Splittings and automorphisms of relatively hyperbolic groups

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Abstract. We study automorphisms of a relatively hyperbolic group G. When G is one-ended, we describe $\mathrm{Out}(G)$ using a preferred JSJ tree over subgroups that are virtually cyclic or parabolic. In particular, when G is toral relatively hyperbolic, $\mathrm{Out}(G)$ is virtually built out of mapping class groups and subgroups of $\mathrm{GL}_n(\mathbb{Z})$ fixing certain basis elements. When more general parabolic groups are allowed, these subgroups of $\mathrm{GL}_n(\mathbb{Z})$ have to be replaced by McCool groups: automorphisms of parabolic groups acting trivially (i.e. by conjugation) on certain subgroups.

Given a malnormal quasiconvex subgroup P of a hyperbolic group G, we view G as hyperbolic relative to P and we apply the previous analysis to describe the group $\mathrm{Out}(P \nearrow G)$ of automorphisms of P that extend to G: it is virtually a McCool group. If $\mathrm{Out}(P \nearrow G)$ is infinite, then P is a vertex group in a splitting of G. If P is torsion-free, then $\mathrm{Out}(P \nearrow G)$ is of type VF, in particular finitely presented.

We also determine when $\mathrm{Out}(G)$ is infinite, for G relatively hyperbolic. The interesting case is when G is infinitely-ended and has torsion. When G is hyperbolic, we show that $\mathrm{Out}(G)$ is infinite if and only if G splits over a maximal virtually cyclic subgroup with infinite center. In general we show that infiniteness of $\mathrm{Out}(G)$ comes from the existence of a splitting with infinitely many twists, or having a vertex group that is maximal parabolic with infinitely many automorphisms acting trivially on incident edge groups.

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Contents

1. Introduction

This paper studies automorphisms of hyperbolic and relatively hyperbolic groups in relation with their splittings. The first result in this direction is due to Paulin [54]: if G is a hyperbolic group with $\mathrm{Out}(G)$ infinite, then G has an action on an $\mathbb{R}$-tree with virtually cyclic (possibly finite) arc stabilizers. Rips theory then implies that G splits over a virtually cyclic group.

Understanding the global structure of $\mathrm{Out}(G)$ requires techniques which depend on the number of ends of G. If G is one-ended, there is an $\mathrm{Out}(G)$-invariant JSJ decomposition, and its study leads to Sela's description of $\mathrm{Out}(G)$ as a virtual extension of a direct product of mapping class groups by a virtually abelian group [59, 45]. If G has infinitely many ends, one does not get such a precise description because there is no $\mathrm{Out}(G)$-invariant splitting. One may study $\mathrm{Out}(G)$ by letting it act on a suitable space of splittings, the most famous being Culler–Vogtmann's outer space for $\mathrm{Out}(F_n)$.

Before moving on to relatively hyperbolic groups, here is a basic problem about which we get new results in the context of hyperbolic, and even free, groups. Given a finitely generated subgroup P of a group G, consider the group $\mathrm{Out}(P \nearrow G) \subset \mathrm{Out}(P)$ consisting of outer automorphisms of P which extend to automorphisms of G. What can one say about this group ? For instance, is it finitely generated? finitely presented? This question was asked by D. Calegari for automorphisms of free groups, and we answer it when P is malnormal.

The answer is related to splittings through the following simple remark: if P is a vertex group in a splitting of G, say $G = A *_{C_1} P *_{C_2} B$, then any automorphism of P which acts *trivially* (i.e. as conjugation by some element $p_i \in P$) on each

incident edge group C_i, extends to G (this is the algebraic translation of the fact that any self-homeomorphism of a closed subset $Y \subset X$ which is the identity on the frontier of Y extends by the identity to a homeomorphism of X).

The following theorem says that this phenomenon accounts for almost all of $\text{Out}(P \nearrow G)$.

Theorem 1.1 (see Corollary 6.3). *Let P be a quasiconvex malnormal subgroup of a hyperbolic group G. If* $\text{Out}(P \nearrow G)$ *is infinite, then P is a vertex group in a splitting of G, and the group of outer automorphisms of P acting trivially on incident edge groups has finite index in* $\text{Out}(P \nearrow G)$.

We call the group of outer automorphisms of P acting trivially on a family of subgroups a *McCool group of P*, because of McCool's paper about subgroups of $\text{Out}(F_n)$ fixing a finite set of conjugacy classes [50]. In this language, Theorem 1.1 says that $\text{Out}(P \nearrow G)$ is virtually a McCool group of P. It is a theme of this paper that many groups of automorphisms may be understood in terms of McCool groups, and that many results concerning the full group $\text{Out}(G)$ also apply to McCool groups (see also [37]).

The groups considered by McCool are finitely presented [50]. In fact, they have a finite index subgroup with a finite classifying space [13]. In [37] we extend these results to all McCool groups of torsion-free hyperbolic groups (and more generally of toral relatively hyperbolic groups). From this, one deduces that $\text{Out}(P \nearrow G)$ has a finite index subgroup with a finite classifying space when G and P are as in Theorem 1.1, with G torsion-free.

Our hypotheses for Theorem 1.1, namely quasiconvexity and malnormality of P, imply that G is hyperbolic relative to P (see [6]). In fact, Theorem 1.1 is just a special case of a result describing $\text{Out}(P \nearrow G)$ as a virtual McCool group when G is relatively hyperbolic and P is a maximal parabolic subgroup (see Theorem 6.2 for a precise statement).

This paper also addresses the question of whether $\text{Out}(G)$ is finite or infinite. It turns out that the answer is much simpler when G is torsion-free, owing to the fact that $\text{Out}(G)$ is then infinite whenever G has infinitely many ends (see Lemma 7.1).

Things are more complicated when torsion is allowed. For instance, characterizing virtually free groups with $\text{Out}(G)$ infinite is a non-trivial problem which was solved by Pettet [56]. The following theorem gives a different characterization. We say that a subgroup of G is $\mathcal{Z}_{\max}$ if it is maximal for inclusion among virtually cyclic subgroups with infinite center.

Theorem 1.2 (see Theorems 7.14 and 7.15). *Let G be a hyperbolic group. Then* $\mathrm{Out}(G)$ *is infinite if and only G splits over a $\mathbb{Z}_{\max}$ subgroup C; in this case, any element $c \in C$ of infinite order defines a Dehn twist which has infinite order in* $\mathrm{Out}(G)$.

Moreover, one may decide algorithmically whether $\mathrm{Out}(G)$ *is finite or infinite.*

The first assertion answers a question asked by D. Groves. See [10] for a related result proved independently by M. Carette, and [45, 17] for the one-ended case.

Let us now consider (in)finiteness of $\mathrm{Out}(G)$ when G is relatively hyperbolic. Suppose that G is hyperbolic with respect to a finite family $\mathcal{P}$ of finitely generated subgroups P_i. Since automorphisms of G need not respect $\mathcal{P}$ (for instance, G may be free and $\mathcal{P}$ may consist of any finitely generated malnormal subgroup), we consider the group $\mathrm{Out}(G; \mathcal{P})$ consisting of automorphisms mapping each P_i to a conjugate (in an arbitrary way). Note that $\mathrm{Out}(G; \mathcal{P})$ has finite index in the full group $\mathrm{Out}(G)$ when the groups P_i are small but not virtually cyclic, more generally when they are not themselves relatively hyperbolic in a nontrivial way [51].

Given a splitting of G, we have already pointed out that any automorphism of a vertex group acting trivially on incident edge groups extends to an automorphism of G. *Twists* around edges of the splitting also provide automorphisms of G. For instance, if $G = A *_C B$, and $a \in A$ centralizes C, there is an automorphism of G equal to conjugation by a on A and to the identity on B. Note that we do not require that C be virtually cyclic or that $a \in C$ (see Subsection 2.6).

The following result says that infiniteness of $\mathrm{Out}(G; \mathcal{P})$ comes from twists or from a McCool group of a parabolic group.

Theorem 1.3 (see Corollary 7.13). *Let G be hyperbolic relative to a family $\mathcal{P} = \{P_1, \ldots, P_n\}$ of finitely generated subgroups. Then $\mathrm{Out}(G; \mathcal{P})$ is infinite if and only if G has a splitting over virtually cyclic or parabolic subgroups, with each P_i contained in a conjugate of a vertex group, such that one of the following holds*:

- *the group of twists of the splitting is infinite*;
- *some P_i is a vertex group and there are infinitely many outer automorphisms of P_i acting trivially on incident edge groups.*

As mentioned above, one can get similar results characterizing the infiniteness of McCool groups of G. We refer to Section 7, in particular Theorem 7.6 and Corollary 7.13, for more detailed statements.

Let us now discuss the techniques that we use. We assume that G is hyperbolic relative to $\mathcal{P}$, and we distinguish two cases according to the number of ends

(technically, we consider relative one-endedness, but we will ignore this in the introduction).

When G is one-ended, we use a canonical $\mathrm{Out}(G;\mathcal{P})$-invariant decomposition Γ_{can} of G, namely (see Subsection 3.3) the JSJ decomposition over *elementary* (i.e. parabolic or virtually cyclic) subgroups relative to parabolic subgroups (i.e. parabolic subgroups have to be contained in conjugates of vertex groups).

One may thus generalize the description of $\mathrm{Out}(G)$ given by Sela for G hyperbolic, and express $\mathrm{Out}(G;\mathcal{P})$ in terms of mapping class groups, McCool groups of maximal parabolic subgroups, and a group of twists $\mathcal{T}$. For simplicity we restrict to a special case here (see Section 4 for a general statement).

Theorem 1.4 (see Corollary 4.4). *Let G be toral relatively hyperbolic and one-ended. Then some finite index subgroup $\mathrm{Out}^1(G)$ of $\mathrm{Out}(G)$ fits in an exact sequence*

$$1 \longrightarrow \mathcal{T} \longrightarrow \mathrm{Out}^1(G) \longrightarrow \prod_{i=1}^{p} MCG^0(\Sigma_i) \times \prod_{k=1}^{m} \mathrm{GL}_{r_k,n_k}(\mathbb{Z}) \longrightarrow 1$$

where $\mathcal{T}$ is finitely generated free abelian, $MCG^0(\Sigma_i)$ is the group of isotopy classes of homeomorphisms of a compact surface Σ_i mapping each boundary component to itself in an orientation-preserving way, and $\mathrm{GL}_{r,n}(\mathbb{Z})$ is the group of automorphisms of $\mathbb{Z}^{r+n}$ fixing the first n generators.

More generally, McCool groups of a one-ended toral relatively hyperbolic group G have a similar description (see Corollary 4.9). A more general statement (without restriction on the parabolic subgroups) is given in Theorems 4.3 and 4.6.

We also show that the *modular group* of G, introduced by Sela [60, 61] and usually defined by considering all suitable splittings of G, may be seen on the single splitting Γ_{can}. We refer to Section 5 for details.

To prove Theorem 1.1 when G is one-ended, one applies the previous analysis, viewing G as hyperbolic relative to P. Note that we use a JSJ decomposition which is relative (to P), and over subgroups which are not small (any subgroup of P is allowed).

Another example of the usefulness of relative JSJ decompositions is to prove the Scott conjecture about fixed subgroups of automorphisms of free groups. The proof that we give in Section 8, though not really new, is simplified by the use of the cyclic JSJ decomposition relative to the fixed subgroup.

We therefore work consistently in a relative context. We fix another family of finitely generated subgroups $\mathcal{H} = \{H_1, \ldots, H_q\}$, and we define $\mathrm{Out}(G;\mathcal{P},\mathcal{H}^{(\mathrm{t})})$

as the group of automorphisms mapping P_i to a conjugate (in an arbitrary way) and acting trivially on H_j (i.e. as conjugation by an element g_j of G).

In order to understand the structure of the automorphism group of a one-ended relatively hyperbolic group from its canonical JSJ decomposition, one needs to control automorphisms of rigid vertex groups. This is made possible by the following generalisation of Paulin's theorem mentioned above:

Theorem 1.5 (see Theorem 3.9). *Let G be hyperbolic relative to $\mathcal{P} = \{P_1, \ldots, P_n\}$, with P_i finitely generated and $P_i \neq G$. Let $\mathcal{H} = \{H_1, \ldots, H_q\}$ be another family of finitely generated subgroups.*

If $\mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(t)})$ *is infinite, then G splits over an elementary* (*virtually cyclic or parabolic*) *subgroup relative to* $\mathcal{P} \cup \mathcal{H}$.

Note that there is no quasiconvexity or malnormality assumption on groups in $\mathcal{H}$, but the automorphisms that we consider have to act trivially on them (see also Remark 9.3).

The theorem is proved using the Bestvina-Paulin method (extended to relatively hyperbolic groups in [1]) to get an action on an $\mathbb{R}$-tree T, and then applying Rips theory as developed in [3] to get a splitting. There are technical difficulties in the second step because G may fail to be finitely presented (the P_i's are not required to be finitely presented), and the action on T may fail to be stable if the P_i's are not slender; it only satisfies a weaker property which we call *hypostability*, and in the last section we generalize [3] to hypostable actions of relatively finitely presented groups.

Theorem 1.5 explains why McCool groups appear in Theorems 1.3 and 1.4. Indeed, given a rigid vertex group G_v in a JSJ decomposition of a one-ended group, Theorem 1.5 implies that only finitely many outer automorphisms of G_v extend to automorphisms of G. In turn, this implies that, after passing to a finite index subgroup, automorphisms of G act trivially on edge groups of the JSJ decomposition. See Subsection 4.1 for details.

When G is not one-ended, one has to consider splittings over finite groups. We do not have an exact sequence as in Theorem 1.4 because there is no $\mathrm{Out}(G; \mathcal{P})$-invariant splitting. In order to prove Theorems 1.2 and 1.3, we use the tree of cylinders introduced in [34] to obtain a non-trivial splitting over finite groups which is invariant or has an infinite group of twists (Corollary 7.11).

The paper is organized as follows. Section 2 consists of preliminaries (JSJ decompositions, automorphisms of a tree, trees of cylinders). Section 3 contains generalities about relatively hyperbolic groups. We point out that vertex groups of a splitting over relatively quasiconvex subgroups are relatively quasi-

convex, and that the canonical JSJ decomposition Γ_{can} has finitely generated edge groups. In Section 4 we study the structure of the automorphism group of a one-ended relatively hyperbolic group. Section 5 is devoted to the modular group. In Section 6 we study extendable automorphisms; Theorem 1.1 is a special case of Theorem 6.2. Section 7 is devoted to the question of whether groups like $\mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})})$ and $\mathrm{Out}(G; \mathcal{P}^{(\mathrm{t})}, \mathcal{H}^{(\mathrm{t})})$ are finite or infinite. Section 8 contains a proof of the Scott conjecture, and a partial generalization to relatively hyperbolic groups. Theorem 1.5 is proved in Section 9.

2. Preliminaries

Unless mentioned otherwise, G will always be a finitely generated group.

Given a group A and a subgroup B, we denote by $Z(A)$ the center of A, by $Z_A(B)$ the centralizer of B in A, and by $N_A(B)$ the normalizer of B in A. We write B^g for gBg^{-1}.

A subgroup $B \subset A$ is *malnormal* if $B^g \cap B$ is trivial for all $g \notin B$, *almost malnormal* if $B^g \cap B$ is finite for all $g \notin B$.

A group is *virtually cyclic* if it has a cyclic subgroup of finite index; it may be finite or infinite. Its outer automorphism group is finite.

A group G is *slender* if G and all its subgroups are finitely generated. We say that G is *small* if it contains no non-abelian free group (see [2] for a slightly weaker definition).

Let $\mathcal{P}$ be a family of subgroups P_i. In most cases, $\mathcal{P}$ will be a finite collection of finitely generated groups $\mathcal{P} = \{P_1, \ldots, P_n\}$.

The group G is *finitely presented relative to* $\mathcal{P} = \{P_1, \ldots, P_n\}$ if it is the quotient of $P_1 * \cdots * P_n * F$ by the normal closure of a finite subset, with F a finitely generated free group. If G is finitely presented relative to $\mathcal{P} = \{P_1, \ldots, P_n\}$, and if $g_1, \ldots, g_n \in G$, then G is also finitely presented relative to $\{P_1^{g_1}, \ldots, P_n^{g_n}\}$.

Let H be a finitely generated subgroup. If G is finitely presented relative to $\mathcal{P}$, then G is finitely presented relative to $\mathcal{P} \cup \{H\}$; the converse is not true in general.

2.1. Relative automorphisms. Given G and $\mathcal{P}$, we denote by $\mathrm{Aut}(G; \mathcal{P})$ the subgroup of $\mathrm{Aut}(G)$ consisting of automorphisms mapping each P_i to a conjugate, and by $\mathrm{Out}(G; \mathcal{P})$ its image in $\mathrm{Out}(G)$. If $\mathcal{P} = \{P_1, \ldots, P_n\}$, we use the notations $\mathrm{Aut}(G; P_1, \ldots, P_n)$ and $\mathrm{Out}(G; P_1, \ldots, P_n)$. We also write $\mathrm{Out}(G; \mathcal{P}, \mathcal{Q})$ for $\mathrm{Out}(G; \mathcal{P} \cup \mathcal{Q})$.

We define $\mathrm{Out}(G; \mathcal{P}^{(\mathrm{t})}) \subset \mathrm{Out}(G; \mathcal{P})$ by restricting to automorphisms whose restriction to each P_i agrees with conjugation by some element g_i of G (we call them *marked automorphisms*, or *automorphisms acting trivially on* $\mathcal{P}$). An outer automorphism Φ belongs to $\mathrm{Out}(G; \mathcal{P}^{(\mathrm{t})})$ if and only if, for each i, it has a representative $\alpha_i \in \mathrm{Aut}(G)$ equal to the identity on P_i.

The group $\mathrm{Out}(G; \mathcal{P}^{(\mathrm{t})})$ is denoted by PMCG in [45], by $\mathrm{Out}_m(G; \mathcal{P})$ in [17], and is called a (generalized) McCool group in [37].

Note that $\mathrm{Out}(G; G^{(\mathrm{t})})$ is trivial, and that $\mathrm{Out}(G; \mathcal{P}^{(\mathrm{t})})$ has finite index in $\mathrm{Out}(G; \mathcal{P})$ if $\mathcal{P}$ is a finite collection of finite groups (more generally, of groups P with $\mathrm{Out}(P)$ finite).

If $\mathcal{H} = \{H_1, \ldots, H_q\}$ is another family of subgroups, we define

$$\mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})}) = \mathrm{Out}(G; \mathcal{P}) \cap \mathrm{Out}(G; \mathcal{H}^{(\mathrm{t})}).$$

We allow $\mathcal{P}$ or $\mathcal{H}$ to be empty, in which case we do not write it.

The groups defined above do not change if we replace each P_i or H_j by a conjugate, or if we add conjugates of the P_i's to $\mathcal{P}$ or conjugates of the H_j's to $\mathcal{H}$.

2.2. Splittings. A *splitting* of a group G is an isomorphism between G and the fundamental group of a graph of groups Γ. Equivalently, using Bass-Serre theory, we view a splitting of G as an action of G on a simplicial tree T, with $T/G = \Gamma$. This tree is well-defined up to equivariant isomorphism, and two splittings are considered equal if there is an equivariant isomorphism between the corresponding Bass-Serre trees.

The group $\mathrm{Out}(G)$ acts on the set of splittings of G (by changing the isomorphism between G and $\pi_1(\Gamma)$, or precomposing the action on T).

Trees will always be simplicial trees with an action of G without inversion. We usually assume that the splitting is *minimal* (there is no proper G-invariant subtree). Since G is assumed to be finitely generated, this implies that Γ is a finite graph.

A splitting is *trivial* if G fixes a point in T (minimality then implies that T is a point).

A splitting is *relative to* $\mathcal{P}$ if every P_i is conjugate to a subgroup of a vertex group, or equivalently if P_i is *elliptic* (i.e. fixes a point) in T. If $\mathcal{P} = \{P_1, \ldots, P_n\}$, we also say that the splitting is relative to $P_1, \ldots, P_n$.

The group G *splits over a subgroup A* (relative to $\mathcal{P}$) if there is a non-trivial minimal splitting (relative to $\mathcal{P}$) such that A is an edge group. The group is *one-ended relative to $\mathcal{P}$* if it does not split over a finite group relative to $\mathcal{P}$.

If Γ is a graph of groups, we denote by V its set of vertices, and by E its set of oriented edges. The origin of an edge $e \in E$ is denoted by $o(e)$. A vertex v or an edge e carries a group G_v or G_e, and there is an inclusion $i_e \colon G_e \to G_{o(e)}$

2.3. Trees and deformation spaces [24, 32]. In this subsection we consider trees rather than graphs of groups. We denote by G_v or G_e the stabilizer of a vertex or an edge.

We often restrict edge stabilizers of T by requiring that they belong to a family $\mathcal{A}$ of subgroups of G which is stable under taking subgroups and under conjugation. We then say that T is an *$\mathcal{A}$-tree*. For instance, $\mathcal{A}$ may be the set of finite subgroups, of cyclic subgroups, of abelian subgroups, of elementary subgroups of a relatively hyperbolic group (see Section 3). We then speak of cyclic, abelian, elementary trees (or splittings).

Besides restricting edge stabilizers, we also often restrict to trees T *relative to $\mathcal{P}$*: every P_i is elliptic in T. We then say that T is an *$(\mathcal{A}, \mathcal{P})$-tree*.

A tree T' is a *collapse* of T if it is obtained from T by collapsing each edge in a certain G-invariant collection to a point; conversely, we say that T *refines* T'. In terms of graphs of groups, one passes from $\Gamma = T/G$ to $\Gamma' = T'/G$ by collapsing edges; for each vertex $v' \in \Gamma'$, the vertex group $G_{v'}$ is the fundamental group of the graph of groups occuring as the preimage of v' in Γ.

Given two trees T and T', we say that T *dominates T'* if there is a G-equivariant map $f \colon T \to T'$, or equivalently if every subgroup which is elliptic in T is also elliptic in T'. In particular, T dominates any collapse T'.

Two trees belong to the same *deformation space* if they dominate each other. In other words, a deformation space $\mathcal{D}$ is the set of all trees having a given family of subgroups as their elliptic subgroups. We denote by $\mathcal{D}(T)$ the deformation space containing a tree T, and by $\mathrm{Out}(\mathcal{D}) \subset \mathrm{Out}(G)$ the group of automorphisms leaving $\mathcal{D}$ invariant. The set of $\mathcal{A}$-trees contained in a deformation space is called a deformation space over $\mathcal{A}$ (and usually denoted by $\mathcal{D}$ also).

A tree is *reduced* if $G_e \neq G_v, G_w$ whenever an edge e has its endpoints v, w in different G-orbits. Equivalently, no tree obtained from T by collapsing the orbit of an edge belongs to the same deformation space as T. If T is not reduced, one may collapse edges so as to obtain a reduced tree in the same deformation space.

Any two reduced trees in a deformation space over finite groups may be joined by slide moves (see [24, 32] for definitions). In particular, they have the same set of edge and vertex stabilizers.

2.4. Induced structures

Definition 2.1 (Incident edge groups Inc_v). Given a vertex v of a graph of groups Γ, we denote by Inc_v the collection of all subgroups $i_e(G_e)$ of G_v, for e an edge with origin v. We call Inc_v the set of *incident edge groups*. We also use the notation Inc_{G_v}.

Similarly, if v is a vertex of a (minimal) tree, there are finitely many G_v-orbits of incident edges, and Inc_v is the family of stabilizers of some representatives of these orbits. This is a finite collection of subgroups of G_v, each well-defined up to conjugacy.

Any splitting Γ_v of G_v relative to Inc_v extends (non-uniquely) to a splitting Λ of G, whose edges are those of Γ_v together with those of Γ; an edge e of Γ incident to v is attached to a vertex of Γ_v whose group contains G_e (up to conjugacy). We call this *refining* Γ at v using Γ_v. One recovers Γ from Λ by collapsing edges of Γ_v.

Lemma 2.2. *Consider subgroups $H \subset K \subset G$ such that, if $g \in G$ and $H^g \subset K$, then $g \in K$ (this holds in particular when K is a vertex stabilizer of a tree T, and H is a subgroup which fixes no edge of T).*

- *If $H' \subset K$ is conjugate to H in G, it is conjugate to H in K.*
- *If $\alpha \in \mathrm{Aut}(G)$ leaves K invariant and maps H to gHg^{-1}, then $g \in K$.*

This lemma is trivial, but very useful. Given a vertex stabilizer G_v of a tree T, it allows us to define a family $\mathcal{P}_{|G_v}$ as follows (like Inc_v, it is a finite set of subgroups of G_v, each well-defined up to conjugacy).

Definition 2.3 (Induced structure $\mathcal{P}_{|G_v}$). Let $\mathcal{P} = \{P_i\}$ be a collection of subgroups of G, and let G_v be a vertex stabilizer in a tree T relative to $\mathcal{P}$. For each i such that P_i fixes a point in the orbit of v, but fixes no edge of T, let $\widetilde{P}_i \subset G_v$ be a conjugate of P_i. When defined, $\widetilde{P}_i$ is unique up to conjugacy in G_v by Lemma 2.2. We define $\mathcal{P}_{|G_v}$ as this collection of subgroups $\widetilde{P}_i \subset G_v$; we define $\mathcal{P}_{|G_v}$ similarly if G_v is a vertex group of $\Gamma = T/G$.

Remark 2.4. Given v and i, one of the following always holds: P_i fixes a vertex of T not in the orbit of v, or some conjugate of P_i fixes v and an edge incident to v, or P_i is conjugate to a group in $\mathcal{P}_{|G_v}$.

Remark 2.5. In this definition, $\mathcal{P}_{|G_v}$ depends not only on $\mathcal{P}$ and G_v, but also on the incident edge groups of G_v. In practice, we will not work with $\mathcal{P}_{|G_v}$ alone, but with $\mathcal{Q}_v = \mathcal{P}_{|G_v} \cup \mathrm{Inc}_v$. This is the case for instance in the following lemma.

Lemma 2.6. *If Γ_v is a splitting of G_v relative to $\mathcal{Q}_v = \text{Inc}_v \cup \mathcal{P}_{|G_v}$, refining Γ at v using Γ_v yields a splitting Λ of G which is relative to $\mathcal{P}$.*

Proof. The refinement is possible because Γ_v is relative to Inc_v. Each P_i is elliptic in Λ by Remark 2.4. ◻

2.5. JSJ decompositions [35]. Fix $\mathcal{A}$ as in Subsection 2.3, and a (possibly empty) family $\mathcal{P}$ of subgroups. All trees considered here are $(\mathcal{A}, \mathcal{P})$-trees.

A subgroup H is *universally elliptic* if it is elliptic (fixes a point) in every tree. A tree is universally elliptic if its edge stabilizers are. A tree T is a *JSJ tree* over $\mathcal{A}$ relative to $\mathcal{P}$ if it is universally elliptic and dominates every universally elliptic tree (see Section 4 of [35]). JSJ trees exist if G is finitely presented relative to $\mathcal{P}$. The set of JSJ trees, if non-empty, is a deformation space called the *JSJ deformation space* over $\mathcal{A}$ relative to $\mathcal{P}$. When $\mathcal{A}$ is the set of cyclic (abelian, elementary...) subgroups, we refer to the cyclic (abelian, elementary...) JSJ.

When $\mathcal{A}$ is the set of finite subgroups, and $\mathcal{P} = \emptyset$, the JSJ deformation space is the *Stallings–Dunwoody deformation space*, characterized by the property that its trees have vertex stabilizers with at most one end (see Section 6 of [35]). Because it is a deformation space over finite groups, all its reduced trees have the same edge and vertex stabilizers (see Subsection 2.3). This space exists if and only if G is accessible, in particular when G is finitely presented [22]. If G is torsion-free, the Stallings–Dunwoody deformation space is the *Grushko deformation space*; edge stabilizers are trivial, vertex stabilizers are freely indecomposable and non-cyclic.

More generally, the JSJ deformation space over finite subgroups relative to $\mathcal{P}$ will be called the Stallings–Dunwoody deformation space relative to $\mathcal{P}$. We will also consider JSJ spaces over finite subgroups of cardinality bounded by some k; these exist whenever G is finitely generated by Linnell's accessibility [49].

If T is a tree (in particular, if it is a JSJ tree), a vertex stabilizer G_v of T not belonging to $\mathcal{A}$ (or v itself) is *rigid* if it is universally elliptic. Otherwise, G_v (or v) is *flexible*. In many situations, flexible vertex stabilizers G_v of JSJ trees are quadratically hanging subgroups (see Section 7 of [35]).

Definition 2.7 (QH vertex). A vertex stabilizer G_v (or v) is *quadratically hanging*, or *QH*, (relative to $\mathcal{P}$) if there is a normal subgroup $F \lhd G_v$ (called the *fiber* of G_v) such that G_v/F is isomorphic to the fundamental group $\pi_1(\Sigma)$ of a compact hyperbolic 2-orbifold Σ (usually with boundary); moreover, if $H \subset G_v$ is an incident edge stabilizer, or is the intersection of G_v with a conjugate of a group in $\mathcal{P}$, then the image of H in $\pi_1(\Sigma)$ is finite or contained in a *boundary subgroup* (a subgroup conjugate to the fundamental group of a boundary component).

Definition 2.8 (Full boundary subgroups). Let G_v be a QH vertex stabilizer. For each boundary component of Σ, we select a representative for the conjugacy class of its fundamental group in $\pi_1(\Sigma)$, and we consider its full preimage in G_v. This defines a finite family $\mathcal{B}_v$ of subgroups of G_v.

If G_v is QH with finite fiber, every infinite incident edge stabilizer is virtually cyclic and (up to conjugacy in G_v) contained with finite index in a group of $\mathcal{B}_v$. If G is one-ended relative to $\mathcal{P}$, then every incident edge stabilizer is infinite, so is contained in a group of $\mathcal{B}_v$.

Remark 2.9. If G_v is a *flexible* QH vertex stabilizer with finite fiber, and $H \subset G_v$ is universally elliptic, then the image of H in $\pi_1(\Sigma)$ is finite or contained in a boundary subgroup (see Proposition 7.6 of [35]; this requires a technical assumption on $\mathcal{A}$, which holds in all cases considered in the present paper). In particular, H is virtually cyclic.

2.6. The automorphism group of a tree [45]. Let T be a tree with a minimal action of G. We assume that T is not a line with G acting by translations.

We denote by $\mathrm{Aut}(T) \subset \mathrm{Aut}(G)$ the group of automorphisms α *leaving T invariant*: there exists an isomorphism $f_\alpha \colon T \to T$ which is α-equivariant in the sense that $f_\alpha(gx) = \alpha(g) f_\alpha(x)$ for $g \in G$ and $x \in T$.

Following [45], we describe the image $\mathrm{Out}(T)$ of $\mathrm{Aut}(T)$ in $\mathrm{Out}(G)$ in terms of the graph of groups $\Gamma = T/G$. Our assumptions on T imply that Γ is minimal, and is not a mapping torus (as defined in [45]).

The group $\mathrm{Out}(T)$ acts on the finite graph $\Gamma = T/G$, and we define $\mathrm{Out}^0(T)$ as the finite index subgroup acting trivially. We use the notations

$$\mathrm{Out}(T; \mathcal{P}, \mathcal{H}^{(\mathrm{t})}) = \mathrm{Out}(T) \cap \mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})}),$$

and

$$\mathrm{Out}^0(T; \mathcal{P}, \mathcal{H}^{(\mathrm{t})}) = \mathrm{Out}^0(T) \cap \mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})})$$

(see Section 2.1).

Action on vertex groups. If $v \in V$ is a vertex of Γ, there is a natural map

$$\rho_v \colon \mathrm{Out}^0(T) \longrightarrow \mathrm{Out}(G_v)$$

defined as follows. Let $\Phi \in \mathrm{Out}^0(T)$. When $N_G(G_v)$ acts on G_v by inner automorphisms, in particular when G_v fixes a unique point in T (in this case $N_G(G_v) = G_v$), one defines $\rho_v(\Phi)$ simply by choosing any representative $\alpha \in \mathrm{Aut}(T)$ of Φ

leaving G_v invariant and considering its restriction to G_v. In general, one has to choose α more carefully: one fixes a vertex $\tilde{v}$ of T mapping to v such that the stabilizer of $\tilde{v}$ is G_v, one chooses α so that f_α fixes $\tilde{v}$, and $\rho_v(\Phi)$ is represented by the restriction $\alpha_{|G_v}$.

If e is an edge of Γ, one may define

$$\rho_e \colon \operatorname{Out}^0(T) \longrightarrow \operatorname{Out}(G_e)$$

similarly.

Let

$$\rho \colon \operatorname{Out}^0(T) \longrightarrow \prod_{v \in V} \operatorname{Out}(G_v)$$

be the product map. As observed in Subsection 2.3 of [45], the image $\rho(\operatorname{Out}^0(T))$ contains $\prod_{v \in V} \operatorname{Out}(G_v; \operatorname{Inc}_v^{(\mathrm{t})})$ and is contained in $\prod_{v \in V} \operatorname{Out}(G_v; \operatorname{Inc}_v)$. More precisely:

Lemma 2.10. *Let $\mathcal{P}, \mathcal{H}$ be two families of subgroups of G. Let T be a tree relative to $\mathcal{P} \cup \mathcal{H}$. Then*

$$\begin{aligned}
\operatorname{Out}(G_v; \operatorname{Inc}_v^{(\mathrm{t})}, \mathcal{P}_{|G_v}, \mathcal{H}_{|G_v}^{(\mathrm{t})}) &\subset \rho_v(\operatorname{Out}^0(T; \mathcal{P}, \mathcal{H}^{(\mathrm{t})})) \\
&\subset \operatorname{Out}(G_v; \operatorname{Inc}_v, \mathcal{P}_{|G_v}, \mathcal{H}_{|G_v}^{(\mathrm{t})})
\end{aligned} \tag{1}$$

for every $v \in V$, and

$$\begin{aligned}
\prod_{v \in V} \operatorname{Out}(G_v; \operatorname{Inc}_v^{(\mathrm{t})}, \mathcal{P}_{|G_v}, \mathcal{H}_{|G_v}^{(\mathrm{t})}) &\subset \rho(\operatorname{Out}^0(T; \mathcal{P}, \mathcal{H}^{(\mathrm{t})})) \\
&\subset \prod_{v \in V} \operatorname{Out}(G_v; \operatorname{Inc}_v, \mathcal{P}_{|G_v}, \mathcal{H}_{|G_v}^{(\mathrm{t})}).
\end{aligned}$$

If $\operatorname{Out}(G_e)$ is finite for all edges (resp. for all edges e incident to v), all inclusions (resp. all inclusions in (1)) have images of finite index.

Recall that $\mathcal{P}_{|G_v}$ was defined in Definition 2.3.

Proof. The inclusion $\prod_{v \in V} \operatorname{Out}(G_v; \operatorname{Inc}_v^{(\mathrm{t})}) \subset \rho(\operatorname{Out}^0(T))$ is proved in [45] by extending any $\Phi_v \in \operatorname{Out}(G_v; \operatorname{Inc}_v^{(\mathrm{t})})$ "by the identity" to get $\Phi \in \operatorname{Out}^0(T)$, with $\Phi_v = \rho_v(\Phi)$, acting as a conjugation on each edge group and on each G_w for $w \neq v$. The left hand side inclusions in the lemma follow from Remark 2.4.

The inclusion $\rho_v(\operatorname{Out}^0(T)) \subset \operatorname{Out}(G_v; \operatorname{Inc}_v)$ follows from the fact that, given an edge e of T containing the lift $\tilde{v}$ of v used to define ρ_v, any $\Phi \in \operatorname{Out}^0(T)$ has

a representative α such that f_α fixes e; this representative induces an automorphism of $G_{\tilde{v}}$ leaving G_e invariant. To prove the right hand side inclusions, apply Lemma 2.2 with $K = G_{\tilde{v}}$, recalling that groups in $\mathcal{H}_{|G_v}$ or $\mathcal{P}_{|G_v}$ fix a unique point in T.

If $\mathrm{Out}(G_e)$ is finite for all incident edge groups, $\mathrm{Out}(G_v; \mathrm{Inc}_v^{(\mathrm{t})})$ has finite index in $\mathrm{Out}(G_v; \mathrm{Inc}_v)$ (see Proposition 2.3 in [45]). Considering the intersection with $\mathrm{Out}(G_v; \mathcal{P}_{|G_v}, \mathcal{H}_{|G_v}^{(\mathrm{t})})$, we get that $\mathrm{Out}(G_v; \mathrm{Inc}_v^{(\mathrm{t})}, \mathcal{P}_{|G_v}, \mathcal{H}_{|G_v}^{(\mathrm{t})})$ has finite index in $\mathrm{Out}(G_v; \mathrm{Inc}_v, \mathcal{P}_{|G_v}, \mathcal{H}_{|G_v}^{(\mathrm{t})})$. This concludes the proof. $\qquad \square$

Remark 2.11. We can also consider automorphisms which do not leave T invariant, but only leave some vertex stabilizer G_v invariant. Assuming that G_v equals its normalizer, there is a natural map

$$\rho_v \colon \mathrm{Out}(G; G_v) \longrightarrow \mathrm{Out}(G_v)$$

and

$$\mathrm{Out}(G_v; \mathrm{Inc}_v^{(\mathrm{t})}, \mathcal{P}_{|G_v}, \mathcal{H}_{|G_v}^{(\mathrm{t})}) \subset \rho_v(\mathrm{Out}(G; G_v, \mathcal{P}, \mathcal{H}^{(\mathrm{t})})) \subset \mathrm{Out}(G_v; \mathcal{P}_{|G_v}, \mathcal{H}_{|G_v}^{(\mathrm{t})}).$$

Twists.   As in Subsection 2.5 of [45], we now consider the kernel of the product map $\rho \colon \mathrm{Out}^0(T) \to \prod_{v \in V} \mathrm{Out}(G_v)$. It consists of automorphisms in $\mathrm{Out}^0(T)$ having, for each v, a representative in $\mathrm{Aut}(T)$ whose restriction to G_v is the identity. If T is relative to $\mathcal{P}$, the group $\ker \rho$ is contained in $\mathrm{Out}(T; \mathcal{P}^{(\mathrm{t})})$.

To study $\ker \rho$, we need to introduce the *group of twists* $\mathcal{T}$ associated to T or equivalently to Γ (we write $\mathcal{T}(T)$ or $\mathcal{T}(\Gamma)$ if there is a risk of confusion).

Let e be a separating edge of Γ with origin v and endpoint w. Then $G = A *_{G_e} B$ with $G_v \subset A$ and $G_w \subset B$. Given $g \in Z_{G_v}(G_e)$, one defines the *twist by g around e near v* as the (image in $\mathrm{Out}(G)$ of the) automorphism of G equal to the identity on B and to conjugation by g on A. There is a similar definition in the case of an HNN extension $G = A*_C = \langle A, t \mid tct^{-1} = \varphi(c), c \in C \rangle$: given $g \in Z_A(C)$, the twist by g is the identity on A and sends t to tg.

The group $\mathcal{T}$ is the subgroup of $\mathrm{Out}(G)$ generated by all twists. It is a quotient of $\prod_{e \in E} Z_{G_{o(e)}}(G_e)$ and is contained in $\ker \rho$. The following facts follow directly from Section 2 of [45].

Lemma 2.12. (1) *If every* $\mathrm{Out}(G_e)$ *is finite, then* $\mathcal{T}$ *has finite index in* $\ker \rho$.

(2) *Assume that every non-oriented edge e of Γ has an endpoint v such that $N_{G_v}(G_e)$ acts on G_e by inner automorphisms* (*this holds in particular if G_v is abelian, or if G_e is malnormal in G_v, or if G_e is infinite and almost malnormal in G_v*). *Then* $\mathcal{T} = \ker \rho$.

The kernel of the epimorphism from $\prod_{e \in E} Z_{G_{o(e)}}(G_e)$ to $\mathcal{T}$ is the image of a natural map

$$ j \colon \prod_{v \in V} Z(G_v) \times \prod_{e \in \mathcal{E}} Z(G_e) \longrightarrow \prod_{e \in E} Z_{G_{o(e)}}(G_e) $$

where $\mathcal{E}$ is the set of non-oriented edges of Γ (see Proposition 3.1 of [45]). The image of an element of $Z(G_v)$ is called a *vertex relation* at v, the image of an element of $Z(G_e)$ is an *edge relation*.

For instance, if Γ is a non-trivial amalgam $G = A *_C B$, then $\mathcal{T}$ is the image of the map $p \colon Z_A(C) \times Z_B(C) \to \mathrm{Out}(G)$ sending (a, b) to the class of the automorphism acting on A as conjugation by a and on B as conjugation by b. The kernel of p is generated by the elements $(a, 1)$ with $a \in Z(A)$ and $(1, b)$ with $b \in Z(B)$ (vertex relations), together with the elements (c, c) with $c \in Z(C)$ (edge relations).

Lemma 2.13. *Let e be an edge of Γ with origin v. If $Z(G_e)$ and $Z(G_v)$ are finite, but $Z_{G_v}(G_e)$ is infinite, then the image of $Z_{G_v}(G_e)$ in $\mathcal{T}$ is infinite. In particular, $\mathcal{T}$ is infinite.*

Note that $Z_{G_v}(G_e)$ is infinite if $N_{G_v}(G_e)$ is infinite and G_e is finite.

Proof. It is pointed out in [47, Lemma 3.2] that the image of $Z_{G_v}(G_e)$ in $\mathcal{T}$ maps onto the quotient $Z_{G_v}(G_e)/\langle Z(G_v), Z(G_e)\rangle$. Since $Z(G_v)$ and $Z(G_e)$ are commuting finite subgroups, this quotient is infinite. $\square$

Let Γ be a graph of groups with fundamental group G, and $\Gamma_0 \subset \Gamma$ a connected subgraph. We view Γ_0 as a graph of groups, with fundamental group $G_0 \subset G$ and associated group of twists $\mathcal{T}(\Gamma_0) \subset \mathrm{Out}(G_0)$.

Lemma 2.14. *If $\mathcal{T}(\Gamma_0)$ is infinite, then so is $\mathcal{T}(\Gamma)$.*

Proof. Let E_0 be the set of oriented edges of Γ_0. Consider the projection from $\prod_{e \in E} Z_{G_{o(e)}}(G_e)$ to $\prod_{e \in E_0} Z_{G_{o(e)}}(G_e)$ obtained by keeping only the factors with $e \subset \Gamma_0$. It is compatible with the vertex and edge relations, so induces an epimorphism from $\mathcal{T}(\Gamma)$ to $\mathcal{T}(\Gamma_0)$. $\square$

Lemma 2.15. *If Γ is a graph of groups with $\mathcal{T}(\Gamma)$ infinite, there is an edge e such that the graph of groups Γ_e obtained by collapsing every edge except e has $\mathcal{T}(\Gamma_e)$ infinite.*

Proof. There is an edge e such that $Z_{G_{o(e)}}(G_e)$ has infinite image in $\mathrm{Out}(G)$. Twists of Γ around e are also twists of Γ_e. $\square$

2.7. Trees of cylinders [34]. We recall some basic properties of the tree of cylinders (see Section 4 of [34] for details). Besides $\mathcal{A}$ (and possibly $\mathcal{P}$), we have to fix a conjugacy-invariant subfamily $\mathcal{E} \subset \mathcal{A}$ and an admissible equivalence relation $\sim$ on $\mathcal{E}$. Rather than giving a general definition, we describe the examples that will be used in this paper (with $\mathcal{A}$ consisting of all finite, elementary, or abelian subgroups respectively):

(1) $\mathcal{E}$ consists of all finite subgroups of a fixed order k, and $\sim$ is equality;

(2) G is relatively hyperbolic (see Section 3), $\mathcal{E}$ consists of all infinite elementary subgroups (parabolic or loxodromic), and $\sim$ is co-elementarity: $A \sim B$ if and only if $\langle A, B \rangle$ is elementary;

(3) G is a torsion-free CSA group, $\mathcal{E}$ consists of all infinite abelian subgroups, and $\sim$ is commutation: $A \sim B$ if and only if $\langle A, B \rangle$ is abelian (recall that G is CSA if centralizers of non-trivial elements are abelian and malnormal).

Let T be a tree with edge stabilizers in $\mathcal{E}$. We declare two (non-oriented) edges e and f to be equivalent if $G_e \sim G_f$. The union of all edges in an equivalence class is a subtree Y, called a *cylinder* of T. Two distinct cylinders meet in at most one point. The *tree of cylinders* T_c of T is the bipartite tree such that $V_0(T_c)$ is the set of vertices x of T which belong to at least two cylinders, $V_1(T_c)$ is the set of cylinders Y of T, and there is an edge $\varepsilon = (x, Y)$ between x and Y in T_c if and only if $x \in Y$. In other words, one obtains T_c from T by replacing each cylinder Y by the cone on its boundary (defined as the set of vertices of Y belonging to some other cylinder). Note that T_c may be trivial even if T is not.

The tree T_c is dominated by T (in particular, it is relative to $\mathcal{P}$ if T is). It only depends on the deformation space $\mathcal{D}$ containing T (we sometimes say that it is the tree of cylinders of $\mathcal{D}$). In particular, T_c is invariant under any automorphism of G leaving $\mathcal{D}$ invariant.

The stabilizer of a vertex $x \in V_0(T_c)$ is the stabilizer of x, viewed as a vertex of T. The stabilizer G_Y of a vertex $Y \in V_1(T_c)$ is the stabilizer $G_{\mathcal{C}}$ of the equivalence class $\mathcal{C} \in \mathcal{E}/\sim$ containing stabilizers of edges in Y, for the action of G on $\mathcal{E}$ by conjugation (see Subsection 5.1 of [34]). It is the normalizer of a finite subgroup in case 1, a maximal elementary (resp. abelian) subgroup in case 2 (resp. 3). Note that $A \subset G_{\mathcal{C}}$ if $A \in \mathcal{E}$ and $\mathcal{C}$ is its equivalence class.

The stabilizer of an edge $\varepsilon = (x, Y)$ of T_c is $G_\varepsilon = G_x \cap G_Y$; it is elliptic in T. In cases 2 and 3, G_ε belongs to $\mathcal{A}$. But, in case 1, it may happen that edge stabilizers of T_c are not in $\mathcal{A}$, so we also consider the *collapsed tree of cylinders* T_c^* obtained from T_c by collapsing each edge whose stabilizer does not belong to $\mathcal{A}$ (see Subsection 5.2 of [34]). It is an $(\mathcal{A}, \mathcal{P})$-tree if T is, and $(T_c^*)_c^* = T_c^*$.

3. Relatively hyperbolic groups

In this section we assume that G is hyperbolic relative to a finite family $\mathcal{P} = \{P_1, \ldots, P_n\}$ of finitely generated subgroups; we say that $\mathcal{P}$ is the *parabolic structure*.

The group G is finitely generated. It is not necessarily finitely presented, but it is finitely presented relative to $\mathcal{P}$ [53], so JSJ decompositions relative to $\mathcal{P}$ exist. In particular, there is a deformation space of relative Stallings–Dunwoody decompositions (see Subsection 2.5).

3.1. Generalities. We refer to [39] for equivalent definitions of relative hyperbolicity. In particular, G acts properly discontinuously on a proper geodesic δ-hyperbolic space X (which may be taken to be a graph [28]), the action is cocompact in the complement of a G-invariant union $\mathcal{B}$ of disjoint horoballs, and the P_i's are representatives of conjugacy classes of stabilizers of horoballs. Any horoball $B \in \mathcal{B}$ has a unique point at infinity ξ, and the stabilizer of ξ (for the action of G on ∂X) coincides with the stabilizer of B.

For each constant $M > 0$, one can change the system of horoballs so that any two distinct horoballs are at distance at least M. Indeed, for each horoball B with stabilizer P defined by a horofunction h, the function $h'(x) = \sup_{g \in P} h(gx)$ is another (well-defined) horofunction which is P-equivariant; then $B' = h'^{-1}([R, \infty))$ is a new P-invariant horoball such that $d(B', X \setminus B) \geq M$ for R large enough. Doing this for a chosen horoball in each orbit, and extending by equivariance, one gets a system of horoballs at distance at least M from each other.

A subgroup of G is *parabolic* if it is contained in a conjugate of some P_i, *loxodromic* if it is infinite, virtually cyclic, and not parabolic, *elementary* if it is parabolic or virtually cyclic (finite or loxodromic). Any small subgroup is elementary. The group G itself is elementary if it is virtually cyclic or equal to a P_i. We say that A is an elementary subgroup of B if it is elementary and contained in B.

One may remove any virtually cyclic subgroup from $\mathcal{P}$, without destroying relative hyperbolicity (see e.g. [20, Cor 1.14]). Conversely, one may add to $\mathcal{P}$ a finite subgroup or a maximal loxodromic subgroup (see e.g. [52]). These operations do not change the set of elementary (or relatively quasiconvex, as defined below) subgroups, and it is sometimes convenient (as in [39]) to assume that every P_i is infinite. Any infinite P_i is a maximal elementary subgroup.

The following lemma is folklore, but we have not found it in the literature.

Lemma 3.1. *Given a relatively hyperbolic group G, there exists a number M such that any elementary subgroup $H \subset G$ of cardinality $> M$ is contained in a unique maximal elementary subgroup $E(H)$. There are finitely many conjugacy classes of non-parabolic finite subgroups.*

Proof. We may assume that every P_i is infinite. Let H be elementary. The existence of $E(H)$ is well-known if H is infinite (see for instance [53]), so assume H is finite. We also assume that the distance between any two distinct horoballs in $\mathcal{B}$ is bigger than 6δ. Given $r > 0$, let X_r be the set of points of X which are moved less than r by H. It follows from Lemma 3.3 p. 460 of [7] (existence of quasi-centres) that $X_{5\delta}$ is nonempty.

Arguing as in the proof of Lemma 1.15 page 407 and Lemma 3.3 page 428 in [7], one sees that any geodesic joining two points of $X_{5\delta}$, or a point of $X_{5\delta}$ to a fixed point of H in ∂X, is contained in $X_{9\delta}$.

If $X_{9\delta}$ meets $X \setminus \mathcal{B}$, properness of the action of G on $X \setminus \mathcal{B}$ implies that there are only finitely many possibilities for H up to conjugacy, so we can choose M to ensure $X_{9\delta} \subset \mathcal{B}$. This implies that $X_{5\delta}$ is contained in a unique horoball B_0 of $\mathcal{B}$. This horoball is H-invariant since horoballs are 6δ-apart, so H fixes the point at infinity ξ_0 of B_0 and is contained in the maximal parabolic subgroup $E(H) = \mathrm{Stab}(B_0)$. In particular, H is parabolic.

There remains to prove uniqueness of $E(H)$. It suffices to check that H cannot fix any point $\xi \neq \xi_0$ in ∂X. If it did, a geodesic joining ξ to a point of $X_{5\delta}$ would be contained in $X_{9\delta}$ and meet $X \setminus \mathcal{B}$. This contradicts our choice of M. $\square$

Since maximal elementary subgroups are equal to their normalizer, we get:

Corollary 3.2 ([21, Lemma 4.20]). *Maximal elementary subgroups E are uniformly almost malnormal: if $E \cap gEg^{-1}$ has cardinality $> M$, then $g \in E$.*

3.2. Quasiconvexity

Definition 3.3 (Hruska [39]). Let X and $\mathcal{B}$ be as above, and $C > 0$. A subspace $Y \subset X$ is relatively C-quasiconvex if, given $y, y' \in Y$, any geodesic $[y, y'] \subset X$ has the property that $[y, y'] \setminus \mathcal{B}$ lies in the C-neighbourhood of Y. The space Y is relatively quasiconvex if it is relatively C-quasiconvex for some C. A subgroup $H < G$ is relatively quasiconvex if some (equivalently, every) H-orbit is relatively quasiconvex in X.

Proposition 3.4. *Let G be hyperbolic relative to $\mathcal{P} = \{P_1, \ldots, P_n\}$, with P_i finitely generated. If G acts on a simplicial tree T relative to $\mathcal{P}$ with relatively quasiconvex edge stabilizers, then vertex stabilizers are relatively quasiconvex.*

The proposition applies in particular if edge stabilizers are elementary, since elementary subgroups are relatively quasiconvex. It was proved by Bowditch [5, Proposition 1.2] and Kapovich [41, Lemma 3.5] for G hyperbolic. We generalize Bowditch's argument.

Proof. As usual, we assume that G acts on T minimally and without inversion. Since G is finitely generated, the graph T/G is finite. We also assume that X is a connected graph, and edges of T have length 1.

Since P_i is elliptic in T, and stabilizers of points of X are finite, hence elliptic in T, there exists an equivariant map $f : X \rightarrow T$ sending vertices to vertices, mapping each edge linearly to an edge path, and constant on each horoball of $\mathcal{B}$.

For each edge e of T, let m_e be the midpoint of e, and $Q_e = f^{-1}(m_e)$. Let v be a vertex of T, and let E_v be the set of edges of T with origin v. Let $Q_v \subset X$ be the preimage under f of the closed ball of radius $\frac{1}{2}$ around v in T. Note that $Q_e \subset Q_v$ for all $e \in E_v$ and $Q_e \cap \mathcal{B} = \emptyset$. Also note that $Q_e \neq \emptyset$ by minimality of T and connectedness of X.

If $f(x) = f(hx)$ for $x \in X$ and $h \in G$, then h fixes $f(x)$. Since G acts cocompactly on $X \setminus \mathcal{B}$, it follows that G_v acts cocompactly on $Q_v \setminus \mathcal{B}$ and G_e acts cocompactly on $Q_e = Q_e \setminus \mathcal{B}$. In particular, Q_e is the G_e-orbit of a finite set. Relative quasiconvexity of G_e implies that Q_e is relatively quasiconvex. Since T/G is a finite graph, there exists a common constant C such that all subsets Q_e are relatively C-quasiconvex.

We now fix a vertex v, and we show that G_v is relatively quasiconvex. Choose $x \in Q_v \setminus \mathcal{B}$. Since G_v acts cocompactly on $Q_v \setminus \mathcal{B}$, the Hausdorff distance between $Q_v \setminus \mathcal{B}$ and the G_v-orbit of x is finite, so it suffices to prove that Q_v is relatively quasiconvex. Let γ be a geodesic of X joining two points of Q_v, and let γ_0 be a maximal subgeodesic contained in $\gamma \setminus Q_v$. Considering the image of γ in T, we see that both endpoints of γ_0 belong to the same Q_e, for some $e \in E_v$. Thus $\gamma_0 \setminus \mathcal{B}$ is C-close to Q_e, hence to Q_v. This shows that Q_v is relatively C-quasiconvex. $\square$

A relatively quasiconvex subgroup is relatively hyperbolic in a natural way ([39, Theorem 9.1]). In particular:

Lemma 3.5. *If G_v is an infinite vertex stabilizer of a tree with finite edge stabilizers, it is hyperbolic relative to the family $\mathcal{P}_{|G_v}$ of Definition 2.3.*

Proof. This follows from Theorem 9.1 of [39], adding finite groups belonging to $\mathcal{P}_{|G_v}$ to the parabolic structure if needed. $\square$

3.3. The canonical JSJ decomposition. In this section we recall the description of the canonical relative JSJ decomposition. The content of the word canonical is that the JSJ tree (not just the JSJ deformation space) is invariant under automorphisms.

Let G be hyperbolic relative to $\mathcal{P}$, and denote by $\mathcal{A}$ the family of elementary subgroups. In this subsection we fix another (possibly empty) family of finitely generated subgroups $\mathcal{H} = \{H_1, \ldots, H_q\}$ and we assume that G is one-ended relative to $\mathcal{P} \cup \mathcal{H}$.

We consider the canonical $\mathrm{Out}(G; \mathcal{P} \cup \mathcal{H})$-invariant JSJ tree T_{can} defined (under the name T_c^*) in Theorem 13.1 of [36] (see also Theorem 7.5 of [34]). It is the tree of cylinders (see Subsection 2.7) of the JSJ deformation space $\mathcal{D}$ over elementary subgroups relative to $\mathcal{P} \cup \mathcal{H}$, and it belongs to $\mathcal{D}$. It is $\mathrm{Out}(G; \mathcal{P} \cup \mathcal{H})$-invariant because the JSJ deformation space $\mathcal{D}$ is.

Being a tree of cylinders, T_{can} is bipartite, with vertices $x \in V_0(T_{\mathrm{can}})$ and $Y \in V_1(T_{\mathrm{can}})$. Stabilizers of vertices in $V_0(T_{\mathrm{can}})$ are non-elementary, and stabilizers of vertices in $V_1(T_{\mathrm{can}})$ are maximal elementary subgroups. Non-elementary vertex stabilizers may be rigid or flexible (see Subsection 2.5), and flexible vertex stabilizers are QH with finite fiber (see Theorem 13.1 of [36]). Elementary vertex stabilizers are infinite by one-endedness, they may be parabolic or loxodromic. Thus there are exactly four possibilities for a vertex $v \in T_{\mathrm{can}}$:

0.a. *rigid*: G_v is non-elementary and is elliptic in every $(\mathcal{A}, \mathcal{P} \cup \mathcal{H})$-tree.

0.b. *(flexible) QH*: G_v is non-elementary and not universally elliptic. Then v is a flexible QH vertex with finite fiber as in Subsection 2.5.

1.a. *maximal parabolic*: G_v is conjugate to a P_i.

1.b. *maximal loxodromic*: G_v is a maximal virtually cyclic subgroup of G, and G_v is not parabolic.

Remark 3.6. A QH vertex $v \in V_0(T_{\mathrm{can}})$ is flexible, except in a few cases; for instance, if G is torsion-free, the only exceptional case is when the underlying surface is a pair of pants (thrice punctured sphere). In these cases we view v as rigid rather than QH. This should not cause confusion. In particular, Propositions 4.1 and 4.7 would remain valid with v viewed as QH.

If $\varepsilon = (x, Y)$ is an edge, then $G_\varepsilon = G_x \cap G_Y$ is an infinite maximal elementary subgroup of G_x (but G_ε may fail to be maximal elementary in G and G_Y). In particular, G_ε is always almost malnormal in G_x, so that Assertion 2 of Lemma 2.12 applies to T_{can}, showing $\mathcal{T} = \ker \rho$.

Let now v be a QH vertex. We claim that, *if $\mathcal{H} = \emptyset$ and every P_i is infinite, then* $\mathcal{B}_v = \mathrm{Inc}_v \cup \mathcal{P}_{|G_v}$, *and* $\mathcal{B}_v = \mathrm{Inc}_v$ *if no P_i is virtually cyclic* (see Definitions 2.1, 2.3 and 2.8).

Groups in $\mathrm{Inc}_v \cup \mathcal{P}_{|G_v}$ are infinite maximal elementary subgroups of G_v, and groups in $\mathcal{B}_v$ are virtually cyclic, so $\mathrm{Inc}_v \cup \mathcal{P}_{|G_v} \subset \mathcal{B}_v$ by Definition 2.7. Conversely, because of one-endedness, every boundary component of Σ is *used* by $\mathrm{Inc}_v \cup \mathcal{P}_{|G_v}$ [36, Subsection 2.5 and Theorem 13.1]: for any full boundary subgroup $B \in \mathcal{B}_v$, there exists a subgroup $H \in \mathrm{Inc}_v \cup \mathcal{P}_{|G_v}$ such that some G_v-conjugate of H is a finite index subgroup of B (hence equals B). This proves the converse. Since groups in $\mathcal{B}_v$ are virtually cyclic, $\mathcal{B}_v = \mathrm{Inc}_v$ if no P_i is virtually cyclic. This proves the claim.

When $\mathcal{H} \neq \emptyset$, we still have $\mathrm{Inc}_v \cup \mathcal{P}_{|G_v} \subset \mathcal{B}_v$. If H_j is infinite, the intersection of any of its conjugates with G_x is contained in a full boundary subgroup, in particular is virtually cyclic. Conversely, a group of $\mathcal{B}_v$ belongs to $\mathrm{Inc}_v \cup \mathcal{P}_{|G_v}$ or contains with finite index a G_v-conjugate of a group $H \in \mathcal{H}_{|G_v}$.

This analysis implies that a group P_i which is not virtually cyclic is contained in a rigid G_x or is equal to some G_Y (which may be contained in a rigid G_x). A group H_j which is not virtually cyclic is contained in a rigid G_x or in a G_Y.

Lemma 3.7. T_{can} *has finitely generated edge* (*hence vertex*) *stabilizers.*

We do not assume that the P_i's are slender, so there may exist infinitely generated elementary subgroups.

Proof. Since G is finitely presented relative to $\mathcal{P} \cup \mathcal{H}$, there is an elementary JSJ tree T_J relative to $\mathcal{P} \cup \mathcal{H}$ having finitely generated edge stabilizers ([35], Theorem 5.1). The tree T_{can} is the tree of cylinders of T_J.

Consider an edge $\varepsilon = (x, Y)$ of T_{can}, and $G_\varepsilon = G_x \cap G_Y$. We view Y as a subtree of T_J containing x. If G_Y is virtually cyclic, then G_ε is obviously finitely generated, so we can assume that G_Y is a maximal parabolic group P_i.

Since $G_Y = P_i$ is elliptic in T_J and leaves Y invariant, it fixes a vertex $y \in Y$. If $y = x$, then $G_Y \subset G_x$, so $G_\varepsilon = G_Y = P_i$ is finitely generated. If $y \neq x$, let e be the initial edge of the segment $[x, y]$ in T_J. It is contained in Y, so $G_e \subset G_Y$, hence $G_e \subset G_x \cap G_Y \subset G_x \cap G_y \subset G_e$. It follows that $G_\varepsilon = G_x \cap G_Y = G_e$ is finitely generated. $\square$

By Proposition 3.4, vertex groups of T_{can} are relatively quasiconvex, hence relatively hyperbolic. We make the parabolic structure explicit.

Lemma 3.8. *If* $x \in V_0(T_{\mathrm{can}})$, *the group* G_x *is hyperbolic relative to the finite family of finitely generated subgroups* $\mathcal{Q}_x = \mathrm{Inc}_x \cup \mathcal{P}_{|G_x}$, *where* Inc_x *is a set of representatives of conjugacy classes of incident edge stabilizers and* $\mathcal{P}_{|G_x}$ *is the induced structure (see Definitions* 2.1 *and* 2.3*).*

Proof. By Theorem 9.1 of [39], we have to consider infinite groups of the form $G_x \cap gP_ig^{-1}$. Recall that stabilizers of vertices adjacent to x are maximal elementary subgroups, and distinct maximal elementary subgroups have finite intersection. Thus gP_ig^{-1} must be the stabilizer of an adjacent vertex, or have x as unique fixed point, so $G_x \cap gP_ig^{-1}$ is conjugate in G_x to a group in $\mathrm{Inc}_x \cup \mathcal{P}_{|G_x}$. Conversely, a group in $\mathrm{Inc}_x \cup \mathcal{P}_{|G_x}$ which is not an infinite group of the form $G_x \cap gP_ig^{-1}$ is finite or is a loxodromic maximal virtually cyclic subgroup of G_x. Such groups may be added to the parabolic structure. $\square$

3.4. Rigid groups have finitely many automorphisms.

Theorem 3.9. *Let* G *be hyperbolic relative to finitely generated subgroups* $\mathcal{P} = \{P_1, \ldots, P_n\}$, *with* $P_i \neq G$. *Let* $\mathcal{H} = \{H_1, \ldots, H_q\}$ *be another family of finitely generated subgroups. If* $\mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})})$ *is infinite, then* G *splits over an elementary subgroup relative to* $\mathcal{P} \cup \mathcal{H}$.

The hypothesis means that there are infinitely many (classes of) automorphisms which map each P_i to a conjugate (in an arbitrary way) and act on each H_j as conjugation by an element of G.

The proof of the theorem has two steps. First, using the Bestvina-Paulin method (see [54]), extended by Belegradek-Szczepański [1] to relatively hyperbolic groups, one constructs an action of G on an $\mathbb{R}$-tree T. Rips theory then yields a splitting. This is fairly standard but there are technical difficulties, in particular because the action on T is not necessarily stable if the P_i's are not assumed to be slender. Details are in Section 9.

4. Automorphisms of one-ended relatively hyperbolic groups

Let G be hyperbolic relative to $\mathcal{P} = \{P_1, \ldots, P_n\}$, with P_i infinite and finitely generated. We assume that G is one-ended relative to $\mathcal{P}$. In Subsection 4.1 we study $\mathrm{Out}(G; \mathcal{P})$ through its action on the canonical JSJ tree. This leads to the main results of this section, which are stated in Subsection 4.2. In Subsection 4.3 we study automorphisms which act trivially on another family $\mathcal{H}$.

4.1. Automorphisms of the canonical JSJ splitting. Let T_{can} be the canonical $\mathrm{Out}(G;\mathcal{P})$-invariant JSJ tree as in Subsection 3.3 (with $\mathcal{H} = \emptyset$), and $\Gamma_{\mathrm{can}} = T_{\mathrm{can}}/G$. Edge stabilizers are infinite elementary subgroups. Vertex stabilizers may be rigid (non-elementary), QH with finite fiber, maximal parabolic (conjugate to a P_i), or maximal loxodromic (virtually cyclic). A rigid or QH stabilizer G_x fixes a unique point in T_{can}, hence is equal to its normalizer; incident edge stabilizers are maximal elementary subgroups of G_x.

We study $\mathrm{Out}(G;\mathcal{P})$ through its action on T_{can} as in Subsection 2.6. In general, $\mathrm{Out}(G;\mathcal{P})$ is a proper subgroup of $\mathrm{Out}(T_{\mathrm{can}})$.

We define finite index subgroups $\mathrm{Out}^0(G;\mathcal{P})$ and $\mathrm{Out}^0(G;\mathcal{P}^{(\mathrm{t})})$ by taking the intersection of $\mathrm{Out}(G;\mathcal{P})$ and $\mathrm{Out}(G;\mathcal{P}^{(\mathrm{t})})$ with the group $\mathrm{Out}^0(T_{\mathrm{can}})$ consisting of automorphisms acting trivially on the graph $\Gamma_{\mathrm{can}} = T_{\mathrm{can}}/G$.

By the second assertion of Lemma 2.12, the kernel of

$$\rho\colon \mathrm{Out}^0(T_{\mathrm{can}}) \longrightarrow \prod_{v \in V} \mathrm{Out}(G_v)$$

is the group of twists $\mathcal{T}$. Note that $\mathcal{T} \subset \mathrm{Out}^0(G;\mathcal{P}^{(\mathrm{t})}) \subset \mathrm{Out}^0(G;\mathcal{P})$ since every P_i is elliptic in T_{can} and a twist acts as a conjugation on any vertex stabilizer.

The group $\mathcal{T}$ is the image in $\mathrm{Out}(G)$ of a finite direct product $\prod_{e \in E} Z_{G_{o(e)}}(G_e)$. Each factor is virtually cyclic or contained in a conjugate of some P_i.

We now consider the image of $\mathrm{Out}^0(G;\mathcal{P})$ and $\mathrm{Out}^0(G;\mathcal{P}^{(\mathrm{t})})$ by

$$\rho_v\colon \mathrm{Out}^0(T_{\mathrm{can}}) \longrightarrow \mathrm{Out}(G_v),$$

for v a vertex of Γ_{can} (viewed as a vertex of T_{can} with stabilizer G_v). Using Theorem 3.9, we shall show that both images are finite if G_v is rigid. If $G_v = P_i$, the index of $\mathrm{Out}(G_v;\mathrm{Inc}_v^{(\mathrm{t})})$ in the image of $\mathrm{Out}^0(G;\mathcal{P})$ is finite because w is rigid whenever $e = vw$ is an edge with $\mathrm{Out}(G_e)$ infinite. More precisely:

Proposition 4.1. *The images of* $\mathrm{Out}^0(G;\mathcal{P})$ *and* $\mathrm{Out}^0(G;\mathcal{P}^{(\mathrm{t})})$ *by*

$$\rho_v\colon \mathrm{Out}^0(T_{\mathrm{can}}) \longrightarrow \mathrm{Out}(G_v)$$

may be described as follows:

- *if G_v is virtually cyclic or rigid, both images are finite*;
- *if G_v is a QH vertex stabilizer, both images contain $\mathrm{Out}(G_v;\mathcal{B}_v^{(\mathrm{t})})$ with finite index*;
- *if G_v is (conjugate to) P_i, the image of $\mathrm{Out}^0(G;\mathcal{P}^{(\mathrm{t})})$ is trivial. The image of $\mathrm{Out}^0(G;\mathcal{P})$ contains $\mathrm{Out}(G_v;\mathrm{Inc}_v^{(\mathrm{t})})$ with finite index.*

If e is any edge of T_{can}, the images of $\mathrm{Out}^0(G; \mathcal{P})$ *and* $\mathrm{Out}^0(G; \mathcal{P}^{(\mathrm{t})})$ *by*

$$\rho_e \colon \mathrm{Out}^0(T_{\mathrm{can}}) \longrightarrow \mathrm{Out}(G_e)$$

are finite.

See Definition 2.8 for the definition of the family of full boundary subgroups $\mathcal{B}_v$, and recall that $\mathcal{B}_v = \mathrm{Inc}_v$ if no P_i is virtually cyclic.

Proof. If G_v is virtually cyclic, $\mathrm{Out}(G_v)$ is finite. If G_v is rigid, finiteness follows from Theorem 3.9, as we now explain. We have seen (Lemma 3.8) that G_v is hyperbolic relative to a finite family $\mathcal{Q}_v = \mathrm{Inc}_v \cup \mathcal{P}_{|G_v}$ consisting of incident edge groups and conjugates of the P_i's having v as unique fixed point. These groups are finitely generated by Lemma 3.7. By Lemma 2.10, the group $\rho_v(\mathrm{Out}^0(G; \mathcal{P}))$ is contained in $\mathrm{Out}(G_v; \mathcal{Q}_v)$. If it is infinite, G_v splits over an elementary subgroup relative to $\mathcal{Q}_v$ by Theorem 3.9 (applied with $\mathcal{P} = \mathcal{Q}_v$ and $\mathcal{H} = \emptyset$). By Lemma 2.6, this splitting may be used to refine T_{can}, yielding an elementary splitting of G relative to $\mathcal{P}$ in which G_v is not elliptic. This contradicts rigidity.

If G_v is QH, first note that

$$\mathrm{Out}(G_v; \mathcal{B}_v^{(\mathrm{t})}) = \mathrm{Out}(G_v; \mathrm{Inc}_v^{(\mathrm{t})}, \mathcal{P}_{|G_v}^{(\mathrm{t})})$$

and

$$\mathrm{Out}(G_v; \mathcal{B}_v) = \mathrm{Out}(G_v; \mathrm{Inc}_v, \mathcal{P}_{|G_v})$$

because $\mathcal{B}_v = \mathrm{Inc}_v \cup \mathcal{P}_{|G_v}$ (see Subsection 3.3, recalling that groups in $\mathcal{P}$ are assumed to be infinite).

Lemma 2.10 then yields

$$\mathrm{Out}(G_v; \mathcal{B}_v^{(\mathrm{t})}) \subset \rho_v(\mathrm{Out}^0(G; \mathcal{P}^{(\mathrm{t})})) \subset \rho_v(\mathrm{Out}^0(G; \mathcal{P})) \subset \mathrm{Out}(G_v; \mathcal{B}_v).$$

We conclude by observing that $\mathrm{Out}(G_v; \mathcal{B}_v^{(\mathrm{t})})$ has finite index in $\mathrm{Out}(G_v; \mathcal{B}_v)$ because groups in $\mathcal{B}_v$ are virtually cyclic, hence have finite outer automorphism group.

If G_v is (conjugate to) P_i, the image of $\mathrm{Out}^0(G; \mathcal{P}^{(\mathrm{t})})$ is clearly trivial. Since $\mathcal{P}_{|G_v}$ equals $\{G_v\}$ or is empty, the image of $\mathrm{Out}^0(G; \mathcal{P})$ contains $\mathrm{Out}(G_v; \mathrm{Inc}_v^{(\mathrm{t})}) = \mathrm{Out}(G_v; \mathrm{Inc}_v^{(\mathrm{t})}, \mathcal{P}_{|G_v})$, and we have to show that the index is finite. If $\mathrm{Out}(G_\varepsilon)$ is finite for every incident edge ε, this follows from Lemma 2.10. In general, we have to control the action of automorphisms on G_ε for incident edges ε with $\mathrm{Out}(G_\varepsilon)$ infinite. Note that there is no natural map from $\mathrm{Out}(G_v; \mathrm{Inc}_v)$ to $\mathrm{Out}(G_\varepsilon)$ if $N_{G_v}(G_\varepsilon)$ acts non-trivially on G_ε.

Infiniteness of $\mathrm{Out}(G_\varepsilon)$ implies that the other endpoint of ε is a rigid vertex x: it cannot be QH since G_ε would then be virtually cyclic. As explained above, the image of $\mathrm{Out}^0(G; \mathcal{P})$ in $\mathrm{Out}(G_x)$ is finite. Any $\Phi \in \mathrm{Out}^0(G; \mathcal{P})$ has a representative α leaving G_x and G_v invariant (the associated map f_α fixes ε). Replacing $\mathrm{Out}^0(G; \mathcal{P})$ by a finite index subgroup, we may suppose that α acts on G_x as conjugation by some element g. This g must be in G_x since G_x equals its normalizer, and in fact in G_ε because G_ε is almost malnormal in G_x. This shows that Φ maps into $\mathrm{Out}(G_v; G_\varepsilon^{(\mathrm{t})})$. Arguing in this way for each incident edge proves the result for the image of ρ_v.

Since any edge of T_{can} has a vertex v with G_v virtually cyclic or conjugate to a P_i, the previous argument also shows finiteness for images by ρ_e.   □

Definition 4.2. Let Σ be a compact 2-dimensional hyperbolic orbifold. The *extended mapping class group* $MCG^*(\Sigma)$ is the group of outer automorphisms of $\pi_1(\Sigma)$ preserving the set of boundary subgroups.

If v is a QH vertex of T_{can} with underlying orbifold Σ and finite fiber F, we define $MCG^0_{T_{\mathrm{can}}}(\Sigma)$ as the group $\mathrm{Out}(G_v; \mathcal{B}_v^{(\mathrm{t})})$. This group maps with finite kernel onto a finite index subgroup of $MCG^*(\Sigma)$ [17, pp. 240 and 268].

As noted in [26, 17], one can understand the extended mapping class group of an orbifold with mirrors in terms of the extended mapping class group of a suborbifold without mirrors.

If G is torsion free, Σ is a surface, $MCG^*(\Sigma)$ is the group of isotopy classes of homeomorphisms, and $MCG^0_{T_{\mathrm{can}}}(\Sigma)$ is the group of isotopy classes of homeomorphisms which map each boundary component to itself in an orientation-preserving way (in this case it only depends on Σ since the fiber is trivial).

Mapping class groups are usually infinite, but there are exceptions. In the torsion-free case, the exceptions are the pair of pants and the twice punctured projective plane [44, Cor 4.6]; all other hyperbolic surfaces contain an essential 2-sided simple closed curve not bounding a Möbius band, so there is a Dehn twist of infinite order. As a QH vertex, a pair of pants is rigid; every simple closed curve is homotopically trivial or boundary parallel. A twice punctured projective plane is flexible, but every 2-sided simple closed curve is homotopically trivial, boundary parallel, or bounds a Möbius band, so there is no non-trivial Dehn twist.

4.2. Automorphisms of G. Motivated by the previous subsection, we define a subgroup $\mathrm{Out}^1(G; \mathcal{P}^{(\mathrm{t})}) \subset \mathrm{Out}^0(G; \mathcal{P}^{(\mathrm{t})})$ as the set of Φ such that $\rho_v(\Phi)$ is trivial if G_v is virtually cyclic, rigid, or conjugate to a P_i, and

$$\rho_v(\Phi) \in \mathrm{Out}(G_v; \mathcal{B}_v^{(\mathrm{t})}) = MCG^0_{T_{\mathrm{can}}}(\Sigma)$$

if G_v is QH. Proposition 4.1 shows that this subgroup has finite index. We define a finite index subgroup $\mathrm{Out}^1(G; \mathcal{P}) \subset \mathrm{Out}^0(G; \mathcal{P})$ similarly, allowing $\rho_v(\Phi) \in \mathrm{Out}(G_v; \mathrm{Inc}_v^{(\mathrm{t})})$ if G_v is conjugate to a P_i.

We may now sum up the discussion in Subsection 4.1 as:

Theorem 4.3. *Let G be hyperbolic relative to $\mathcal{P} = \{P_1, \ldots, P_n\}$, with P_i infinite and finitely generated. Assume that G is one-ended relative to $\mathcal{P}$.*

Then $\mathrm{Out}(G; \mathcal{P}^{(\mathrm{t})})$ and $\mathrm{Out}(G; \mathcal{P})$ have finite index subgroups $\mathrm{Out}^1(G; \mathcal{P}^{(\mathrm{t})})$ and $\mathrm{Out}^1(G; \mathcal{P})$ which fit in exact sequences

$$1 \longrightarrow \mathcal{T} \longrightarrow \mathrm{Out}^1(G; \mathcal{P}^{(\mathrm{t})}) \longrightarrow \prod_{i=1}^{p} MCG^0_{T_{\mathrm{can}}}(\Sigma_i) \longrightarrow 1$$

and

$$1 \longrightarrow \mathcal{T} \longrightarrow \mathrm{Out}^1(G; \mathcal{P}) \longrightarrow \prod_{i=1}^{p} MCG^0_{T_{\mathrm{can}}}(\Sigma_i) \times \prod_{j} \mathrm{Out}(P_j; \mathrm{Inc}_{P_j}^{(\mathrm{t})}) \longrightarrow 1,$$

where:

- *$\mathcal{T}$ is the group of twists of the canonical elementary JSJ decomposition T_{can} relative to $\mathcal{P}$; it is a quotient of a finite direct product where each factor is a subgroup of G which is virtually cyclic or contained in a P_i;*

- *$\Sigma_1, \ldots, \Sigma_p$ are the 2-orbifolds occuring in flexible QH vertices v of T_{can}, and $MCG^0_{T_{\mathrm{can}}}(\Sigma_i)$ maps with finite kernel onto a finite index subgroup of the extended mapping class group $MCG^*(\Sigma_i)$;*

- *the last product is taken only over those P_j's which occur as vertex stabilizers of T_{can}, and Inc_{P_j} is the set of incident edge groups.*

Note that $\mathcal{T}$ is slender (resp. small, virtually solvable, virtually nilpotent, virtually abelian) if the P_i's are. Also note that $\mathrm{Out}(G; \mathcal{P})$ has finite index in $\mathrm{Out}(G)$ if the P_i's are small but not virtually cyclic, since they may be characterized up to conjugacy as maximal among the subgroups of G which are small but not virtually cyclic. More generally, this holds if no P_i is relatively hyperbolic [51, Lemma 3.2].

Recall that G is *toral relatively hyperbolic* if it is torsion-free and hyperbolic relative to a finite family $\mathcal{P}$ of finitely generated abelian subgroups. Limit groups, and more generally groups acting freely on $\mathbb{R}^n$-trees, are toral relatively hyperbolic [15, 30].

Corollary 4.4. *Let G be toral relatively hyperbolic and one-ended. Then some finite index subgroup $\mathrm{Out}^1(G)$ of $\mathrm{Out}(G)$ fits in an exact sequence*

$$1 \longrightarrow \mathcal{T} \longrightarrow \mathrm{Out}^1(G) \longrightarrow \prod_{i=1}^{p} MCG^0(\Sigma_i) \times \prod_{k=1}^{m} \mathrm{GL}_{r_k,n_k}(\mathbb{Z}) \longrightarrow 1$$

where $\mathcal{T}$ is finitely generated free abelian, $MCG^0(\Sigma_i)$ is the group of isotopy classes of homeomorphisms of a compact surface Σ_i mapping each boundary component to itself in an orientation-preserving way, and

$$\mathrm{GL}_{r,n}(\mathbb{Z}) = M_{r,n}(\mathbb{Z}) \rtimes \mathrm{GL}_r(\mathbb{Z})$$

is the group of automorphisms of $\mathbb{Z}^{n+r}$ fixing the first n generators.

See Theorem 5.3 of [9] and Theorem 6.5 of [33] for the case of limit groups, based on results from [43] and [9].

Proof. We may assume that no $P_i \in \mathcal{P}$ is cyclic, so $\mathrm{Out}(G; \mathcal{P})$ has finite index in $\mathrm{Out}(G)$. If P_j is isomorphic to $\mathbb{Z}^a$, then $\mathrm{Out}(P_j; \mathrm{Inc}_{P_j}^{(\mathrm{t})})$ is isomorphic to some $\mathrm{GL}_{r,n}(\mathbb{Z})$ with $r + n = a$, so the exact sequence follows from Theorem 4.3. We know that the group of twists $\mathcal{T}$ of T_{can} is finitely generated and abelian. There remains to check that it is torsion-free.

Recall from Subsection 2.6 that $\mathcal{T}$ is generated by the product of all $Z_{G_{o(e)}}(G_e)$, subject to edge and vertex relations. Denoting an edge of $\Gamma_{\mathrm{can}} = T_{\mathrm{can}}/G$ by $\varepsilon = (x, Y)$ with $x \in V_0(\Gamma_{\mathrm{can}})$ and $Y \in V_1(\Gamma_{\mathrm{can}})$, there is no relation at the vertex x since $Z(G_x)$ is trivial. Moreover, $Z_{G_x}(G_\varepsilon) = G_\varepsilon$, so the edge relation identifies the twists around ε near x with twists near Y. Thus $\mathcal{T}$ is the direct product, over vertices v of Γ_{can} carrying an abelian group, of $(\prod_{e \in E_v} Z_{G_v}(G_e))/Z(G_v)$ where E_v is the set of oriented edges with origin v and $Z(G_v)$ is embedded diagonally. Since G_v is abelian, $Z_{G_v}(G_e) = Z(G_v) = G_v$, so $\mathcal{T}$ is isomorphic to a finite direct product $\prod(G_v)^{|E_v|-1}$ of abelian vertex groups. It is therefore torsion-free. $\square$

Corollary 4.5. *If G is a toral relatively hyperbolic group, $\mathrm{Out}(G)$ is virtually torsion-free and has a finite index subgroup with a finite classifying space.*

Proof. This follows from Corollary 4.4 if G is one-ended. In general, we write $G = G_1 * \cdots * G_q * F$ with G_ℓ one-ended and F free. All groups G_ℓ and $G_\ell/Z(G_\ell)$ have a finite classifying space [14], so we can apply Theorem 5.2 of [33]. (We mention here that the arguments given in [33] are insufficient to get a finite classifying space: there should exist finite classifying spaces for the groups $\mathrm{Out}^S(G)$ themselves (rather than for finite index subgroups); this is achieved by restricting to some finite index subgroup of $\mathrm{Out}(G)$, see [37] for details.) $\square$

4.3. The relative case. We generalize the analysis of Subsection 4.1 to a relative situation.

Let $G, \mathcal{P}$ be as above, and fix another family of finitely generated subgroups $\mathcal{H} = \{H_1, \ldots, H_q\}$. Assume that G is one-ended relative to $\mathcal{P} \cup \mathcal{H}$, and let now T_{can} be the canonical elementary JSJ tree relative to $\mathcal{P} \cup \mathcal{H}$.

Theorem 4.6. *Under these hypotheses, both* $\text{Out}(G; \mathcal{P}^{(t)}, \mathcal{H}^{(t)})$ *and* $\text{Out}(G; \mathcal{P}, \mathcal{H}^{(t)})$ *have finite index subgroups* $\text{Out}^1(G; \mathcal{P}^{(t)}, \mathcal{H}^{(t)})$ *and* $\text{Out}^1(G; \mathcal{P}, \mathcal{H}^{(t)})$ *which fit in exact sequences*

$$1 \longrightarrow \mathcal{T} \longrightarrow \text{Out}^1(G; \mathcal{P}^{(t)}, \mathcal{H}^{(t)}) \longrightarrow \prod_{i=1}^{p} MCG_{T_{\text{can}}}^1(\Sigma_i) \longrightarrow 1$$

and

$$1 \longrightarrow \mathcal{T} \longrightarrow \text{Out}^1(G; \mathcal{P}, \mathcal{H}^{(t)})$$
$$\longrightarrow \prod_{i=1}^{p} MCG_{T_{\text{can}}}^1(\Sigma_i) \times \prod_{j} \text{Out}(P_j; \text{Inc}_{P_j}^{(t)}, \mathcal{H}_{|P_j}^{(t)}) \longrightarrow 1,$$

as in Theorem 4.3.

The group $MCG_{T_{\text{can}}}^1(\Sigma_i)$ *equals* $\text{Out}(G_v; \mathcal{B}_v^{(t)}, \mathcal{F}_v^{(t)})$, *where* $\mathcal{F}_v$ *is a set of representatives of conjugacy classes of finite subgroups in* G_v; *it is a finite index subgroup of* $MCG_{T_{\text{can}}}^0(\Sigma_i) = \text{Out}(G_v; \mathcal{B}_v^{(t)})$.

Note that $\mathcal{F}_v$ is a finite set since the QH vertex group G_v maps onto a 2-orbifold group with finite kernel. The family $\mathcal{F}_v$ is not needed if all groups in $\mathcal{H}$ are infinite (see the proof below).

The theorem is proved as in the absolute case, replacing Proposition 4.1 by the following result.

Proposition 4.7. *The images of* $\text{Out}^0(G; \mathcal{P}, \mathcal{H}^{(t)})$ *and* $\text{Out}^0(G; \mathcal{P}^{(t)}, \mathcal{H}^{(t)})$ *by*

$$\rho_v : \text{Out}^0(T_{\text{can}}) \longrightarrow \text{Out}(G_v)$$

may be described as follows;

- *if G_v is virtually cyclic or rigid, both images are finite:*
- *if G_v is a QH vertex stabilizer, both images contain* $\text{Out}(G_v; \mathcal{B}_v^{(t)}, \mathcal{F}_v^{(t)})$ *with finite index (where $\mathcal{F}_v$ is as in Theorem 4.6 and $\mathcal{B}_v$ is as in Definition 2.8);*
- *if G_v is (conjugate to) P_i, the image of* $\text{Out}^0(G; \mathcal{P}^{(t)}, \mathcal{H}^{(t)})$ *is trivial. The image of* $\text{Out}^0(G; \mathcal{P}, \mathcal{H}^{(t)})$ *contains* $\text{Out}(G_v; \text{Inc}_v^{(t)}, \mathcal{H}_{|G_v}^{(t)})$ *with finite index.*

Proof. We only mention the differences with the proof of Proposition 4.1.

If v is a (non-elementary) rigid vertex, the images of $\mathrm{Out}^0(G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})})$ and $\mathrm{Out}^0(G; \mathcal{P}^{(\mathrm{t})}, \mathcal{H}^{(\mathrm{t})})$ by ρ_v are contained in $\mathrm{Out}(G_v; \mathcal{Q}_v, \mathcal{H}^{(\mathrm{t})}_{|G_v})$ by Lemma 2.10. It follows that the images are finite since, otherwise, Theorem 3.9 would yield a splitting relative to $\mathcal{Q}_v \cup \mathcal{H}_{|G_v}$, which extends to a splitting of G relative to $\mathcal{P} \cup \mathcal{H}$ by Lemma 2.6.

When G_v is conjugate to a parabolic group P_j, Lemma 2.10 says that the image of $\mathrm{Out}^0(G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})})$ contains $\mathrm{Out}(G_v; \mathrm{Inc}^{(\mathrm{t})}_v, \mathcal{H}^{(\mathrm{t})}_{|G_v})$, and the index is finite for the same reason as before.

When G_v is QH, we write

$$\mathrm{Out}(G_v; \mathrm{Inc}^{(\mathrm{t})}_v, \mathcal{P}_{|G_v}, \mathcal{H}^{(\mathrm{t})}_{|G_v}) \subset \rho_v(\mathrm{Out}^0(T; \mathcal{P}, \mathcal{H}^{(\mathrm{t})}))$$
$$\subset \mathrm{Out}(G_v; \mathrm{Inc}_v, \mathcal{P}_{|G_v}, \mathcal{H}^{(\mathrm{t})}_{|G_v})$$

using Lemma 2.10.

The proof in the non-relative case relied on the equality

$$\mathcal{B}_v = \mathrm{Inc}_v \cup \mathcal{P}_{|G_v}.$$

Here (see Subsection 3.3) we have $\mathrm{Inc}_v \cup \mathcal{P}_{|G_v} \subset \mathcal{B}_v$, and a group $B \in \mathcal{B}_v$ not in $\mathrm{Inc}_v \cup \mathcal{P}_{|G_v}$ contains with finite index a group H' conjugate to some $H \in \mathcal{H}_{|G_v}$. Since B is the only maximal elementary subgroup of G_v containing H', any automorphism preserving H' preserves B, so

$$\mathrm{Out}(G_v; \mathrm{Inc}_v, \mathcal{P}_{|G_v}, \mathcal{H}^{(\mathrm{t})}_{|G_v}) \subset \mathrm{Out}(G_v; \mathcal{B}_v).$$

If all groups in $\mathcal{H}$ are infinite, the intersection of any conjugate of H_j with G_v is contained in a full boundary subgroup, so

$$\mathrm{Out}(G_v; \mathcal{B}^{(\mathrm{t})}_v) \subset \mathrm{Out}(G_v; \mathrm{Inc}^{(\mathrm{t})}_v, \mathcal{P}_{|G_v}, \mathcal{H}^{(\mathrm{t})}_{|G_v}).$$

Otherwise $\mathcal{H}_{|G_v}$ may contain finite groups (fixing v but no other vertex of T_{can}) and we can only write

$$\mathrm{Out}(G_v; \mathcal{B}^{(\mathrm{t})}_v, \mathcal{F}^{(\mathrm{t})}_v) \subset \mathrm{Out}(G_v; \mathrm{Inc}^{(\mathrm{t})}_v, \mathcal{P}_{|G_v}, \mathcal{H}^{(\mathrm{t})}_{|G_v}).$$

The proposition follows because the index of $\mathrm{Out}(G_v; \mathcal{B}^{(\mathrm{t})}_v, \mathcal{F}^{(\mathrm{t})}_v)$ in $\mathrm{Out}(G_v; \mathcal{B}_v)$ is finite. $\qquad\square$

Remark 4.8. Because we use Theorem 3.9 to control automorphisms of rigid groups, we do not have a similar result concerning $\mathrm{Out}(G; \mathcal{P}^{(\mathrm{t})} \cup \mathcal{H})$ or $\mathrm{Out}(G; \mathcal{P} \cup \mathcal{H})$: we have to impose that automorphisms act trivially on $\mathcal{H}$. We also need finite generation of groups in $\mathcal{H}$.

Arguing as in the previous subsection, one gets:

Corollary 4.9. *Let G be toral relatively hyperbolic, one-ended relative to a family $\mathcal{H} = \{H_1, \ldots, H_q\}$ of finitely generated subgroups. Then $\mathrm{Out}(G; \mathcal{H}^{(t)})$ has a finite index subgroup $\mathrm{Out}^1(G; \mathcal{H}^{(t)})$ fitting in an exact sequence as in Corollary 4.4.*

5. The modular group

The goal of this section is to show that the modular group, usually defined by considering all suitable splittings of a group G, may be seen on a single splitting, namely the canonical JSJ decomposition.

5.1. Definitions and examples. Let G be hyperbolic relative to $\mathcal{P} = \{P_1, \ldots, P_n\}$, where each P_i is finitely generated. Without loss of generality, we assume that no P_i is virtually cyclic (in particular, P_i is infinite). Let $\mathcal{H}$ be another finite family of finitely generated subgroups H_j such that every P_i which contains a free group F_2 is contained in a group of $\mathcal{H}$.

In particular, we may take $\mathcal{H} = \mathcal{P}$, or $\mathcal{H} = \emptyset$ if every P_i is small. We will assume that G is one-ended relative to $\mathcal{H}$ (equivalently, relative to $\mathcal{P} \cup \mathcal{H}$ since every P_i is one-ended or contained in a group of $\mathcal{H}$).

We consider trees T with elementary edge stabilizers, which are relative to $\mathcal{H}$ (universal ellipticity will be with respect to these trees, unless indicated otherwise). They are not necessarily relative to P_i if P_i is small, but our assumption on $\mathcal{H}$ implies that elementary subgroups which are not small have a conjugate contained in a group in $\mathcal{H}$, so are universally elliptic. We shall associate a modular group $\mathrm{Mod}(T) \subset \mathrm{Out}(T) \subset \mathrm{Out}(G)$ to such a tree T.

Lemma 5.1. *If v is a flexible QH vertex with finite fiber, then any elementary subgroup of G_v is virtually cyclic.*

Recall (Definition 2.7) that G_v maps onto $\pi_1(\Sigma)$ with finite kernel F; flexibility of G_v is equivalent to the 2-orbifold Σ containing an essential 1-suborbifold. Since T is only assumed to be relative to $\mathcal{H}$, Definition 2.7 only restricts intersections of G_v with conjugates of groups in $\mathcal{H}$.

Proof. Assume on the contrary that some subgroup $E < G_v$ is elementary, but not virtually cyclic. Then E contains F_2 and is parabolic. As pointed out before, our assumption on $\mathcal{H}$ implies that E is universally elliptic. This contradicts Remark 2.9 saying that such a group has to be virtually cyclic. □

Remark. If G_v is a flexible QH vertex stabilizer with elementary fiber F, then F is finite. Indeed, if F is infinite, then G_v is elementary by almost malnormality of maximal elementary subgroups (Corollary 3.2). Since it contains F_2, it is universally elliptic, contradicting flexibility.

Definition 5.2. We say that a vertex v of T (or of $\Gamma = T/G$) is *modular* if G_v is flexible and QH (relative to $\mathcal{H}$) with finite fiber, or G_v is elementary. Note that G_v cannot be both.

Recall (Subsection 2.6) the maps $\rho_v \colon \mathrm{Out}^0(T) \to \mathrm{Out}(G_v)$ defined on the finite index subgroup of $\mathrm{Out}(T)$ consisting of automorphisms acting trivially on $\Gamma = T/G$.

Definition 5.3. We define $\mathrm{Mod}(T)$ by saying that $\Phi \in \mathrm{Out}^0(T)$ belongs to $\mathrm{Mod}(T)$ if it satisfies the following conditions:

- if v is not modular, $\rho_v(\Phi)$ is trivial;
- if G_v is elementary, $\rho_v(\Phi) \in \mathrm{Out}(G_v; \mathrm{Inc}_v^{(t)})$; in other words, $\rho_v(\Phi)$ acts on each incident edge group as a conjugation;
- if G_v is QH with finite fiber, and flexible, then $\rho_v(\Phi) \in \mathrm{Out}(G_v; \mathcal{B}_v^{(t)})$, with $\mathcal{B}_v$ consisting of full preimages of boundary subgroups of $\pi_1(\Sigma)$ as in Definition 2.8.

Note that $\mathrm{Mod}(T)$ contains the group of twists $\mathcal{T}(T)$, and that automorphisms in $\mathrm{Mod}(T)$ need not preserve $\mathcal{H}$.

We have assumed that G is one-ended relative to $\mathcal{H}$, so we can consider the canonical elementary JSJ tree T_{can} relative to $\mathcal{P} \cup \mathcal{H}$ as in Subsection 3.3. Note that $\mathrm{Mod}(T_{\mathrm{can}})$ has finite index in $\mathrm{Out}(G; \mathcal{P})$ when $\mathcal{H} = \mathcal{P}$ (it contains the group $\mathrm{Out}^1(G; \mathcal{P})$ defined in Subsection 4.2, possibly strictly because of vertices with G_v virtually cyclic).

Theorem 5.4. *Let G be hyperbolic relative to $\mathcal{P} = \{P_1, \ldots, P_n\}$, with each P_i finitely generated, not virtually cyclic. Let $\mathcal{H}$ be a family of subgroups such that every P_i which contains F_2 is contained in a group of $\mathcal{H}$, and G is one-ended relative to $\mathcal{H}$.*

If T is any elementary splitting of G relative to $\mathcal{H}$, then $\mathrm{Mod}(T) \subset \mathrm{Mod}(T_{\mathrm{can}})$, where T_{can} is the canonical elementary JSJ tree relative to $\mathcal{P} \cup \mathcal{H}$.

This applies in particular if G is one-ended and no P_i contains F_2 (taking $\mathcal{H} = \emptyset$), or if G is one-ended relative to an arbitrary $\mathcal{P}$ and we restrict to splittings relative to $\mathcal{P}$ (taking $\mathcal{H} = \mathcal{P}$). The theorem will be proved in the next subsection.

Remark 5.5. Rather than defining $\mathrm{Mod}(T)$ by imposing conditions on the action on vertex groups, as we just did, one could define it by giving generators: twists around edges, and certain automorphisms of vertex groups. This would yield a slightly smaller group $\mathrm{Mod}'(T)$: its intersection with $\ker \rho$ is $\mathcal{T}$, whereas $\mathrm{Mod}(T)$ contains all of $\ker \rho$. Theorem 5.4 (and Theorem 5.6 below) also hold with this more restrictive definition, since $\mathrm{Mod}'(T_{\mathrm{can}}) = \mathrm{Mod}(T_{\mathrm{can}})$ by Assertion 2 of Lemma 2.12.

There is a similar statement for torsion-free CSA groups (recall that G is CSA if centralizers of non-trivial elements are abelian and malnormal). We now consider abelian splittings of G. A vertex v is modular if G_v is either abelian or QH as above (in this case F is trivial and Σ is a surface). The definition of $\mathrm{Mod}(T)$ is the same (with elementary replaced by abelian). The tree T_{can} is the canonical abelian JSJ tree relative to non-cyclic abelian subgroups; it is also the tree of cylinders of the (non-relative) abelian JSJ deformation space (see Theorem 11.1 of [36]).

Theorem 5.6. *Let G be a finitely generated, torsion-free, one-ended, CSA group. If T is any splitting of G over abelian groups, then $\mathrm{Mod}(T) \subset \mathrm{Mod}(T_{\mathrm{can}})$, where T_{can} is the tree of cylinders of the abelian JSJ deformation space.*

Example 5.7. Let G be the Baumslag–Solitar group

$$\mathrm{BS}(2, 4) = \langle a, b \mid ba^2 b^{-1} = a^4 \rangle.$$

Any splitting of G as a graph of infinite cyclic groups is a cyclic JSJ decomposition of G [25, 35]. Its modular group coincides with its group of twists, and is a finite abelian group (see [47]). But the JSJ deformation space of $\mathrm{BS}(2, 4)$ is quite large [11], and JSJ splittings of $\mathrm{BS}(2, 4)$ may have modular groups of arbitrarily large order. In particular, there is no splitting whose modular group contains all others.

Example 5.8. Even if G is as in Theorem 5.4, one cannot replace T_{can} by an arbitrary tree in its deformation space: there exists such trees whose modular group is not maximal (even up to finite index).

Indeed, let $G = A_1 *_{C_1} B *_{C_2} A_2$, where:

- A_1 and A_2 are torsion-free hyperbolic groups with no cyclic splitting;
- C_i is a maximal infinite cyclic subgroup of A_i;
- B is torsion-free, hyperbolic relative to a subgroup

$$\hat{C} = C_1 \oplus C_2 \oplus \mathbb{Z} \simeq \mathbb{Z}^3,$$

and does not split over an abelian group.

The group G is hyperbolic relative to $\hat{C}$ by [15].

The graph of groups

$$
\begin{array}{c}
B \\
\big| C_1 \oplus C_2 \\
A_1 \; \underset{C_1}{\rule{1.5em}{0.4pt}} \; C_1 \oplus C_2 \; \underset{C_2}{\rule{1.5em}{0.4pt}} \; A_2
\end{array}
$$

is an elementary JSJ decomposition of G (both absolute and relative to $\widehat{C}$) because its vertex groups are universally elliptic (see Lemma 4.7 of [35]). Given any $z \in \widehat{C} \setminus (C_1 \oplus C_2)$, the automorphism τ defined as the identity on A_1 and B and as conjugation by z on A_2 is not an automorphism of this graph of groups. But τ is a twist of T_{can}, which is the Bass-Serre tree of the graph of groups below.

$$
\begin{array}{c}
B \\
\big| \widehat{C} \\
A_1 \; \underset{C_1}{\rule{1.5em}{0.4pt}} \; \widehat{C} \; \underset{C_2}{\rule{1.5em}{0.4pt}} \; A_2
\end{array}
$$

5.2. Proof of Theorem 5.4. We prove Theorem 5.4. The proof of Theorem 5.6 is similar and left to the reader. The main difference in the context of a general CSA group is that we have no version of Theorem 3.9 saying that a rigid vertex group only has finitely many outer automorphisms. But the proof given below does not use Theorem 3.9.

By Theorem 13.1 of [36], the trees T_{can} and T are compatible: they have a common refinement $\widehat{T}$ (as defined in Subsection 2.3). We may assume that no edge of $\widehat{T}$ is collapsed in both T_{can} and T (so $\widehat{T}$ is the least common multiple of T_{can} and T as defined in Section 3 of [36]). The tree $\widehat{T}$ has elementary edge stabilizers and is relative to $\mathcal{H}$ since T_{can} and T are (Proposition 3.22 of [36]).

We first claim that T_{can} is $\mathrm{Mod}(T)$-invariant. To see this, it suffices to show that the image of an infinite group $J \in \mathcal{P} \cup \mathcal{H}$ by a modular automorphism Φ has a finite index subgroup which is contained (up to conjugacy) in a group belonging to $\mathcal{P} \cup \mathcal{H}$.

If J is a small P_i, its image is elementary and not virtually cyclic, so is parabolic. If J is a P_i containing F_2, it is contained in a group of $\mathcal{H}$, so we only have to consider groups $J \in \mathcal{H}$. Such a group fixes a vertex v in T, and G_v is Φ-invariant. We distinguish three cases.

If v is non-modular, Φ acts trivially on J. If v is a flexible QH vertex, then J is contained in a group of $\mathcal{B}_v$, hence Φ-invariant. Now suppose that J is contained in an elementary G_v. If G_v contains F_2, it is a Φ-invariant group contained in a group of $\mathcal{H}$, so the image of J is contained in a group of $\mathcal{H}$. The case when G_v is

small but not virtually cyclic has been dealt with before. If G_v is virtually cyclic, $\Phi(J)$ has a finite index subgroup contained in J. This completes the proof of the claim.

Let $\Phi \in \mathrm{Mod}(T)$. The heart of the proof of the theorem is to study the action of Φ on a non-elementary vertex stabilizer G_v of T_{can} (it is rigid or QH). In particular, given an edge $e = vw$ in T_{can}, we show that Φ has a representative α leaving G_v invariant and equal to the identity on G_e.

We have defined T_{can} as a JSJ tree relative to $\mathcal{P} \cup \mathcal{H}$. When T_{can} is viewed as relative to $\mathcal{H}$ only, a flexible QH vertex remains flexible QH. It follows from Sections 8 and 13 of [36] that a rigid (non-elementary) vertex stabilizer G_v of T_{can} remains universally elliptic relative to $\mathcal{H}$. The argument goes as follows: if T is an elementary splitting relative to $\mathcal{H}$, then G_v is elliptic in its tree of cylinders T_c because T_c is relative to $\mathcal{H} \cup \mathcal{P}$ and G_v is rigid relative to $\mathcal{H} \cup \mathcal{P}$; since groups elliptic in T_c but not in T are elementary (see Subsection 2.7), and G_v is not, this implies that G_v is elliptic in T.

We distinguish two cases.

Case 1: v is rigid. Then G_v is universally elliptic, so fixes a point u in T, which is unique because edge stabilizers are elementary and G_v is not. The group G_u cannot be elementary. By Remark 2.9, it cannot be flexible QH because its subgroup G_v is universally elliptic and non-elementary. Thus u is not modular and Φ has a representative α equal to the identity on G_u, hence on G_v.

Case 2: v is a flexible QH vertex of T_{can}. Let e be an adjacent edge and $\hat{e}$ its lift to $\widehat{T}$. Recall that $G_e = G_{\hat{e}}$ is a maximal elementary subgroup of G_v. We define a point $\hat{v} \in \widehat{T}$ as follows. If G_v is elliptic in $\widehat{T}$, we call $\hat{v}$ its unique fixed point. If it is not elliptic, its action on its minimal subtree in $\widehat{T}$ is dual to a family of 1-suborbifolds of Σ (see Lemma 7.4 of [35]). We let $\hat{v}$ be the point of that subtree closest to $\hat{e}$ (possibly an endpoint of $\hat{e}$).

The stabilizer of $\hat{v}$ is QH, associated to a suborbifold $\widehat{\Sigma}$ of Σ (if $\widehat{\Sigma}$ contains no essential 1-suborbifold, $\hat{v}$ is a rigid vertex of $\widehat{T}$). Note that $G_{\hat{v}}$ is non-elementary by Lemma 5.1. The stabilizer of $\hat{e}$, and also of edges between $\hat{e}$ and $\hat{v}$ if any, is contained in $G_{\hat{v}}$, in fact in the preimage of a boundary subgroup of $\pi_1(\Sigma)$ and $\pi_1(\widehat{\Sigma})$.

Let u be the image of $\hat{v}$ in T.

Subcase 2a: u is not modular. Then Φ has a representative α equal to the identity on G_u, hence on $G_{\hat{v}}$ and on $G_e = G_{\hat{e}}$. Note that α leaves G_v invariant because α is an automorphism of T_{can} and v is the only vertex of T_{can} fixed by $G_{\hat{v}}$.

Subcase 2b: G_u is QH with finite fiber. Then G_u is elliptic in T_{can} (see Proposition 7.13 of [35], which is valid in a relative setting), hence in $\widehat{T}$, cf. [36, Proposi-

tion 3.22], so $G_u = G_{\hat{v}}$. Unfortunately, this argument says nothing about incident edge groups at u.

First suppose that $\hat{v}$ is an endpoint of $\hat{e}$. Choose an edge path with origin $\hat{v}$, starting with $\hat{e}$, such that all edges except the last one get collapsed to u in T (this path consists of the single edge $\hat{e}$ if $\hat{e}$ is not collapsed to a point in T). Call this last edge $\hat{e}'$, and its initial vertex a. We have $G_{\hat{e}'} \subset G_a \subset G_u = G_{\hat{v}}$, so $G_{\hat{e}'} \subset G_{\hat{e}}$.

The group G_u is QH with finite fiber, and $G_{\hat{e}'}$ is an incident edge group. It is infinite by one-endedness, so is contained in a unique maximal elementary subgroup C of G_u (the preimage of a boundary subgroup of the underlying 2-orbifold). Since $G_{\hat{e}'} \subset G_{\hat{e}}$, we have $G_{\hat{e}} \subset C$. By definition of $\mathrm{Mod}(T)$, there is a representative α of Φ leaving G_u invariant and equal to the identity on C, hence on $G_e = G_{\hat{e}}$. As above, α leaves G_v invariant.

If there are edges between $\hat{v}$ and $\hat{e}$, call $\hat{e}'$ the edge that contains $\hat{v}$. It is not collapsed to a point in T, since it is collapsed in T_{can}, so $G_{\hat{e}'}$ is an incident edge group of G_u. We now have $G_{\hat{e}} \subset G_{\hat{e}'} \subset C$ and we argue as in the previous case. This completes the analysis of the action of Φ on G_e in case 2.

Still in case 2 (i.e. assuming that v is a flexible QH vertex stabilizer of T_{can}), we also need to understand the action of Φ on an element $B \in \mathcal{B}_v$ which is not an incident edge stabilizer. Such a B contains a conjugate of an H_j with finite index.

By minimality of $\hat{T}$, the group B fixes a QH vertex $\hat{v} \in \hat{T}$. We then argue as above. In subcase 2b, we have $B \in \mathcal{B}_u$ (up to conjugacy) because B contains a conjugate of H_j, so we can find α leaving G_u invariant and equal to the identity on B since $\Phi \in \mathrm{Mod}(T)$. This finishes case 2.

We can now conclude. Consider $\Gamma_{\mathrm{can}} = T_{\mathrm{can}}/G$, and recall that T_{can} is a tree of cylinders, so Γ_{can} is bipartite, with edges joining a vertex $x \in V_0(\Gamma_{\mathrm{can}})$ carrying a non-elementary group to a vertex $Y \in V_1(\Gamma_{\mathrm{can}})$ carrying an elementary group. We know that Φ fixes each vertex $x \in V_0(\Gamma_{\mathrm{can}})$, and its action on G_x is trivial if x is not modular, in $\mathrm{Out}(G_x; \mathcal{B}_x^{(\mathrm{t})})$ if G_x is QH.

Since T_{can} is a tree of cylinders, distinct edges of Γ_{can} with origin x in $V_0(\Gamma_{\mathrm{can}})$ carry groups which are not conjugate in G_x. As $\rho_x(\Phi) \in \mathrm{Out}(G_x; \mathrm{Inc}_x^{(\mathrm{t})})$, we deduce that Φ acts as the identity on edges of Γ_{can} with origin x, hence on the whole of Γ_{can}. Thus $\Phi \in \mathrm{Out}^0(T_{\mathrm{can}})$.

There remains to check that $\rho_Y(\Phi) \in \mathrm{Out}(G_Y; \mathrm{Inc}_Y^{(\mathrm{t})})$ for $Y \in V_1(\Gamma_{\mathrm{can}})$. If $\varepsilon = (x, Y)$ is an adjacent edge, we have seen that Φ has a representative α equal to the identity on G_ε. Since G_ε is infinite, G_Y is the unique maximal elementary subgroup containing it, so α leaves G_Y invariant. This completes the proof.

6. Induced automorphisms

In this section G is hyperbolic relative to $\mathcal{P} = \{P_1, \dots, P_n\}$, and we study automorphisms of P_1 which are induced by automorphisms of G. We then apply this to the case when G is hyperbolic and H is a malnormal quasiconvex subgroup, viewing G as hyperbolic relative to H.

Definition 6.1. Given families of subgroups $\mathcal{P}$ and $\mathcal{H}$, and a subgroup Q, we say that $\alpha \in \mathrm{Aut}(Q)$ is *extendable to G relative to $\mathcal{P}$ and $\mathcal{H}^{(\mathrm{t})}$* if it is the restriction to Q of an automorphism of G representing an element of $\mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})})$.

Being extendable only depends on the image of α in $\mathrm{Out}(Q)$, so we define the *group of extendable automorphisms*

$$\mathrm{Out}(Q \wr (G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})})) \subset \mathrm{Out}(Q).$$

We write $\mathrm{Out}(Q \wr (G; \mathcal{P}))$ when $\mathcal{H} = \emptyset$, and $\mathrm{Out}(Q \wr G)$ for $\mathrm{Out}(Q \wr (G; \emptyset)) = \mathrm{Out}(Q \wr (G; \{Q\}))$.

If Q equals its normalizer (for instance if Q is an infinite maximal parabolic subgroup), there is a map $\mathrm{Out}(G; Q) \to \mathrm{Out}(Q)$, and $\mathrm{Out}(Q \wr G)$ is its image.

Suppose that $P_1 = G_v$ is a vertex group of a splitting of G relative to $\mathcal{P} = \{P_1, \dots, P_n\}$, and P_1 contains no conjugate of P_i for $i > 1$ (this is automatic if no P_i is finite). Then $\mathrm{Out}(P_1 \wr (G; \mathcal{P}))$ contains $\mathrm{Out}(G_v; \mathrm{Inc}_v^{(\mathrm{t})})$ (see Lemma 2.10). The following theorem says that virtually all extendable automorphisms occur in this fashion.

Theorem 6.2. *Let G be hyperbolic relative to $\mathcal{P} = \{P_1, \dots, P_n\}$, with P_i finitely generated and infinite, and $P_i \neq G$. Let $\mathcal{H}$ be a finite family of finitely generated subgroups of G. If $\mathrm{Out}(P_1 \wr (G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})}))$ is infinite, then:*

(1) *P_1 is a vertex group G_v in an elementary JSJ decomposition Γ relative to $\mathcal{P} \cup \mathcal{H}$. Edge groups of Γ are finitely generated;*

(2) *The group $\mathrm{Out}(P_1 \wr (G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})})) \subset \mathrm{Out}(P_1)$ has a finite index subgroup equal to $\mathrm{Out}(P_1; \mathcal{K}^{(\mathrm{t})})$, where $\mathcal{K} = \mathrm{Inc}_v \cup \mathcal{H}_{|G_v}$ is a finite family of finitely generated subgroups of P_1 (the family of incident edge groups Inc_v, and $\mathcal{H}_{|G_v}$, are defined in Subsection 2.4).*

Since we do not assume that G is one-ended relative to $\mathcal{P} \cup \mathcal{H}$, there is no *canonical* JSJ decomposition.

Proof. • First assume that G is one-ended relative to $\mathcal{P} \cup \mathcal{H}$. Consider the canonical elementary JSJ tree T_{can} relative to $\mathcal{P} \cup \mathcal{H}$ as in Subsection 3.3.

Let G_v be a vertex stabilizer containing P_1. It cannot be flexible QH because P_1 is not virtually cyclic (see Remark 2.9). If it is rigid (non-elementary), we have seen in Subsection 4.3 that the image of $\text{Out}^0(G; \mathcal{P}, \mathcal{H}^{(\text{t})})$ in $\text{Out}(G_v)$ is finite (recall that $\text{Out}^0(G; \mathcal{P}, \mathcal{H}^{(\text{t})})$ is the finite index subgroup of $\text{Out}(G; \mathcal{P}, \mathcal{H}^{(\text{t})})$ acting trivially on T_{can}/G. Since P_1 equals its normalizer, this implies that $\text{Out}(P_1 \wr (G; \mathcal{P}, \mathcal{H}^{(\text{t})}))$ is finite, a contradiction. Thus G_v is elementary, so we deduce $G_v = P_1$. This proves Assertion 1 in the one-ended case (edge stabilizers of T_{can} are finitely generated by Lemma 3.7). Assertion 2 is also clear since $\text{Out}(P_1 \wr (G; \mathcal{P}, \mathcal{H}^{(\text{t})}))$ is virtually $\text{Out}(P_1; \text{Inc}_v^{(\text{t})}, \mathcal{H}_{|G_v}^{(\text{t})})$ by Proposition 4.7.

• We now consider the general case, first assuming that G is torsion-free. Let $F = G_u$ be the vertex stabilizer containing P_1 in a Grushko decomposition S relative to $\mathcal{P} \cup \mathcal{H}$ (see Subsection 2.5), and let $\mathcal{P}_{|F}, \mathcal{H}_{|F}$ be the induced structures (see Definition 2.3); if $\mathcal{P} \cup \mathcal{H} = \{P_1\}$, then F is simply the smallest free factor containing P_1.

Since F is hyperbolic relative to $\mathcal{P}_{|F}$ by Lemma 3.5, and

$$\text{Out}(P_1 \wr (G; \mathcal{P}, \mathcal{H}^{(\text{t})})) = \text{Out}(P_1 \wr (F; \mathcal{P}_{|F}, \mathcal{H}_{|F}^{(\text{t})}))$$

because F is a free factor (or by Remark 2.11), the results of the previous case apply. The group P_1 is a vertex group G_v of a splitting Γ_F of F, which may be used to refine S to an elementary JSJ decomposition Γ of G having G_v as a vertex group (see Subsection 8.1 of [35]). The families Inc_v and $\mathcal{H}_{|G_v}$ are the same for Γ_F and Γ.

• If G has torsion, we define $F = G_u$ and $\mathcal{P}_{|F}, \mathcal{H}_{|F}$ as above, using a Stallings–Dunwoody tree S relative to $\mathcal{P} \cup \mathcal{H}$. The proof is technically more complicated because we cannot neglect the incident edge groups Inc_u.

All Stallings–Dunwoody trees S have a unique vertex stabilizer $G_{u(S)}$ equal to F, but the incident edge groups may vary. This was studied in Section 4 of [32], where we defined a "peripheral structure" for F. To state the relevant result, we choose a Stallings–Dunwoody tree S for which the valence of $u(S)$ in the quotient graph of groups S/G is minimal. Since no edge stabilizer is properly contained in a conjugate, it follows from Proposition 4.9 of [32] that the incident structure $\text{Inc}_{u(S)}$ does not depend on the choice of such an S (in trees with non-minimal valence, there may be more incident edge groups; such a group is contained in a group belonging to $\text{Inc}_{u(S)}$). We fix S, and from now on we write u rather than $u(S)$, so $F = G_u$.

Any automorphism representing an element of $\mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})})$ and leaving P_1 invariant also leaves F invariant. Since P_1 and F are equal to their normalizers, $\mathrm{Out}(P_1 \nearrow (G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})}))$ is the image of the map

$$p \colon \mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})}) \longrightarrow \mathrm{Out}(P_1),$$

and p factors through

$$\rho_u \colon \mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})}) \longrightarrow \mathrm{Out}(F).$$

By Remark 2.11, the image of ρ_u contains $\mathrm{Out}(F; \mathrm{Inc}_u^{(\mathrm{t})}, \mathcal{P}_{|F}, \mathcal{H}_{|F}^{(\mathrm{t})})$ and is contained in $\mathrm{Out}(F; \mathcal{P}_{|F}, \mathcal{H}_{|F}^{(\mathrm{t})})$.

Our choice of S implies that automorphisms in the image of ρ_u preserve Inc_u globally. Since Inc_u consists of finitely many finite subgroups of F (well defined up to conjugacy), the index of $\mathrm{Out}(F; \mathrm{Inc}_u^{(\mathrm{t})}, \mathcal{P}_{|F}, \mathcal{H}_{|F}^{(\mathrm{t})})$ in $\mathrm{Out}(F; \mathcal{P}_{|F}, \mathcal{H}_{|F}^{(\mathrm{t})})$ is finite. It therefore suffices to study the image of

$$q \colon \mathrm{Out}(F; \mathrm{Inc}_u^{(\mathrm{t})}, \mathcal{P}_{|F}, \mathcal{H}_{|F}^{(\mathrm{t})}) \longrightarrow \mathrm{Out}(P_1),$$

and to show that it is virtually $\mathrm{Out}(P_1; \mathcal{K}^{(\mathrm{t})})$.

The group $F = G_u$ is hyperbolic relative to the family $\mathcal{P}_{|F}$ (see Lemma 3.5), and one-ended relative to $\mathcal{P}_{|F} \cup \mathcal{H}_{|F}$. Since the image of q is infinite, we have seen that P_1 is a vertex group G_v in the canonical elementary JSJ decomposition Γ_{can} of F relative to $\mathcal{P}_{|F} \cup \mathcal{H}_{|F}$. One obtains an elementary JSJ tree T of G relative to $\mathcal{P} \cup \mathcal{H}$ by refining S using JSJ decompositions of vertex groups (see Subsection 8.1 of [35]), so Assertion 1 is proved.

Moreover, $\mathrm{Out}(P_1 \nearrow (G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})}))$ is virtually $\mathrm{Out}(P_1; \mathcal{K}'^{(\mathrm{t})})$, where $\mathcal{K}'$ is the union of Inc_v (the incident edge groups of P_1 in T_{can}) and $(\mathrm{Inc}_u \cup \mathcal{H}_{|F})_{|P_1}$. We now show that $\mathcal{K}' = \mathcal{K}$ if we construct T carefully.

When S is refined to yield T, the vertex u is replaced by T_{can}. There is some freedom in the way edges of S containing u are attached to T_{can}: an edge e may be attached to any vertex of T_{can} which is fixed by G_e. We may therefore assume that, if e is an edge of $T \setminus T_{\mathrm{can}}$ attached to v, then v is the only fixed point of G_e in T_{can}.

The family $(\mathcal{H}_{|F})_{|P_1}$ is defined viewing F as a vertex group of S, and then P_1 as a vertex group of T_{can}. Since groups in $\mathcal{H}$ are infinite and edge stabilizers of S are finite, $(\mathcal{H}_{|F})_{|P_1}$ equals $\mathcal{H}_{|P_1}$, defined viewing P_1 as a vertex group of T. We complete the proof by showing that $\mathrm{Inc}_v \cup (\mathrm{Inc}_u)_{|P_1}$ is the family of incident edge groups in $P_1 = G_v$ viewed as a vertex stabilizer of T.

There are two types of incident edge groups of G_v in T. Those fixing edges in T_{can} are precisely those in Inc_v. Because of the way we constructed T, those fixing edges in $T \setminus T_{\text{can}}$ have v as unique fixed point in T_{can}, they are the groups in $(\text{Inc}_u)_{|P_1}$ (see Definition 2.3). □

If G is (absolutely) hyperbolic, and P is a subgroup, then G is hyperbolic relative to $\{P\}$ if (and only if) P is quasiconvex and almost malnormal, see [6, Theorem 7.11] or [52]. If so, Theorem 6.2 applies and describes $\text{Out}(P \nearrow G)$, the automorphisms of P which extend to G.

Corollary 6.3. *Let P be a quasiconvex, almost malnormal subgroup of a hyperbolic group G, with $P \neq G$.*

- *If $\text{Out}(P \nearrow G)$ is infinite, then P is a vertex group in a splitting of G with finitely generated edge groups, and $\text{Out}(P \nearrow G)$ is virtually $\text{Out}(P; \mathcal{K}^{(t)})$ with $\mathcal{K}$ the family of incident edge groups (a finite family of finitely generated subgroups of P).*

- *If P is torsion-free, then $\text{Out}(P \nearrow G)$ has a finite index subgroup with a finite classifying space.*

Proof. The first assertion follows from Theorem 6.2. Being quasiconvex, P is a hyperbolic group. It is proved in [37] that, if P is a torsion-free hyperbolic group and $\mathcal{K}$ is an arbitrary family of subgroups, then $\text{Out}(P; \mathcal{K}^{(t)})$ has a finite index subgroup with a finite classifying space. □

If $G = F_n$, every finitely generated subgroup is quasiconvex (it is a virtual retract by [38]), so we get:

Corollary 6.4. *If $P \subset F_n$ is finitely generated and malnormal, then $\text{Out}(P \nearrow F_n)$ is virtually $\text{Out}(P; \mathcal{K}^{(t)})$ for some finite family $\mathcal{K}$ of finitely generated subgroups of P. It has a finite index subgroup with a finite classifying space.*

This is a partial answer to a question that was asked by D. Calegari. Note that the proof uses JSJ decompositions over groups which are not small.

Example 6.5. Let $P \subset F_n$ be a characteristic subgroup of finite index, with $n \geq 3$. Then $\text{Out}(P \nearrow G)$ is not virtually of the form $\text{Out}(P; \mathcal{K}^{(t)})$ because there exist automorphisms of F_n with no nontrivial periodic conjugacy class. There are similar examples with P of infinite index.

7. Groups with infinitely many automorphisms

In this section, we characterize those relatively hyperbolic groups whose automorphism group is infinite.

In the first subsection, we point out that determining whether $\text{Out}(G)$ is infinite or not is relatively easy when G is torsion-free or one-ended. In particular, we give a complete answer for toral relatively hyperbolic groups.

The most interesting case is thus when G has torsion and splits over a finite group. For instance, virtually free groups with Out finite were determined by M. Pettet [56]. We will give a different characterization (see Example 7.7).

If G is hyperbolic relative to $\mathcal{P}$, we will show in Subsection 7.2 that the group $\text{Out}(G; \mathcal{P}^{(t)})$ of automorphisms which act trivially on each parabolic subgroup is infinite if and only if G has an elementary splitting relative to $\mathcal{P}$ whose group of twists is infinite.

In Subsection 7.3, we get a characterization for the full group $\text{Out}(G; \mathcal{P})$ being infinite: this happens if and only if G has an elementary splitting relative to $\mathcal{P}$ whose group of twists is infinite, or in which a maximal parabolic subgroup P occurs as a vertex group and P has infinitely many outer automorphisms acting trivially on incident edge groups (such automorphisms extend to G).

When G is hyperbolic, we show in Subsection 7.4 that $\text{Out}(G)$ being infinite is equivalent to G having a splitting over a maximal virtually cyclic group with infinite center; this is decidable algorithmically.

7.1. Torsion free groups. We first note:

Lemma 7.1. *If a torsion-free, finitely generated, group G is a non-trivial free product, then* $\text{Out}(G)$ *is infinite.*

Proof. Write $G = A * B$. If $a \in A$ is not central, the automorphism of G equal to conjugation by a on A and to the identity on B has infinite order in $\text{Out}(G)$. Assuming that $\text{Out}(G)$ is finite, we deduce that $Z(A)$ has finite index in A, so A is abelian because $[A, A]$ is finite by a result due to Schur [57, 10.1.4]. Similarly, B is abelian. Moreover, $\text{Out}(A)$ and $\text{Out}(B)$ are finite, so $A = B = \mathbb{Z}$. This is a contradiction since $\text{Out}(\mathbb{Z} * \mathbb{Z})$ is infinite. $\qquad\square$

Thus, for torsion-free groups, infiniteness of $\text{Out}(G)$ is only interesting for one-ended groups. One can get a similar result in a relative setting.

Proposition 7.2. *Let G be a finitely generated, non-cyclic, torsion-free group, and $\mathcal{H}$ a finite collection of finitely generated subgroups. If the Grushko decomposition of G relative to $\mathcal{H}$ is non-trivial, and not an amalgam $G = A_1 * A_2$ with A_1, A_2 abelian, then $\mathrm{Out}(G; \mathcal{H}^{(\mathrm{t})})$ is infinite.*

Remark 7.3. If the Grushko decomposition Γ relative to $\mathcal{H}$ is $G = A_1 * A_2$ with A_1, A_2 abelian, then $\mathrm{Out}(G; \mathcal{H}^{(\mathrm{t})})$ is finite if and only if, for $i = 1, 2$, the subgroup of A_i generated by subgroups conjugate to a group in $\mathcal{H}$ has finite index. This is because Γ is $\mathrm{Out}(G; \mathcal{H}^{(\mathrm{t})})$-invariant by [24] (its Bass-Serre tree is the unique reduced tree in its deformation space). Twists are trivial because A_1 and A_2 are abelian, so $\mathrm{Out}(G; \mathcal{H}^{(\mathrm{t})})$ is infinite if and only if A_1 or A_2 has infinitely many automorphisms acting trivially on $\mathcal{H}_{|A_i}$.

Proof. Assume that $\mathrm{Out}(G; \mathcal{H}^{(\mathrm{t})})$ is finite, and let Γ be a reduced Grushko decomposition of G relative to $\mathcal{H}$. We assume that Γ is non-trivial and we show that it is an amalgam as in the proposition.

We first note that G cannot split relative to $\mathcal{H}$ as an HNN extension $G = A*_{\{1\}}$ over the trivial group. Indeed, the group of twists of this HNN extension is isomorphic to $(A \times A)/Z(A)$, with $Z(A)$ embedded diagonally, so contains the infinite group A, a contradiction. It follows that Γ is a tree of groups.

The proof of Lemma 7.1 shows that, whenever G splits as a free product $A * B$ relative to $\mathcal{H}$, then A and B are abelian: otherwise the group of twists of the splitting is infinite. Since Γ is reduced, it follows that it is an amalgam $G = A_1 * A_2$ with A_1, A_2 abelian: if Γ has more than one edge, collapsing an edge provides a decomposition with a non-abelian vertex group. $\qed$

Let now G be hyperbolic relative to $\mathcal{P} = \{P_1, \ldots, P_n\}$, with P_i finitely generated, not virtually cyclic.

If G is one-ended relative to $\mathcal{P}$, one can read infiniteness of $\mathrm{Out}(G; \mathcal{P})$ from the JSJ decomposition thanks to Theorem 4.3: $\mathrm{Out}(G; \mathcal{P})$ is finite if and only if the canonical elementary JSJ decomposition relative to $\mathcal{P}$ has no flexible QH vertex with infinite mapping class group, the parabolic subgroups P_j appearing as vertex stabilizers have $\mathrm{Out}(P_j; \mathrm{Inc}_{P_j}^{(\mathrm{t})})$ finite, and the group of twists is finite. We may be more specific under additional conditions on the parabolic subgroups.

Proposition 7.4. *Let G be a non-abelian toral relatively hyperbolic group. The following are equivalent.*

(1) *$\mathrm{Out}(G)$ is finite.*

(2) *Every non-trivial abelian one-edge splitting of G is an amalgam $A *_C B$ with C of finite index in A or B.*

(3) *G has no non-trivial splitting over an abelian subgroup stable under taking roots.*

(4) *G is freely indecomposable and its canonical abelian JSJ decomposition* Γ_{can} *relative to non-cyclic abelian subgroups satisfies the following*:

- Γ_{can} *consists of a central vertex, possibly connected to terminal vertices carrying an abelian group*;
- *the central vertex is rigid, or QH with underlying surface* Σ *homeomorphic to a pair of pants or a twice punctured projective plane*;
- *at each terminal vertex, the incident edge group has finite index in the vertex group*.

A subgroup A is stable under taking roots if $g \in A$ whenever $g^n \in A$ for some $n \geq 2$ (this is also called pure, or isolated).

The pair of pants and the twice punctured projective plane appear in this statement because they are the only compact hyperbolic surfaces with finite mapping class group (see the end of Subsection 4.1). The fundamental group of a pair of pants is rigid. The fundamental group of a twice punctured projective plane has two (incompatible) cyclic splittings relative to the boundary (it is flexible), but none over a maximal cyclic subgroup.

Automorphisms of toral relatively hyperbolic groups were considered in [16], and some of the equivalences in Proposition 7.4 follow from their results (note for instance that a splitting as in (3) is an essential splitting in the sense of their Definition 3.30).

Proof. If G is a free product, (2) and (3) are false, and so is (1) by Lemma 7.1. We therefore assume that G is freely indecomposable.

We prove (1) $\Rightarrow$ (2) by assuming that (2) does not hold, and showing that $\text{Out}(G)$ is infinite. If G is an HNN extension over an abelian group, or an amalgam with A and B non-abelian, the group of twists of the splitting is infinite. If $G = A *_C B$ with A abelian containing C with infinite index, $\text{Out}(G)$ is infinite because A has nontrivial automorphisms equal to the identity on C.

It is clear that (2) implies (3). To prove that (3) implies (2), first assume that $G = A *_C B$ with C abelian of infinite index in both A and B. Let $\hat{C}$ be the set of all roots of elements of C, an abelian subgroup containing C with finite index (recall that all abelian subgroups of G are cyclic or parabolic, hence finitely generated). Since $\hat{C}$ is elliptic in the amalgam, up to exchanging the role of A and B, we can assume that $\hat{C} < A$. Then $G = A *_{\hat{C}} \langle B, \hat{C} \rangle$ is a decomposition

contradicting (3). The case of an HNN extension is similar, but we do not need the hypothesis that C has infinite index in A.

If (4) does not hold, we construct a splitting contradicting (2). If Γ_{can} has a flexible QH vertex, and if the underlying surface is not a twice punctured projective plane, then one simply considers the cyclic splitting dual to a 2-sided essential simple closed curve not bounding a Möbius band. The other possibility is that Γ_{can} has a vertex v carrying an abelian group such that either v has valence ≥ 2, or v is terminal with the edge group of infinite index in G_v. One gets the required splitting by collapsing edges of Γ_{can}.

We have proved (1) $\Rightarrow$ (2) $\Rightarrow$ (4) and (2) $\Leftrightarrow$ (3) . We conclude by deducing from the exact sequence of Corollary 4.4 that $\text{Out}(G)$ is finite if (4) holds. The groups $\text{GL}_{r_k, n_k}(\mathbb{Z})$ are trivial because $r_k = 0$ by the finiteness assumption at terminal vertices. The mapping class group of Σ is finite. Twists are trivial because terminal vertices carry an abelian group. $\qquad\square$

Proposition 7.5. *Let G be torsion-free, hyperbolic relative to nilpotent subgroups. Assume that G is not nilpotent. If $\text{Out}(G)$ is finite, then G is freely indecomposable and its canonical JSJ decomposition Γ_{can} over nilpotent groups relative to non-cyclic nilpotent subgroups consists of a central vertex which is rigid, or QH with underlying surface Σ homeomorphic to a pair of pants or a twice punctured projective plane, possibly connected to terminal vertices carrying a nilpotent group.*

Proof. We may assume that no P_i is cyclic, so $\text{Out}(G) = \text{Out}(G; \mathcal{P})$. As above, G is freely indecomposable and there is no QH vertex other than those mentioned. Recall that an infinite torsion-free nilpotent group has infinite center. As in the proof of Corollary 4.4, the group of twists of Γ_{can} contains the direct product $\prod (Z(G_v))^{|E_v|-1}$ taken over vertices carrying a nilpotent group, so is infinite as soon as there is a vertex with valence $|E_v| \geq 2$. $\qquad\square$

7.2. Infinity of marked automorphisms

Theorem 7.6. *Let G be hyperbolic relative to $\mathcal{P} = \{P_1, \ldots, P_n\}$, with each P_i finitely generated. Then $\text{Out}(G; \mathcal{P}^{(\text{t})})$ is infinite if and only if there is a splitting of G over finitely generated elementary subgroups, relative to $\mathcal{P}$, with an infinite group of twists $\mathcal{T}$ (see Subsection 2.6).*

More generally, if $\mathcal{Q}$ is a finite family of finitely generated subgroups with $\mathcal{P} \subset \mathcal{Q}$, the same characterization holds for $\text{Out}(G; \mathcal{Q}^{(\text{t})})$, with a splitting relative to $\mathcal{Q}$.

By Lemma 2.15, the splittings may be assumed to have only one edge. The proof will be given at the end of the subsection.

Example 7.7. Consider the virtually free group $G = D_4 *_C D_6 *_C D_4$, where D_n is the dihedral group of order n and C has order 2 (note that C is central in D_4, but equal to its centralizer in D_6). For this two-edge splitting Γ, and any splitting obtained by collapsing an edge, the group of twists is finite. The one-edge splitting Δ given by the amalgam $G = D_6 *_C (D_4 *_C D_4)$, however, has a twist of infinite order. The Bass-Serre tree of Δ is the tree of cylinders of the Bass-Serre tree T of Γ, and Assertion 1 of Proposition 7.10 holds in this case. Compare [56].

When G is hyperbolic, the group of twists of the splitting provided by Theorem 7.6 contains an element of infinite order (see also Theorem 7.14). The following example shows that this does not hold for general relatively hyperbolic groups.

Example 7.8. Let $G = B_1 * B_2$ be the free product of two infinite torsion groups with trivial center. It is hyperbolic relative to $\mathcal{P} = \{B_1, B_2\}$, and $\mathrm{Out}(G; \mathcal{P}^{(t)})$ is infinite. But no splitting over elementary subgroups has a twist of infinite order, as we now show. By way of contradiction, suppose that some $Z_{G_{o(e)}}(G_e)$ contains an element of infinite order. The group G_e is trivial, or contains a torsion element $g \neq 1$, or is isomorphic to $\mathbb{Z}$. It cannot be trivial since $G_{o(e)}$ would then be a torsion group. The existence of a torsion element g also leads to a contradiction since the centralizer of such a g is a torsion group. We conclude by observing that G does not split over $\mathbb{Z}$: if it does, an equivariant map from the Bass-Serre tree of the amalgam $B_1 * B_2$ to that of the splitting maps vertices to vertices and must be locally injective, hence globally injective, a contradiction.

Before proving Theorem 7.6, we note the following fact, which follows from the presentation of $\mathcal{T}$ recalled in Subsection 2.6.

Lemma 7.9. *Let G be a relatively hyperbolic group. Let Γ be a one-edge splitting of G over a virtually cyclic group G_e with infinite center, with G_e not parabolic. Any element of infinite order in $Z(G_e)$ defines a twist which has infinite order in* $\mathrm{Out}(G)$, *unless Γ is an amalgam and one of the vertex groups is virtually cyclic with infinite center.*

The following result is a key step in the proof of Theorem 7.6.

Proposition 7.10. *Let T be a non-trivial tree with edge stabilizers of constant finite cardinality k. Let T_c be its tree of cylinders for the equality equivalence relation (see Subsection 2.7). Then at least one of the following holds:*

(1) *T_c is nontrivial and its edge stabilizers are finite;*

(2) *T has a collapse T' which is nontrivial and has an infinite group of twists $\mathcal{T}(T')$;*

(3) *T has a collapse T' which is nontrivial and invariant under $\mathrm{Out}(\mathcal{D}(T))$.*

$\mathcal{D}(T)$ denotes the deformation space of T over groups of cardinality $\leq k$. All reduced trees in $\mathcal{D}(T)$ have edge stabilizers of order k (see Subsection 2.3). Also note that T_c is invariant under $\mathrm{Out}(\mathcal{D}(T))$, and dominated by T, so we get:

Corollary 7.11. *T has a collapse which is nontrivial and has an infinite group of twists, or T dominates a nontrivial tree T' with finite edge stabilizers which is invariant under $\mathrm{Out}(\mathcal{D}(T))$.*

Proof of Proposition 7.10. We can assume that T is reduced. We first consider the case when T_c is trivial. Since edges of T belong to the same cylinder if and only if they have the same stabilizer, all edges of T have the same stabilizer, a finite normal subgroup A.

If there is only one orbit of edges, T is the unique reduced tree in $\mathcal{D}(T)$ because no edge stabilizer may be properly contained in another [46]. It follows that $T' = T$ is invariant under $\mathrm{Out}(\mathcal{D}(T))$.

Assume that there is more than one orbit of edges. Since A is normal and finite, its centralizer in G has finite index, so for any vertex v in an arbitrary collapse T' of T (including T itself), and any edge e of T' incident to v, the group $Z_{G_v}(G_e)$ is infinite as soon as G_v is infinite. By Lemma 2.13, if Assertion 2 does not hold, then all vertex stabilizers of nontrivial collapses T' are either finite or infinite with infinite center. Since T is reduced, this is possible only if T has one orbit of vertices, all vertex stabilizers equal to A, and only two orbits of edges (this is respectively because a non-trivial amalgam $H_1 *_A H_2$, an HNN extension H_1*_A with $H_1 \neq A$, and a double HNN extension $(A*_A)*_A$, have finite center). In particular, G/A is free of rank 2. One easily checks that collapsing the orbit of any edge gives a tree T' with $\mathcal{T}(T')$ infinite, though Lemma 2.13 does not apply. This concludes the case when T_c is trivial.

From now on, we assume that T_c is nontrivial and some edge $\varepsilon = (x, Y)$ of T_c has infinite stabilizer. View the cylinder Y as a subtree of T containing x, and consider an edge $e \subset Y$ with origin x. Then we have $G_Y = N_G(G_e)$ and

$G_\varepsilon = G_Y \cap G_x = N_{G_x}(G_e)$. We know that $Z_{G_x}(G_e)$ is infinite, but we cannot apply Lemma 2.13 since we do not know that $Z(G_x)$ is finite.

Assume for a moment that Y contains edges from at least two G-orbits. Consider T' obtained from T by collapsing all edges in the orbit of e, and denote by x' the image of x in T'. Since T is reduced, $G_x \subsetneq G_{x'}$. By the assumption on Y, there is an edge e' of T' incident to x' with $G_{e'} = G_e$. Since $Z_{G_{x'}}(G_{e'})$ is infinite, we are done if $Z(G_{x'})$ is finite. We now show that $Z(G_{x'})$ being infinite leads to a contradiction. Edge stabilizers of T being finite, x is the unique point of T fixed by G_ε. Since $G_\varepsilon \subset G_x \subset G_{x'}$, the group $Z(G_{x'})$ normalizes G_ε, hence also fixes x, and only x since it is infinite. This implies that $G_{x'}$ fixes x, a contradiction to $G_x \subsetneq G_{x'}$.

Returning to the general case, there is a G-invariant partition of the set of cylinders: those for which there is an edge (x, Y) of T_c with infinite stabilizer, and the others. Thanks to the previous argument, we may assume that all edges contained in a given cylinder of the first type belong to the same G-orbit. Let now T' be the tree obtained from T by collapsing all edges in cylinders of the second type. It is nontrivial (but $T' = T$ is possible). We show that T' does not change if we replace T by another reduced tree T_1 in $\mathcal{D}(T)$. This implies that T' is invariant under $\mathrm{Out}(\mathcal{D}(T))$.

One may join T and T_1 by slide moves (see Subsection 2.3). In a slide move, an edge e slides over an edge f belonging to a different orbit, with $G_e \subset G_f$. Here one must have $G_e = G_f$, so e and f belong to the same cylinder, necessarily of the second type. The slide move does not change T' since the cylinder gets collapsed.   □

Proof of Theorem 7.6. We prove the "only if " direction (the other direction is clear since twists act by conjugations on vertex groups). We may assume that all groups in $\mathcal{Q}$ are infinite. We first suppose that G is one-ended relative to $\mathcal{Q}$. Let T be the canonical elementary JSJ tree relative to $\mathcal{Q} = \mathcal{P} \cup \mathcal{H}$ as in Subsection 3.3. Its edge stabilizers are finitely generated by Lemma 3.7. By Theorem 4.6, if $\mathrm{Out}(G; \mathcal{Q}^{(t)})$ is infinite, then either $\mathcal{T}(T)$ is infinite and we are done, or T has at least a non-rigid QH vertex v with $\mathrm{Out}(G_v; \mathcal{B}_v)$ infinite.

The underlying orbifold group $\pi_1(\Sigma)$ splits, relative to its boundary subgroups, over a maximal virtually cyclic subgroup with infinite center (see [17], Proposition 3.1). This induces an elementary splitting of G_v which extends to a splitting of G. By Lemma 2.6, this splitting of G is relative to $\mathcal{Q}$ (because the intersection of G_v with a conjugate of a group in $\mathcal{Q}$ projects into a boundary subgroup in $\pi_1(\Sigma)$), and has an infinite group of twists by Lemmas 7.9 and 2.14. This proves the theorem if G is one-ended relative to $\mathcal{Q}$.

In the general case, let $k \in \mathbb{N}$ be the smallest number such that G splits relative to $\mathcal{Q}$ over a group of cardinality k. Let T be a reduced JSJ tree relative to $\mathcal{Q}$ over subgroups of cardinality k (see Subsection 2.5). Its deformation space is invariant under $\mathrm{Out}(G; \mathcal{Q}^{(\mathrm{t})})$. By Corollary 7.11, either some collapse of T has an infinite group of twists (and we are done), or T dominates a nontrivial tree T' with finite edge stabilizers which is invariant under $\mathrm{Out}(G; \mathcal{Q}^{(\mathrm{t})})$. Note that the groups in $\mathcal{Q}$ are elliptic in T', since they are elliptic in T and T dominates T'.

We may assume that $\mathcal{T}(T')$ is finite. Let $A_0 \subset \mathrm{Out}(G; \mathcal{Q}^{(\mathrm{t})})$ be the finite index subgroup acting trivially on the graph T'/G. By Assertion 1 of Lemma 2.12, there exists a vertex group G_v of T' such that $\rho_v(A_0) \subset \mathrm{Out}(G_v)$ is infinite. In particular, G_v is infinite.

The group G_v is hyperbolic relative to the family $\mathcal{P}_{|G_v}$ (see Lemma 3.5). By Lemma 2.10, we have $\rho_v(A_0) \subset \mathrm{Out}(G_v; \mathcal{Q}^{(\mathrm{t})}_{|G_v})$.

If the theorem holds for G_v, we get a graph of groups decomposition Γ_0 of G_v relative to $\mathcal{Q}_{|G_v}$ having an infinite group of twists. Since T' has finite edge stabilizers, Γ_0 is relative to Inc_v and one may refine T'/G to a graph of groups Λ by using Γ_0. By Lemma 2.14, the splitting Λ has an infinite group of twists. It is relative to $\mathcal{Q}$ by Lemma 2.6.

If the theorem does not hold for G_v, we repeat the construction. If the process stops after finitely many steps, we get a splitting Λ as in the previous case. If it does not stop, we get an infinite sequence of trees T_i relative to $\mathcal{Q}$ with finite edge stabilizers, with T_{i+1} strictly dominating T_i. Since G is finitely presented relative to $\mathcal{Q}$, there is a Stallings–Dunwoody decomposition relative to $\mathcal{Q}$ (see Subsection 2.5), and we reach a contradiction (see [19, p. 130]). □

7.3. Infinity of unmarked automorphisms.

Using Theorem 6.2, we now characterize relatively hyperbolic groups for which $\mathrm{Out}(G; \mathcal{P})$ is infinite.

We first note the following consequence of Theorem 7.6.

Corollary 7.12. *Let G be hyperbolic relative to $\mathcal{P} = \{P_1, \ldots, P_n\}$, where each P_i is infinite and finitely generated, and let $\mathcal{H}$ be a finite family of finitely generated subgroups.*

If $\mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})})$ is infinite, there is an elementary splitting relative to $\mathcal{P} \cup \mathcal{H}$ with an infinite group of twists, or there is an i such that the natural map

$$\mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})}) \longrightarrow \mathrm{Out}(P_i)$$

(defined because P_i equals its normalizer) has infinite image.

Proof. If all maps $\mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(t)}) \to \mathrm{Out}(P_i)$ have finite image, then the intersection of their kernels, namely $\mathrm{Out}(G; \mathcal{P}^{(t)}, \mathcal{H}^{(t)})$, is infinite. We can now apply Theorem 7.6. ∎

Corollary 7.13. *Assume furthermore that G is non-elementary (i.e. $P_i \neq G$). Then $\mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(t)})$ is infinite if and only if G has an elementary splitting as a graph of groups Λ relative to $\mathcal{P} \cup \mathcal{H}$ such that one of the following holds:*

- *the group of twists of Λ is infinite,*

- *or Λ has a vertex v such that $G_v = P_i$ is a maximal parabolic subgroup and $\mathrm{Out}(G_v; \mathrm{Inc}_v^{(t)}, \mathcal{H}_{|G_v}^{(t)})$ is infinite.*

Proof. As in Theorem 7.6, the "if" direction is clear, so we can assume that $\mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(t)})$ is infinite. By Corollary 7.12, and up to renumbering, we can assume that $\mathrm{Out}(P_1 \nearrow (G; \mathcal{P}, \mathcal{H}^{(t)}))$ is infinite. By Theorem 6.2, P_1 is a vertex group in some elementary decomposition of G relative to $\mathcal{P} \cup \mathcal{H}$ and $\mathrm{Out}(P_1; \mathrm{Inc}_v^{(t)}, \mathcal{H}_{|G_v}^{(t)})$ has finite index in $\mathrm{Out}(P_1 \nearrow (G; \mathcal{P}, \mathcal{H}^{(t)}))$. In particular, $\mathrm{Out}(P_1; \mathrm{Inc}_v^{(t)}, \mathcal{H}_{|G_v}^{(t)})$ is infinite. ∎

7.4. Hyperbolic groups. We apply the results of the previous subsections to the case when G is a hyperbolic group. We say that a subgroup of G is $\mathcal{Z}_{\max}$ if it is maximal for inclusion among virtually cyclic subgroups with infinite center. Note that any virtually cyclic subgroup C with infinite center is contained in a unique $\mathcal{Z}_{\max}$ subgroup $\widehat{C}$ (the pointwise stabilizer of ∂C).

Given any splitting of a hyperbolic group over a $\mathcal{Z}_{\max}$ subgroup C, any central element $c \in C$ of infinite order defines a twist of infinite order in $\mathrm{Out}(G)$.

Theorem 7.14. *If G is hyperbolic, $\mathrm{Out}(G)$ is infinite if and only if there is a nontrivial splitting of G over a $\mathcal{Z}_{\max}$ subgroup (such a splitting always has a twist of infinite order).*

If $\mathcal{H}$ is a finite family of finitely generated subgroups, $\mathrm{Out}(G; \mathcal{H}^{(t)})$ is infinite if and only if there is a non-trivial splitting over a $\mathcal{Z}_{\max}$ subgroup relative to $\mathcal{H}$.

It was proved independently by M. Carette [10] that $\mathrm{Out}(G)$ is infinite if and only if G has a splitting over a finite group or a (maybe non maximal) virtually cyclic group with infinite center, with a twist of infinite order (see [45, 17] for the one-ended case).

Proof. The "if" direction is clear, so we assume that $\mathrm{Out}(G; \mathcal{H}^{(t)})$ is infinite. All splittings considered in this proof will be relative to $\mathcal{H}$.

Theorem 7.6 and Lemma 2.15 say that G has a one-edge splitting over a (possibly finite) virtually cyclic group C, whose group of twists is infinite. We assume that this splitting is an amalgam $G = A *_C B$; the case of an HNN extension is similar. We first explain how to replace this amalgam over C by one over a (possibly non-maximal) virtually cyclic subgroup C' with infinite center.

If C is infinite with finite center, its centralizer in G is finite and this forces the group of twists to be finite. So assume that C is finite.

If both $Z_A(C)/Z(A)$ and $Z_B(C)/Z(B)$ are finite, the group of twists is finite, so assume for instance that $Z_A(C)/Z(A)$ is infinite. Note that $Z(A)$ has to be finite, since otherwise A would be virtually cyclic, and $Z_A(C)/Z(A)$ would be finite. Consider an element of infinite order $t \in Z_A(C)$, and perform a fold to get $G = A *_{\langle C,t \rangle} \langle B, t \rangle$. This is a splitting over a virtually cyclic subgroup C' with infinite center, and it is relative to $\mathcal{H}$. The twist defined by t has infinite order in $\mathrm{Out}(G)$ by Lemma 7.9.

Now suppose that an amalgam $G = A' *_{C'} B'$ has an infinite group of twists, with C' virtually cyclic with infinite center. Then A', B' have finite center. The $\mathcal{Z}_{\max}$ subgroup $\widehat{C}'$ containing C' is elliptic in the amalgam, and one can perform a fold to get an amalgam over $\widehat{C}'$. This splitting is non-trivial because $\widehat{C}'$ is not conjugate to A' or B' since they have finite center. The group of twists of the new splitting is clearly infinite. □

Theorem 7.15. *There is an algorithm which, given a hyperbolic group G, decides whether $\mathrm{Out}(G)$ is infinite or not. More generally, if $\mathcal{H}$ is a finite family of finitely generated subgroups, one may decide whether $\mathrm{Out}(G; \mathcal{H}^{(\mathrm{t})})$ is infinite.*

Proof. We start with the first assertion.

We first construct an algorithm that stops if and only if $\mathrm{Out}(G)$ is finite. By Theorem 8.1 of [17], one can compute a finite generating set of $\mathrm{Out}(G)$. Moreover, one can solve the word problem in $\mathrm{Out}(G)$ as this amounts to solving (uniformly) the simultaneous conjugacy problem in G (see [8] for a solution). Thus, for each $R > 0$, one can determine the ball B_R of radius R in the Cayley graph of $\mathrm{Out}(G)$. Checking whether $B_R = B_{R+1}$ for some R gives the required algorithm.

It now suffices to construct an algorithm that stops if $\mathrm{Out}(G)$ is infinite. By [17, Lemma 2.8], one can decide whether a subgroup of G (given by generators) is $\mathcal{Z}_{\max}$ or not. One can therefore enumerate all decompositions of G as an amalgam or HNN extension over $\mathcal{Z}_{\max}$ subgroups. By Corollary 7.14, this provides an algorithm that stops if $\mathrm{Out}(G)$ is infinite.

The argument to decide whether $\mathrm{Out}(G; \mathcal{H}^{(\mathrm{t})})$ is infinite is similar. The first algorithm is the same since Theorem 8.1 of [17] provides generators for $\mathrm{Out}(G; \mathcal{H}^{(\mathrm{t})})$.

For the second algorithm, one has to restrict to splittings relative to $\mathcal{H}$, so one needs an algorithm that, given a splitting, stops if the splitting is relative to $\mathcal{H}$. This is done by choosing a generating set S_i for each $H_i \in \mathcal{H}$, enumerating all conjugates of S_i, and comparing them with words written using the generators of a vertex group. □

In general, we do not know how to decide whether $\mathrm{Out}(G; \mathcal{H})$ is infinite (see Remark 4.8). The following is an answer when G is hyperbolic relative to $\mathcal{H}$.

Proposition 7.16. *There is an algorithm which, given a torsion-free hyperbolic group G, a finite family $\mathcal{P}$ of finitely generated locally quasiconvex subgroups P_i such that G is hyperbolic relative to $\mathcal{P}$, and a finite family $\mathcal{H}$ of finitely generated subgroups, decides whether $\mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})})$ is infinite.*

Since G is assumed to be hyperbolic relative to $\mathcal{P}$, each P_i is quasiconvex in G. In particular, P_i is itself a hyperbolic group. Local quasiconvexity of P_i means that its finitely generated subgroups are quasiconvex (in P_i, hence also in the hyperbolic group G).

Proof. First, using Touikan's algorithm [63, Theorem A], one can decide whether G splits as a free product relative to $\mathcal{P} \cup \mathcal{H}$. If it does, it is easy to decide whether $\mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})})$ is infinite using Proposition 7.2 and Remark 7.3.

So assume that G is one-ended relative to $\mathcal{P} \cup \mathcal{H}$. We may also assume that no P_i is cyclic. We use [63, Theorem C] to decide whether G splits in a suitable way. For this, we need our parabolic groups P_i to be algorithmically tractable in the sense of [63, Definition 1.13].

Since P_i is locally quasiconvex, it is hyperbolic and the conjugacy problem is solvable in P_i. Moreover, local quasiconvexity of P_i implies that one can decide whether a finite subset $S \subset P_i$ generates P_i or not, by checking whether a given generating set of P_i lies in the quasiconvex subgroup $\langle S \rangle$ [40]. This says that P_i is algorithmically tractable.

Applying [63, Theorem C], one can decide whether there exists an elementary splitting of G (viewed as a relatively hyperbolic group) relative to $\mathcal{P} \cup \mathcal{H}$ with finitely generated edge groups, and if so find one. By local quasiconvexity of P_i, edge groups of the splitting are quasiconvex in the hyperbolic group G, and so are vertex groups (see Subsection 3.2).

Iterating this process, one can compute a maximal elementary splitting Γ of G relative to $\mathcal{P} \cup \mathcal{H}$ (i.e. a splitting that cannot be refined non-trivially into an elementary splitting relative to $\mathcal{P} \cup \mathcal{H}$). Arguing as in [16, Section 6] and [17, Lemma 2.34],

one may then recognize the QH subgroups in Γ, and find the canonical elementary JSJ decomposition of G relative to $\mathcal{P} \cup \mathcal{H}$ (see also [18, Theorem 3.12]).

By Theorem 4.6, $\mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(t)})$ is infinite if and only if the group of twists $\mathcal{T}$ is infinite, or there is a vertex v such that $\mathrm{Out}(G_v; \mathrm{Inc}_v^{(t)}, \mathcal{H}_{|G_v}^{(t)})$ is infinite.

Recall that $\mathcal{T}$ is isomorphic to a quotient of $\prod_{e \in E} Z_{G_{o(e)}}(G_e)$ by edge and vertex relations. Each group $Z_{G_{o(e)}}(G_e)$ is either trivial or infinite cyclic, and is computable. Edge relations and vertex relations are generated by embeddings of groups $Z(G_e)$ and $Z(G_v)$ in this product. Since one can compute the corresponding subgroups of the abelian group $\prod_{e \in E} Z_{G_{o(e)}}(G_e)$, one can decide whether the group of twists is infinite or not.

To decide whether $\mathrm{Out}(G_v; \mathrm{Inc}_v^{(t)}, \mathcal{H}_{|G_v}^{(t)})$ is infinite, we can apply [16] (or more explicitly [17, Corollary 3.4]), or Theorem 7.15, but we need to determine $\mathcal{H}_{|G_v}$ (see Definition 2.3). Given $H \in \mathcal{H}$ generated by a finite set S, one can decide whether there is $g \in G$ such that $S^g \subset G_v$ in the same way as above because G_v is quasiconvex. One can similarly decide whether H fixes an edge, which allows to compute $\mathcal{H}_{|G_v}$. $\qquad\square$

8. Fixed subgroups

In this section, we use JSJ decompositions to study fixed subgroups of automorphisms. This is inspired by arguments due to Sela [58]. The proof of the Scott conjecture (Theorem 8.1) given below is not really new, but using the relative JSJ decomposition makes the argument more direct. Using Theorem 8.2, we will prove in [37] that, given a toral relatively hyperbolic group G, there are only finitely many possibilities for fixed subgroups of automorphisms of G, up to isomorphism. This was proved by Shor [62] for G torsion-free hyperbolic.

Theorem 8.1 ([4]). *Let α be an automorphism of a free group F_n. Its fixed subgroup* Fix α *has rank at most n.*

Proof. The smallest free factor containing Fix α is α-invariant. Replacing G by this free factor, we may assume that F_n is one-ended (freely indecomposable) relative to Fix α. We also assume that Fix α is not cyclic.

Let T be the canonical cyclic JSJ tree relative to Fix α (see Subsection 3.3), and let G_v be the vertex stabilizer containing Fix α. It is α-invariant because T is invariant and v is the only vertex fixed by Fix α, and abelianizing shows that it has rank $\leq n$. By Remark 2.9 it cannot be flexible QH because Fix α is not cyclic, so it is rigid. By standard arguments due to Paulin and Rips, α has finite

order in $\mathrm{Out}(G_v)$: otherwise, applying Theorem 3.9 with $\mathcal{P}$ consisting of incident edge groups and $\mathcal{H}$ consisting of Fix α (which is finitely generated) yields a cyclic splitting of G_v which contradicts rigidity. By Dyer-Scott [23], Fix α is a free factor of G_v so has rank $\leq n$.

If we do not wish to use Gersten's result that Fix α is finitely generated [27], we argue by contradiction as follows. Let H be a finitely generated free factor of Fix α of rank $> n$. We claim that F_n is one-ended relative to H (see [55] and Lemma 7.6 of [36] for more general statements). Otherwise, let $\hat{H}$ be the smallest free factor of F_n containing H. Then $\hat{H}$ is α-invariant, and Fix $\alpha \cap \hat{H}$ has rank at most $n - 1$ (assuming that the theorem holds in $\hat{H}$ by induction on n), a contradiction since Fix $\alpha \cap \hat{H}$ retracts onto H.

Define G_v as above, using the cyclic JSJ splitting of F_n relative to H. The fixed subgroup of $\alpha_{|G_v}$ is a subgroup of Fix α which has rank $\leq n$ and contains H. This is a contradiction since H is a retract of Fix α. □

This proof uses [23], which is specific to free groups. Applying the same argument to (relatively) hyperbolic groups only yields:

Theorem 8.2. *Let G be hyperbolic relative to a finite family $\mathcal{P}$ of slender subgroups. Consider $\alpha \in \mathrm{Aut}(G; \mathcal{P})$ such that Fix α is not elementary (i.e. not virtually cyclic or parabolic). Then Fix α is contained in an α-invariant vertex group G_v of a splitting of G over elementary subgroups relative to $\mathcal{P}$, and $\alpha_{|G_v}$ has finite order in $\mathrm{Out}(G_v)$.*

Proof. The proof is similar to the one above. Note that Fix α is finitely generated by [39, Cor. 9.2] because it is relatively quasiconvex [51] and groups in $\mathcal{P}$ are slender. First assume that G is one-ended relative to $\mathcal{P} \cup \{\mathrm{Fix}\,\alpha\}$. Let T_{can} be the canonical elementary JSJ decomposition of G relative to $\mathcal{P} \cup \{\mathrm{Fix}\,\alpha\}$. Let v be the unique vertex of T_{can} fixed by Fix α. As above, G_v is α-invariant; it cannot be QH because it contains the universally elliptic subgroup Fix α which is not virtually cyclic, so G_v is rigid: it has no elementary splitting relative to $\mathrm{Inc}_v \cup \mathcal{P}_{|G_v} \cup \{\mathrm{Fix}\,\alpha\}$. By Lemma 3.8, G_v is hyperbolic relative to $\mathrm{Inc}_v \cup \mathcal{P}_{|G_v}$. By Theorem 3.9, the group $\mathrm{Out}(G_v; \mathrm{Inc}_v, \mathcal{P}_{|G_v}, \{\mathrm{Fix}\,\alpha\}^{(\mathrm{t})})$ is finite. It contains the class of $\alpha_{|G_v}$ by Lemma 2.10, so $\alpha_{|G_v}$ has finite order in $\mathrm{Out}(G_v)$.

If G is not relatively one-ended, we consider a reduced Stallings–Dunwoody tree S relative to $\mathcal{P} \cup \{\mathrm{Fix}\,\alpha\}$ (see Subsection 2.5). Since Fix α is infinite, it fixes a unique vertex $u \in S$. The deformation space of S is α-invariant, so $\alpha(G_u)$ fixes a vertex $u' \in S$. Since Fix $\alpha \subset \alpha(G_u)$ fixes only u, we have $u' = u$ and $\alpha(G_u) = G_u$. We now apply the previous analysis to the restriction of α to G_u,

which is hyperbolic relative to $\mathcal{P}_{|G_u}$ by Lemma 3.5. We get a splitting Λ of G_u relative to $\mathcal{P}_{|G_u}$, and we obtain the desired splitting of G by refining S/G using Λ (see Lemma 2.6). $\qquad\square$

9. Rigid groups have finitely many automorphisms

The goal of this section is to prove Theorem 3.9. Let us first recall its statement.

Theorem 3.9. *Let G be hyperbolic relative to finitely generated subgroups $\mathcal{P} = \{P_1, \dots, P_n\}$, with $P_i \neq G$. Let $\mathcal{H} = \{H_1, \dots, H_q\}$ be another family of finitely generated subgroups. If $\mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})})$ is infinite, then G splits over an elementary subgroup relative to $\mathcal{P} \cup \mathcal{H}$.*

As mentioned earlier, the proof uses $\mathbb{R}$-trees. All actions on $\mathbb{R}$-trees considered here are by isometries. An arc is a subset isometric to an interval $[a, b] \subset \mathbb{R}$ with $a \neq b$. As in the simplicial case, an action on an $\mathbb{R}$-tree T is *relative* to subgroups H_i if each H_i is elliptic (fixes a point) in T.

Because the parabolic groups are not assumed to be slender, we will need to analyze actions on $\mathbb{R}$-trees which are not quite stable.

9.1. Constructing an $\mathbb{R}$-tree

Theorem 9.1 ([1]). *Let $G, \mathcal{P}, \mathcal{H}$ be as in Theorem 3.9. If $\mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})})$ is infinite, then G has a non-trivial action on an $\mathbb{R}$-tree T relative to $\mathcal{P} \cup \mathcal{H}$ such that arc stabilizers are elementary.*

The proof is essentially in [1], noting that a locally elementary subgroup is elementary by Lemma 3.1. We also add the remark that the groups P_i, H_j are elliptic in T.

Proof. Let φ_k be automorphisms representing distinct elements of $\mathrm{Out}(G; \mathcal{P}, \mathcal{H}^{(\mathrm{t})})$. Let X be a δ-hyperbolic space on which G acts as in Subsection 3.1. Consider a finite generating set S of G, and the minimal displacement

$$d_k = \inf_{x \in X} \max_{s \in S} d_X(x, \varphi_k(s) \cdot x).$$

Choose a point $x_k \in X$ where $\max_{s \in S} d_X(x_k, \varphi_k(s).x_k) \leq d_k + \frac{1}{k}$.

Using the Bestvina–Paulin method, it is shown in [1] that d_k goes to infinity, the rescaled pointed metric spaces $X_k = (\frac{1}{d_k} X, x_k)$ converge to an $\mathbb{R}$-tree T (after taking a subsequence), and the action of G on X_k twisted by φ_k converges to a

non-trivial isometric action of G on T with locally elementary (hence elementary) arc stabilizers.

We now prove that the action is relative to $\mathcal{P} \cup \mathcal{H}$. Since groups in $\mathcal{P} \cup \mathcal{H}$ are finitely generated, it suffices to show that any element g belonging to P_i or H_j is elliptic in T.

Suppose g acts hyperbolically in T. Then there exists $a \in T$ such that

$$d_T(a, g^2 a) = 2 d_T(a, ga) > 0.$$

If a_k is an approximation point of a in $X_k = \frac{1}{d_k} X$, then

$$\frac{d_X(a_k, \varphi_k(g^2)a_k) - d_X(a_k, \varphi_k(g)a_k)}{d_k}$$

converges to

$$d_T(a, g^2 a) - d_T(a, ga) = d_T(a, ga) > 0,$$

so for k large enough

$$d_X(a_k, \varphi_k(g^2)a_k) - d_X(a_k, \varphi_k(g)a_k) > 2\delta + \frac{d_k}{2} d_T(a, ga).$$

Lemme 9.2.2 of [12] implies that $\varphi_k(g)$ acts loxodromically on X, with translation length going to infinity since $d_k \to \infty$. This is a contradiction if $g \in P_i$, since every $\varphi_k(g)$ is parabolic in this case. If $g \in H_j$, all elements $\varphi_k(g)$ are conjugate so have the same translation length in X, also a contradiction. □

Remark 9.2. One can show that a group $H \in \mathcal{H}$ is elliptic in T, even if it is not assumed to be finitely generated. We know that every $h \in H$ is elliptic. If H is not elliptic, it fixes an end of T, so every finitely generated subgroup of H fixes a ray. This implies that finitely generated subgroups of H are parabolic, so H is parabolic and therefore elliptic in T.

On the other hand, Theorem 9.4 below requires finite generation.

Remark 9.3. The hypothesis that automorphisms act trivially on H_j may be weakened. It is sufficient to assume that their growth under iteration is slower on H_j than on G.

9.2. Hypostability. To deduce a splitting as in Theorem 3.9 from the action on the $\mathbb{R}$-tree of Theorem 9.1, we will generalize the following basic fact (see Theorem 9.9):

Theorem 9.4 ([3, Thm 9.6]). *Let G be a finitely presented group, and let $\mathfrak{Q}$ be a finite family of finitely generated subgroups. Assume that G has a non-trivial stable action on an $\mathbb{R}$-tree T relative to $\mathfrak{Q}$. Then G splits relative to $\mathfrak{Q}$ over a group K which is an extension $1 \to A \to K \to \mathbb{Z}^k \to 1$, where A fixes an arc of T and $k \geq 0$.*

Recall that an arc J is stable if any subarc of J has the same stabilizer as J. An action is *stable* if every arc I contains a stable subarc J.

Corollary 9.5. *Theorem 3.9 holds if every P_i is slender and G is finitely presented.*

Proof. Assume that every P_i is slender. In this case a subgroup of G is elementary if and only if it is slender. In particular, elementary subgroups satisfy the ascending chain condition, so the action on the $\mathbb{R}$-tree T provided by Theorem 9.1 is stable. If furthermore G is finitely presented, Theorem 9.4 (applied with $\mathfrak{Q} = \mathcal{P} \cup \mathcal{H}$) gives a splitting that satisfies the conclusion of Theorem 3.9 (note that K is slender because A is). $\square$

In general, however, G is only finitely presented relative to $\mathcal{P}$, and the action on T only satisfies a weaker property than stability, which we call hypostability (see [42] for a different property called semistability).

Definition 9.6. Let G be a group acting on an $\mathbb{R}$-tree T. The action is *hypostable* if, for each arc $I \subset T$, there exists a subarc $J \subset I$ satisfying the following hypostability condition: if $g \in G$ acts hyperbolically in T and $gJ \cap J$ is an arc, then $\mathrm{Stab} J = \mathrm{Stab}(gJ)$ (equivalently, g normalizes $\mathrm{Stab} J$).

Hypostability is weaker than stability, because any stable arc J satisfies the hypostability condition: if $gJ \cap J$ is an arc, $\langle \mathrm{Stab}(J), \mathrm{Stab}(gJ) \rangle$ is contained in the stabilizer of $gJ \cap J$, which coincides with $\mathrm{Stab}(J)$ and $\mathrm{Stab}(gJ)$ by stability of J and gJ.

Lemma 9.7. *Let G be hyperbolic relative to finitely generated subgroups $\mathcal{P} = \{P_1, \ldots, P_n\}$. Any action of G on an $\mathbb{R}$-tree T relative to $\mathcal{P}$ with elementary arc stabilizers is hypostable.*

Proof. Let C be such that any elementary subgroup H of G of cardinality $> C$ is contained in a unique maximal elementary subgroup $E(H)$ (see Lemma 3.1). Let $I \subset T$ be an arc. If the stabilizer of every subarc has cardinality at most C, then a subarc $J \subset I$ whose stabilizer has the greatest cardinality is stable and we are done.

Otherwise, consider $J \subset I$ whose stabilizer $H = \text{Stab}(J)$ has cardinality greater than C. The stabilizer of every subarc of J is elementary, so is contained in $E(H)$. If subgroups of $E(H)$ satisfy the ascending chain condition, in particular if $E(H)$ is virtually cyclic, then J contains a stable subarc. Thus we can assume that $E(H)$ is parabolic.

We prove hypostability by showing that any g such that $gJ \cap J$ contains an arc is elliptic in T. Indeed, $\langle H, H^g \rangle$ fixes an arc in T, so is elementary. It follows that $\langle H, H^g \rangle \subset E(H)$, so $E(H) = E(H^g) = E(H)^g$. Since $E(H)$ is its own normalizer, we get $g \in E(H)$. But T is relative to $\mathcal{P}$, so g is elliptic. $\qquad\square$

Example 9.8. We sketch the construction of an action as in Lemma 9.7 which is not stable. Let G be the free product $G = P * \mathbb{Z}$, with P a (non-slender) finitely generated group containing a copy of the free abelian group on a countable basis $\mathbb{Z}^{(\mathbb{Q})}$. Note that G is hyperbolic relative to $\{P, \mathbb{Z}\}$. Informally, identifying the edge of the free product with $[0, 1]$, one can produce an $\mathbb{R}$-tree from the Bass-Serre tree of this splitting by folding the group $\mathbb{Z}^{([0, \frac{p}{q}] \cap \mathbb{Q})}$ on a length $\frac{p}{q}$ for all $0 < \frac{p}{q} < 1$. The stabilizer of an arc $[a, b] \subset [0, 1]$ is then $\mathbb{Z}^{([0, b] \cap \mathbb{Q})}$. This action is hypostable but unstable.

Theorem 9.9. *Theorem* 9.4 *holds if the action on the* $\mathbb{R}$*-tree is only assumed to be hypostable, and the group* G *is only assumed to be finitely presented relative to* $\mathcal{Q}$.

The proof will be given in the next subsection. The following corollary is an immediate consequence of Lemma 9.7 and Theorem 9.9.

Corollary 9.10. *Let* G *be a relatively hyperbolic group, with* $\mathcal{P}, \mathcal{H}$ *as in Theorem* 3.9. *If* G *acts non-trivially on an* $\mathbb{R}$*-tree* T *relative to* $\mathcal{P} \cup \mathcal{H}$ *with elementary arc stabilizers, then* G *splits over an elementary subgroup relative to* $\mathcal{P} \cup \mathcal{H}$.

Theorem 3.9 follows immediately from this corollary, using the $\mathbb{R}$-tree provided by Theorem 9.1. A refinement of Corollary 9.10 will be given in Subsection 9.4.

Remark 9.11. Let G and T be as in Theorem 9.4. If T is not a line, one can get a splitting over a group K which is an extension of $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ by a group A fixing an arc in T (see [3]). One can also approximate T (in the equivariant Gromov topology) by simplicial trees with controlled edge stabilizers, as in [29]. The same facts are true under the assumptions of Theorem 9.9.

9.3. Proof of Theorem 9.9. Recall that a subtree $Y \subset T$ is *indecomposable* [31] if, given arcs $I, L \subset Y$, there exist $g_1, \ldots, g_n \in G$ such that $L \subset g_1 I \cup \cdots \cup g_n I$, and every $g_i I \cap g_{i+1} I$ is an arc. We call $g_1, \ldots, g_n$ an *I-covering* of L.

Lemma 9.12. *Assume that $Y \subset T$ is an indecomposable subtree. Given two arcs $I, L \subset Y$ with $I \subset L$, there exists an I-covering $g_1, \ldots, g_r$ of L such that $g_1 = 1$ and every $g_i g_{i-1}^{-1}$ is hyperbolic in T.*

Proof. Given any I-covering of L, there exists i such that $I \cap g_i I$ is an arc, and therefore $1, g_i, g_{i-1}, \ldots, g_1, g_2, \ldots, g_n$ is an I-covering starting with 1. From now on, we only consider coverings starting with 1. The interval covered will always be L.

We fix an orientation of I. It induces an orientation of gI for any $g \in G$. If $I \cap gI$ is an arc, the orientations of I and gI may agree or disagree on this arc. We first claim that there exists an I-covering $1 = a_1, \ldots, a_p$ of L such that, for each i, the orientation of $a_i I$ agrees with that of $a_{i+1} I$ on their intersection (we say that such an I-covering is orientation-preserving).

If not, we can find arcs $gI, g'I$ whose intersection is an arc on which the orientations disagree. Define $g_0 = g'^{-1} g$ and $J = I \cap g_0 I$; the orientations of I and $g_0 I$ disagree on J. Now consider a J-covering $1 = b_1, \ldots, b_p$ of L. It is also an I-covering of L. Since $J = I \cap g_0 I$, we get another I-covering of L if we replace some of the b_i's by $b_i g_0$. Starting with $a_1 = b_1 = 1$, we then define a_i inductively as either b_i or $b_i g_0$, making sure that orientations agree. This proves the claim.

Next note that there exists a hyperbolic element $h \in G$ mapping an arc $J' \subset I$ to a different arc $h(J') \subset I$ in an orientation-preserving way. To see this, first choose h_1 mapping an arc $J_1 \subset I$ to a disjoint arc $J_2 \subset I$. If orientation is reversed, choose h_2 mapping an arc $J_3 \subset J_2$ to an arc $J_4 \subset I$ different from J_1 and J_3. Then take h equal to h_2 or $h_2 h_1$.

We can now conclude. Let $1 = a_1, \ldots, a_r$ be an orientation-preserving J'-covering of L. Since $J' \subset I \cap h^{-1} I$, we get another orientation-preserving I-covering if we replace some of the a_i's by $a_i h^{-1}$. If $a_i a_{i-1}^{-1}$ is not hyperbolic, it is the identity on $a_{i-1} J' \cap a_i J'$. We therefore get the required I-covering of L by defining g_i inductively as a_i or $a_i h^{-1}$ so that $g_i g_{i-1}^{-1}$ is not the identity on $a_{i-1} J' \cap a_i J'$. $\qquad\qquad\square$

Corollary 9.13. *Let T be hypostable, and let $Y \subset T$ be an indecomposable subtree. Any element $g \in G$ fixing an arc in Y fixes the whole of Y. In particular, any arc in Y is stable.*

Proof. Assume that g fixes an arc $I \subset Y$. Given $x \in Y$, we aim to prove that g fixes x. After making I smaller, we can assume that there is an arc L containing x and I. Definition 9.6 provides a subarc $J \subset I$ satisfying the hypostability condition. Consider a J-covering $1 = g_1, \ldots, g_r$ of L as in Lemma 9.12. Since $g_{i+1}g_i^{-1}$, hence also $g_i^{-1}g_{i+1}$, is hyperbolic, hypostability of J implies that all arcs $g_i J$ have the same stabilizer. The element g fixes $g_1 J = J$, so it fixes every $g_i J$ and therefore L. In particular, g fixes x. □

Proof of Theorem 9.9. We explain how to adapt the arguments in [3, 29].

Let $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$. Let $\langle S_i \mid \mathcal{R}_i \rangle$ be a presentation of Q_i, with S_i a finite generating set and $\mathcal{R}_i$ a possibly infinite set of relators. Let $\langle S \mid \mathcal{R} \rangle$ be a presentation of G such that S is a finite generating set of G containing each S_i, and $\mathcal{R}$ is the union of $\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_q$ with finitely many additional relators.

Consider a finite subtree $K \subset T$, i.e. the convex hull of finitely many points. We explain how to choose K large enough so as to yield a resolution of T as in [29, Definition 2.2], even though G is only relatively finitely presented.

For each $s \in S$, we consider $K_s = K \cap s^{-1}K$ and the restriction $\varphi_s \colon K_s \to sK_s$ of s (we may assume that no K_s is empty). We then define the suspension Σ as the foliated 2-complex obtained by gluing foliated bands $K_s \times [0, 1]$ to K, where we glue $(x, 0)$ to x and $(x, 1)$ to $\varphi_s(x)$. Note that $\pi_1(\Sigma)$ is naturally identified with the free group on S.

Next, we need all relators of $\mathcal{R}$ to be represented by loops contained in leaves of Σ. Since each Q_i fixes a point p_i in T, and S contains S_i, requiring that K contains $p_1, \ldots, p_q$ takes care of $\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_q$. There remain finitely many other relators, and, as in [3, 29], one can choose K so that they also are represented by loops contained in leaves.

The complex Σ provides a resolution of T in the sense of Definition 2.2 of [29] (as pointed out in [29], the set $\mathcal{C}$ of curves contained in leaves mentioned in Definition 2.2 is not assumed to be finite). Obtaining a resolution is the only place where finite presentation is used in [29].

As for stability, it is used only in Proposition 4.3 of [29] to prove that, if an element fixes an arc in the subtree $T_{\Gamma_v} \subset T$ corresponding to a minimal component of Σ, then it fixes the whole of T_{Γ_v}. By [31, Proposition 1.25], the geometric $\mathbb{R}$-tree dual to a minimal component of Σ is indecomposable, and by Lemma 1.19(1) of [31] its image $T_{\Gamma_v} \subset T$ is an indecomposable subtree of T. Corollary 9.13 then replaces Proposition 4.3 of [29] under our hypostability assumption.

The rest of the argument of [29] applies without modification. As in Proposition 4.1 of [29], the tree dual to Σ is a graph of actions on $\mathbb{R}$-trees $T(\mathcal{G}')$, such that arc stabilizer of the vertex actions lie in the kernel of these actions. Apply-

ing Propositions 5.2, 7.2 and 8.1 of [29], one can replace these vertex actions by actions on simplicial trees whose stabilizers are abelian modulo the kernel. By Bass-Serre theory, this provides a splitting of G over the extension of an abelian group A by the kernel K of a vertex action. This splitting is relative to $\mathcal{Q}$ as in the Reduction Lemma in [29, §4]. □

9.4. $\mathcal{Z}_{\mathbf{max}}$ splittings. Say that a subgroup of a relatively hyperbolic group is $\mathcal{Z}_{\mathrm{max}}$ if it is maximal for inclusion among non-parabolic virtually cyclic subgroups with infinite center.

Theorem 9.14. *Let G be hyperbolic relative to $\mathcal{P} = \{P_1, \dots, P_n\}$, with P_i finitely generated. Let $\mathcal{H} = \{H_1, \dots H_q\}$ be a (possibly empty) family of finitely generated subgroups. Assume that G acts non-trivially on an $\mathbb{R}$-tree T relative to $\mathcal{P} \cup \mathcal{H}$, and that arc stabilizers are either finite, parabolic or $\mathcal{Z}_{\mathrm{max}}$.*

Then G splits relative to $\mathcal{P} \cup \mathcal{H}$ over a finite, parabolic or $\mathcal{Z}_{\mathrm{max}}$ subgroup.

Corollary 9.10 provides a splitting over an elementary group A. Here we assume that every loxodromic arc stabilizer of T is $\mathcal{Z}_{\mathrm{max}}$, and we claim that the same is true for A: if it is loxodromic, then it is $\mathcal{Z}_{\mathrm{max}}$.

Proof. • Assume first that the foliated 2-complex Σ constructed in the proof of Theorem 9.9 has a minimal component Σ_v. Let G_v be the image of its fundamental group in G, and let $T_{G_v} \subset T$ be the corresponding subtree of T. In particular, G_v is not elliptic in T. By [3, Theorem 5.13] or [29, Theorem 3.1], G_v is a vertex group in a decomposition of G as a graph of groups Γ relative to $\mathcal{P} \cup \mathcal{H}$. All arcs in T_{G_v} have the same stabilizer F, a normal subgroup of G_v.

We claim that F is finite. Otherwise, there are two cases. If F is non-parabolic, hence virtually cyclic, it has finite index in its normalizer and therefore in G_v, so G_v is elliptic, a contradiction. If F is infinite and contained in a maximal parabolic group P, then almost malnormality of P implies that the normalizer of F is contained in P, so $G_v \subset P$. Since P is elliptic in T, this is also a contradiction. Thus F is finite.

We now distinguish several cases, depending on the nature of the minimal component Σ_v.

First, Σ_v cannot be a homogeneous (axial, toral) component since G_v would then be virtually $\mathbb{Z}^k$ for some $k \geq 2$ ([3, Theorem 9.4(2)] or [29, section 5.1]), hence parabolic, contradicting ellipticity of parabolic groups in T.

If Σ_v is an exotic (Levitt, thin) minimal component, one obtains a splitting of G over F, and we are done ([29, Proposition 7.2], [3, Theorem 9.4(3)]).

If Σ_v is a surface (IET) component, then by [3, Theorem 9.4(1)] or [29, section 8], after performing some moves, one can assume that Σ_v is a surface with boundary, and G_v/F is the fundamental group of a 2-orbifold with conical singularities supporting a measured foliation with dense leaves. Moreover, G_v is a QH vertex group (with fiber F) of Γ.

Let $A \subset G_v$ be the preimage of the fundamental group of an essential two-sided simple closed curve not bounding a Möbius band, and not boundary parallel. Then G splits over A relative to $\mathcal{P} \cup \mathcal{H}$. We check that A is $\mathcal{Z}_{\max}$. Clearly, A is virtually cyclic with infinite center, and is maximal among virtually cyclic subgroups of G_v. Since G_v is QH and A is not conjugate into a boundary subgroup, it easily follows that A is $\mathcal{Z}_{\max}$ in G.

• The remaining case is when Σ has no minimal component. In this case, all leaves of Σ are finite, and the dual tree T_Σ is simplicial. Its edge stabilizers fix an arc in T, so we are done if one of these edge stabilizers is finite or parabolic. Otherwise, arc stabilizers of T_Σ are virtually cyclic with infinite center but may fail to be $\mathcal{Z}_{\max}$. If this happens, we have to enlarge the finite tree K used to construct Σ.

By [48], we can find an exhaustion of T by an increasing sequence of finite subtrees K_k such that the corresponding dual trees T_{Σ_k} strongly converge to T. We refer to [48] for the definition of strong convergence; we will only use the fact that, if A is a finitely generated group fixing an arc in T, then A fixes an edge in T_{Σ_k} for large enough k.

We can assume that all dual trees T_{Σ_k} are simplicial, and that their edge stabilizers are infinite and not parabolic. Let A_0 be an edge stabilizer of T_{Σ_0}. Then A_0 fixes an arc I in T. By the hypothesis on arc stabilizers of T, the stabilizer of I is a $\mathcal{Z}_{\max}$ subgroup $A \supset A_0$, which fixes an edge e of T_{Σ_k} for k large enough. Since G_e contains A and fixes an arc in T, it is equal to A, so T_{Σ_k} provides the desired splitting. □

A similar proof yields the following results.

Theorem 9.15. *Let G be hyperbolic relative to $\mathcal{P} = \{P_1, \ldots, P_n\}$, with P_i slender, and let $\mathcal{H} = \{H_1, \ldots H_q\}$ be a family of finitely generated subgroups. Assume that G acts non-trivially on an $\mathbb{R}$-tree T relative to $\mathcal{P} \cup \mathcal{H}$, with elementary arc stabilizers.*

Then G splits relative to $\mathcal{P} \cup \mathcal{H}$ over a $\mathcal{Z}_{\max}$ subgroup or over the stabilizer of an arc of T.

Corollary 9.16. *Let G be a toral relatively hyperbolic group. Consider a non-trivial action of G on an* $\mathbb{R}$*-tree relative to non-cyclic abelian subgroups. If arc stabilizers are abelian and stable under taking roots, then G splits* (*relative to non-cyclic abelian subgroups*) *over an abelian subgroup stable under taking roots.*

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