i) $f|_H \equiv 0$;

(ii) *the action of H on $(X, \preceq)$ is not unbounded*;

(iii) *for any compatible halfspace filtration $\{H_n\}$ with height function h and any $x \in X$ the function $g \mapsto h(gx)$ is bounded on H.*

If the poset $(X, \preceq)$ is well-understood, this allows one to study subgroups with vanishing quasimorphism. In the study of quasimorphisms on Lie groups a refinement of this idea can be used. We can formulate this refinement in some generality. For this we recall that a poset $(X, \leq)$ is called *globally hyperbolic* if X comes equipped with a topology such that the finite order intervals

$$[x, y] := \{z \in X \mid x \leq z \leq y\}$$

are compact and the infinite order intervals

$$[x, \infty) := \{z \in X \mid x \leq z\}, \quad (-\infty, x] := \{z \in X \mid z \leq x\}.$$

are closed. Let us call a subset $B \subset X$ *bounded* if its closure is compact. We will prove the following result in Theorem 3.12.

Theorem 1.10. *Let H be a group, $(X, \leq)$ a globally hyperbolic poset with a quasi-total H-action, $x_0 \in X$ and $f = \gamma_{x_0}$ be an associated growth function. Then for a subgroup $G < H$ the following are equivalent*:

(i) $f|_G \equiv 0$;

(ii) *every G-orbit in X is bounded*;

(iii) *there exists a bounded G-orbit in X.*

Thus if we manage to find a realization of a quasimorphism by a globally hyperbolic quasi-total triple, then we obtain valuable geometric information about subgroups of vanishing quasimorphism.

1.5. Smooth quasi-total triples and Lie groups. In order to illustrate the above machinery and to provide an application, let us discuss the case of quasimorphisms of finite-dimensional connected Lie groups. For this let let $\mathcal{D}$ be an irreducible bounded symmetric domain with Shilov boundary $\check{S}$ and let G be the identity component of the isometry group of $\mathcal{D}$ with respect to the Bergman metric. Then any infinite covering of G admits a (unique up to multiples) homogeneous quasimorphism, called Guichardet–Wigner quasimorphism [18, 7]. These quasimorphisms are the basic building blocks for quasimorphisms on Lie groups, in the sense that every quasimorphism on a finite-dimensional connected Lie group arises as a pull-back of a sum of Guichardet–Wigner quasimorphisms and homomorphisms to $\mathbb{R}$ (see [5] for details). For general $\mathcal{D}$ we do not know whether the associated Guichardet–Wigner quasimorphism can be realized using a globally hyperbolic quasi-total triple. However, if the domain $\mathcal{D}$ happens to be of tube type, then by a result of Kaneyuki [23] there is a unique (up to inversion) G-invariant causal structure on $\check{S}$, which gives rise to a partial order $\preceq$ on the universal covering $\check{R}$ of $\check{S}$ (see Section 5.4 for details of this construction). It was observed by Clerc and Koufany [9] that this order is closely related to the Guichardet–Wigner quasimorphism. In the language of the present paper their results can be reinterpreted as follows. Denote by T the unique deck transformation of the covering $\check{R} \to \check{S}$ which is non-decreasing with respect to $\preceq$. Also denote by $\check{G}$ the unique cyclic covering of G which acts transitively and effectively on $\check{R}$. Then we have

Theorem 1.11. *The triple $(\check{R}, \preceq, T)$ is a globally hyperbolic quasi-total triple, on which $\check{G}$ acts by automorphisms. In particular, the growth functions of this action are proportional to the Guichardet–Wigner quasimorphism on $\check{G}$.*

Theorem 1.11 follows by combining Corollary 5.14 and Corollary 5.15 below. Concerning the latter, it is important to remark that the order $\preceq$ used in the present article is the *closure* of the order considered in [9]. The combination of Theorem 1.11 and Theorem 1.10 yields the following result.

Corollary 1.12. *Let $H < \check{G}$ be a subgroup. Then the Guichardet–Wigner quasimorphism vanishes along H if and only if some (hence any) H-orbit in $\check{R}$ is bounded.*

To the best of our knowledge this is the first characterization of groups with vanishing Guichardet–Wigner quasimorphism in the literature.

1.6. Left-orderable groups and circular quasimorphisms. We should mention at this point that in the special case of totally left-ordered group, our main theorem is related to classical results in the field. Indeed, let G be a countable group and $\preceq$ a total order on G, which is preserved by the left action of G on itself. (The existence of such an order implies in particular, that G is torsion-free.) As explained e.g. in [25], there exists an order-preserving G-action on the real line which admits an equivariant order-preserving embedding of G, which is unique up to conjugation. Note that by construction, the two actions of G on $(\mathbb{R}, \leq)$ and $(G, \preceq)$ give rise to the same growth functions. If $\preceq$ admits a dominant, then the G-action on $\mathbb{R}$ can be chosen to commute with the $\mathbb{Z}$-action; in this case the associated growth functions are proportional to the restriction of the translation number, hence homogeneous quasimorphisms. To summarize, growth functions of left-invariant total orders (admitting a dominant) on countable groups are homogeneous quasimorphisms. The resulting quasimorphisms are known to be of a very special sort (see e.g. [14, 15, 1]). Since they arise from lifts of circle actions, we will call them *circular*.

It turns out that we can use the machinery of quasi-total triples to decide whether a given quasimorphism $f : G \to \mathbb{R}$ is circular. Let us assume for simplicity that f is unbounded on the center of G. This assumption is satisfied in many examples of interest and can anyway always be achieved by passing to a central extension. Then we have

Theorem 1.13. *Let G be a torsion-free countable group and $f : G \to \mathbb{R}$ be a homogeneous quasimorphism, which is unbounded on the center of G. Then the following are equivalent*:

(i) *f is circular*;

(ii) *f is proportional to the growth functions of a quasi-total triple of the form $(X, \preceq, T)$ with $X = G$ and $\preceq$ a total order.*

For proofs and more precise results (avoiding the assumption that f is unbounded on the center) see Section 6.

1.7. Structure of the paper. This paper is organized as follows. In Section 2 we collect various preliminary results and notions. In particular, we provide a dictionary between groups actions on posets and partially bi-ordered groups, which allows us to translate some key results from [6] into our language. As a warm up, we also reprove a simple special case of our main theorem, due to Hölder. Section 3 is the technical heart of this paper. All the general machinery concerning quasi-total actions is established in this section. The final three sections are essentially independent and concern the three main classes of examples studied in this article.

Section 4 discusses some examples related to countable groups. Section 5 deals with smooth quasi-total triples and Lie groups. Finally, Section 6 discusses circular quasimorphisms on torsion-free groups. The appendix collects some basic results and definition concerning smooth orders, which are used in Section 5.

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2. Preliminaries

2.1. Poset actions and bi-invariant orders. Effective group actions on posets are closely related to bi-invariant orders on groups. In the current framework the poset point of view appears more naturally. Since, however, some of the results we use are formulated in the language of bi-invariant orders, we will provide a brief dictionary here. In the sequel we will then use both points of view interchangeably.

Let us begin by recalling some standard terminology concerning orders on groups. A partial order $\leq$ on a group G is called *bi-invariant* if $g \leq h$ implies both $gk \leq hk$ and $kg \leq kh$ for all $g, h, k \in G$. Equivalently, the associated *order semigroup*

$$G^+ := \{g \in G \mid g \geq e\}$$

is a conjugation-invariant monoid satisfying the *pointedness condition*

$$G^+ \cap (G^+)^{-1} = \{e\}.$$

In this case we call the pair $(G, \leq)$ a *partially bi-ordered group*. Every effective action of G on a poset $(X, \preceq)$ induces a bi-invariant partial order on G, called the *induced order* by setting

$$g \leq h \iff (kg).x \preceq (kh).x \text{ for all } k \in G, x \in X.$$

Note that every bi-invariant partial order is induced from itself via the left action of G. In the sequel, a bi-invariant partial order induced by a quasi-total action will be called a *quasi-total order*.

In [11] the *dominant set* of a partially bi-ordered group is defined as

$$G^{++} := \{g \in G^+ \setminus \{e\} \mid \text{for all } h \in G \text{ there exists } n \in \mathbb{N}_0 \text{ such that } g^n \geq h\},$$

while the *relative growth function*

$$\gamma_{\leq} \colon G^{++} \times G \longrightarrow \mathbb{R}$$

is defined by

$$\gamma_{\leq}(g,h) := \lim_{n\to\infty} \frac{\inf\{p \in \mathbb{Z} \mid g^p \geq h^n\}}{n}.$$

Our definitions, as stated in the introduction, are chosen to match these definitions. Given an effective G-action on a poset $(X, \preceq)$ with induced order $\leq$, the notions of dominants for $\preceq$ and $\leq$ coincide. Given an effective G-action on a poset $(X, \leq)$ with induced order $\leq$ and a dominant g, the growth function γ_g of $\preceq$ is just the function $\gamma_{\leq}(g, \cdot)$.

2.2. Relative growth and the sandwich lemma. The link between actions on posets and bi-invariant partial orders makes the results from [6] available in our context. One such result, which will enter crucially into the proof of both directions of Theorem 1.8, is the so-called *sandwich lemma*. Because of its relevance we recall it here. Given a partially bi-ordered group $(G, \leq)$ with order semigroup G^+ and a homogeneous quasimorphism $f : G \to \mathbb{R}$, we say that f *sandwiches* G^+ if for some $C > 0$

$$\{g \in G \mid f(g) \geq C\} \subset G^+. \tag{4}$$

(In this case automatically $G^+ \subset \{g \in G \mid f(g) \geq 0\}$, hence the name.) Then the sandwich lemma can be stated as follows ([6, Proposition 3.3]).

Proposition 2.1. *Suppose that $(G, \leq)$ is a partially bi-ordered group and that $f : G \to \mathbb{R}$ is a non-zero homogeneous quasimorphism. If f sandwiches $\leq$, then $G^{++} \neq \emptyset$ and for all $g \in G^{++}$, $h \in G$ we have*

$$\gamma(g,h) = \frac{f(h)}{f(g)}.$$

2.3. A toy example: Hölder's theorem. Before we turn to the study of general quasi-total actions, it seems instructive to consider a special class of such actions, for which the arguments are particularly simple and transparent. Thus we will study growth functions of bi-invariant total orders. The result is as follows.

Proposition 2.2. *Every bi-invariant total order on a group G, which admits a dominant, is induced from a quasi-total triple. Moreover, the associated growth functions are homomorphisms.*

According to [25] an equivalent result was first established by Hölder. Since the proof is short we reproduce it there; this gives us the opportunity to recall the following trivial but useful fact to be used throughout.

Lemma 2.3. *Let G be a group and $\leq$ be a bi-invariant partial order on G. Then for all $f_1, f_2, g_1, g_2 \in \Gamma$ we have*

$$f_1 \geq g_1,\ f_2 \geq g_2 \implies f_1 f_2 \geq g_1 g_2.$$

Proof of Proposition 2.2. Concerning the first statement, choose $h \in G^{++}$ and set

$$(X, \preceq, T) := (G, \leq, \rho_h),$$

where ρ_h denotes right-multiplication by h. Then it is easy to check that $(X, \preceq, T)$ is a quasi-total triple and that the induced quasi-total order coincides with $\leq$. We now turn to the proof of the second statement. We fix $g \in G^{++}$ and show that γ_g is a homomorphism. For this let $a, b \in G$. We may assume without loss of generality that $ab \leq ba$. From bi-invariance we then obtain for all $w_1, w_2 \in G$ the inequality

$$w_1 ab w_2 \leq w_1 ba w_2. \tag{5}$$

We claim that this implies that for every $n \in \mathbb{N}$,

$$a^n b^n \leq (ab)^n \leq b^n a^n. \tag{6}$$

Indeed, using (5) repeatedly we obtain

$$\begin{aligned}(ab)^n &= abab \cdot \dots \cdot ab \geq a^2 bbab \cdot \ldots \cdot ab \\ &\geq a^2 babb \cdot \ldots \cdot ab \geq a^3 b^3 ab \cdot \ldots \cdot ab \\ &\geq a^n b^n,\end{aligned}$$

and the other inequality is proved similarly. If we abbreviate

$$\gamma_n(g,h) := \inf\{p \in \mathbb{Z} \mid g^p \geq h^n\}, \quad h \in G,$$

then we obtain

$$\gamma_1(g, a^n b^n) \leq \gamma_n(g, ab) \leq \gamma_1(g, b^n a^n). \tag{7}$$

On the other hand totality of $\leq$ yields for every $n \in \mathbb{N}$,

$$g^{\gamma_n(g,a)-1} \leq a^n \leq g^{\gamma_n(g,a)},$$

$$g^{\gamma_n(g,b)-1} \leq b^n \leq g^{\gamma_n(g,b)}$$

and thus by Lemma 2.3

$$g^{\gamma_n(g,a)+\gamma_n(g,b)-2} \leq a^n b^n \leq g^{\gamma_n(g,a)+\gamma_n(g,b)+2},$$

$$g^{\gamma_n(g,a)+\gamma_n(g,b)-2} \leq b^n a^n \leq g^{\gamma_n(g,a)+\gamma_n(g,b)+2}.$$

We deduce that

$$|\gamma_1(g, a^n b^n) - \gamma_n(g,a) - \gamma_n(g,b)| \leq 2$$

$$|\gamma_1(g, b^n a^n) - \gamma_n(g,a) - \gamma_n(g,b)| \leq 2.$$

Combining this with (7) we obtain

$$\gamma_n(g,a) + \gamma_n(g,b) - 2 \leq \gamma_n(g,ab) \leq \gamma_n(g,a) + \gamma_n(g,b) + 2$$

Dividing by n and passing to the limit $n \to \infty$ we get

$$\gamma(g,a) + \gamma(g,b) \leq \gamma(g,ab) \leq \gamma(g,a) + \gamma(g,b).$$

This shows that γ_g is a homomorphism. □

The condition of admitting a bi-invariant total order is rather restrictive. We refer the reader to [25] and the references therein for various characterizations and properties of totally bi-orderable groups.

3. Foundations of quasi-total actions

3.1. The translation number of a quasi-total action. Let G be a group acting quasi-totally on a poset $(X, \preceq)$. Fix a compatible halfspace filtration $\{H_n\}$ with underlying height function h and a constant $d > 0$ such that (2) holds.

Proposition 3.1. *The functions* $\{f_a\colon G \to X\}_{a\in X}$ *given by* $f_a(g) := h(ga, a)$ *are mutually equivalent quasimorphisms of defect* $\le d$*. In fact, their mutual distances are uniformly bounded by* d*.*

Proof. We first show that the f_a are mutually at bounded distance:

$$\begin{aligned} |f_a(g) - f_b(g)| &= |h(ga) - h(a) - h(gb) + h(b)| \\ &= |h(ga) - h(gb) - (h(a) - h(b))| \\ &= |h(ga, gb) - h(a, b)| < d. \end{aligned}$$

Now let us use this fact to show that they are quasimorphisms:

$$\begin{aligned} &|f_a(gk) - f_a(g) - f_a(k)| \\ &\quad \le |f_a(gk) - f_{ka}(g) - f_a(k)| + d \\ &\quad = |h(gka) - h(a) - h(gka) + h(ka) - h(ka) + h(a)| + d \\ &\quad = d. \end{aligned}$$

∎

The assumption that the action of G on X is unbounded implies immediately that each of the functions f_a is unbounded. Then standard properties of homogeneization (see e.g. [8]) yield the following results

Corollary 3.2. *There exists a nonzero homogeneous quasimorphism*

$$T_{(X,\preceq,\{H_n\})}\colon G \longrightarrow \mathbb{R}$$

of defect $\le 2d$ *such that for all* $a \in X$*,*

$$T_{(X,\preceq,\{H_n\})}(g) = \lim_{n\to\infty} \frac{h(g^n a, a)}{n}.$$

A priori the quasimorphism $T_{(X,\preceq,\{H_n\})}$ depends on the choice of halfspace filtration. We will see below that different choices result in proportional quasimorphisms.

Definition 3.3. The quasimorphism $T_{(X,\preceq,\{H_n\})}\colon G \to \mathbb{R}$ is called the *translation number* associated with the action of G on $(X, \preceq)$ and the filtration $\{H_n\}$.

3.2. Translation numbers as growth functions. Let $(X, \preceq)$, G, $\{H_n\}$, d be defined as in the last subsection and assume that the width of $\{H_n\}$ is bounded by w. Denote by $\le$ the bi-invariant order induced on G via the action on $(X, \preceq)$. Our goal is to establish the following result.

Theorem 3.4. *The growth functions of $\leq$ are multiples of $T_{(X,\preceq,\{H_n\})}$, in particular, they are mutually proportional nonzero homogeneous quasimorphisms.*

The theorem also shows that up to a positive multiple the translation number $T_{(X,\preceq,\{H_n\})}$ does not depend on the choice of half-space filtration. In view of Proposition 2.1 the proof of Theorem 3.4 is actually reduced to establishing the following proposition.

Proposition 3.5. *With notation as above, the quasimorphism $T_{(X,\preceq,\{H_n\})}$ sandwiches the partial order $\leq$.*

Proof. The quasimorphisms f_a are mutually at uniformly bounded distance d, hence at distance d from $T_{(X,\preceq,\{H_n\})}$. Now assume $T_{(X,\preceq,\{H_n\})}(g) > w + 2d$. Then for all $k \in G$, $x \in X$ we have

$$h(kg.x, k.x) \geq h(g.x, x) - d \geq f_x(g) - d \geq T_{(X,\preceq,\{H_n\})}(g) - 2d > w,$$

hence $kg.x \succeq k.x$ by definition of w. This in turn implies $g \geq e$. □

This finished the proof of Theorem 3.4 and thereby establishes Part (i) of Theorem 1.8.

3.3. Quasi-total triples vs. quasi-total orders. Throughout this subsection, let $(X, \preceq, T)$ be a complete quasi-total triple. We choose a basepoint $x_0 \in X$ and define halfspaces H_n for $n \in \mathbb{Z}$ by

$$H_n := [T^n.x_0, \infty).$$

Since T is dominating, these form indeed a halfspace filtration for X. Let us describe the height function of $(X, \{H_n\})$ in terms of T and x_0. Since the order is complete, there exists an absolute constant $C(X)$ such that for all $a, b \in X$ we have

$$(a \preceq T^{C(X)}b) \vee (b \preceq T^{C(X)}a).$$

Assume $h(a) = n$. Then $a \succeq T^n.x_0$ and $a \not\succeq T^{n+k}.x_0$ for $k > 0$, whence $a \preceq T^{n+(2C(X)+1)}.x_0$. Thus we have established the following result.

Lemma 3.6. *If $h(a) = n$, then*

$$T^n.x_0 \preceq a \preceq T^{n+(2C(X)+1)}.x_0.$$

This property determines h up to a bounded error. Moreover, we can deduce the following result.

Lemma 3.7. *The pair $(X, \preceq)$ is a halfspace order, and $\{H_n\}$ is a compatible halfspace filtration.*

Proof. Assume
$$h(a,b) > 2C(X) + 1,$$
and let $n := h(a)$, $m := h(b)$. By the lemma we then have
$$a \succeq T^n.x_0 \succeq T^{m+(2C(X)+1)}.x_0 \succeq b. \qquad \square$$

Now assume G acts by automorphisms on the triple $(X, \preceq, T)$. Then we have the following result.

Proposition 3.8. *For all $g \in G$, $|h(ga, gb) - h(a,b)| < 4C(X) + 2$.*

Proof. Assume $h(a) = n$ and $h(b) = m$. Then we have $h(a,b) = n - m$, and
$$T^n.x_0 \preceq a \preceq T^{n+(2C(X)+1)}.x_0,$$
$$T^m.x_0 \preceq b \preceq T^{m+(2C(X)+1)}.x_0,$$
and consequently
$$T^n.gx_0 \preceq ga \preceq T^{n+(2C(X)+1)}.gx_0,$$
$$T^m.gx_0 \preceq gb \preceq T^{m+(2C(X)+1)}.gx_0.$$
Now we find $k \in \mathbb{Z}$ such that
$$T^k x_0 \preceq g.x_0 \preceq T^{k+(2C(X)+1)}.x_0;$$
inserting this into the previous set of inequalities we obtain
$$T^{n+k}.x_0 \preceq ga \preceq T^{n+k+(4C(X)+2)}.x_0,$$
$$T^{m+k}.x_0 \preceq gb \preceq T^{m+(4C(X)+2)}.x_0.$$
We deduce that
$$n + k \leq h(ga) \leq n + k + (4C(X) + 2),$$
$$m + k \leq h(gb) \leq m + k + (4C(X) + 2),$$
and hence
$$(n-m) - (4C(X) + 2) \leq h(ga) - h(gb) \leq (n-m) + (4C(X) + 2),$$
which is to say
$$|h(ga, gb) - h(a,b)| \leq 4C(X) + 2. \qquad \square$$

Corollary 3.9. *If $G < \mathrm{Aut}_T(X, \preceq)$ contains a dominant, then it acts quasi-totally on $(X, \preceq)$.*

We will denote the translation number of the triple $(X, \preceq, \{H_n\})$ defined as above by

$$T_{(X,\preceq,T)} := T_{(X,\preceq,\{H_n\})}.$$

This is justified by the observation that while the sets $\{H_n\}$ depend on T and a basepoint x_0, the resulting translation number is independent of the choice of basepoint. Indeed, given $a, b \in X$ define their *relative T-height* by the formula

$$h_T : X^2 \longrightarrow \mathbb{Z}, \quad h_T(a,b) := \inf\{m \in \mathbb{Z} \mid T^m b \geq a\}.$$

Then unravelling definitions yields

$$|h_T(a,b) - h(a,b)| < 2C(X) + 2.$$

Thus we obtain the following result.

Proposition 3.10. *The translation number $T_{(X,\preceq,T)}$ satisfies*

$$T_{(X,\preceq,T)}(g) = \lim_{n\to\infty} \frac{h_T(g^n a, a)}{n};$$

for all $a \in X$.

This yield in particular the desired independence of the basepoint.

3.4. Tautological realization. We now turn to the proof of Part (ii) of Theorem 1.8. We will actually establish the following slightly stronger version.

Proposition 3.11. *Let G be a group. Every nonzero quasimorphisms on G arises as the multiple of a growth functions of a quasi-total action of G. In fact, it arises from a quasi-total triple of the form $(G, \preceq, T)$ where G acts by left-multiplication on itself.*

Proof. Let G be a group and f a nonzero (hence unbounded) homogeneous quasimorphism on G. Set

$$x \prec_f y \iff f(gx) < f(gy), \text{ for all } g \in G. \tag{8}$$

and let $h \in G$ be an element with

$$f(h) > 10D(f) + 5.$$

Denote by ρ_h the right-multiplication by h. Then $(G, \preceq_f, \rho_h)$ is a quasi-total triple. In view of Proposition 2.1 it remains to show that f sandwiches the partial order $\leq_f$ induced by $(G, \preceq_f, \rho_h)$ on G. For this let $g \in G$ with $f(g) > 10D(f) + 5$; then for all $x \in G$ we have

$$f(hgx) \geq f(g) + f(h) + f(x) - 2D(f) > f(hx), \qquad \text{for all } h \in G,$$

hence $gx \succeq_f x$ and thus $g \geq_f e$, finishing the proof □

This completes the proof of Theorem 1.8.

3.5. Global hyperbolicity. It is well-known that the classical translation number contains valuable information about orbits of subgroups of $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ on the real line and circle. For example, the question whether a subgroup $H <$ $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ has a bounded orbit in $\mathbb{R}$ can be decided from the translation number. Indeed, such a bounded orbit exists if and only if $T_{\mathbb{R}}|_H \equiv 0$; in this case, in fact all orbits are bounded. In order to obtain similar results for other types of quasimorphisms additional topological assumptions are necessary; concerning the existence of bounded orbits, global hyperbolicity is the key property. Indeed, we have the following result, which was stated as Theorem 1.10 in the introduction.

Theorem 3.12. *Let $(X, \leq)$ be a globally hyperbolic halfspace order. Let $\{H_n\}$ be a compatible halfspace filtration and let H be a group acting quasi-totally on $(X, \leq)$. Denote by $T_X : H \to \mathbb{R}$ the associated translation number. Then the following are equivalent for a subgroup $G < H$:*

(i) $(T_X)|_G \equiv 0$;

(ii) *every G-orbit in X is bounded*;

(iii) *there exists a bounded G-orbit in X.*

Before we turn to the proof we observe that global hyperbolicity can be characterized in terms of halfspaces.

Lemma 3.13. *In the situation of the theorem the order $(X, \preceq)$ is hyperbolic if and only if order intervals are closed and the sets $\overline{H_n \setminus H_{n+1}}$ are compact.*

Proof. Denote by w the width of $(X, \leq, \{H_n\})$. Given $n \in \mathbb{N}$ let $x_n^+ \in H_{n+w+1}$ and $x_n^- \in H_{n-w-1} \setminus H_{n-w}$. Then

$$H_n \setminus H_{n+1} \subset [x_n^-, x_n^+],$$

hence global hyperbolicity implies compactness of the sets $\overline{H_n \setminus H_{n+1}}$. For the converse observe that if $x \in H_n$, $y \in H_m$, then

$$[x, y] \subset \bigcup_{k=n-w-1}^{m+w+1} \overline{H_k \setminus H_{k+1}} \qquad \square$$

Proof of Theorem 3.12. (i) $\implies$ (ii). We claim that (i) implies that there are no $g \in G, n \in \mathbb{N}, x \in X$ satisfying

$$h(g^n x, x) \geq 2d.$$

Indeed, otherwise, we had for all $m \in \mathbb{N}$ the inequality

$$h(g^{nm}.x, g^{n(m-1)}.x) \geq h(g^n x, x) - d \geq d,$$

whence inductively

$$h(g^{nm}.x, x) \geq h(g^{nm}.x, g^{n(m-1)x}) + h(g^{n(m-1)}.x, x) \geq md,$$

which leads to

$$\frac{h(g^{nm}.x, x)}{nm} \geq \frac{d}{n},$$

and thus $T_X(g) \geq \frac{d}{n}$ by passing to the limit $m \to \infty$. This contradiction shows that

$$h(g^n x, x) \leq 2d \quad \text{for all } g \in G, n \in \mathbb{N}, and x \in X,$$

Applying the same argument to the reverse order, we can strengthen this to

$$|h(g^n x, x)| \leq 2d \quad \text{for all } g \in G, n \in \mathbb{N}, x \in X.$$

This implies that each orbit is contained in a finite number of strips of the form $H_n \setminus H_{n+1}$, hence bounded by the lemma.

(ii) $\implies$ (iii). Obvious.

(iii) $\implies$ (i). Assume that $\overline{G.x}$ is compact and let $g \in G$. Consider the sequence $x_n := g^n.x$. We claim that there exist n_-, n_+ (possibly depending on g and x) such that

$$\{x_n\} \subset H_{n_+} \setminus H_{n_-}.$$

Observe first that the claim implies that $h(g^n x, x)$ is bounded, whence $T_X(g) = 0$; it thus remains to establish the claim. Assume that the claim fails; replacing the order by its reverse if necessary we may assume that $h(g^n x, x)$ is not bounded above. We thus find a subsequence n_k such that for every $y \in X$ there exists $k(y)$ such that for all $k > k(y)$ we have $g^{n_k}x \succeq y$. By compactness of $\overline{G.x}$ there exists an accumulation point x_∞ of x_n and since order intervals are closed we have $x_\infty \succeq y$ for all $y \in X$. However, a halfspace order does not admit a maximum. □

4. Elementary examples

4.1. Basic constructions. The notion of a quasi-total triple is closed under various elementary constructions. The following three persistence properties are immediate from the definition.

Proposition 4.1. (i) Lexicographic products. *Let $(X_0, \preceq_0, T_0)$ be a complete quasi-total triples and $(X_i, \preceq_i)_{i \in \mathbb{N}}$ be a family of arbitrary posets. On*

$$X := \prod_{i=0}^{\infty} X_i$$

define the lexicographic ordering by

$$(x_i) \prec (y_i) \iff \text{there exists } j \in \mathbb{N}_0 \text{ such that } x_i = y_i \text{ for all } i < j \text{ and } x_j \prec_j y_j$$

and define

$$T\colon X \longrightarrow X$$

by

$$T(x_i) := (T_0 x_0, x_1, x_2, \ldots).$$

Then $(X, \preceq, T)$ is a quasi-total triple.

(ii) Subtriples. *Let $(X, \preceq, T)$ be a complete quasi-total triple. Let $Y \subset X$ be a subset and suppose there exists $S \in$ Aut$(Y, \preceq)$ such that for all $y \in Y$ we have $Sy \succeq Ty$. Then $(Y, \preceq, S)$ is quasi-total.*

(iii) Refinement. *Let $(X, \preceq, T)$ be a quasi-total triple and $\preceq'$ be a refinement of $\preceq$. Then $(X, \preceq', T)$ is a quasi-total triple.*

Example 4.2. Let $(X, \preceq, T) = (\mathbb{R}, \leq, x \mapsto x + 1)$ be the standard quasi-total triple, which realizes the classical Poincaré translation number $T_{\mathbb{R}}$. Then for any set X (which we consider as a trivial poset $(X, =)$) we obtain a new quasi-total triple $(C_X := \mathbb{R} \times X, \preceq, T)$ as the lexicographic product. Explicitly, we have

$$(\lambda, x) \prec (\mu, y) \iff \lambda < \mu$$

and

$$T(\lambda, x) := (\lambda + 1, x).$$

This quasi-total triple induces a quasimorphism

$$T_{C_X} = T_{(C_X, \preceq, T)}$$

on

$$G := \mathrm{Aut}(C_X, \preceq, T),$$

hence on any subgroup of G.

The reader might have the impression that these quasimorphisms are a trivial variation of $T_{\mathbb{R}}$, but this is not the case. In fact, we will provide non-trivial examples of the above type of construction in Section 5.4. At this point, let us just point out that T_{C_X} cannot be the pullback of $T_{\mathbb{R}}$ via any embedding $\mathrm{Aut}(C_X, \preceq, T) \to \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$, even for $X := S^1$. Indeed, such an embedding does not exist, since $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ is torsion-free.

The notion of a halfspace order is even more flexible. The following trivial observation is important.

Lemma 4.3. *Let $(X, \preceq, \{H_n\})$ be a halfspace order and $Y \subset X$ a subset such that $Y \cap (H_n \setminus H_{n+1}) \neq \emptyset$ for all n. Then $(Y, \preceq|_{Y \times Y}, \{H_n \cap Y\})$ is a halfspace order.*

In the next section we will apply the following special case.

Corollary 4.4. *Let Y be a set. Then an embedding $\iota\colon Y \hookrightarrow \mathbb{R}^n$ induces a halfspace order on Y by setting*

$$y_1 \prec y_2 \iff \iota(y_1) = (x_1, \ldots, x_n), \iota(y_2) = (z_1, \ldots, z_n), x_1 < z_1$$

and

$$H_n := \{y \in Y \mid \iota(y) = (x_1, \ldots, x_n), x_1 \geq n\},$$

provided $H_n \setminus H_{n+1} \neq \emptyset$ for all $n \in \mathbb{Z}$.

4.2. Cayley graph embeddings. In view of the last corollary, a good strategy to construct quasimorphisms on groups is making the group act on a subset of $\mathbb{R}^n$. One way to achieve this is to embed the group itself, or its Cayley graph (or presentation 2-complex) into $\mathbb{R}^n$. If the embedding is chosen in such a way that the action of G induces an unbounded action on the image by quasi-automorphisms, then we obtain a quasimorphism. We provide two examples of such Cayley graph embedding into $\mathbb{R}^3$. For visualization we will depict the projections of these embeddings into the (x_1, x_2) plane along the x_3-coordinate. The precise choice of x_3 coordinates will not matter, as long as the x_3-coordinate is chosen in such a way as to obtain an embedding. In both examples we will choose the edges to be of uniformly bounded x_1-length. Under this assumption, quasi-totality of the action will follow from quasi-totality of the action on the vertex set on the graph.

Example 4.5 (Rademacher quasimorphism). Let

$$G = \mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}.$$

Denote by S a generator of $\mathbb{Z}/2\mathbb{Z}$ and by R a generator of $\mathbb{Z}/3\mathbb{Z}$, so that

$$G = \langle S, R \mid S^2, R^3 \rangle.$$

We observe that the translations

$$T_1 := SR \quad \text{and} \quad T_2 := SR^2$$

are of infinite order in G and generate a free semigroup G_0 in G. Every element of G can be written uniquely as either w, Sw, wS or SwS, where $w \in G_0$. Now the Rademacher quasimorphism f on G can be described as follows (see [1] and also [29, Corollary 4.3]). Given $g \in G$, let w be the element in G_0 such that $g \in \{w, Sw, wS, SwS\}$. Then $f(g)$ is the number of T_1s in w minus the number of T_2s in w. A planar embedding of a piece of the Cayley graph of G with generating set $\{S, R, R^2\}$ is depicted in Figure 1, and continuing the picture as indicated provides a map from the full Cayley graph into $\mathbb{R}^2$, which can be deformed into an embedding into $\mathbb{R}^3$ without changing the first two coordinates. We claim that the action of G on the vertex set is quasi-total. Indeed, one immediately reduces to showing that G_0 acts quasi-totally. However, in the above embedding of the Cayley graph, T_1 acts by increasing the x_1-coordinate by 1, while T_2 acts by decreasing the x_1-coordinate by 1, whence G_0 even preserves the relative height function. To see that the resulting quasimorphism is indeed the Rademacher quasimorphism, just observe that every $g \in G$ with $f(g) > 5$ maps every point in the Cayley graph to the right and consequently the induced order on G is sandwiched by the Rademacher quasimorphism.

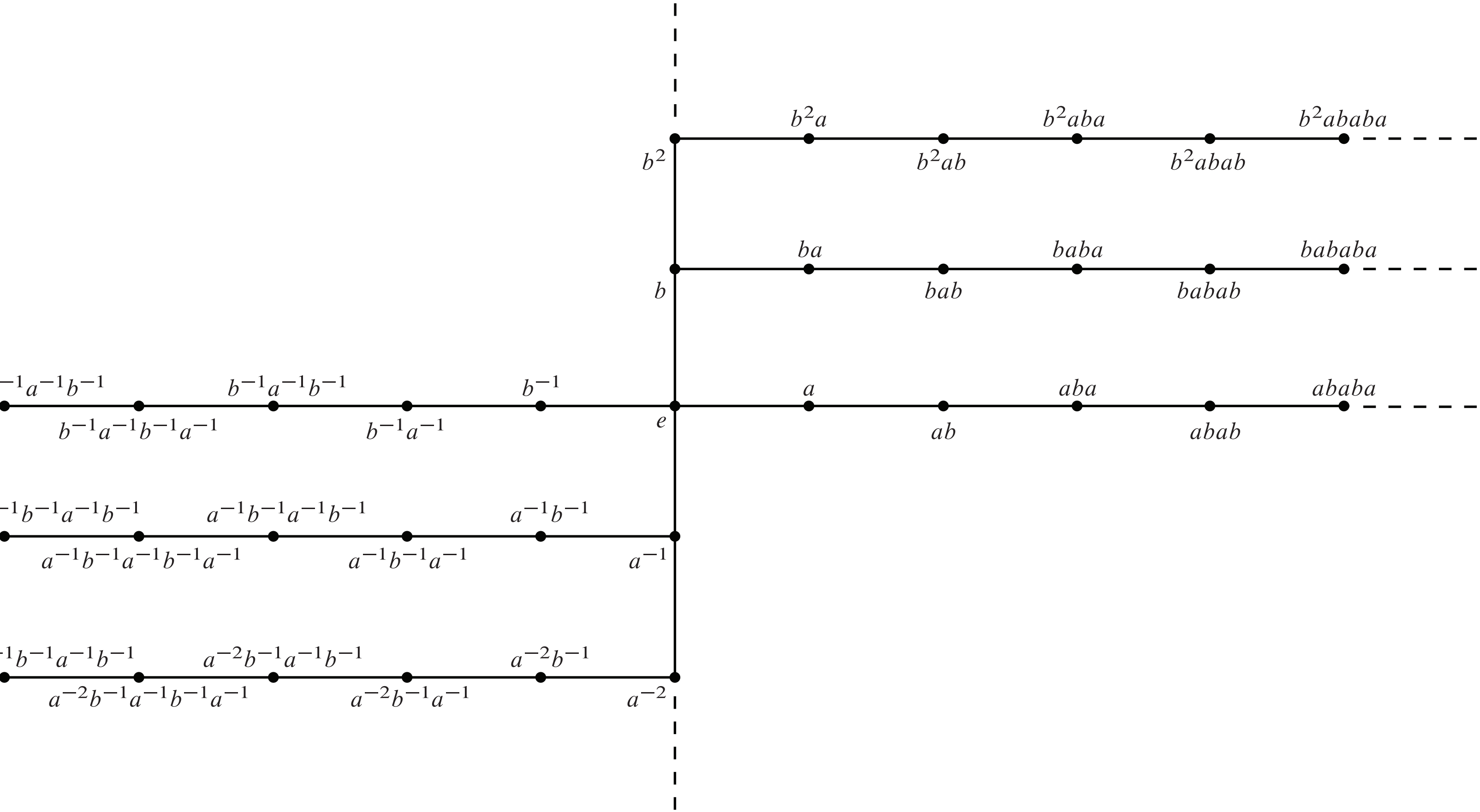

Figure 2. A planar subgraph of the Cayley graph of the free group.

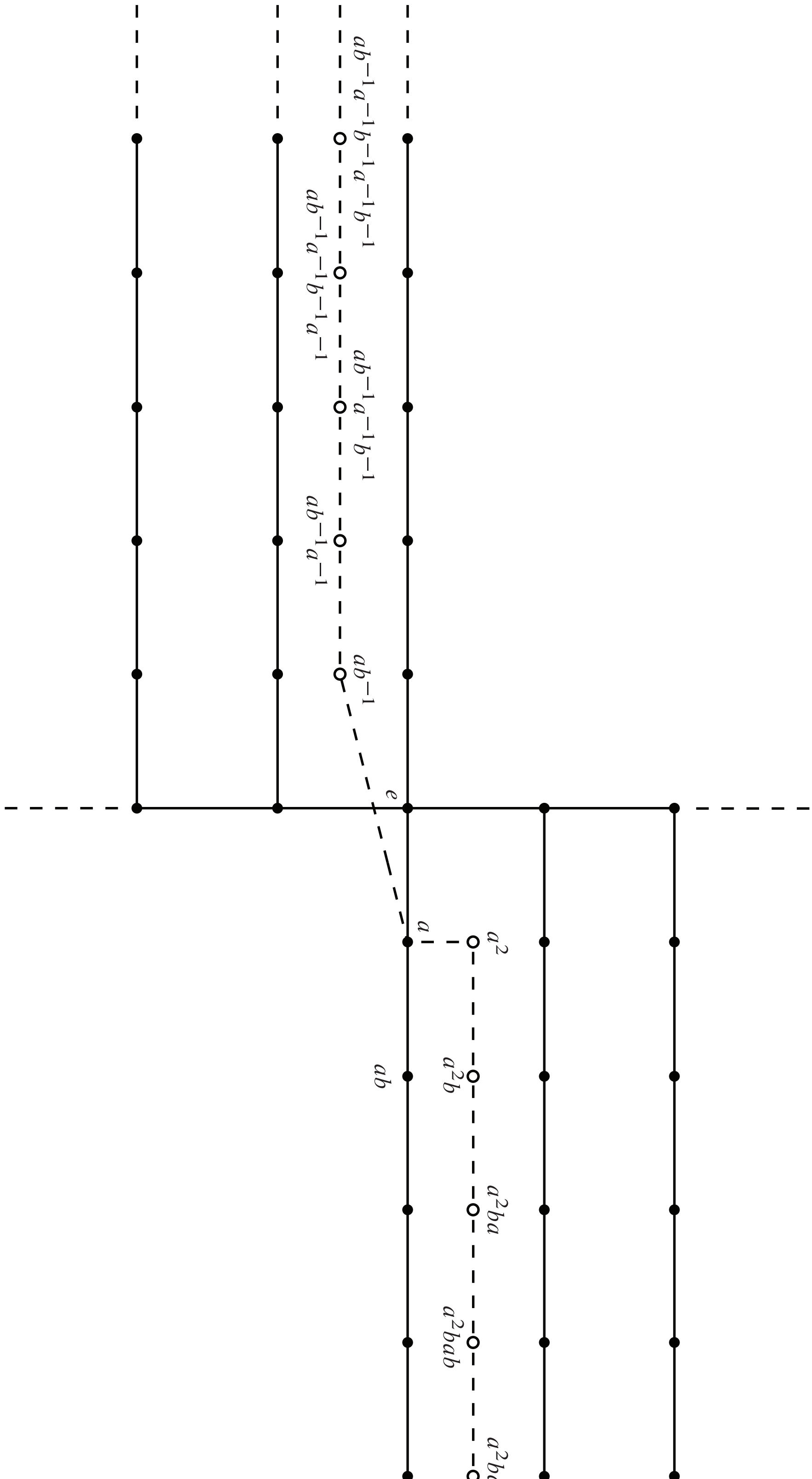

Figure 3. Growing two hairs at a.

Example 4.6 (Brooks quasimorphism). We now give a similar construction for the Cayley graph of the free group G on two generators a and b with respect to the standard generating set $\{a^{\pm 1}, b^{\pm 1}\}$. We first label the intersection of the lattice $\mathbb{Z}^2$ with the first and third quadrant as in Figure 2, thereby embedding a subset of the Cayley graph into the plane; in a second step we will extend this embedding to an embedding of the whole Cayley graph into $\mathbb{R}^3$ *growing hair*. (As before, the x_3 coordinate is only used to avoid self-intersections). To explain this procedure, let w be a vertex in the graph in Figure 2 with the property that at least one of its four neighbours in the Cayley graph of G with respect to $\{a^{\pm 1}, b^{\pm 1}\}$ does not yet appear. We will then add the missing neighbour(s) and some further vertices by the following rules.

- Assume that the a-neighbour wa is missing; this can only happen if the last letter of w is either a or b^{-1}. We then add the vertices wa, wab, $waba$, $wabab$ etc. to the graph. Where precisely we place the first new vertex wa depends on the last letter of w. If it is an a, then we place wa above w (at a height that has not yet been taken); if it is a b^{-1} we place it two to the right of w at a yet available height. Once wa has been placed, we add wab one to the right of wa at the same height, $waba$ one to the right of wab at the same height etc. For $w = a$ this is depicted in Figure 3.
- Similar rules apply to the other types of missing neighbours. If the b-neighbour of w is missing, then the last letter of w is either b or a^{-1}. In both cases we place wb above w and add wba, $wbab$, $wbaba$, etc. to the right. If the a^{-1}-neighbour of w is missing, then the last letter of w is b or a^{-1}. In both cases we place wb above w and add wa^{-1}, $wa^{-1}b^{-1}$, $wa^{-1}b^{-1}a^{-1}$, etc. to the left. Finally, if the b^{-1}-neighbour of w is missing, then the last letter of w is either a or b^{-1} and we place wb^{-1} two steps left below, respectively straight below w accordingly. We then add $wb^{-1}a^{-1}$, $wb^{-1}a^{-1}b^{-1}$ etc. to the left. (See again Figure 3.)

After applying this procedure once, every vertex in the original embedding has four neighbours, but the newly added vertices have only two neighbours each; we thus continue by growing hair to them according to the same rules. Repeating this procedure ad infinitum we finally obtain an embedding of the full Cayley graph into $\mathbb{R}^3$. Similarly as in the last example it can be checked that the action of G is quasi-total using the following key observation. If a word $w \in G$ contains ab, respectively $b^{-1}a^{-1}$ as a subword n_+, respectively n_--times, then the action of w is at uniformly bounded height-distance from a translation by $2(n_+ - n_-)$. This fact can be used to show not only that the action is quasi-total, but also that the quasimorphism corresponding to the embedding is given (up to a multiple) by the Brooks quasimorphism associated with the word ab, which assigns to w as above the difference $n_+ - n_-$. This construction can easily be modified to apply to Brooks quasimorphisms for other words than ab.

More generally, one may wonder whether general quasimorphisms on countable groups can be realised by special classes of quasi-total actions on countable subsets of $\mathbb{R}^3$. We refer the reader to [4] for a universal construction in this direction.

4.3. A characterization result. As a consequence of Theorem 1.8, a group admits a nonzero homogeneous quasimorphism if and only if admits a quasi-total action. A more refined version of this characterization was stated in Corollary 1.6. We will now derive this characterization from Theorem 1.4 and Proposition 1.5. These imply that a group G admits a non-zero homogeneous quasimorphism if and only if it admits an *effective* action containing a dominant on a *complete* quasi-total triple $(X, \preceq, T)$. In fact, the completeness assumption can be dropped. For this we observe that T is automatically fixed point-free and non-decreasing, i.e.

$$T^m a \not\preceq a \quad \text{for all } a \in X, m \in \mathbb{N}.$$

In general, however, T need not be strictly increasing. This defect can be repaired as follows. Define a new partial order $\preceq_T$ by setting

$$a \preceq_T b \iff \text{there exists } k \geq 0 \text{ such that } T^k a \preceq b.$$

Then T is strictly increasing with respect to $\preceq_T$, hence $(X, \preceq_T, T)$ is a complete quasi-total triple. We refer to $(X, \preceq_T, T)$ as the *completion* of $(X, \preceq, T)$. The following simple observation explains why the passage from an incomplete to a complete quasi-total triple does not affect the corresponding quasimorphisms.

Proposition 4.7. *A quasi-total triple $(X, \leq, T)$ and its completion $(X, \leq_T, T)$ define the same T-height on X, hence give rise to the same translation number.*

Proof. This follows from

$$\begin{aligned} h_{\leq_T, T}(a, b) &= \inf\{m \in \mathbb{Z} \mid \text{there exists } k \in \mathbb{N}_0 \text{ such that } T^{m-k} b \geq a\} \\ &= \inf\{m \in \mathbb{Z} \mid T^m b \geq a\} \\ &= h_{\leq, T}(a, b). \end{aligned}$$

□

Note that if the G-action on a quasi-total triple admits a dominant, then so does its completion.

We now show that the effectiveness assumption on the action can also be dropped. Indeed, if G acts dominatingly, but not necessarily effectively on some quasi-total triple $(X, \preceq, T)$, then we obtain an effective dominating diagonal action of G on the quasi-total triple $(G \times X, \leq, T')$, where $(g, x) < (g', x')$ if and only if $x < x'$ and $T'(g, x) := (g, Tx)$. Altogether we have established

Lemma 4.8. *If a group acts dominatingly on some quasi-total triple $(X, \preceq, T)$, then it acts dominatingly and effectively on some complete quasi-total triple $(X', \preceq', T')$.*

Now Corollary 1.6 follows by combining the lemma with Theorem 1.4 and Proposition 1.5.

5. Lie groups and smooth quasi-total triples

5.1. Causal coverings. In this section we study quasi-total triples induced by smooth partial orders on manifolds. Since the notion of a smooth partial order is not used consistently in the literature we have collected the definitions that we are going to use in the appendix of this paper together with some basic facts relating them. From now on we will assume the terminology and results of the appendix.

Throughout we will denote by $(\widetilde{M}, \widetilde{\mathcal{C}})$ a causal manifold in the sense of Definition A.6 and by $\preceq$ the associated partial order. We also denote by $G(\widetilde{M}, \widetilde{\mathcal{C}})$ the associated automorphism group (see Definition A.4). The main problem of this section can then be formulated as follows.

Problem 1. Given a causal manifold $(\widetilde{M}, \widetilde{\mathcal{C}})$, is there an automorphism $T \in G(\widetilde{M}, \widetilde{\mathcal{C}})$, which turns $(\widetilde{M}, \preceq, T)$ into a quasi-total triple?

Remark 5.1. For the purpose of this subsection we could as well consider a weakly causal manifold in the sense of Definition A.6 and study the associated strict causality $\preceq_s$ instead of $\preceq$. We would then ask for an automorphism T of $(\widetilde{M}, \widetilde{\mathcal{C}})$ turning $(\widetilde{M}, \preceq_s, T)$ into a quasi-total triple. All results of this subsection remain valid in this setting; the difference between $\preceq_s$ and $\preceq$ will only become important when we discuss global hyperbolicity in Subsection 5.3 below.

We now fix a causal manifold $(\tilde{M}, \tilde{\mathcal{C}})$. We observe that if T as in Problem 1 exists, then it has to be of infinite order. Hence, fix $T \in \mathrm{Aut}(\tilde{M}, \tilde{\mathcal{C}})$ of infinite order and assume moreover that the group $\Gamma \cong \mathbb{Z}$ generated by T in $G(\tilde{M}, \tilde{\mathcal{C}})$ acts properly discontinuously on $\tilde{M}$; then $M := \Gamma\backslash\tilde{M}$ is a manifold and

$$p\colon \tilde{M} \longrightarrow M$$

is a covering projection. We will denote by $\check{G}$ the centralizer of T (hence Γ) in $G(\tilde{M}, \tilde{\mathcal{C}})$. Then $G := \check{G}/\Gamma$ acts on M and $\check{G}$ is a central extension of G by Γ. We refer to the central extension

$$0 \longrightarrow \Gamma \longrightarrow \check{G} \longrightarrow G \longrightarrow 1$$

as the central extension *associated* with the covering $p\colon \tilde{M} \to M$.

Definition 5.2. The covering $p\colon \tilde{M} \to M$ is called a *causal covering* if M is totally acausal (in the sense of Definition A.6).

Lemma 5.3. *Assume that $p\colon \tilde{M} \to M$ is a causal covering, Then either T or T^{-1} is dominant.*

Proof. Denote by $p\colon \tilde{M} \to M$ the covering projection and let $a, b \in \tilde{M}$. Since M is totally acausal there exists a closed causal loop $\gamma_{p(a),p(b)}$ at $p(a)$ through $p(b)$. We can lift this loop to a curve $\gamma_{a,b}$ with initial point a; the result is a causal curve through a and some T-translate of b. Thereby we find integers $l(a,b) \in \mathbb{Z}$ with $a \preceq T^{l(a,b)}b$. Since $\tilde{M}$ does not contain causal loops, we may assume $l(a,a) > 0$ for some given basepoint a upon possibly replacing T by its inverse. We claim that this implies $l(b,b) > 0$ for all b. Indeed, suppose otherwise, say $b \succeq T^k b$ with $k > 0$. We then find $m > 0$ with

$$T^{-mk}a \preceq b \preceq T^{mk}a,$$

$$T^{-mk}b \succeq b \succeq T^{mk}b,$$

hence $b \succeq T^{mk}(T^{-mk}a) = a$ and $b \preceq T^{-mk}(T^{mk}a) = a$. This yields $a = b$ and $l(a,a) < 0$, which is a contradiction. We see in particular that we can choose $l(a,b)$ positive by adding a suitable multiple of $l(b,b)$. This implies that T is dominant. □

Thus replacing T by T^{-1} if necessary we will assume from now on that T is the unique dominant generator of Γ. We see from the proof of the last proposition that for any pair $a, b \in \tilde{M}$ there exists $n(a,b) := \min\{l(a,b), l(b,a)\} \in \mathbb{N}$ such that $a \preceq T^{n(a,b)}b$ or $b \preceq T^{n(a,b)}a$. However, the number n is in general not uniformly bounded.

Definition 5.4. A causal covering $\widetilde{M} \to M$ is called *quasi-total* if the number $n(a,b)$ is uniformly bounded.

Equivalently, $(\widetilde{M}, \preceq, T)$ is a quasi-total triple. In the next section we will provide two different criteria which guarantee this property. Before, let us give some elementary examples of quasi-total causal coverings. Firstly, the classical translation number $T_{\mathbb{R}}$ is associated with the causal covering $\mathbb{R} \to S^1$. The following example can be considered as a smooth twisting of this trivial example; as in Example 4.2 it is easy to argue that this sort of twisting produces fundamentally different quasimorphisms.

Example 5.5. Let $\widetilde{M} := \mathbb{R}\times]-1,1[$ be a strip of bounded diameter with basepoint $x_0 := (0,0)$ and let $C \subset \mathbb{R}^2$ be a closed regular cone which contains the positive x-axis in its interior. Then the translation invariant cone field on $\mathbb{R}^2$ modelled on C restricts to a conal structure $\widetilde{\mathcal{C}}$ on $\widetilde{M}$, and the conal manifold $(\widetilde{M}, \widetilde{\mathcal{C}})$ is in fact causal, since every non-constant causal curve is strictly monotone in the x-coordinate. Since the cone C contains the positive x-axis in its interior we find $x_\pm \in \mathbb{R}$ such that

$$x_\pm \in \mathbb{R} \times \{\pm 1\} \cap C.$$

Choose $x_\pm$ minimal with this property and set $x_0 := 2\max\{x_+, x_-\}$. Let T be the translation along the x-axis by x_0, i.e. $T(x,y) := (x + x_0, y)$ and let $M := \widetilde{M}/\langle T\rangle$. Then $M \cong S^1\times]-1,1[$ and $\widetilde{M} \to M$ is a quasi-total causal covering.

Similar twists can also be defined for the examples from Lie groups as discussed below.

5.2. Criteria for quasi-totality. Before we can discuss further examples, we need to develop criteria which guarantee quasi-totality. Throughout this section we fix a causal covering $\widetilde{M} \to M$ and denote by

$$0 \to \Gamma \to \check{G} \to G \to 1$$

the associated central extension of automorphism groups. The easiest way to guarantee quasi-totality is to demand enough transitivity of G on M.

Definition 5.6. An action of a group G on a space X is *almost 2-transitive* it there exists a G-orbit $X^{(2)} \subset X^2$ with the property that for all $x, y \in X$ there exists $z \in X$ such that

$$\{(x,z),(z,y)\} \subset X^{(2)}.$$

In this case we call X *almost* 2*-homogeneous* and write $x \pitchfork y$ to indicate that $(x, y) \in X^{(2)}$.

Then we obtain the following result.

Theorem 5.7. *Let $p\colon \widetilde{M} \to M$ be a causal covering. If $G := \check{G}/\Gamma$ acts almost 2-transitively on M, then p is quasi-total.*

Note that the almost 2-transitivity of G on M implies automatically that M is totally acausal. We prepare the proof of Theorem 5.7 by the following lemma.

Lemma 5.8. *In the situation of Theorem 5.7 there exists a constant $N \in \mathbb{N}$, depending only on M, and a point $c \in \widetilde{M}$ such that for all $b \in \widetilde{M}$ there exists $l \in \mathbb{Z}$ such that $b \preceq T^l c \preceq T^N b$.*

Proof. Let $a \in \widetilde{M}$ be some basepoint and $x := p(a)$. Then we find $z \in M$ with $z \pitchfork x$ and a closed causal loop $\gamma_x\colon [0, 1] \to M$ at x with $z = \gamma_x(1/2)$. Let $\hat{\gamma}_a$ be the lift of γ_x with initial point a. Then $\hat{\gamma}_a(0) \preceq \hat{\gamma}_a(1)$; we thus find $N_0 > 0$ such that $\hat{\gamma}_a(1) = T^{N_0}\hat{\gamma}_a(0)$. If we define $c := \hat{\gamma}_a(1/2)$, then

$$a \preceq c \preceq T^{N_0} a.$$

Now let $d \in \widetilde{M}$, $w := p(d)$ and assume $w \pitchfork z$. Then we find $g_0 \in G_0$ with $g_0.x = w$ and $g_0.z = z$. Thus $\gamma_w = g_0.\gamma_x$ is a closed causal loop at w through z. Now let $\hat{\gamma}_d$ be a lift of γ_w with initial point d. Let $g \in G$ be a lift of g_0; then g maps a to a point in the fiber of d, and modifying g by a deck transformation if necessary we can assume $g.a = d$. Then $g.\hat{\gamma}_a$ is a lift of γ_w with initial point d, hence $\hat{\gamma}_d = g.\hat{\gamma}_a$ by uniqueness. Now we have

$$\hat{\gamma}_d(1) = g\hat{\gamma}_a(1) = gT^{N_0}\hat{\gamma}_a(0) = T^{N_0}g\hat{\gamma}_a(0) = T^{N_0}\hat{\gamma}_d(0).$$

Now since $\gamma_w(1/2) = z$ we find $l \in \mathbb{Z}$ such that $\hat{\gamma}_d(1/2) = T^l c$. We thus have established

$$d \preceq T^l c \preceq T^{N_0} d \tag{9}$$

under the assumption $p(d) \pitchfork z$. Now consider the case of an arbitrary $b \in \widetilde{M}$ and let $y := p(b)$. We then find $d \in \widetilde{M}$ such that $w := p(d)$ satisfies

$$y \pitchfork w \pitchfork z.$$

Then (9) holds for some $l \in \mathbb{Z}$. Moreover, we find $h_0 \in G_0$ with $h_0.(x,z) = (y,w)$. Define $\gamma_y := h_0.\gamma_x$ and denote by $\hat{\gamma}_b$ the lift of γ_y with initial point b. By the same argument as before we then show $\hat{\gamma}_b(1) = T^{N_0}\hat{\gamma}_b(0)$. Now $\gamma_y(1/2) = w$, so we find $l' \in \mathbb{Z}$ with $\hat{\gamma}_b(1/2) = T^{l'}d$. Thus

$$b \preceq T^{l'}d \preceq T^{N_0}b \tag{10}$$

Combining (9) and (10) we obtain

$$b \preceq T^{l'+l}c \preceq T^{2N_0}b$$

We may thus choose $N := 2N_0$. ☐

Now we deduce the following result.

Proof of Theorem 5.7. Let c and N be as in the lemma. Given $a, b \in \widetilde{M}$ we find $l, l' \in \mathbb{Z}$ such that $b \preceq T^l c \preceq T^N b$ and $a \preceq T^{l'}c \preceq T^N a$. We may assume w.l.o.g. that $l' \geq l$. Now for all $k \geq 1$ we have

$$T^{l'-l}b \preceq T^{l'}c \preceq T^N a \preceq T^{kN}a,$$

hence $T^{l'-l-kN}b \preceq a$. Now choosing k appropriately we can ensure that $-N \preceq l'-l-kN \preceq N$. Thus for all $a, b \in \widetilde{M}$ there exists $k \in \{-N, \dots, N\}$ such that

$$a \preceq T^k b \quad \text{or} \quad b \preceq T^k a.$$

Now (2) follows for $N(\widetilde{M}) := 2N$. ☐

The almost 2-transitivity condition of Theorem 5.7 is rather strong and not always easy to check in practice. We are thus looking for an alternative condition that ensures quasi-totality. One consequence of quasi-totality is that $T^k.x \succeq x$ for all x and a uniformly bounded k. Here we shall assume the slightly stronger condition

$$T.x \succeq x, \tag{11}$$

i.e. completeness of the triple $(\widetilde{M}, \preceq, T)$. We then call the covering $p\colon \widetilde{M} \to M$ a *complete* causal covering. This terminology understood we have the following useful criterion.

Theorem 5.9. *Assume that M is compact. Then any complete causal covering $p\colon \widetilde{M} \to M$ is quasi-total.*

Proof. We first observe that there exist points $a, b \in \widetilde{M}$ and an open subset $U \subset \widetilde{M}$ such that $a \preceq x \preceq b$ for all $x \in U$. Indeed, choosing a and b close enough we can ensure that $\exp(\mathrm{Int}(\widetilde{\mathcal{C}}_a))$ and $\exp(\mathrm{Int}(-\widetilde{\mathcal{C}}_b))$ have open intersection. By compactness of M there exists finally many elements $g_1, \dots, g_l$ such that

$$M = p\Big(\bigcup_{j=1}^{l} g_j U\Big). \tag{12}$$

Thus if we set

$$H^{\pm} := \bigcup_{k \geq 0} T^{\pm k}\Big(\bigcup_{j=1}^{l} g_j U\Big),$$

then $\widetilde{M} = H^- \cup H^+$. For $j = 1, \dots, l$ we set

$$a_j := g_j a \quad \text{and} \quad b_j := g_j b.$$

Let m_{ij} be integers such that

$$b_i \leq T^{m_{ij}} a_j$$

and set

$$N := \max m_{ij}.$$

Then $x \leq T^N y$ for all $x, y \in \bigcup g_j U$, hence $x \leq T^N y$ for all $x \in H^-$, $y \in H^+$. Now let $x, y \in \widetilde{M}$ be arbitrary. We distinguish three cases.

- If one of them is in H^- and the other is contained in H^+, then $y \preceq T^N y$ or $y \preceq T^N x$.
- If none of them is in H^+, apply T until the first of them is. We may assume $T^k x \in H^+$ and $T^{k-1} y \notin H^+$, hence $T^{k-1} y \in H^-$. Then $T^{k-1} y \leq T^{k+N} x$, hence $y \leq T^{N+1} x$.
- If none of them is in H^- we argue dually.

We thus obtain $x \preceq T^{N+1} y$ or $y \preceq T^{N+1} x$ in all possible cases. □

5.3. A criterion for global hyperbolicity. We have seen in the last section how compactness of the base manifold can be used to obtain quasi-totality of a given causal covering. This sort of compactness assumption also has implications to global hyperbolicity, which we briefly want to outline here. More precisely, we will establish the following result.

Theorem 5.10. *Let $p\colon \widetilde{M} \to M$ be a causal covering and assume that M is compact. Then the partial order $\preceq$ on $\widetilde{M}$ and its completion $\preceq_T$ are globally hyperbolic.*

While up to this point we could have worked with the strict causality $\preceq_s$ instead of the closed causality $\preceq$, closedness of $\preceq$ is clearly necessary for Theorem 5.10 to hold.

Concerning the proof of Theorem 5.10 we first observe that the order intervals of $\preceq$ are closed by construction; since

$$[a,b]_{\preceq_T} = \bigcup_{k=0}^{N(\widetilde{M})-1} \bigcup_{l=0}^{N(\widetilde{M})-1} [T^k a, T^{-l} b]_{\preceq},$$

we see that also $\preceq_T$ is closed. It thus remains only to show that finite order intervals of $\preceq_T$ are bounded. From now on, all order intervals will be with respect to $\preceq_T$. Our starting point is the following trivial observation.

Lemma 5.11. *Let $a \in \widetilde{M}$ and $N \in \mathbb{N}$. Then for all $x \in M$ there exists $b_0 \in p^{-1}(x)$ such that*

$$p^{-1}(x) \cap [a, T^N a] \subset \{b_0, Tb_0, \dots, T^N b_0\}.$$

Proof. Let $x \in M$ and $b \in \mathcal{F}_x := p^{-1}(x)$. Consider

$$E_b := \{k \in \mathbb{Z} \mid T^k b \succeq_T a\}.$$

We claim that E_b has a minimal element. Indeed, since T is dominating we have $T^l a \succeq_T b$ for some $l \in \mathbb{Z}$ and hence $a \succeq_T T^{-l} b$. Now if $T^k b \succeq_T a$, then $k \geq -l$, for otherwise $T^k b \succeq_T a \succeq_T T^{-l} b$, hence $T^{k+l} b \succeq_T b$ and thus $0 > k + l \geq 0$. Now let $k \in E_b$ be minimal and $b_0 := T^k b$. Then $T^k b_0 \geq a$ implies $k \geq 0$, while $T^k b_0 \leq T^N a$ implies $k \leq N$. Thus

$$p^{-1}(x) \cap [a, T^N a] \subset \{b_0, \dots, T^N b_0\}. \qquad \square$$

Now the key to the proof of Theorem 5.10 is the following lemma.

Lemma 5.12. *For every $x_0 \in \widetilde{M}$ there exists a bounded subset $M_{x_0} \subset \widetilde{M}$ such that $p(M_{x_0}) = M$ and $x \succeq x_0$ for all $x \in M_{x_0}$.*

Proof. Let $a \in M$ be some basepoint. Then there exists an open subset $U \subset M$ with the following properties.

- there exists a compact neighbourhood V of $\overline{U}$, which is evenly covered under p;
- U is contractible;

- $a \in \partial U$;
- for every $b \in U$ there exists a causal curve $\gamma_b \colon [0,1] \to M$ with $\gamma_b(0) = a$, $\gamma_b(1) = b$ and $\gamma_b(t) \in U$ for all $t \in (0,1]$.

Indeed, we can take a sufficiently small open subset of $\exp(\mathrm{Int}(C_a))$. Since M is compact and weakly homogeneous there exist $g_1, \dots, g_r \in G(M, \mathcal{C})$ such that

$$M = \bigcup_{j=1}^{r} g_j U.$$

Now choose $x_j \in p^{-1}(g_j a)$ with $x_j \succeq x_0$ and let V_j be the connected component of $p^{-1}(U_j)$ containing x_j in its boundary. Then we define

$$M_{x_0} := \bigcup_{j=1}^{r} V_j.$$

This is clearly bounded (since the closure of each V_j is compact) and covers M. It remains to show that $x \succeq x_j$ for every $x \in V_j$; this however follows easily by lifting the curve $\gamma_{p(x)}$ to V_j with basepoint x_j; the resulting lift is then a causal curve joining x_j with x. □

Now we can easily deduce the theorem.

Proof of Theorem 5.10. Let $a \in \widetilde{M}$. We use the lemma to obtain a bounded subset M_a of $\widetilde{M}$ such that $p(M_a) = M$ and $x \succeq a$ for all $x \in M_a$. In particular we have $x \succeq_T a$ for all $x \in M_a$. We claim that there exists some $N_0 \in \mathbb{N}$ such that

$$M_a \subset [a, T^{N_0} a]. \tag{13}$$

Indeed, suppose otherwise. Then there exists a sequence $x_n \in M_a$ with $x_n \succeq T^n a$. Since M_a is bounded there exists a subsequence n_k such that $x_{n_k} \to x \in \widetilde{M}$. Now for every l there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ we have $n_k \geq l$ and thus $x_{n_k} \succeq_T T^l a$. This implies that $x_{n_k} \succeq T^{l'} a$ for some $l' \in \{l - N, \cdots, l\}$ for some fixed constant N. By passing to another subsequence we can thus ensure that $x_{n_{k_m}} \succeq T^{l'} a$ for $k_m \geq k_0$. Since $\succeq$ is closed this yields $x \succeq T^{l'} a$ and thus $x \succeq_T T^{l-N} a$. Since l was arbitrary, this contradicts the fact that T is dominating. This contradiction establishes (13).

Now let $b \in \widetilde{M}$. We then find $N \in \mathbb{N}$ such that $N \geq N_0$ and $[a, b] \subset [a, T^N a]$. It remains to show that $[a, T^N a]$ is bounded. We claim that

$$[a, T^N a] \subset \bigcup_{n=-N}^{N} T^n M_a,$$

which implies the desired boundedness. Indeed, let $c \in [a, T^N a]$ and let

$$x := p(c).$$

We consider the fiber

$$\mathcal{F}_x := p^{-1}x.$$

By Lemma 5.11 we have

$$\mathcal{F}_x \cap [a, T^N a] \subset \{b_0, Tb_0, \dots, T^N b_0\}$$

for some $b_0 \in \mathcal{F}_x$. In particular, we find $k_1 \in \mathbb{N}$ with $0 \leq k \leq N$ and $c = T^{k_1} b_0$. On the other hand we have we have

$$M_a \subset [a, T^N a] \cap \mathcal{F}_x \cap [a, T^N a] \neq \emptyset$$

in view of (13). We thus find $k_2 \in \mathbb{N}$ with $0 \leq k \leq N$ and $T^{k_2} b_0 \in M_a$. Then

$$c = T^{k_1} b_0 = T^{k_1 - k_2} T^{k_2} b_0 \in T^{k_1 - k_2} M_a.$$

Since $-N \leq k_1 - k_2 \leq N$ we obtain

$$c \in \bigcup_{n=-N}^{N} T^n M_a,$$

and since $c \in [a, T^N a]$ was chosen arbitrarily, we obtain the desired boundedness result. □

5.4. Examples from Lie groups. We now explain how the Clerc–Koufany construction of the Guichardet–Wigner quasimorphisms on a 1-connected simple Hermitian Lie groups of tube type [9] can be reinterpreted in the language of the present paper.

Let G be an adjoint simple Lie group with maximal compact subgroup K. Then G is called *Hermitian* if the associated symmetric space G/K admits a G-invariant complex structure J, and *of tube type* if $(G/K, J)$ is biholomorphic to a complex tube. From now on G will always denote an adjoint simple Hermitian Lie group of tube type. We then find a Euclidean Jordan algebra V such that G/K can be identified with the unit ball $\mathcal{D}$ in $V^{\mathbb{C}}$ with respect to the spectral norm. We will fix such an identification once and for all. The action of G on $\mathcal{D}$ extends continuously to the Shilov boundary of $\check{S}$. Since $\check{S}$ is a generalized flag manifold, we obtain a notion of transversality on $\check{S}$ from the associated Bruhat decomposition. It then follows from the abstract theory of generalized flag manifolds that

G acts almost 2-transitively on $\check{S}$ (see e.g. [31, Lemma 3.30]). If e_V denotes the unit element of the Jordan algebra V, then $e_V \in \check{S}$ and the set $\check{S}_{e_V}$ of points in $\check{S}$ transverse to e_V is Zariski open in $\check{S}$. The Cayley transform of $V^{\mathbb{C}}$ identifies $\check{S}_{e_V}$ and hence $T_{-e_V}\check{S}$ with V. Thus the (closed) cone of squares in V gives rise to a closed cone $\Omega \subset T_{-e_V}\check{S}$. By a result of Kaneyuki [23] there exists a unique G-invariant causal structure $\mathcal{C}$ on $\check{S}$ with $\mathcal{C}_{-e_V} = \Omega$.

The universal covering $(\check{R}, \widetilde{\mathcal{C}})$ of the causal manifold $(\check{S}, \mathcal{C})$ is described in [9]. Namely, it turns out that $\pi_1(\check{S}) \cong \mathbb{Z}$, so that $p\colon \check{R} \to \check{S}$ is an infinite cyclic covering. The universal covering $\widetilde{G}$ of G acts transitively on $\check{R}$; the kernel of this action can be identified with $\pi_1(G)_{\text{tors}}$. Thus

$$\check{G} := \widetilde{G}/\pi_1(G)_{\text{tors}}$$

acts transitively and effectively on $\check{R}$. Now we claim:

Proposition 5.13. *The covering $p\colon \check{R} \to \check{S}$ is a complete quasi-total causal covering.*

Proof. Identify the tangent space of $T_{-e_V}\check{S}$ with V and choose an inner product $\langle \cdot, \cdot \rangle$ on V such that Ω is a symmetric cone with respect to $\langle \cdot, \cdot \rangle$ [13]. Since e_V is contained in the interior of Ω it follows from the self-duality of the latter that there exists $\epsilon > 0$ such that, for all $x \in \Omega$,

$$\langle x, e_V \rangle \geq \epsilon \cdot \|v\|. \tag{14}$$

Now identify $\check{S}$ with the compact symmetric space K/M, where M denotes the stabilizer of $-e_V$. Since the stabilizer action of M preserves both e_V and the inner product, there exists a K-invariant 1-form α on $\check{S}$ with

$$\alpha_{-e_V}(v) = \langle x, e_V \rangle, \quad v \in T_{-e_V}\check{S}.$$

Since K/M is symmetric, this form is closed. It then follows from (14) that α is a uniformly positive 1-form in the sense of Definition A.7. This implies that the pullback $\beta := p^*\alpha$ is a uniformly positive 1-form on $\check{R}$. In particular, $\check{R}$ is causal by Proposition A.8. Since $\check{S}$ is a flag variety, the action of G on $\check{S}$ is almost 2-transitive; just take $\check{S}^{(2)}$ to be the set of transverse pairs in $\check{S}$ (see e.g. [31]). This almost 2-transitivity implies immediately that $\check{S}$ is totally acausal, whence $p\colon \check{R} \to \check{S}$ is a causal covering; in view of Theorem 5.7 it also implies that this causal covering is quasi-total. It remains to show that this covering is total, i.e. $Tx \succeq x$ for all $x \in \check{R}$. For this it suffices to construct a causal curve joining x and Tx; this is established in [9]. □

In view of the pioneering work in [23] we refer to the partial order $\preceq$ on $\check{R}$ as the *Kaneyuki order*. It was established in [23], that $G(\check{S}, \check{C}) = G$ unless $G \cong \mathrm{PSL}_2(\mathbb{R})$. Thus let us assume $G \not\cong \mathrm{PSL}_2(\mathbb{R})$ from now on. Then the central extension associated with the causal covering $p\colon \check{R} \to \check{S}$ is precisely

$$0 \longrightarrow \mathbb{Z} \longrightarrow \check{G} \longrightarrow G \longrightarrow \{e\}.$$

We thus obtain a non-trivial quasimorphism $T_{\check{R}}$ on the simple Lie group $\check{G}$. It follows from the classification of such quasimorphisms in [30] that $T_{\check{R}}$ is necessarily a multiple of the Guichardet–Wigner quasimorphism on $\check{G}$; see [18]. We have thus proved the following result.

Corollary 5.14. *The growth functions of the order $\leq$ on $\check{G}$ induced from the Kaneyuki order on $\check{R}$ are multiples of the Guichardet–Wigner quasimorphism.*

Guichardet–Wigner quasimorphisms are very well understood; see [7] for an explicit formula. The observation that Guichardet–Wigner quasimorphisms are related to the causal structure on the corresponding Shilov boundaries was first made in [9] (see also [8] for an English introduction to their work). However, their precise formulation of this phenomenon is different from ours. Corollary 5.14 allows us to characterize subgroups of $\check{G}$ with vanishing Guichardet–Wigner quasimorphism. Indeed, as as special case of Theorem 5.10 we obtain the following result.

Corollary 5.15. *The Kaneyuki order is globally hyperbolic.*

Combining this observation with Theorem 3.12 we deduce the following result.

Corollary 5.16. *Let $H < \check{G}$ be a subgroup. Then the following are equivalent*:

(i) *the Guichardet–Wigner quasimorphism vanishes on H*;

(ii) *H has a bounded orbit in $\check{R}$*;

(iii) *every H-orbit in $\check{R}$ is bounded.*

6. Total triples and circular quasimorphisms

6.1. Total triples. A very special class of examples of quasi-total triples $(X, \preceq, T)$ is given by *totally ordered spaces* $(X, \preceq)$ together with a dominating automorphism T. We then say that $(X, \preceq T)$ is a *total triple*. In this situation the theory simplifies considerably. For instance, the T-height function (see Proposition 3.10 and the preceding definition) admits the following simpler description.

Proposition 6.1. *Let* $(X, \preceq, T)$ *denote a complete total triple and let* $a, b \in X$*. Then* $h_T(a,b)$ *is the unique integer such that*

$$T^{h_T(a,b)-1}.b \prec a \preceq T^{h_T(a,b)}.b.$$

Now let us specialize further to the case where X coincides with G. In this case $\preceq$ is a left-invariant order on G and we have a distinguished basepoint given by $a = e$. Given $g \in G$ define

$$n := n(g)$$

to be the unique integer satisfying

$$T^{n-1}.e \preceq g \preceq T^n.e;$$

Then, as a special case of the last proposition we see that the function $g \mapsto n(g)$ is at bounded distance from the translation number $T_{G,\preceq,T}$ associated with the triple $(G, \preceq, T)$. From this description we see in particular that our construction generalizes a construction of Ito [22].

Corollary 6.2. *Let* G *be a group,* $\preceq$ *a left-invariant total order on* G*,* $x \in G$ *and*

$$\rho_x(g) := gx.$$

Assume that $(G, \preceq, \rho_x)$ *is a total triple. Then the translation number* $T_{G,\preceq,\rho_x}$ *is the homogeneization of the quasimorphism* $\rho^G_{x,\preceq}$ *constructed in* [22].

A particular example seems worth mentioning at this point.

Example 6.3 (Ito). Let B_n be again the n-string braid group and denote by $\sigma_1, \dots, \sigma_{n-1}$ its canonical (Artin) generators. There is a canonical left-ordering $\preceq$ on B_n, which is described e.g. in [10] and sometimes called the *Dehornoy order*. If we choose

$$x := ((\sigma_1 \dots \sigma_{n-1})(\sigma_1\sigma_2)\sigma_1)^2,$$

then x is central in B_n and $(B_n, \preceq, \rho_x)$ is a total triple. Combining the last corollary with [22, Example 1], we see that the translation number $T_{B_n,\preceq,\rho_x}$ is the homogeneization of the Dehornoy floor quasimorphism.

In [22] it is always assumed that

$$T = \rho_x \quad \text{for some } x \in G.$$

If G is assumed countable, then this is not a serious restriction.

Lemma 6.4. *Let G be a countable group and $(G, \preceq, T)$ be a total triple with a dominating G-action. Then there exists a supergroup G_1 of G, a total order $\preceq_1$ on G_1 and an element $x \in G_1$ with the following properties*:

(i) *$\preceq_1$ is a left-invariant, total order on G_1 and $\preceq_1 |_G = \preceq$;*

(ii) *$x \in Z(G_1)$ and x is dominant for $\preceq_1$;*

(iii) *$(G_1, \preceq_1, \rho_x)$ is a total triple with a dominating G_1-action.*

Moreover, G_1 is isomorphic to a quotient of $G \times \mathbb{Z}$ and the embeddings of G into $G \times \mathbb{Z}$ and G_1 are compatible.

Proof. Let G_1 be the subgroup of Aut(G, $\preceq$, T) generated by G and T and set $x := T$. Here G acts on itself by left-multiplication. Since G and T commute, this group is a quotient of $G \times \mathbb{Z}$ and x is central. Note that G_1 acts on G preserving $\preceq$. To define $\preceq_1$ choose an enumeration $\{g_i\}_{i \in \mathbb{N}}$ of G with $g_1 = e$; then define that $g \preceq_1 h$ if and only if $(gg_i) \preceq (hg_i)$ with respect to the lexicographical order on $G^{\mathbb{N}}$. Since G_1 acts effectively on G, this defines a total order and x is dominant, since T is dominant. Also, $\preceq_1$ is G_1-invariant, since $\preceq$ is. Finally, let $g, h \in G$ be distinct; then either $g \prec h$ or $g \succ h$. In the former case we have $g.g_1 \prec h.g_1$ (since $g_1 = e$) and thus $g \prec_1 h$, while in the second case we have $g \succ_1 h$. This shows that $\preceq_1$ restricts to $\preceq$ on G. □

Thus in studying total triples $(G, \preceq, T)$ over a countable group G we may focus on the case, where $T = \rho_x$ for a central dominant $x \in G$.

6.2. From circular quasimorphisms to total triples. In this subsection we study quasimorphisms which arise from lifts of actions on the circle.

Definition 6.5. Let G be a group. A nonzero homogeneous quasimorphism f on G is called *circular* if there exists an injective homomorphism

$$\varphi\colon G \longrightarrow \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$$

such that

$$f = \varphi^* T_{\mathbb{R}}.$$

It turns out that circular quasimorphisms are closely related to total triples. The precise relation is somewhat technical, and we offer three different (essentially equivalent) formulations.

Proposition 6.6. *Let G be a group, and f be a circular homogeneous quasimorphism on G.*

(i) *There exists a left-invariant total order $\preceq$ on G such that the growth functions of the order induced from $\preceq$ via the left-action of G on itself are multiples of f.*

(ii) *There exists a quasi-total triple $(G, \preceq_0, T)$ realizing f with the property that $\preceq_0$ can be refined into a left-invariant total order $\preceq$ on G.*

(iii) *Assume that f is unbounded on the center of G. Then there exists a total triple $(G, \preceq, T)$ realizing f.*

Proof. (i) We first recall [25] that every enumeration $\{q_n\}$ of $\mathbb{Q}$ defines a total order $\preceq$ on $H := \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ by setting $g \preceq h$ if and only if $(gq_n) \leq (hq_n)$ with respect to the lexicographic ordering on $\mathbb{R}^{\mathbb{N}}$. Indeed, this follows from the fact that every $h \in H$ is uniquely determined by its restriction to $\mathbb{Q}$. We fix such an enumeration and the corresponding ordering $\preceq$ once and for all. By construction, $\preceq$ is left-invariant. Denote by $\leq_H$ the bi-invariant order on H induced by $\preceq$. Then $\leq_H$ is sandwiched by $T_{\mathbb{R}}$. Indeed, assume $T_{\mathbb{R}}(h) > 10$. Then for all $q \in \mathbb{R}$ we have $h.q > q$, whence $(hq_n) \succ (eq_n)$ and thus $h \geq_H e$.

Now assume $f: G \to \mathbb{R}$ is circular and nonzero, say $f = \varphi^* T_{\mathbb{R}}$ for some injection $\varphi: G \to H$. For notation's sake let us assume that G is a subgroup of H and φ the inclusion. Then the restriction $\preceq|_G$ defines a left-invariant total order on G. Let $\leq$ be the order on G induced by $\preceq|_G$. From the fact that $T_{\mathbb{R}}$ sandwiches $\leq$ we deduce that f sandwiches $f^* \leq_H$; since $\leq$ is a refinement of $f^* \leq_H$, it also sandwiches $\leq$.

(ii) Argue as in (i), but define $\preceq_0$ to be the bi-invariant order induced by $\preceq$ and choose T to be right multiplication by some element $g \in G$ with $\varphi(g) > 10$.

(iii) Construct $\preceq$ as in (i) and choose T to be multiplication by a central element x with $f(x) > 10D(f) + 5$. ☐

For countable groups we will establish a partial converse to Proposition 6.6 in Theorem 6.7 below.

6.3. From total triples to circular quasimorphisms.

The goal of this section is to establish the following partial converse of Proposition 6.6.

Theorem 6.7. *Let G be a countable group and $(G, \preceq, T)$ be a total triple with a dominating G-action. Denote by $\leq$ the induced bi-invariant order on G. Then the growth functions of $\leq$ are nonzero circular quasimorphisms.*

We will first establish the theorem under the additional hypothesis that $T = \rho_x$ for some central dominant $x \in G$. In a second step we will then reduce the general case to this case by means of Lemma 6.4. The first step of the proof uses crucially the notion of a dynamical realization [25].

Definition 6.8. Let $\preceq$ be a left-invariant total order on G. A *dynamical realization* of $\preceq$ is a pair (φ, t) consisting of an injective homomorphism

$$\varphi\colon G \longrightarrow \mathrm{Homeo}^+(\mathbb{R})$$

and a φ-equivariant embedding

$$t\colon G \longrightarrow \mathbb{R}$$

such that

$$t(e) = 0, \quad \inf_{g \in G} t(g) = -\infty, \quad \sup_{g \in G} t(g) = \infty$$

and

$$f \prec g \iff \varphi(f).0 < \varphi(g).0. \tag{15}$$

A *dynamical realization* of $\preceq$ is *special* if $\varphi(G)$ centralizes the translation

$$T\colon x \longmapsto x + 1;$$

it is called *adapted* to $x \in G$ if

$$\varphi(x) = T.$$

The following fact is well-known.

Proposition 6.9. *Let G be a countable group and $\preceq$ a left-invariant total order on G.*

(i) *$\preceq$ admits a dynamical realization.*

(ii) *There exists a special dynamical realization of $\preceq$ adapted to $x \in G$ if and only if x is both central and dominant for $\preceq$.*

Proof. (i) see [25, Proposition 2.1]. (ii) Assume that such a realization exists. Since $\varphi(x)$ is contained in the centralizer of $\varphi(G)$ and φ is injective we must have $x \in Z(G)$. Also, given any $g \in G$ we find $n \in \mathbb{N}$ with $\varphi(g).0 < n = \varphi(x^n).0$. We deduce that $g \preceq x^n$, which shows that x is a dominant. Thus the conditions are necessary. On the other hand, assume that $x \in Z(G)$ is dominant. Then every element in G may be written uniquely as $g = g_0 x^n$ with $n \in \mathbb{Z}$ and $e \preceq g_0 \preceq x$. Now define the embedding t as follows. Set $t(e) = 0$, $t(x) = 1$ and let $\{g_k\}_{k\in\mathbb{N}}$ be an enumeration of the order interval $[e, x]$ with $g_1 = e$, $g_2 = x$. Inductively assume $t(g_1), \dots, t(g_{i-1})$ have been defined. Then there exists g_m, g_M such that $g_m < g_i < g_M$ and $]g_m, g_M[\cap \{g_1, \dots, g_{i-1}\} = \emptyset$. We then define $t(g_i) := (t(g_m) + t(g_M))/2$. Now extend the map $t\colon [e, x] \to \mathbb{R}$ to all of G by the formula $t(g_0 x^n) = n + t(g_0)$. The action of G on $t(G)$ given by $\varphi(g)t(h) := t(gh)$ extends continuously to the closure of $t(G)$ and can be extended to an action of homeomorphisms on $\mathbb{R}$ in a standard way, see [25]. We have $\varphi(x).t(g) = t(g+1)$, hence $\varphi(x) = T$ on $\overline{t(G)}$. From the construction of the extension in loc. cit. we deduce $\varphi(x) = T$ on all of $\mathbb{R}$. □

We now fix a total triple of the form $(G, \preceq, \rho_x)$ with $x \in Z(G)$ dominant and a dynamical realization (φ, t) of $\preceq$ adapted to x. As before, we denote by $\leq$ the order induced by $\preceq$ on G. We recall that the corresponding dominant set G^{++} is non-empty and that its growth functions are multiples of $T_{(G,\preceq,\rho_x)}(g)$. We now aim to describe these growth functions in terms of the homomorphism $\varphi\colon G \to \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$. To this end we observe that φ allows us to pullback the classical translation number $T_{\mathbb{R}}$ to a homogeneous quasimorphism $\varphi^* T_{\mathbb{R}}$ on G. The following fact was observed in [22].

Proposition 6.10 (Ito). *Let G be a countable group, $\preceq$ a left-invariant total order on G and $x \in G$ a central dominant. Let (φ, t) be a dynamical realization of $\preceq$ adapted to x. Then*

$$T_{(G,\preceq,\rho_x)} - \varphi^* T_{\mathbb{R}}\colon G \longrightarrow \mathbb{R}$$

is a homomorphism.

Proof. By the proof of [22, Theorem 3] the pullback of the bounded Euler class $-e_b := dT_{\mathbb{R}}$ under φ in real bounded cohomology is the class represented by the differential of the quasimorphism denoted $\rho^G_{x,\preceq}$ in [22]. (In fact, this is even true for the corresponding integral bounded cohomology classes, but we do not need this stronger statement here.) Since $T_{G,\preceq,\rho_x}$ is at bounded distance from $\rho^G_{x,\preceq}$ by Corollary 6.2, we deduce that the differential of $f := T_{G,\preceq,\rho_x} - \varphi^* T_{\mathbb{R}}$ represents the trivial class in $H^2_b(G; \mathbb{R})$. Now f is both homogeneous and cohomologically trivial, hence a homomorphism. □

We will strengthen this as follows.

Lemma 6.11. *Let G be a countable group, $\preceq$ a left-invariant total order on G and $x \in G$ a central dominant. Let (φ, t) be a dynamical realization of $\preceq$ adapted to x. Then*

$$T_{(G,\preceq,\rho_x)} = \varphi^* T_{\mathbb{R}}.$$

Proof. Since both quasimorphisms are homogeneous it suffices to show that they are at bounded distance. For this we may replace $T_{\mathbb{R}}$ by the function $g \mapsto \varphi(g).0$ and $T_{(G,\preceq,\rho_x)}(g)$ by $h_T(g,e)$, since those are at bounded distance from the original functions. Now choose n so that

$$\varphi(x)^{n-1}.0 = n - 1 < \varphi(g).0 \le n = \varphi(x)^n.0.$$

This implies both $|\varphi(g).0 - n| < 1$ and $x^{n-1} \prec g \preceq x^{n+1}$, the latter by (15). We may rewrite the last chain of inequalities by

$$\rho(x)^{n-1}.e = x^{n-1} \prec g \prec x^{n+1} = \rho(x)^{n+1}.e.$$

From this we deduce that $|h_T(g,e) - n| < 2$, whence $|\varphi(g).0 - h_T(g,e)| < 3$. □

Now we can deduce the theorem.

Proof of Theorem 6.7. Let $(G, \preceq, T)$ be any total triple with a dominating G-action. We then construct the extended triple $(G_1, \preceq_1, \rho_x)$ as in Lemma 6.4 and denote by $\le_1$ the order induced by $\preceq_1$ on G_1. We then choose a dynamical realization (φ_1, t_1) of $\preceq_1$ adapted to x and deduce from Theorem 1.8 and Lemma 6.11 that $\le_1$ is sandwiched by $T_{G_1,\preceq_1,\rho_x} = \varphi_1^* T_{\mathbb{R}}$. We thus find a constant C such that for $g \in G_1$ with $T_{\mathbb{R}}(\varphi_1(g)) > C$ we have

$$g.x \succeq_1 x \quad \text{for all } x \in G_1. \tag{16}$$

Denote by $\le$ the bi-invariant order induced by $\preceq$ on G and by φ the composition of the inclusion $G \to G_1$ with φ_1. We then claim that $\le$ is sandwiched by $\varphi^* T_{\mathbb{R}}$. Indeed, assume $g \in G$ satisfies $\varphi^* T_{\mathbb{R}}(g) > C$; then (16) holds, and in particular

$$g.x \succeq_1 x \quad \text{for all } x \in G.$$

But since $\succeq_1|_G = \succeq$, this shows that $g \ge e$, which yields the desired sandwiching result. □

For quasimorphisms which are unbounded on the center of G we have obtained a complete characterization of circularity.

Corollary 6.12. *Let $f : G \to \mathbb{R}$ be a quasimorphism, which is unbounded on the center of G. Then the following are equivalent*:

(i) *f is circular*;

(ii) *f can be realized by a total triple $(G, \preceq, T)$.*

For quasimorphism, which are not unbounded on the center of G the situation is slightly more technical, as witnessed by the more complicated formulation of Proposition 6.6 for such quasimorphisms.

A. Smooth partial orders on manifolds

A.1. Causal structures. In various branches of mathematics and physics cone fields in the tangent bundle of a manifold are used to define a *causality* (i.e. a reflexive and transitive relation) on the manifold itself. The precise definitions of such causalities, however, differ widely in the literature; it thus seem worthwhile to elaborate a bit on the definitions we use in the body of text. In the present paper we are mainly interested in invariant cone fields on homogeneous spaces of finite-dimensional Lie groups, and our definitions are adapted to work well in this context. We refer the reader to [20, 19, 24] for sources with a point of view similar to ours.

A convex, $\mathbb{R}^{>0}$-invariant closed subset Ω of a vector space V will be called a *wedge*. A wedge is called a *closed cone* if it is *pointed*, i.e. $\Omega \cap (-\Omega) = \{0\}$. It is called *regular* if its interior is non-empty. Given a closed regular cone $\Omega \subset V$ we denote by $G(\Omega)$ the group

$$G(\Omega) := \{g \in GL(V) \mid g\Omega = \Omega\}.$$

Definition A.1. Let M be a d-dimensional manifold and $\Omega \subset \mathbb{R}^d$ a closed regular cone. Then a *causal structure* on M is a principal $G(\Omega)$-bundle $P \to M$ together with an isomorphism

$$\iota\colon P \times_{G(\Omega)} V \longrightarrow TM$$

(i.e. a reduction of the structure group of TM from $GL_d(\mathbb{R})$ to $G(\Omega)$.) The associated fiber bundle

$$\mathcal{C} := \iota(P \times_{G(\Omega)} \Omega)$$

of P with fiber Ω is called the *cone field* of the causal structure P. We then refer to the pair $(M, \mathcal{C})$ as a *conal manifold*.

We warn the reader that the term *causal manifold* is traditionally reserved for a conal manifold with additional properties, see the definition below. We also remark that [20] uses a more general definition of causal structure, but the present definition is sufficient for our purposes. For us it will be important that cone fields can be lifted along coverings.

Lemma A.2. *Let $(M, \mathcal{C})$ be a conal manifold and $\widetilde{M}$ its universal covering. Then there exists a unique cone field $\widetilde{\mathcal{C}}$ on $\widetilde{M}$ such that $\pi_1(M)$ acts by causal diffeomorphisms on $(\widetilde{M}, \widetilde{\mathcal{C}})$. Conversely, every $\pi_1(M)$-invariant cone field descends to M.*

Proof. The only way to define a causal structure with the desired property is to set

$$\widetilde{\mathcal{C}}_x = (dp_M)^{-1}\mathcal{C}_{p_M(x)},$$

where $dp_M : T_x\widetilde{M} \to T_{p_M(x)}M$ is the derivative of the universal covering projection. This defines indeed a causal structure on $\widetilde{M}$, since the triviality condition is local. The second statement is obvious. □

Given a manifold M and real numbers $a < b$ we call a curve $\gamma\colon [a,b] \to M$ *piecewise smooth* if it is continuous and there exists real numbers $a = a_0 < a_1 < \cdots < a_n = b$ such that $\gamma|_{(a_j,a_{j+1})}$ is of class C^∞ for $j = 0, \ldots, n-1$.

Definition A.3. Let $(M, \mathcal{C})$ be a conal manifold. A piecewise smooth curve

$$\gamma\colon [a,b] \longrightarrow M$$

is called $\mathcal{C}$-*causal* if

$$\dot{\gamma}(t) \in \mathcal{C}_{\gamma(t)}$$

for all but finitely many $t \in [a,b]$. The relation $\preceq_s$ on M obtained by setting $x \preceq_s y$ if there exists a causal curve $\gamma\colon [a,b] \to M$ with $\gamma(a) = x$ and $\gamma(b) = y$, is called the *strict causality* of $(M, \mathcal{C})$.

Since the concatenation of piecewise smooth curves is piecewise smooth, the strict causality is indeed a causality. It may or may not be anti-symmetric, and it may or may not be smooth. Antisymmetry can sometimes be obtained by passing to a suitable covering. Obtaining a closed causality is difficult in general. Indeed, the closure $\preceq$ of $\preceq_s$ in $M \times M$ need no longer be transitive. Fortunately, in homogeneous examples this kind of pathology hardly occurs. We make this precise in the following definition.

Definition A.4. Let $(M, \mathcal{C})$ be a conal manifold. A diffeomorphism φ of M is called *causal* with respect to $\mathcal{C}$ if

$$d\varphi(\mathcal{C}_m) = \mathcal{C}_{\varphi(m)} \quad \text{for all } m \in M.$$

The group of all causal diffeomorphisms of $(M, \mathcal{C})$ is denoted $G(M, \mathcal{C})$. A group action $G \times M \to M$ is *causal* if G acts by causal diffeomorphisms. In this case the causal structure $\mathcal{C}$ is called *G-invariant*. The conal manifold $(M, \mathcal{C})$ is called *uniformly homogeneous* if $G(M, \mathcal{C})$ acts transitively on M and every $x \in M$ has an open neighbourhood U such that for all $x_n \in U$ with $x_n \to x$ there exists a sequence $g_n \in G(M, \mathcal{C})$ such that

$$g_n x_n = x \quad \text{and} \quad g_n \to e$$

in the compact-open topology.

Lemma A.5 (Hilgert–Ólafsson). *Let $(M, \mathcal{C})$ be a uniformly homogeneous conal manifold. Then the closure $\preceq$ of $\preceq_s$ in $M \times M$ is a causality. Moreover, the order intervals of $\preceq$ are the closures of the order intervals of $\preceq_s$.*

Proof. The assumption of uniform homogeneity guarantees that the proof of [20, Proposition 2.2.4] carries over. ☐

Note that if G is a finite-dimensional Lie group and H is a closed subgroup, then every G-invariant cone field on G/H is uniformly homogeneous; indeed, this follows from the existence of local sections of the principal bundle $G \to G/H$. This case is actually all we need. In any case in all our examples $\preceq$ will be a well-defined causality. We follow [19, 20, 32, 26, 24] in calling $\preceq$, rather than $\preceq_s$ the causality associated with the conal manifold $(M, \mathcal{C})$.

A.2. Causal manifolds and positive 1-forms. In this section let $(M, \mathcal{C})$ be a conal manifold. We denote by $\preceq_s$ the strict causality on M and by $\preceq$ its closure.

Definition A.6. The conal manifold $(M, \mathcal{C})$ is called *causal* if $\preceq$ is a partial order and *totally acausal* if $\preceq = M \times M$. It is called *weakly causal* if $\preceq_s$ is a partial order.

Note that for $(M, \mathcal{C})$ to be causal we demand in particular that $\preceq$ is transitive. We now provide a sufficient condition for M which guarantees causality.

Definition A.7. Let $(M, \mathcal{C})$ be a conal manifold. A closed 1-form $\alpha \in \Omega^1(M)$ is called *uniformly positive* with respect to $\mathcal{C}$ if there exists a Riemannian metric on M and $\epsilon > 0$ such that

$$\alpha_x(v) \geq \epsilon \cdot \|v\| \quad \text{for all } x \in M, v \in \mathcal{C}_x.$$

Uniformly positive 1-forms are a special case of positive 1-forms as introduced in [19] refining ideas from [32, 26]. Positive 1-forms were introduced to prove antisymmetry of the strict causality on certain 1-connected manifolds. Uniformly positive 1-forms play a similar role for the closed causality.

Proposition A.8. *Let $(M, \mathcal{C})$ be a simply-connected uniformly homogeneous conal manifold admitting a uniformly positive 1-form α. Then $(M, \mathcal{C})$ is causal.*

Proof. Since M is uniformly homogeneous, $\preceq$ is transitive, and it remains to show that it is antisymmetric. Let $x, y \in M$ be distinct points and assume $x \preceq y \preceq x$. By definition this means that there exist sequences $x_n \to x$, $x_n' \to x$, $y_n \to y$, $y_n' \to y$ in $\check{R}$ such that

$$x_n \preceq_s y_n, \quad y_n' \preceq_s x_n'.$$

Let $G := G(M, \mathcal{C})$ and observe that since $(M, \mathcal{C})$ is uniformly homogeneous there exist sequence $g_n \to e$, $g_n' \to e$ in G such that $y_n = g_n y$, $y_n' = g_n' y$. Now define $a_n := g_n^{-1} x_n$, $b_n := (g_n')^{-1} y_n'$. Then

$$a_n \preceq_s y \preceq_s b_n, \quad a_n \to x, \quad b_n \to x.$$

Denote by d the metric induced by the Riemannian metric, for which α is uniformly positive. Then for any causal curve $c\colon [a, b] \to \widetilde{M}$ we have

$$\int_c \alpha = \int_a^b \alpha_{c(t)}(\dot{c}(t)) \geq L(c) \cdot \epsilon,$$

where $L(c)$ denotes the length of c. Fix a neighbourhood U of x not containing y in its closure and set $\delta := \frac{\epsilon}{2} d(U, y)$. Choose n_0 such that $a_n, b_n \in U$ for all $n \geq n_0$. By shrinking U if necessary we may assume that U is geodesically convex and relatively compact. Now let c_n be a causal curve from a_n to b_n through y; then $L(c_n) \geq \delta/\epsilon$ and thus

$$\int_{c_n} \beta \geq L(c_n) \cdot \epsilon \geq \delta > 0.$$

Now denote by c_n' a geodesic joining a_n to b_n in U (parametrized by arclength) and by $(c_n')^*$ the same curve with the opposite orientation. Then the concatenation $c_n\#(c_n')^*$ is a closed loop in M; since M is simply-connected this loop bounds a disc D and thus

$$\int_{c_n'} \alpha = \int_{c_n} \alpha + \int_{\partial D} \alpha = \int_{c_n} \alpha + \int_D d\alpha = \int_{c_n} \alpha.$$

Now α is bounded on the compact set $\overline{U}$, hence their exists $C > 0$ such that

$$\int_{c_n'} \alpha \leq C \cdot L(c_n') = C \cdot d(a_n, b_n).$$

We have thus established for all $n > n_0$ the inequality

$$0 < \delta \leq C \cdot d(a_n, b_n).$$

Since $d(a_n, b_n) \to 0$, this is a contradiction. □

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