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Indecomposable F_N-trees and minimal laminations

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Abstract. We extend the techniques of [8] to build an inductive procedure for studying actions in the boundary of the Culler–Vogtmann Outer Space, the main novelty being an adaptation of the classical Rauzy–Veech induction for studying actions of *surface type*. As an application, we prove that a tree in the boundary of Outer space is free and indecomposable if and only if its dual lamination is *minimal up to diagonal leaves*. Our main result generalizes [3, Proposition 1.8] as well as the main result of [22].

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Contents

1. Introduction

We consider $\mathbb{R}$-trees T equipped with a minimal very small action of the free group F_N of rank N by isometries: these are points in the closure of the celebrated unprojectivized Culler–Vogtmann Outer Space $\overline{\text{cv}}_N$ [14]. A dual lamination $L(T)$ is associated to such trees [10]. We recall that a lamination for the free group is a closed F_N-invariant, flip invariant, subset of the double Gromov boundary

$\partial^2 F_N = (\partial F_N)^2 \setminus \Delta$, where Δ is the diagonal. The main result result of the paper relates *minimality* properties of the tree to *minimality* properties of its dual lamination.

For an action $F_N \curvearrowright T$ there are several notions of minimality for the dynamics of the action; see Section 2.8. Mixing trees were considered by Morgan [26], and Guirardel introduced the stronger notion of indecomposability in [17].

Definition 1.1. An $\mathbb{R}$-tree $T \in \overline{\text{cv}}_N$ is *indecomposable* if for any non-degenerate segments I and J in T, there exist finitely many elements $u_1, \dots, u_n$ in F_N such that

(1) $I \subseteq u_1 J \cup u_2 J \cup \dots \cup u_n J$

(2) $u_i J \cap u_{i+1} J$ is a non degenerate segment for any $i = 1, \dots, n-1$.

A lamination L is minimal if it does not contain a proper sublamination, or, equivalently, if the orbit of any leaf of L is dense in L. Laminations dual to trees are always diagonally closed: if $L(T)$ contains leaves $l_1 = (X_1, X_2)$, $l_2 = (X_2, X_3), \dots, l_n = (X_{n-1}, X_n)$, then $L(T)$ also contains the leaf $l = (X_1, X_n)$. Oftentimes diagonal leaves are isolated in the dual lamination. Thus we need a slightly modified notion of minimality for laminations dual to $\mathbb{R}$-trees.

Definition 1.2. A lamination L is *minimal up to diagonal leaves* if

(i) there is a unique minimal sublamination $L_0 \subseteq L$, and

(ii) $L \setminus L_0$ consists of finitely many F_N-orbits of leaves, each of which is diagonal over L_0.

In the definition, a leaf $l \in L$ is *diagonal* over a sublamination $L_0 \subseteq L$ if there are leaves $(X_1, X_2), (X_2, X_3), \dots (X_{n-1}, X_n) \in L_0$ such that $l = (X_1, X_n)$. Note that the above notion of minimality coincides with the correct notion of minimality for foliations on a closed surface. Our main result is the following theorem.

Theorem A. *Let T be an $\mathbb{R}$-tree with a free, minimal action of F_N by isometries with dense orbits. The tree T is indecomposable if and only if $L(T)$ is minimal up to diagonal leaves.*

In this case the unique minimal sublamination is the derived sublamination $L(T)'$: the subset of non-isolated leaves.

The third author [27] proved that under the same hypotheses, T is indecomposable if and only if no leaf of the dual lamination $L(T)$ is carried by a finitely generated subgroup of infinite index. We use this characterization in the proof of Theorem A.

Theorem A generalizes the main result of [22]. Indeed, it is shown in [8] that the repelling tree $T_{\Phi^{-1}}$ of a fully irreducible (iwip) outer automorphism Φ of F_N is indecomposable, and $T_{\Phi^{-1}}$ is free exactly when Φ is non-geometric. Moreover, the attracting lamination L_Φ of Φ is minimal [3] and contained in the dual lamination of the repelling tree (see [8]).

Corollary 1.3 ([22]). *Let Φ be a non-geometric iwip outer automorphism of F_N. The dual lamination of the repelling tree $L(T_{\Phi^{-1}})$ is the diagonal closure of the attracting lamination L_Φ.*

Recall that a current on F_N is a positive F_N-invariant, flip invariant, Radon measure μ on $\partial^2 F_N$ [19]. The support $\mathrm{Supp}(\mu)$ of μ is a lamination (see [12]). We consider the intersection form i_{KL} between currents and trees:

$$i_{\mathrm{KL}}\colon \overline{\mathrm{cv}}_N \times \mathrm{Curr}(F_N) \to \mathbb{R}_{\geq 0},$$

see [20]. A current μ is orthogonal to a tree $T \in \overline{\mathrm{cv}}_N$ (i.e $i_{\mathrm{KL}}(T,\mu) = 0$) if and only if the support of μ is a sublamination of the dual lamination $L(T)$ [21]. Theorem A implies the following result.

Corollary 1.4. *Let $T \in \overline{\mathrm{cv}}_N$ be a free, indecomposable F_N-tree, and let μ be a geodesic current. The following are equivalent*:

(i) $\mathrm{Supp}(\mu) \subseteq L(T)$,

(ii) $i_{\mathrm{KL}}(T,\mu) = 0$,

(iii) $\mathrm{Supp}(\mu) = L(T)'$.

Our strategy for proving Theorem A is summarized as follows. We encode a tree $T \in \overline{\mathrm{cv}}_N$ in a system of isometries on a compact $\mathbb{R}$-tree as in [13], and then develop an inductive procedure for studying such systems of isometries. There are two cases to consider: if the tree T is of Levitt type we use Process I of the Rips Machine [8] (see also [1, 16]); if the tree T is of surface type we use a splitting procedure inspired by the Rauzy–Veech induction for flat surfaces. The result is a sequence of systems of isometries encoding the tree T; there is a train track associated to each of these systems of isometries, and the dual lamination $L(T)$ is the inverse limit of this sequence of train tracks. We are able read off from the sequence of train tracks the desired minimality properties of the lamination. The motivation for our procedure comes from surface theory.

1.1. The case of a surface lamination. The proof of Theorem A relies on an adaptation to the free group F_N of *train track expansions* of a measured lamination on a hyperbolic surface, which we now casually recall. First, we mention the correspondence between measured laminations, non-classical interval exchange transformations, and train tracks, as this correspondence suggests the procedure presented in this paper. Let Σ be a non-exceptional hyperbolic surface equipped with a measured geodesic lamination $\mathcal{L} = (L, \mu)$. We suppose that L is minimal and filling. Choosing an arc $I \subseteq \Sigma$ that is transverse to L, one can consider the first return map $f: I \to I$ induced by the holonomy of L. The map f is a (non-classical) *interval exchange transformation*, whose dynamics coincide with those of a finitely generated collection (similar to a *pseudo-group*) S of partial isometries on I: the generators of S are the maximal continuous restrictions of f. Suspending S, see Section 2.3, gives a foliated space $\mathcal{S}$, which is homeomorphic to Σ with finitely many discs removed; after gluing in these discs and extending the foliation to them in the obvious way, one obtains a surface homeomorphic to Σ, carrying a measured foliation which corresponds to measured lamination $\mathcal{L}$; see [23].

The suspension $\mathcal{S}$ of the interval exchange f has the homotopy type of a finite graph – collapsing each band of $\mathcal{S}$ onto one of its leaves gives the desired homotopy equivalence $h: \mathcal{S} \to \Gamma$. The graph Γ can be equipped with a *train track structure* as follows: let e and e' be (oriented) edges of Γ such that the endpoint v of e coincides with the initial point of e'; make e and e' tangent at the vertex v if and only if there is a non-singular leaf of $\mathcal{S}$ that crosses the (ordered) pair of bands corresponding to (e, e'). The widths of the bands of $\mathcal{S}$ give an assignment of weights to the edges of Γ, which clearly satisfy a *switch condition*. In fact, one sees that Γ can be embedded in Σ as a (measured) train track *carrying* L.

We now consider the effect of the *Rauzy–Veech induction*. Recall that given an interval exchange transformation $f: I \to I$ with corresponding partition $I = I_1 \cup \cdots \cup I_r$, one step of the Rauzy–Veech induction consists of: first removing the right most interval I_r, then replacing f with the first return map f' on the interval $I \setminus I_r$. Equivalently, consider the suspension $\mathcal{S}$, and split apart the right-most singularity of the foliation on $\mathcal{S}$ (all singularities lie on I) until I is again reached. Rauzy–Veech induction is defined in the generic case, and extra information is available when it is undefined; see Proposition 4.7. This process gives a foliated space $\mathcal{S}'$, which is the suspension of the interval exchange transformation f'. In the present note, we consider this "splitting" point of view of Rauzy–Veech induction.

As noted before, there is a measured train track Γ corresponding to the space $\mathcal{S}$, such that Γ carries L. If $\mathcal{S}'$ arises from $\mathcal{S}$ via one step of the Rauzy–Veech induction, then we say that the measured train track Γ' corresponding to $\mathcal{S}'$ arises from Γ via *splitting*; there is a homotopy equivalence $\tau\colon \Gamma' \to \Gamma$ (τ is a *fold*). The sequence of train tracks (Γ_i) arising by iterating this procedure is usually called a *splitting sequence*, or, more specifically, a *train track expansion* of L. The key features for us are that each Γ_i carries L and that the branches of Γ_i approximate leaves of L for $i >> 0$.

1.2. The case of a very small F_N-tree. We now outline our translation of the above technology. Let T be an $\mathbb{R}$-tree equipped with a free, minimal action of F_N by isometries, and suppose that the action $F_N \curvearrowright T$ is mixing (see Section 2.8). Let $L(T)$ denote the dual lamination associated to T (see Section 2.5). Choose a basis A for F_N, let X_A denote the Cayley tree of F_N with respect to this basis, and identify $\partial^2 F_N = \partial^2 X_A$ in the obvious way. Let C_A denote the subset of $\partial^2 F_N$ consisting of points represented by geodesics in X_A passing through the identity element. The set C_A is compact (and open), hence $L_A := L(T) \cap C_A$ is compact as well. As $L(T)$ is F_N-invariant, we may consider the restricted action of F_N on L_A; this turns out to be a classical *symbolic flow* (see [10]). We "geometrize" this flow using the tree T.

It is shown in [13] that there is an F_N-equivariant, continuous map

$$Q^2\colon L(T) \longrightarrow \overline{T},$$

where $\overline{T}$ denotes the metric completion of T; consider the compact subspace

$$\Omega_A := Q^2(L(T) \cap C_A);$$

this is the analogue of an arc transverse to a surface lamination. From the equivariance of Q^2, we get a partial action of F_N on Ω_A, which is just the restriction of the action $F_N \curvearrowright T$. It is shown in [8] that there are two possibilities for the structure of Ω_A: either Ω_A is a finite union of compact $\mathbb{R}$-trees, in which case T is called *surface type*; or Ω_A is totally disconnected, in which case T is called *Levitt type*. In either case the suspension of the (partial) action of F_N on Ω_A is a compact foliated space that is a "geometric realization" of L_A. However, in the Levitt type case, it is more convenient to work with a nicer space: put K_A to be the convex hull of Ω_A in $\overline{T}$ (so K_A is a compact $\mathbb{R}$-tree), and suspend the (partial) action of F_N on K_A. The result is a foliated space $\mathcal{S}$ in which some leaves are 1-ended; deleting all leaves with strictly less than two ends gives the suspension of Ω_A (see Sections 2.6, 3.2, 2.3, and 3.1).

We now describe our generalization of Rauzy–Veech induction; our induction has two distinct procedures. In the case that T is surface type, we show (Proposition 4.7) that there exist in the suspension $\mathcal{S}$ singularities which look like singularities which arise in foliated surface, *i.e.* they are "splittable"; splitting apart these singularities as above gives a homotopy equivalence $h_1 : \mathcal{S}' \to \mathcal{S}$ (h_1 "zips up" the splitting). There is a homotopy equivalence $h : \mathcal{S} \to \Gamma$ to a finite graph Γ got by collapsing each band onto a vertical fiber, and splitting induces a homotopy equivalence $\Gamma' \to \Gamma$, where Γ' is the graph associated to $\mathcal{S}'$ (see Section 4).

In the case that T is Levitt type, we operate on $\mathcal{S}$ using Process I of the *Rips Machine*: erase from K_A all points x such that x is an endpoint of a leaf to get $K'_A \subseteq \Omega_A$, and suspend K'_A get $\mathcal{S}'$. Again there are homotopy equivalences $\mathcal{S} \to \Gamma$, $\mathcal{S}' \to \Gamma'$, and $\Gamma' \to \Gamma$.

In either case, we get a sequence of foliated spaces ($\mathcal{S}_i$) and graphs Γ_i with homotopy equivalences $\tau_i : \Gamma_i \to \Gamma_{i-1}$. Any sublamination $L_0 \subseteq L(T)$ defines a *train track structure* on Γ_i just as in Section 1.1, where leaves in L_0 are treated as "non-singular." As one might expect, choosing too large of a sublamination L_0 gives too many legal turns; however, it follows from the results of this paper that for T *indecomposable* (see Section 2.8), the derived set $L(T)'$ consists of leaves that are morally non-singular; see Sections 2.7 and 5.2, and the Appendix, where we handle a more general situation. As before, the main features are that the train tracks Γ_i all carry $L(T)$ and that for $i >> 0$, the branches of Γ_i "approximate" leaves of $L(T)'$. Our train-tracks will have no transverse measure; however, the informed reader will realize that currents are the natural object for measuring these train tracks. See [9], where the first two authors obtain a bound on the number of ergodic currents dual to a very small tree with a free action.

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2. Background

In this section we briefly review the relevant definitions around $\mathbb{R}$-trees, Outer space, and laminations. In what follows F_N denotes the free group of rank N.

2.1. Basics about $\mathbb{R}$-trees. A metric space (T, d) is an $\mathbb{R}$-*tree* (or just a *tree*) if for any two points $x, y \in T$, there is a unique topological arc $p_{x,y} : [0, 1] \to T$ connecting x to y, and the image of $p_{x,y}$ is isometric to the segment $[0, d(x, y)]$. As is usual, we let $[x, y]$ stand for $\mathrm{Im}(p_{x,y})$, and we call $[x, y]$ the *segment* (also called an *arc*) in T from x to y. A segment is *non-degenerate* if it contains strictly more than one point. We let $\overline{T}$ stand for the metric completion of T. Unless otherwise stated, we regard T as a topological space with the metric topology. If T is a tree, and $x \in T$, then x is a *branch point* if the cardinality of $\pi_0(T - \{x\})$ is strictly greater than two. For $x \in T$, the elements of $\pi_0(T - \{x\})$ are the *directions* at x.

In this paper, all the trees we consider are equipped with an isometric (left) action of F_N, i.e. a group morphism $\rho\colon F_N \to \mathrm{Isom}(T)$; as usual, we suppress the morphism ρ and identify F_N with $\rho(F_N)$. A tree T equipped with an isometric action will be called an *F_N-tree*, and we denote this situation by $F_N \curvearrowright T$. Notice that an action $F_N \curvearrowright T$ induces an action of F_N on the sets of directions and branch points of T. We identify two F_N-trees T, T' if there is an F_N-equivariant isometry between them.

There are two sorts of isometries of trees: an isometry g of T is *elliptic* if g fixes some point of T, while an isometry h of T is *hyperbolic* if it is not elliptic. Any hyperbolic isometry h of T leaves invariant a unique isometric copy of $\mathbb{R}$ in T which is the *axis*, $A(h)$, of h. If g is an elliptic isometry, we let $A(g)$ stand for the fixed point set of g, *i.e.*

$$A(g) := \{x \in T | gx = x\}.$$

The *translation length function* of a F_N-tree T is $l_T : F_N \to \mathbb{R}$, where

$$l_T(g) := \inf\{d(x, gx) | x \in T\}.$$

The number $l_T(g)$ is the *translation length* of g, and for any $g \in F_N$, the infimum is always realized on $A(g)$, so that g acts on $A(g)$ as a translation of length $l_T(g)$. If $H \leq F_N$ is a finitely generated subgroup containing a hyperbolic isometry, then H leaves invariant the set

$$T_H^{\min} := \bigcup_{l_T(h)>0} A(h).$$

which is a subtree of T, and is minimal in the set of H-invariant subtrees of T; $T_H^{\min}$ is the *minimal invariant subtree for H*. In the case that $H = F_N$, we omit H and write $T^{\min}$ for $T_{F_N}^{\min}$. An action $F_N \curvearrowright T$ is *minimal* if $T = T^{\min}$; a minimal action $F_N \curvearrowright T$ is *non-trivial* if T contains strictly more than one point.

2.2. System of isometries. A metric space F is a *finite forest* if F has finitely many connected components, each of which is a compact $\mathbb{R}$-tree. A *partial isometry a* of F is an isometry between two closed (hence compact) subtrees of F.

A *system of isometries* is a pair $S = (F, A)$ where F is a finite forest and A is a finite collection of non-empty partial isometries of F. By allowing inverses and composition we get a sort of pseudo-group of partial isometries on F, with the domains of the partial isometries being closed; our convention will be that this pseudo-group acts on F on the right.

To a system of isometries $S = (F, A)$ we associate a graph Γ whose vertices are the connected components of F and such that for each partial isometry $a \in A$ there is an oriented edge starting at the connected component of F containing the domain, $\mathrm{dom}(a)$, and ending at the connected component of F containing the image of a. Denote by $V(\Gamma)$ and $E(\Gamma)$ the sets of vertices and edges of Γ, respectively.

By a *path* in Γ, we mean a finite *edge path*, that is, a path starting and ending at vertices of Γ. A path γ in Γ is called *reduced* if γ is an immersion; any path is homotopic relative to endpoints to a reduced path. A reduced path γ in Γ defines a partial isometry of F, and we say that γ is *admissible* if the corresponding partial isometry has non-empty domain. We abuse notation, identifying an admissible path γ with the partial isometry corresponding to it. An *infinite reduced path* γ is an immersion

$$\gamma\colon \mathbb{R}_{\geq 0} \longrightarrow \Gamma$$

such that

$$\gamma^{-1}(V(\Gamma)) = \mathbb{N}.$$

For an infinite reduced path γ, the *i-prefix* of γ, denoted γ_i, is the restriction $\gamma|_{[0,i]}$. For $i \geq j$, one has that $\mathrm{dom}(\gamma_i) \subseteq \mathrm{dom}(\gamma_j)$, and we put

$$\mathrm{dom}(\gamma) := \bigcap_i \mathrm{dom}(\gamma_i).$$

We say that an infinite path γ is *admissible* if $\mathrm{dom}(\gamma) \neq \emptyset$. A *bi-infinite reduced path* γ is an immersion

$$\gamma\colon \mathbb{R} \longrightarrow \Gamma$$

such that

$$\gamma^{-1}(V(\Gamma)) = \mathbb{Z}.$$

Given a bi-infinite path γ, the *halves* of γ are the restrictions

$$\gamma^- := \gamma|_{\mathbb{R}_{\leq 0}} \quad \text{and} \quad \gamma^+ := \gamma|_{\mathbb{R}_{\geq 0}};$$

we reparametrize γ^- in order to regard it as an infinite path. A bi-infinite reduced path is *admissible* if the domains of its halves have non-empty intersection.

Equivalently a bi-infinite path γ is admissible if and only if for any $i \leq j \in \mathbb{Z}$, the restriction $\gamma|_{[i,j]}$ is admissible.

The set of bi-infinite admissible paths is called the *admissible lamination* of the system of isometries $S = (F, A)$ and is denoted $L(S)$; general *laminations* are defined in Section 2.5.

2.3. The suspension and the dual tree. Let $S = (F, A)$ be a system of isometries, and let $I = [0, 1]$ denote the unit interval. For each $a_i \in A$, let

$$b_i := \mathrm{dom}(a_i);$$

one forms a *band*

$$B_i := b_i \times I.$$

Identify b_i with $b_i \times \{0\} \subseteq B_i$, and denote

$$\tilde{b_i} := b_i \times \{1\}.$$

Say that b_i and $\tilde{b_i}$ are the *bases* of the band B_i.

Definition 2.1. The *suspension* $\mathcal{S}$ of S is the quotient of $F \amalg \coprod_i B_i$, where one identifies b_i with $\mathrm{dom}(a_i)$ and $\tilde{b_i}$ with $\mathrm{im}(a_i) = \mathrm{dom}(a_i) \cdot a_i$.

The suspension of a system of isometries is a compact, Hausdorff space that has the homotopy type of a finite graph Γ. The graph Γ is a deformation retract of $\mathcal{S}$ obtained by contracting each connected component of the forest F to a point and each band $b_i \times I$ to a core $\{pt\} \times I$. The graph Γ is the graph associated to the system of isometries in Section 2.2. In the sequel, we always suppose that $\mathcal{S}$ is connected. Hence, $\pi_1(\mathcal{S})$ is a free group F_N.

We think of each band as being *foliated* by leaves of the form $\{pt\} \times [0, 1]$, and the foliations of the bands of $\mathcal{S}$ give rise to a foliation on $\mathcal{S}$: define a relation R on points of $\mathcal{S}$ by declaring xRy if and only if x and y lie in the same leaf of the foliation on some band. The classes of the smallest equivalence relation containing R are the *leaves* of the foliation on $\mathcal{S}$. For a point $x \in F$, we let $l(x)$ denote the leaf of the foliation on $\mathcal{S}$ containing x.

We consider the path metric on leaves coming from the metric on $[0, 1]$. A finite, infinite, or bi-infinite path γ in $\mathcal{S}$ is an *admissible leaf path* if $\gamma\colon J \to \mathcal{S}$, J a closed subinterval of $\mathbb{R}$ with extremities $\partial J \subseteq \mathbb{Z} \cup \{\pm\infty\}$, is a locally isometric, immersed leaf path, so $\gamma^{-1}(F) = J \cap \mathbb{Z}$. The *lamination* (which is rather a foliation in this setting) $L(\mathcal{S})$ is the set of bi-infinite admissible leaf paths.

Any admissible leaf path γ defines an admissible path in Γ which we also denote by γ. Any bi-infinite admissible leaf path in $L(\mathcal{S})$ defines a bi-infinite admissible path in the admissible lamination $L(S)$. Thus, the admissible lamination $L(S)$ associated to a system of isometries $S = (F, A)$ is a combinatorial version of the foliation on the suspension $\mathcal{S}$.

A *length measure* μ on a tree T is a collection $\{\mu_I\}_{I \subseteq T}$ of finite Borel measures on the compact arcs I of T, such that if $J \subseteq I$, then $\mu_J = \mu_I|_J$. For any tree T, we have the *Lebesgue measure* μ_L on T consisting of the Lebesgue measures on the compact intervals of T. The foliated space $\mathcal{S}$ inherits a *transverse measure* from the Lebesgue measures on the bases.

Let $S = (F, A)$ be a system of isometries, and let $\tilde{\mathcal{S}}$ denote the universal cover of $\mathcal{S}$. Lift the foliation and transverse measure from $\mathcal{S}$ to $\tilde{\mathcal{S}}$. Then $F_N = \pi_1(\mathcal{S})$ acts on $\tilde{\mathcal{S}}$ by deck transformations, and the foliation and transverse measure are preserved. Collapsing to a point each leaf of the foliation of $\tilde{\mathcal{S}}$ gives an $\mathbb{R}$-tree $T_{\mathcal{S}}$. As the action $F_N \curvearrowright \tilde{\mathcal{S}}$ preserves the foliation and transverse measure, one gets an isometric action $F_N \curvearrowright T_{\mathcal{S}}$; see [8] or [1]. The action $F_N \curvearrowright T_{\mathcal{S}}$ is *dual* to the system of isometries S.

2.4. Outer space and its closure. Recall that an action $F_N \curvearrowright T$ is *free* if for any $1 \neq g \in F_N$ one has $l_T(g) > 0$. If $X \subseteq T$, then the *stabilizer* of X is $\mathrm{Stab}(X) := \{g \in F_N | gX = X\}$ – the setwise stabilizer of X. An action $F_N \curvearrowright T$ is *very small* if

(i) $F_N \curvearrowright T$ is minimal,

(ii) for any non-degenerate arc $I \subseteq T$, $\mathrm{Stab}(I) = \{1\}$ or $\mathrm{Stab}(I)$ is a maximal cyclic subgroup of F_N,

(iii) stabilizers of tripods are trivial.

A minimal action $F_N \curvearrowright T$ is *discrete* (or *simplicial*) if the F_N-orbit of any point of T is a discrete subset of T; in this case T is obtained by equivariantly assigning a metric to the edges of a (genuine) simplicial tree. Note that the metric topology is weaker than the simplicial topology if the tree is not locally compact.

The *unprojectivised outer space* of rank N, denoted cv_N, is the topological space whose underlying set consists of free, minimal, discrete, isometric actions of F_N on $\mathbb{R}$-trees. A minimal F_N-tree is completely determined by its translation length function (see, for example, [6]): we can embed $\mathrm{cv}_N \subseteq \mathbb{R}^{F_N}$. The closure $\overline{\mathrm{cv}}_N$ in $\mathbb{R}^{F_N}$ consists of very small isometric actions of F_N on $\mathbb{R}$-trees [7, 2]. For more background on cv_N and its closure, see [29] and the references therein.

2.5. The map Q and the dual lamination. Here we recall dual algebraic laminations associated to F_N-trees; see [10] and [11] for a careful development of the general theory. Let ∂F_N denote the Gromov boundary of F_N – *i.e.* the Gromov boundary of any Cayley graph of F_N; let

$$\partial^2(F_N) := \partial F_N \times \partial F_N \setminus \Delta,$$

where Δ is the diagonal. The left action of F_N on a Cayley graph induces actions by homeomorphisms of F_N on ∂F_N and $\partial^2 F_N$. Let

$$i : \partial^2 F_N \longrightarrow \partial^2 F_N$$

denote the involution that exchanges the factors. A *lamination* is a non-empty, closed, F_N-invariant, i-invariant subset $L \subseteq \partial^2 F_N$.

Remark 2.2. In the setting of Sections 2.2 and 2.3, if the graph Γ is connected, then its fundamental group is a free group F_N. Specifying a marking isomorphism $\pi_1(\Gamma) \simeq F_N$ gives a homeomorphism $\partial\tilde{\Gamma} \simeq \partial F_N$, where $\tilde{\Gamma}$ is the universal cover of Γ. A bi-infinite reduced path in Γ lifts to bi-infinite reduced paths in $\tilde{\Gamma}$ that are completely described by their pairs of ends in $\partial^2 F_N$.

Proposition 2.3 ([25]). *Let $T \in \overline{\mathrm{cv}}_N$ have dense orbits, and suppose that $X \in \partial F_N$. There is a unique point $Q(X) \in \overline{T} \cup \partial T$ such that there exists a sequence u_n in F_N converging to X and a point $P \in T$ such that $u_n P$ converges to $Q(X)$.*

More intuitively, the map Q is the continuous extension of the map

$$\begin{aligned} Q_P : F_N &\longrightarrow T, \\ u &\longmapsto uP \end{aligned}$$

to $\partial F_N \to \widehat{T}^{\mathrm{obs}}$, where

$$\widehat{T} = \overline{T} \cup \partial T$$

is endowed with the weaker observers' topology – the set of directions in $\widehat{T}$ is a basis of open sets for the observers' topology. The space $\widehat{T}^{\mathrm{obs}}$ is Hausdorff and compact.

Proposition 2.4 ([25]). *Let $T \in \overline{\mathrm{cv}}_N$ have dense orbits. The map*

$$Q : \partial F_N \longrightarrow \widehat{T}$$

is F_N-equivariant and surjective; further, points in ∂T have exactly one pre-image by Q.

The crucial property for us is that Q can be used to associate to T a lamination [11].

Definition 2.5. Let $T \in \overline{\mathrm{cv}}_N$ have dense orbits. The *dual lamination* of T is

$$L(T) := \{(X,Y) \in \partial^2(F_N) \mid Q(X) = Q(Y)\}.$$

2.6. The map Q^2 and the compact heart. For a tree $T \in \overline{\mathrm{cv}}_N$ with dense orbits, the map

$$Q : \partial F_N \longrightarrow \widehat{T} = \overline{T} \cup \partial T$$

induces a map

$$\begin{aligned} Q^2 :\ \ L(T) &\longrightarrow \overline{T}, \\ (X,Y) &\longmapsto Q(X) = Q(Y). \end{aligned}$$

The metric topology on T canonically extends to $\overline{T}$, and we have the following result.

Proposition 2.6 ([13]). *The map $Q^2 : L(T) \to \overline{T}$ is continuous.*

The space $\partial^2 F_N$ is not compact, but there are many "nice" coverings of $\partial^2 F_N$ be compact sets. Fixing a basis A for F_N gives an identification of ∂F_N with the space of infinite reduced words in $A^{\pm 1}$, hence an identification of $\partial^2 F_N$ with the space of pairs (X,Y) of distinct infinite reduced words $X \neq Y$. For an infinite word X, we let X_1 stand for the first letter of X. The *unit cylinder* of $\partial^2 F_N$ with respect to A is the subset

$$C_A := \{(X,Y) \in \partial^2 F_N | X_1 \neq Y_1\}.$$

For any basis A of F_N, C_A is an open, compact subspace of $\partial^2 F_N$, and

$$\partial^2 F_N = \bigcup_{g \in F_N} g C_A.$$

For a tree $T \in \overline{\mathrm{cv}}_N$ with dense orbits and a basis A of F_N, the *compact limit set* of T (with respect to A) is $\Omega_A := Q^2(L(T) \cap C_A) \subseteq \overline{T}$. The *heart* of T (with respect to A) is the convex hull K_A of Ω_A in $\overline{T}$. It follows from Proposition 2.6 that Ω_A is a compact subset of $\overline{T}$, so K_A is a compact subtree of $\overline{T}$.

Now, fix a tree $T \in \overline{\mathrm{cv}}_N$ with dense orbits and a basis A for F_N. For a compact subtree $K \subseteq \overline{T}$, let $S = (K, A)$ denote the system of isometries with $a \in A$ the partial isometry got by the (maximal) restriction of the action of a^{-1} to K. Let $\mathcal{S}$ denote the suspension of S, and let $T_{\mathcal{S}}$ denote the F_N-tree dual to $\mathcal{S}$. The following result is essential for the present note.

Proposition 2.7 ([13]). *Let $T \in \overline{\mathrm{cv}}_N$ have dense orbits, and fix a basis A for F_N. Let K_A denote the heart of T with respect to A; let*

$$S = (K_A, A)$$

denote the associated system of isometries, and let $\mathcal{S}$ denote its suspension.

(i) *The tree T is dual to S: $T_{\mathcal{S}}^{\min} = T$,*

(ii) $L(T) \cap C_A = L(S) = L(\mathcal{S})$, *and*

(iii) *for any infinite admissible leaf path X in $\mathcal{S}$,* $\mathrm{dom}(X) = \{Q(X)\}$.

The statement $L(S) = L(\mathcal{S})$ means that the obvious map is a bijection. Proposition 2.7 allows us to fully transfer the problem of understanding the dual lamination of a tree in the boundary of Outer space to the problem of understanding the associated systems of isometries.

If a basis A of F_N is fixed, then given any $T \in \overline{\mathrm{cv}}_N$, we have a well-defined system of isometries $S = (K_A, A)$. If A is understood, then we say that S is *associated to T*.

A tree $T \in \overline{\mathrm{cv}}_N$ is *geometric* if its compact heart K_A for some (hence any) basis A of F_N is a finite tree, that is to say a compact $\mathbb{R}$-tree which is the convex hull of finitely many points, see [13].

2.7. Regular leaves and the derived sublamination. In this Section, we collect some results regarding the derived space of $L(T)$ and its relationship to regular leaves in systems of isometries associated to T.

Definition 2.8. Let T be an F_N-tree in $\overline{\mathrm{cv}}_N$ with dense orbits. A leaf $l \in L(T)$ is *regular* if there exists a sequence $l_n \in L(T)$ of leaves converging to l and such that the $x_n = Q^2(l_n)$ are distinct. The set of regular leaves is the *regular sublamination* $L_r(T)$.

If the action of F_N on T is free, by [8, Theorem 5.3] we have that for all $x \in \overline{T}$, $(Q^2)^{-1}(x)$ is a finite set of uniformly bounded cardinality. Recall that the set of non-isolated points of a topological space, X, is its *derived space*, X'.

Lemma 2.9. *Let $T \in \overline{\mathrm{cv}}_N$ be free with dense orbits. The regular sublamination of T is equal to the derived lamination*:

$$L_r(T) = L(T)'.$$

2.8. Mixing properties for F_N-trees. From the work of V. Guirardel [17] (after J. Morgan [26]), there are several notions of "minimality" for the dynamics of an action of a group on an $\mathbb{R}$-tree. These notions are hierarchized as follows:

(1) *dense orbits*: the F_N-orbit of some (hence any) point P of T is dense in T;

(2) *arc-dense*: every orbit meets every non-degenerate segment of T;

(3) *arc-dense directions*: for each $x \in T$, each direction d at x, and each non-degenerate arc $I \subseteq T$, there exists $g \in F_N$ such that $gx \in I$ and $gd \cap I$ is non-degenerate;

(4) *mixing*: for any non-degenerate segments I and J in T, there exists finitely many elements $u_1, \dots, u_n$ in F_N such that $I \subseteq u_1 J \cup u_2 J \cup \dots \cup u_n J$;

(5) *indecomposable*: for any non-degenerate segments I and J in T, there exists finitely many elements $u_1, \dots, u_n$ in F_N such that

 (a) $I \subseteq u_1 J \cup u_2 J \cup \dots \cup u_n J$

 (b) $u_i J \cap u_{i+1} J$ is a non degenerate segment for any $i = 1, \dots, n-1$.

We remark that this hierarchy is not exactly strict as

Lemma 2.10 ([27, Lemma 12.6]). *Let $T \in \overline{\mathrm{cv}}_N$. The action $F_N \curvearrowright T$ is mixing if and only if it has arc-dense directions.*

In this paper we will use two characterizations of indecomposable trees. A *transverse family* for an action $F_N \curvearrowright T$ of F_N on an $\mathbb{R}$-tree T is an F_N-invariant family $\{T_v\}_{v \in V}$ of non-degenerate, proper subtrees of T such that if $T_v \neq T_{v'}$, then $T_v \cap T_{v'}$ contains at most one point.

Proposition 2.11 ([28, Lemma 4.1]). *Let $F_N \curvearrowright T$ be an action of F_N on an $\mathbb{R}$-tree T. Then $F_N \curvearrowright T$ is indecomposable if and only if there is no transverse family.*

In the proof of Lemma 5.1, we need a refined understanding of transverse families that occur in free F_N-trees. We collect the following:

Proposition 2.12 ([27, Lemma 4.4] and [24, Theorem 5]). *Let $T \in \overline{\mathrm{cv}}_N$ be free with dense orbits. If T is not indecomposable there exists a non-degenerate subtree T_0 of T, such that*

(1) *$\{gT_0 \mid g \in F_N\}$ is a transverse family,*

(2) *$H = \mathrm{Stab}(T_0)$ is a free factor of F_N,*

(3) *H acts on T_0 with dense orbits.*

A finitely generated subgroup H of F_N is quasiconvex, thus there is a natural inclusion of boundaries $\partial H \subseteq \partial F_N$. We say that a line $(X, Y) \in \partial^2 F_N$ is *carried* by H if $(X, Y) \in \partial^2 H$.

Proposition 2.13 ([28, Corollary 4.8]). *Let $T \in \overline{\mathrm{cv}}_N$ be free and indecomposable, and let $H \leq F_N$ be finitely generated. Then H carries a leaf of $L(T)$ if and only if H has finite index in F_N.*

3. The Rips Machine and types of actions

3.1. The Rips Machine. We recall the generalization of Process I of the Rips Machine [16, 1], henceforth called *Rips induction*, that was first studied in the present context in [8].

Let $S = (F, A)$ be a system of isometries. The output of one step of the Rips Machine applied to S is a new system of isometries $S' = (F', A')$ defined as follows:

$$F' := \{x \in F \mid \exists a \neq a' \in A^{\pm 1}, x \in \mathrm{dom}(a) \cap \mathrm{dom}(a')\}$$

Since A is finite and since intersections of domains of isometries are compact $\mathbb{R}$-trees, we have that F' is again a finite forest. We let A' consist of all maximal restrictions of the elements of A to pairs of connected components of F', so $S' = (F', A')$ is indeed a system of isometries, as required.

The suspension $\mathcal{S}'$ of S' is a subspace of the suspension $\mathcal{S}$ of S. We can regard each leaf-path in $\mathcal{S}'$ as a leaf-path in $\mathcal{S}$, in particular, considering bi-infinite admissible leaf paths gives $L(\mathcal{S}') \subseteq L(\mathcal{S})$. On the other hand, the Rips Machine does not modify bi-infinite admissible leaf paths, thus:

Lemma 3.1. *Let $S = (F, A)$ be a system of isometries, and let $S' = (F', A')$ denote the output of the Rips Machine applied to S. The laminations are equal: $L(\mathcal{S}) = L(\mathcal{S}')$.*

3.2. Types of actions. We consider the output of iterating the Rips Machine on a system of isometries $S_0 = (F_0, A_0)$; we denote by S_i the output of the i^{th} iteration of the Rips Machine. If for some i_0, one has that $F_{i_0} = F_{i_0+1}$, *i.e.* the Rips Machine *halts* on S_{i_0}, then we say that the Rips Machine *eventually halts* on S_0.

Definition 3.2. Let S_0 be a system of isometries. If the Rips Machine eventually halts on S_0, then S_0 is called *surface type*.

The *limit set* of S_0 is $\Omega = \bigcap_{i \in \mathbb{N}} F_i$. If S_0 is of surface type then $\Omega = F_{i_0}$. If $S = (K_A, A)$ is a system of isometries associated to a tree $T \in \overline{\mathrm{cv}}_N$ with dense orbits, then, by definition, the limit set Ω of the system of isometries is equal to the compact limit set Ω_A with respect to the basis A defined in Section 2.6.

Definition 3.3. Let S_0 be a system of isometries, and suppose that the Rips Machine does not eventually halt on S_0. If the limit set Ω associated to S_0 is totally disconnected, then S_0 is said to be *Levitt type*.

In [8] it is shown that for $T \in \overline{\mathrm{cv}}_N$ with dense orbits, if for some basis A, the system of isometries associated to K_A is of surface type (resp. Levitt type), then for every basis A', the system of isometries associated to $K_{A'}$ is of surface type (resp. Levitt type). In this case we say that T is of *surface type* (resp. *Levitt type*). It should be noted that there are trees in $\overline{\mathrm{cv}}_N$ that are neither of surface type nor Levitt type; however, we have the following:

Proposition 3.4 ([8, Proposition 5.14]). *Let $T \in \overline{\mathrm{cv}}_N$ have dense orbits. If the action $F_N \curvearrowright T$ is mixing, then T is either of surface type or Levitt type.*

3.3. Levitt type actions.

Let $T \in \overline{\mathrm{cv}}_N$ has dense orbits, and let

$$S_0 = (K_A, A) = (F_0, A_0)$$

be an associated system of isometries. Denote by S_i the output of the i^{th} iteration of the Rips Machine.

Recall the definition of the graph Γ associated to a system of isometries S from Section 2.2. Let Γ_i denote the graph associated to S_i: Γ_i is got by contracting each band of $\mathcal{S}_i$ onto one of its leaves. There are induced graph morphisms

$$\tau_i : \Gamma_i \longrightarrow \Gamma_{i-1}.$$

The following Lemma follows from [8, Propositions 3.12, 3.13, and 5.6].

Lemma 3.5. *Let $T \in \overline{\mathrm{cv}}_N$ have dense orbits; let A a basis for F_N; and let S_0 denote the associated system of isometries. Denote by S_i the output of the i^{th} iteration of the Rips Machine applied to S_0, and let Γ_i be the associated graph.*

(i) *Γ_i has no vertices of valence 0 or 1, and*

(ii) *the maps $\tau_i : \Gamma_i \to \Gamma_{i-1}$ are homotopy equivalences.*

Note that, as $F_0 = K_A$ is connected, Γ_0 is a rose with N petals, so Lemma 3.5 gives a uniform bound $2N-2$ on the number of vertices of valence strictly greater than two in Γ_i.

Lemma 3.6. *Let $T \in \overline{\text{cv}}_N$ be free with dense orbits, and suppose that T is of Levitt type. If $L_0 \subsetneq L_r(T)$ is a proper sublamination, then every leaf of L_0 is carried by a proper free factor of F_N.*

The hypothesis that the action is free is not necessary; see the Appendix.

Proof. Fix a basis A for F_N, and let $S_0 = (K_A, A)$ be the associated system of isometries; let $S_i = (F_i, A_i)$ denote the output of the i^{th} iteration of the Rips Machine, and let Γ_i denote the graph associated to S_i. Recall that since T is of Levitt type, the limit set Ω is totally disconnected, hence the number of vertices of Γ_i goes to infinity with i. Let l be a bi-infinite admissible leaf path in $L_r(\mathcal{S}_0) \setminus L_0$. There exists a sequence l_n in $L(\mathcal{S}_0)$ converging to l such that the $x_n = Q^2(l_n)$ are distinct and distinct from $x = Q^2(l)$. Additionally, we can assume that $l_n|_{[-n,n]} = l|_{[-n,n]}$ (viewed as admissible paths in Γ_0). As Ω is totally disconnected, for any m, there is $i(m)$ such that x_n lie in separate components of F_i for $n \leq m$ and $i \geq i(m)$. Also, since $l \notin L_0$, there is M such that

$$\{l' \in L_0 \mid l'|_{[-M,M]} = l|_{[-M,M]}\} = \emptyset.$$

We now apply Lemma 3.1 to view the leaves l_n as bi-infinite admissible leaf-paths in the suspensions $\mathcal{S}_i$.

For a given m the leaves $l_n, n \leq m$, define distinct admissible paths $l_m|_{[-M;M]}$ in $\Gamma_{i(m)}$ each of length $2M$. As the action of F_N on T is free there exists $j(M)$ such that for $i \geq j(M)$ the size of any reduced loop in Γ_i is strictly bigger than $2M$. Recall that there are at most $2N-2$ vertices of valence strictly bigger than two in Γ_i. Thus for m large enough, for any i bigger than $i(m)$ and $j(M)$, there exists $n \leq m$ such that the admissible reduced path $l_n|_{[-M;M]}$ in Γ_i does not cross any vertex of valence strictly greater than two.

Note that no leaf of L_0 could cross any edge in the image of $l_n|_{[-M,M]}$ in Γ_i (else it crosses every such edge); thus every leaf of L_0 is contained the subgraph $G_0 := (\Gamma_i \setminus Im(l_n|_{[-M,M]}))$. By Lemma 3.5, we have that G_0 corresponds to a proper free factor $H \leq F_N$ and every leaf of L_0 is carried by H. □

We now have our first main result.

Proposition 3.7. *Let $T \in \overline{\text{cv}}_N$, and assume that T is free and indecomposable and of Levitt type. The regular sublamination $L_r(T) \subseteq L(T)$ is minimal.*

Proof. According to Lemma 3.6, if there happened to be a proper sublamination $L_0 \subsetneq L_r(T)$, then every leaf of L_0 would be carried by a proper free factor H of F_N. This is impossible by Proposition 2.13. □

4. Splitting

In this section we define an inductive procedure that allows us to study the dual lamination of a free, surface type tree. To define this procedure, which is a generalization of the classical Rauzy–Veech induction, we need to find "good" singularities in systems of isometries associated to trees of surface type. Toward that end we recall some results regarding indices of trees.

4.1. Q-index and geometric index. Let $T \in \overline{\mathrm{cv}}_N$ have dense orbits, and let

$$Q : \partial F_N \longrightarrow \widehat{T} = \overline{T} \cup \partial T$$

be the map defined in Section 2.5. For $x \in \overline{T}$, let $\mathrm{Stab}(x) \leq F_N$ denote the stabilizer of x. It was shown in [15] that there are finitely many orbits of points in T with non-trivial stabilizer and that $\mathrm{Stab}(x)$ is finitely generated. Note that for $x \in \widehat{T} \setminus T$, $\mathrm{Stab}(x)$ is always trivial. From the definition of Q, one sees that $\partial\mathrm{Stab}(x) \subseteq Q^{-1}(x)$; put

$$Q_r^{-1}(x) := Q^{-1}(x) \setminus \partial\mathrm{Stab}(x).$$

Evidently, $\mathrm{Stab}(x)$ acts on $Q^{-1}(x)$, leaving invariant $Q_r^{-1}(x)$. For $x \in \widehat{T}$, the *Q-index* of x is

$$\mathrm{ind}_Q(x) := |Q_r^{-1}(x)/\mathrm{Stab}(x)| + 2\mathrm{Rank}(\mathrm{Stab}(x)) - 2$$

The Q-index is constant on F_N-orbits in T, and the *Q-index of T* is

$$\mathrm{ind}_Q(T) := \sum_{[x]\in\widehat{T}/F_N} \max\{0, \mathrm{ind}_Q(x)\}$$

As Q is injective on $Q^{-1}(\partial T)$, only points of $\overline{T}$ contribute to the Q-index of T. The following is established in [8]:

Theorem 4.1 ([8, Theorem 5.3]). *Let $T \in \overline{\mathrm{cv}}_N$ have dense orbits. Then*

$$\mathrm{ind}_Q(T) \leq 2N - 2.$$

Moreover, T is surface type if and only if $\mathrm{ind}_Q(T) = 2N - 2$.

Let $T \in \overline{\mathrm{cv}}_N$ have dense orbits, and let $x \in T$. Then $\mathrm{Stab}(x)$ acts on $\pi_0(T \backslash \{x\})$. Following [15], one defines the *geometric index* of x to be

$$\mathrm{ind}_{\mathrm{geom}}(x) := |\pi_0(T \setminus \{x\})/\mathrm{Stab}(x)| + 2\mathrm{Rank}(\mathrm{Stab}(x)) - 2$$

The geometric index is constant on F_N-orbits in T, and one defines the *geometric index* of T to be

$$\mathrm{ind}_{\mathrm{geom}}(T) := \sum_{[x] \in T/F_N} \mathrm{ind}_{\mathrm{geom}}(x)$$

We have the following:

Theorem 4.2 ([15]). *Let $T \in \overline{\mathrm{cv}}_N$. Then*

$$\mathrm{ind}_{\mathrm{geom}}(T) \leq 2N - 2.$$

Moreover, T is geometric if and only if $\mathrm{ind}_{\mathrm{geom}}(T) = 2N - 2$.

4.2. Finding splitting points

Convention 4.3. *If $T \in \overline{\mathrm{cv}}_N$ is of surface type and if B is a basis for F_N, we let $S = (K_B, B)$ denote the associated system of isometries. In the sequel, we assume that that any system of isometries $S = (F, A)$ associated to a surface type action is obtained from some $S = (K_B, B)$ by running the Rips Machine until it halts.*

A point x is *extremal* in a tree K if it is not contained in the interior of an arc contained in K. In this section we are interested in points which are extremal in some bases of a system of isometries but which are non-extremal in the underlying forest.

Let $S = (F, A)$ be a system of isometries. A partial isometry $a \in A^{\pm 1}$ is *defined in direction d* at a point $x \in F$ if d is a direction in $\mathrm{dom}(a)$: $x \in \mathrm{dom}(a)$ and $d \cap \mathrm{dom}(a) \neq \emptyset$.

Proposition 4.4. [8, Proposition 4.3] *Let $T \in \overline{\mathrm{cv}}_N$ be a tree with dense orbits of surface type and $S = (F, A)$ be a system of isometries associated to T. For each direction d at a point x in F there are exactly two partial isometries $a, b \in A^{\pm 1}$ defined in d.*

In the surface type case, we can locally compare the geometric and Q-indices.

Lemma 4.5. *Let $T \in \overline{\mathrm{cv}}_N$ be free with dense orbits of surface type, and let $S = (F, A)$ be a system of isometries associated to T. Let $x \in F$ lie in the intersection of at least three distinct bases. If every point of the S-orbit of x is non-extremal in all bases that contain it, then*

$$\mathrm{ind}_Q(x) \leq \mathrm{ind}_{\mathrm{geom}}(x).$$

The free hypothesis is unnecessary; see the Appendix.

Proof. Let Γ_x be the (infinite) graph with vertices $V(\Gamma_x)$ the S-orbit of x and with an edge labeled by $a \in A^{\pm 1}$ between each pair of vertices $x.u$ and $x.ua$ with $u \in F_N$ a partial isometry defined at x. As T is free, the graph Γ_x is a tree and from Proposition 2.7, its space of ends can be identified with $Q^{-1}(x) \subseteq \partial F_N$. From Theorem 4.1, Γ_x has finitely many ends.

Let Γ_x^d be the (infinite) graph with vertices the directions in F at points in the S-orbit of x and with an edge labeled by $a \in A^{\pm 1}$ between each pair of vertices d and $d.a$ (in particular a is defined in d). The number of connected components in Γ_x^d is $\mathrm{ind}_{\mathrm{geom}}(x) + 2$. From Theorem 4.2, Γ_x^d has finitely many connected components.

By Proposition 4.4, Γ_x^d is a disjoint union of bi-infinite lines. By our hypothesis on x, for each edge labeled by a from $x.u$ to $x.ua$ in Γ_x there are at least two edges labeled by a in Γ_x^d from d_1 to $d_1.a$ and from d_2 to $d_2.a$ where $d_1, d_2 \in V(\Gamma_x^d)$ are directions at $x.u$.

By the previous paragraph, each end of Γ_x is reached by at least two bi-infinite lines in Γ_x^d and a bi-infinite line has two ends, thus the number of lines in Γ_x^d is bounded below by the number of ends in Γ_x:

$$\mathrm{ind}_Q(x) \leq \mathrm{ind}_{\mathrm{geom}}(x). \qquad \square$$

Definition 4.6. A *splitting point* in a system of isometries $S = (F, A)$ is a point x in the connected component K_x in F such that

(S1) x is not extremal in K_x;

(S2) there exists a partial isometry $a_0 \in A^{\pm 1}$ defined at x, such that x is extremal in the base $\mathrm{dom}(a_0)$, and such that $\mathrm{dom}(a_0) \neq \{x\}$. We denote by d_x the unique direction at x which meets $\mathrm{dom}(a_0)$. We call d_x the *splitting direction*

(S3) There exists exactly one other partial isometry $a_1 \in A^{\pm 1} \setminus \{a_0\}$ defined at x and such that $\mathrm{dom}(a_1)$ meets d_x.

(S4) The point x is not extremal in $\mathrm{dom}(a_1)$.

Condition (S3) is guaranteed by our Convention and Proposition 4.4, while Condition (S4) follows from Conditions (S1) and (S3). Lemma 4.5 ensures that that splitting points exists.

Proposition 4.7. *Let $T \in \overline{\mathrm{cv}}_N$ be free, indecomposable, and of surface type. Let $S = (F, A)$ be a system of isometries dual to T. There exists a splitting point x in S.*

Proof. First note that the domain of every element of $A^{\pm}$ necessarily is non-degenerate, as otherwise, the images in the dual tree of lifts of the complement of the corresponding degenerate band would be a transverse family whose elements are stabilized by proper factors of F_N. The main point is to find a point satisfying conditions (S1) and (S2).

If there were no points satisfying conditions (S1) and (S2) then we can use Lemma 4.5 to get

$$\mathrm{ind}_{\mathrm{geom}}(T) = \sum_{[x]\in T/F_N} \mathrm{ind}_{\mathrm{geom}}(x) \geq \sum_{[x]\in T/F_N} \mathrm{ind}_Q(x) = \mathrm{ind}_Q(T).$$

As T is of surface type, from Theorem 4.1, $\mathrm{ind}_Q(T) = 2N - 2$ and by Theorem 4.2 we get that T is geometric. By definition of geometric trees the forest F has finitely many extremal points and as conditions (S1) and (S2) fail, partial isometries send extremal points to extremal points, and the action is not free, a contradiction.

Thus, there exists a point x in F and a partial isometry $a_0 \in A^{\pm 1}$ satisfying conditions (S1) and (S2). As T is of surface type, according to Proposition 4.4 condition (S3) is satisfied. If condition (S4) does not hold then x locally separates the suspension $\mathcal{S}$ of S. In this case there is a proper free factor F' of F_N that carries every leaf of $L_r(\mathcal{S})$, contradicting Proposition 2.13. ☐

The argument at the end of the second paragraph of the proof uses freeness; here is how to argue for general actions with dense orbits. If T has dense orbits, is geometric, and is of surface type, then $\mathcal{S}$ contains a minimal component, and all minimal components can be transformed by the Rips Machine into surfaces carrying arational measured foliations. In this case, the existence of a splitting point is a straightforward consequence of the surface theory; see [1, 5]. Since we will use it later, we record the following observation made in the above proof.

Corollary 4.8. *Let $S = (F, A)$ be a surface type system of isometries. If Conditions (S1)–(S4) do not hold for any x, then every leaf of $L_r(\mathcal{S})$ is carried by a proper free factor of F_N.*

4.3. Splitting. Let $S = (F, A)$ be a system of isometries, and let Γ be its associated graph. Assume that x is a splitting point for S. We assume no element of A has degenerate domain (this is satisfied in all cases of interest), and we use the notation of Definition 4.6.

We split the connected component K_x of x into two new compact $\mathbb{R}$-trees: $K' = d_x \cup \{x\}$ and $K'' = K_x \setminus d_x$. We denote by F' the finite forest obtained by replacing K_x by two disjoint compact $\mathbb{R}$-trees K' and K'' (in particular there are two copies of x in F'). Define

$$A_0 = \{a \in A \mid x \notin \operatorname{dom}(a) \cup \operatorname{dom}(a^{-1})\}.$$

Let $a_0, a_1 \in A^{\pm}$ be as in the Definition 4.6, so $a_0 \neq a_1$, and a_1 is the unique element of $A^{\pm} \setminus \{a_0\}$ whose domain meets d_x and contains x. Note that there may be other elements of $A^{\pm}$ whose domains contain x; any such element, as well as the element a_0, is redefined in the obvious way to have its domain contained in K''. The domain of any other element of $A \setminus \{a_1\}$ is naturally identified with a subset of either K' or K'', and these elements are defined in the obvious way on F'. Finally, the domain of a_1 is split apart at the point x, and a_1 gives rise to two partial isometries, a_1' and a_1'', defined the obvious ways on domains contained in K' and K'', respectively. Use $S' = (F', A')$ for the new system of isometries; we say that S' is obtained from S by *splitting* at x in the direction d_x.

The suspension $\mathcal{S}'$ of S' can be "zipped-up" to recover the suspension $\mathcal{S}$ of S: the map $z\colon \mathcal{S}' \to \mathcal{S}$ which identifies the leaves $\{(x,t) \mid t \in [0;1]\}$ in the band $K' \times [0;1]$ with $\{(x,t) \mid t \in [0;1]\}$ in the band $K'' \times [0,1]$ is a homotopy equivalence.

Lemma 4.9. *A regular bi-infinite admissible leaf l in $\mathcal{S}$ can be lifted by z to a regular bi-infinite admissible leaf in $\mathcal{S}'$.*

Proof. There exists a sequence l_n of bi-infinite admissible leaves in $\mathcal{S}$ converging to l such that the $x_n = Q^2(l_n)$ are distinct and distinct from x. We can assume that for each n the finite admissible leaf path $l_n|_{[-n,n]}$ does not cross x. From the definition of splitting, $l_n|_{[-n,n]}$ can be lifted to a finite admissible leaf path γ_n in $\mathcal{S}'$. The paths γ_n converges to a bi-infinite admissible leaf path l' in $\mathcal{S}'$ which is a lift of l. □

We can now iteratively split our system of isometries dual to a tree of surface type to get the analogue of Lemma 3.5. Though there might be more than one splitting point, there are at most finitely many – each splitting point is the intersection of domains of three distinct partial isometries, and such an intersection contains at most one point. In order to have a canonical procedure, we should include algorithm for selecting which splits to perform; the natural choice would be to split at splitting points and splitting directions "simultaneously." This is not quite well-defined, but it is well-defined in all cases we consider here. Indeed, it is evident from the definition of splitting that the only way that two splits applied to S do not commute is if performing one split renders the other split undefined (*c.f* a saddle connection on a surface).

More precisely, given splitting points and splitting directions x, x' and $d_x, d_{x'}$ for S with associated partial isometries a_0, a_0', a_1, a_1' as in Definition 4.6, the only way that the two splits do not commute is if $xa_1 = x'$ and $d_x a_1 \cap d_{x'}$ is non-degenerate. In this case after applying the x, d_x-split to S to obtain $S' = (F', A')$, we have that x' locally separates a component of F', hence every regular leaf $\mathcal{S}$ is carried by a proper free factor of F_N. It follows that if some leaf of the regular lamination of S is not carried by any proper free factor of F_N, then all splits defined on S commute, and there is a well-defined system of isometries S' got by splitting at all splitting points in all splitting directions. Our convention in the sequel is that all possible splits are performed.

Lemma 4.10. *Let $T \in \overline{\mathrm{cv}}_N$ be free with dense orbits, and suppose that T is of surface type. Let S_0 denote a system of isometries associated to T. Denote by S_i the output of splitting S_{i-1}, and let Γ_i be the graph associated to S_i.*

(i) *Γ_i has no vertices of valence 0 or 1, and*

(ii) *the maps $\tau_i : \Gamma_i \to \Gamma_{i-1}$ are homotopy equivalences.*

4.4. Surface type actions. We now establish analogues of Lemma 3.6 and Proposition 3.7 for actions of surface type.

Lemma 4.11. *Let $T \in \overline{\mathrm{cv}}_N$ be free and mixing, and suppose that T is of surface type. If $L_0 \subsetneq L_r(T)$ is a proper sublamination, then every leaf of L_0 is carried by a proper free factor of F_N.*

We assume that the action is mixing only for simplicity. Again, the hypothesis that the action is free is not necessary; see the Appendix.

Proof. Let $S = (K, A)$ be a system of isometries associated to T. By definition of surface type after finitely many steps the Rips Machine starting on S halts on a surface type system of isometries S_0. According to Proposition 4.7, Lemma 4.8, and Corollary 4.10, either we can then perform splittings on S_0, or else every leaf of $L_r(T)$ is carried by a proper free factor of F_N. Let $S_i = (F_i, A_i)$ denote the result of the i^{th} iteration of splitting applied to S_0.

By Lemma 2.10, directions are arc dense in T, and so by statement (i) of Proposition 2.7, directions are arc dense in F_i under the action of the pseudogroup S_i. In particular if d_x is the first splitting direction, for any non-degenerate arc $[y, y'] \subseteq F$, there is a finite admissible path γ in the graph Γ_0 associated to S_0 such that $x.\gamma_0 \in [y, y']$ and $d_x.\gamma_0$ meets $[y, y']$. For all $i \geq i(y, y') = |\gamma|$, the images of y and y' in F_i lie in different components of F_i.

Using Lemma 4.9 and Proposition 4.10, we may now conclude exactly as in the proof of Lemma 3.6. □

We have:

Proposition 4.12. *Let $T \in \overline{\text{cv}}_N$, and assume that T is free, indecomposable and of surface type. The regular sublamination $L_r(T) \subseteq L(T)$ is minimal.*

Proof. According to Lemma 4.11, if there happened to be a proper sublamination $L_0 \subsetneq L_r(T)$, then every leaf of L_0 would be carried by a proper free factor $H \leq F_N$. This is impossible by Proposition 2.13. □

5. Diagonal leaves

Recall from the introduction that a lamination L is minimal up to diagonal leaves if L contains a unique minimal sublamination L_0, such that $L \setminus L_0$ consists of finitely many F_N-orbits of leaves that are diagonal over L_0.

In this section we use both the Rips Machine and the splitting induction to reach our main result.

5.1. Decomposable trees

Lemma 5.1. *Let $T \in \overline{\text{cv}}_N$ be free with dense orbits. If $L(T)$ is minimal up to diagonal leaves, then T is indecomposable.*

Proof. We argue the contrapositive. Let $T \in \overline{\text{cv}}_N$ have dense orbits, and suppose that T is not indecomposable. Following Proposition 2.12, there exists a non-degenerate transverse family $\{gT_0 \mid g \in F_N\}$, where T_0 is a closed non-degenerate subtree of T. The stabilizer $H = \text{Stab}(T_0)$ is a proper free factor of F_N and acts on T_0 with dense orbits. The dual lamination $L_H(T_0) \subseteq \partial^2 H$ of T_0 is non-empty, and $L_H(T_0)$ is diagonally closed. Recall that H is quasi-convex in F_N and thus there is an embedding $\partial^2 H \subseteq \partial^2 F_N$. Moreover, as H is a free factor we have

$$g\partial H \cap \partial H \neq \emptyset \text{ for all } g \in F_N \iff g \in H.$$

The lamination generated by $L(T_0)$, $L_0 = F_N.L(T_0)$, is a sublamination of $L(T)$ closed by diagonal leaves.

Assume first that there exists $g \in F_N \setminus H$ such that $g\overline{T}_0 \cap \overline{T}_0 \neq \emptyset$, and let $x, y \in \overline{T}_0$ such that $x = gy$. By Proposition 2.4 the map $Q\colon \partial H \to \widehat{T}_0$ is onto, thus there exist, $X, Y \in \partial H$ such that $Q(X) = x$ and $Q(Y) = y$. By definition of the dual lamination, (X, gY) is a leaf of $L(T)$; by construction (X, gY) is not diagonal over L_0.

We now assume, that, for all $g \in F_N \setminus H$, $g\overline{T}_0 \cap T_0 = \emptyset$. The action of F_N on T has dense orbits, thus there exists a sequence $g_n \in F_N \setminus H$ such that $d(T_0, g_n T_0) < \frac{1}{n}$. Fixing a basis $B = \{a_1, \dots a_r\}$ of H which is completed to a basis $A = \{a_1, \dots, a_N\}$ of F_N, we can write $g_n = h_n \cdot g'_n$ in reduced form with $h_n \in H$ and g'_n starting with a letter in $A \setminus B$. Of course $d(T_0, g'_n T_0) < \frac{1}{n}$. The action of H on T_0 has dense orbits thus for any point $y \in T_0$ there exist $h'_n \in H$ such that $d(T_0, g'_n h'_n y) < \frac{1}{n}$. By our assumption the sequence $|g'_n h'_n|$ goes to infinity and there is a subsequence converging to $Y \in \partial F_N$. The first letter of Y written as an infinite reduced word is in $A \setminus B$ thus $Y \notin \partial H$. Now we use the weaker observers' topology so that $\widehat{T}^{\text{obs}}$ is compact and, we extract again a subsequence to have $g'_n h'_n y$ converging to a point $x \in \widehat{T}_0^{\text{obs}}$. We get that $Q(Y) = x$, but as $x \in \hat{T}_0$ there exists $X \in \partial H$ such that $Q(X) = x$. By definition of the dual lamination, (X, Y) is a leaf in $L(T)$; by construction (X, Y) is not diagonal over L_0. □

Considering Propositions 3.7 and 4.12 and Lemma 5.1, to establish Theorem A, we need understand diagonal leaves in $L(T)$ for T free and indecomposable.

5.2. Train tracks and the main result. Let T be a free, indecomposable tree in $\overline{\text{cv}}_N$. Let A be a basis for F_N, and let $S = (K_A, A)$ be the associated system of isometries. By Proposition 3.4, T is either surface or Levitt type. If T is surface, we run the Rips Machine on S until it halts. In either case, we get a system of isometries $S_0 = (F_0, A_0)$ (which is equal to S if T is Levitt type), and we denote by $S_i = (F_i, A_i)$ the result of running either the Rips Machine or splitting on S_0 for i steps. There are homotopy equivalences $S_i \to \Gamma_i$ and $\tau_i\colon \Gamma_i \to \Gamma_{i-1}$.

A *turn* in Γ_i is a pair $\{e, e'\}$ of directed edges with the same initial vertex. We give the graph Γ_i a *train track structure* by declaring a turn *legal* if it is crossed by a regular leaf, *i.e.* a regular leaf contains the subpath $\bar{e}e'$. Train track structures on graphs were introduced in [4].

Remark 5.2. From Propositions 3.5 and 4.9, our inductive procedure (either the Rips Machine or splitting) applied to $\mathcal{S}_i$ has the effect of "splitting" an illegal turn in Γ_i: in other words, the graph morphisms τ_i only fold at illegal turns.

For a vertex v of Γ_i, let $\mathrm{Leg}(v)$ denote the set of legal turns in Γ_i at v, and let $I(v)$ denote the set of edges of Γ_i with initial vertex v. Following [4] again, we define the *Whitehead graph*, $\mathrm{Wh}(v, \Gamma_i)$, associated to the vertex v of Γ_i. The vertex set of $\mathrm{Wh}(v, \Gamma_i)$ is $I(v)$ and there is an edge from e to e' if the turn $\{e, e'\}$ is legal.

Lemma 5.3. *For every $v \in V(\Gamma_i)$, the Whitehead graph $\mathrm{Wh}(v, \Gamma_i)$ is connected.*

Proof. Toward contradiction suppose that there is i and $v \in V(\Gamma_i)$ such that $\mathrm{Wh}(v, \Gamma_i)$ is not connected. Following the proof of [4, Proposition 4.5] this proves that every regular leaf of $L(T)$ is carried by a proper free factor of F_N, contradicting Proposition 2.13. □

Lemma 5.4. *For any $x \in \Omega_A$, there exists a regular leaf l such that $Q^2(l) = x$. In particular, if $l \in L(T)$ is such that $(Q^2)^{-1}(Q^2(l)) = \{l\}$ then l is regular.*

Proof. Let $x \in \Omega_A$. For each i let v_i be the connected component of F_i containing x. By Lemma 5.3, there is a bi-infinite regular admissible leaf-path l_i passing through v_i. Up to passing to a subsequence, l_i converge to a bi-infinite regular admissible leaf path l. By the continuity of Q^2 and arguing as in the proof of Lemma 4.11 and 3.6 we get that $Q^2(l) = x$. □

Proposition 5.5. *Let $T \in \overline{\mathrm{cv}}_N$ be free and indecomposable. Every leaf in $L(T) \setminus L_r(T)$ is diagonal over $L_r(T)$, and there are finitely many F_N-orbits of such leaves.*

Proof. Let l be a leaf in $L(T)$, and let $Q^2(l) = x$. If $(Q^2)^{-1}(x) = \{l\}$, then by Lemma 5.4, l is regular. Assume now that $|(Q^2)^{-1}(x)| > 1$. From [8], there are finitely many orbits of such points x in $\overline{T}$ and as the action is free, $(Q^2)^{-1}(x)$ is finite and there are finitely many orbits of such leaves l. By Lemma 5.4, there are regular leaves in $(Q^2)^{-1}(x)$, and we now proceed to prove that l is in the diagonal closure of the regular leaves in $(Q^2)^{-1}(x)$.

Let A be a basis of F_N and let $S = (K_A, A) = (F_0, A_0)$ be the system of isometries associated to T, and let $S_i = (F_i, A_i)$ denote the output of i iterations of the appropriate inductive procedure (either the Rips Machine or splitting, depending on the type of T). Let Γ_i denote the graph associated to S_i.

Let Γ_x be the (infinite graph) with vertex set the orbit of x under the pseudogroup S (equivalently it is the intersection of the orbit of x in $\overline{T}$ with $F_0 = K_A$), and such that there is an edge labeled by $a \in A^{\pm 1}$ between $x.u$ and $x.ua$, where $u \in F_N$ is a partial isometry defined at x.

Note that $(Q^2)^{-1}(x) = \partial^2\Gamma_x$, which is finite, and so $l = (X, uY)$, where $(X, X'), (Y, Y') \in Q^2(l)$ are regular leaves. The turn between X and u, the turn between Y and u, and the turns taken by u may be illegal; to finish we need to see that l can be finitely subdivided into legal paths. Run the induction for i steps to get that the legal structure on Γ_i has the maximum number of illegal turns. It follows that any turn crossed by l that is not crossed by a regular leaf is illegal in Γ_i and that any legal path in Γ_i is a subpath of a regular leaf.

A turn $\{e, e'\}$ at the vertex $y = x.u$ in Γ_x is legal if there exists a regular bi-infinite leaf-path $l' \in (Q^2)^{-1}(y)$ such that $l'([0,1]) = e$ and $l'([0,-1]) = e'$. As we may have performed splittings, the point y may lie in more than one connected component of F_i. The Whitehead graph $\mathrm{Wh}(y, \Gamma_x)$ at y is the superposition of the Whitehead graphs $\mathrm{Wh}(v, \Gamma_i)$ for all components v of F_i which contain a copy of y. But, Lemma 5.3 gives that $\mathrm{Wh}(y, \Gamma_x)$ is connected. By the previous paragraph, it follows that l is in the diagonal closure of the set of regular leaves in $(Q^2)^{-1}(x)$. □

Combining Propositions 3.7, 4.12, and 5.5 and Lemmata 5.1 and 2.9, we get our main result.

Theorem A. *Let T be an $\mathbb{R}$-tree with a free, minimal action of F_N by isometries with dense orbits. The tree T is indecomposable if and only if $L(T)$ is minimal up to diagonal leaves. In this case the unique minimal sublamination of $L(T)$ is the regular sublamination, which is equal to derived sublamination of $L(T)$.*

6. Appendix: non-free actions

We now explain how to handle non-free actions, and we prove a result that we expect to have nice applications. Let $T \in \overline{\mathrm{cv}}_N$ have dense orbits, and let $x \in T$ have non-trivial stabilizer $H = \mathrm{Stab}(x)$. By Definition 2.5, the dual lamination $L(T)$ contains $\partial^2 H$. It follows from [18], see also [15], that H is finitely generated, so by the Marshall Hall Theorem, $\bigcup_{g \in F_N} \partial^2 H^g$ is a sublamination of $L(T)$. By [15] there are finitely many orbits of points in T with non-trivial stabilizer (arc stabilizers in T are trivial), and so the collection $L_p(T) \subseteq L(T)$ of all leaves carried by a point stabilizer in T is a sublamination of $L(T)$; we all $L_p(T)$ the *peripheral sublamination* of $L(T)$.

Put

$$\lambda(T) := L(T) \setminus L_p(T),$$

and set

$$\Lambda(T) := (\lambda(T))';$$

so $\Lambda(T)$ is an F_N-invariant, flip-invariant subspace of $\partial^2 F_N$. Notice that if T is free and indecomposable, then $\Lambda(T) = L(T)'$. One should think of $\Lambda(T)$ as the "dynamical part" of $L(T)$. In contrast to the case of a measured lamination on a surface, $\Lambda(T)$ need not by closed (it is diagonally closed), and so $\Lambda(T)$ need not be a lamination. Nonetheless, our techniques are sufficiently robust to study $\Lambda(T)$, so we record the following, which will be useful in future applications.

Theorem 6.1. *Let T be a very small tree with dense orbits. If $\Lambda(T)$ contains a leaf l that is not carried by a proper free factor of F_N, then the subspace $\Lambda^{\text{top}}(T) \subseteq \Lambda(T)$ consisting of all leaves not carried by a proper free factor contains no isolated points and is dynamically minimal: if $l, l' \in \Lambda^{\text{top}}(T)$, then the smallest lamination containing l coincides with the smallest lamination containing l'.*

Note that if T is indecomposable, then [28] ensures that $\Lambda^{\text{top}}(T)$ is non-empty, so Theorem 6.1 generalizes Theorem A. The details of the following proof are left as an exercise; we provide a complete sketch.

Proof. As the rank of stabilizers are taken in account in the definition of the geometric and Q-indices, Lemma 4.5 remains true if we remove the free action hypothesis. From this Lemma we can deduce that Proposition 4.7 is also true without that hypothesis: the argument on extremal points in the geometric case can be replaced by an argument on the direction at extremal points and the fact that very small actions with dense orbits have trivial arc stabilizers.

In the proofs of Lemma 3.6 and 4.11 we used that the action is free to infer that the graphs Γ_i do not have short loops for $i >> 0$. In fact we only need that there are finitely many short loops, which is true as the geometric index is bounded above by $2N - 2$. □

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