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Groups, Geometry, and Dynamics

Centralizers of C^1-contractions of the half line

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Abstract. A subgroup $G \subset \mathrm{Diff}^1_+([0,1])$ is C^1*-close to the identity* if there is a sequence $h_n \in \mathrm{Diff}^1_+([0,1])$ such that the conjugates $h_n g h_n^{-1}$ tend to the identity for the C^1-topology, for every $g \in G$. This is equivalent to the fact that G can be *embedded in the* C^1*-centralizer* of a C^1-contraction of $[0,+\infty)$(see [6] and Theorem 1.1).

We first describe the topological dynamics of groups C^1-close to the identity. Then, we show that the class of groups C^1-close to the identity is invariant under some natural dynamical and algebraic extensions. As a consequence, we can describe a large class of groups $G \subset \mathrm{Diff}^1_+([0,1])$ whose topological dynamics implies that they are C^1-close to the identity.

This allows us to show that the free group $\mathbb{F}_2$ admits faithful actions which are C^1-close to the identity. In particular, the C^1-centralizer of a C^1-contraction may contain free groups.

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Contents

1. Introduction

1.1. Groups C^1-close to the identity and centralizers of contractions. The main motivation of this paper is the study of centralizers of the C^1-contractions of the half line $[0,+\infty)$. A diffeomorphism f of $[0,+\infty)$ is called a *contraction* if $f(x) < x$ for every $x \neq 0$. Unless it is explicitly indicated, a contraction will now refer to a C^1-diffeomorphism. When f is at least C^2, Szekeres, in [17] (see also [16]), proved that f is the time-one map of the flow of a C^1-vector field X, and Kopell's Lemma (see [11]) implies that the C^1-centralizer of f is precisely the flow $\{X_t, t \in \mathbb{R}\}$ of the Szekeres vector field. When f is only required to be C^1, Szekeres result does not hold anymore and neither does Kopell's Lemma. Actually, the C^1-centralizer of a C^1-contraction f may be very different according to f. Generically it is trivial (i.e. equal to $\{f^n, n \in \mathbb{Z}\}$, see [18]) but it can also be very large (non abelian and non countable). We will see that there are at the same time many algebraic and dynamical restrictions on the possible groups, but also a large variety of dynamical properties which allows a group to be included in a centralizer of a contraction.

We consider groups G of diffeomorphisms of a compact segment $I \subset (0,1)$. We say that G is *embeddable in the centralizer of a contraction* if there exists a contraction f of $[0,+\infty)$ and a subgroup $\tilde{G}$ of the C^1-centralizer of f which induces G by restriction to I. [6, Theorem 3] shows that G is embeddable in the centralizer of a contraction if and only if there is a C^1-continuous path of diffeomorphisms $h_t \in \mathrm{Diff}^1_+(I)$ such that $h_t g h_t^{-1}$ tends to the identity for every $g \in G$.

As a direct consequence one deduces that, if G is embeddable in the centralizer of a contraction, then G is also embeddable in the centralizer of a diffeomorphism f of $[0,1]$ without fixed point in $(0,1)$.

Finally, an argument by A. Navas proves that:

Theorem 1.1. *Let I be a compact segment. Given a group $G \subset \mathrm{Diff}^1(I)$, the two following properties are equivalent*:

- *there is a path of diffeomorphisms $h_t \in \mathrm{Diff}^1_+(I)$, $t \in [0,1)$, continuous for the C^1-topology, and such that $h_t g h_t^{-1}$ tends to the identity in the C^1-topology when t tends to 1, for every $g \in G$;*
- *there is a sequence of diffeomorphisms $h_n \in \mathrm{Diff}^1_+(I)$, $n \in \mathbb{N}$, such that $h_n g h_n^{-1}$ tends to the identity in the C^1-topology, for every $g \in G$ as $n \to +\infty$.*

This fact is not trivial at all and is specific to the identity map: [6] provides examples of pairs of diffeomorphisms $f, g \in \mathrm{Diff}^1([0,1])$ such that there are sequences $h_n \in \mathrm{Diff}^1([0,1])$ leading g to f by conjugacy, that is

$$h_n g h_n^{-1} \xrightarrow[n\to\infty]{C^1} f,$$

but such that there is no continuous path h_t leading g to f by conjugacy. The proof of Theorem 1.1 is presented in Section 2.

Therefore, we have four equivalent notions which induce a well defined class of subgroups G of $\mathrm{Diff}^1_+([0,1])$:

- G is embeddable in the centralizer of a contraction;
- G is embeddable in the centralizer of a diffeomorphism $f \in \mathrm{Diff}^1_+(I)$ without fixed point in the interior of I;
- there exists a path of diffeomorphisms $h_t \in \mathrm{Diff}^1_+([0;1])$ leading by conjugacy every $g \in G$ to the identity;
- there exists a sequence of diffeomorphisms $h_n \in \mathrm{Diff}^1_+([0;1])$ leading by conjugacy every $g \in G$ to the identity.

Definition 1.1. Let I be a segment. We say that a group $G \subset \mathrm{Diff}^1_+([0,1])$ is *C^1-close to the identity* if it satisfies one of the four equivalent properties above, that is, for instance, if there is a sequence of diffeomorphisms $h_n \in \mathrm{Diff}^1_+([0;1])$ such that, for every $g \in G$,

$$h_n g h_n^{-1} \xrightarrow[n\to\infty]{C^1} \mathrm{id}\,.$$

We denote by $\mathcal{C}^1_{\mathbf{id}}(I)$ the class consisting of these groups; when $I = [0,1]$, we simply denote it by $\mathcal{C}^1_{\mathrm{id}}$.

The aim of this paper is to describe this class of groups C^1-close to the identity, up to group isomorphisms, up to topological conjugacy, and/or up to smooth conjugacy.

In other words, we try to answer to the following questions:

Question 1. *What groups G admit a faithful representation*

$$\varphi\colon G \longrightarrow \mathrm{Diff}^1([0,1)$$

such that $\varphi(G)$ is C^1-close to the identity?

Remark 1.2. (1) B. Farb and J. Franks [7] build examples of C^1-actions of nilpotent groups on the interval, so that the generators can be chosen in a prescribed neighborhood of the identity. However, this is not enough for ensuring that the action is C^1-close to the identity in our meaning: to each neighborhood of the identity corresponds an action. It is clear from their construction that all these actions are topologically conjugated, but they are not conjugated by diffeomorphisms. So, [7] does not prove that these actions they build are C^1-close to the identity in our meaning.

(2) In the same way, Navas [15] considers the Grigorchuk–Maki group G (whose growth is larger than polynomial but less than exponential). [15] shows that, for every C^1-neighborhood V of the identity, there is a group isomorphism from G to $\mathrm{Diff}^1_+([0,1])$ such that the generators of G are mapped to elements of V. This also does not imply that this group is C^1-close to the identity as we defined above.

(3) In [14], Navas shows that any finitely generated nilpotent subgroup $G \subset \mathrm{Diff}^1_+([0,1])$ is topologically conjugated to a group whose generators are arbitrarily C^1-close to the identity. Here again, as the conjugacy is a topological conjugacy, this does not state that G is C^1-close to the identity in our meaning. However, if we understood correctly the proof, [14] should prove that, if a finitely generated nilpotent subgroup $G \subset \mathrm{Diff}^1_+([0,1])$ is without hyperbolic fixed points, then it is C^1-close to the identity.

(4) In [12], McCarthy proves that Baumslag–Solitar groups cannot be C^1-close to the identity.

(5) In contrast, Theorem 1.9 shows that the free group $\mathbb{F}_2$ admits actions C^1-close to the identity. This shows in particular that our class of groups $G \subset \mathrm{Diff}^1_+([0,1])$ C^1-close to the identity contains groups with exponential growth, in contrast to the groups considered in [7], [15], [14].

Question 2. *What is the topological dynamics of a group $\mathcal{C}^1$-close to the identity?*

In other words, given a group $G \subset \mathrm{Homeo}_+([0,1))$, under what hypotheses does there exist $h \in \mathrm{Homeo}_+([0,1])$ such that hGh^{-1} is contained in $\mathcal{C}^1_{\mathrm{id}} \subset \mathrm{Diff}^1_+([0,1])$?

Theorem 1.7 presents a large class of groups $G \subset \mathrm{Diff}^1_+([0,1])$, called *elementary groups*, whose topological dynamics implies that they are C^1-close to the identity. Thus, every group $G' \subset \mathrm{Diff}^1_+([0,1])$ topologically conjugated to an elementary group is C^1-close to the identity.

Question 3. *Given a group* $G \subset \mathrm{Diff}^1_+([0,1])$*, under what conditions does it belong to* $\mathcal{C}^1_{\mathrm{id}}$?

Let us illustrate this question by another one, which is more precise. An immediate obstruction for a group $G \subset \mathrm{Diff}^1_+([0,1])$ to be C^1-close to the identity is the existence of a hyperbolic fixed point for an element $g \in G$. We will say that G is *without hyperbolic fixed point* if, for every $g \in G$ and every $x \in \mathrm{Fix}(g)$, $Dg(x) = 1$. A. Navas suggested that it could be a necessary and sufficient condition, maybe for finitely presented groups. [6] proved that this is true for cyclic groups.

Question 4. *Let G be a subgroup of* $\mathrm{Diff}^1_+([0,1])$ (*maybe assuming countable, or finitely presented, or any other natural hypothesis*)*. We wonder if*

$$G \text{ without hyperbolic fixed point } \overset{?}{\iff} G\ C^1\text{-close to identity.}$$

We denote by $\mathcal{C}^1_{\mathrm{nonhyp}}$ the class of groups $G \subset \mathrm{Diff}^1_+([0,1])$ without hyperbolic fixed points. For now we know

$$\mathcal{C}^1_{\mathrm{id}} \subset \mathcal{C}^1_{\mathrm{nonhyp}}$$

The rest of the introduction expounds the statements of the results above, and proposes some directions for answering these questions.

1.2. Structure results: description of the topological dynamics. This section gives necessary conditions on the topological dynamics of a group $G \subset \mathrm{Diff}^1_+([0,1])$, so that G is C^1-close to the identity.

Given a group $G \subset \mathrm{Homeo}^+([0,1])$, a *pair of successive fixed points of G* is a pair $\{a,b\}$ such that (a,b) is a connected component of $[0,1] \setminus \mathrm{Fix}(g)$ for some $g \in G$. We say that two pairs of successive fixed points $\{a,b\}$ and $\{c,d\}$ are *linked* if either $(a,b) \cap \{c,d\}$ or $(c,d) \cap \{a,b\}$ consists of exactly one point.

Definition 1.3. A group $G \subset \mathrm{Homeo}^+([0,1])$ is *without linked fixed points* if there are no linked pairs of successive fixed points.

The main topological restriction for a group $G \subset \mathrm{Diff}^1_+([0,1])$ to be C^1-close to identity is:

Theorem 1.2. *If* $G \subset \mathrm{Diff}^1_+([0,1])$ *is* C^1*-close to the identity, then G is without linked fixed points.*

Using completely different methods due to A. Navas, we prove a slightly stronger result:

Theorem 1.3 ([5]). *Any group $G \subset \mathrm{Diff}^1_+([0,1])$ without hyperbolic fixed point is without linked fixed points.*

Thus, the intervals of successive fixed points form a nested family. As a consequence, we will see that the family of pairs of successive fixed points is at most countable (Proposition 4.9).

Another consequence of being without linked fixed points is that, for every interval (a,b) of successive fixed points and any $g \in G$, either $g([a,b]) = [a,b]$ or $g([a,b]) \cap (a,b) = \emptyset$. Considering the stabilizers $G_{[a,b]}$ of the segments $[a,b]$, this provides a stratified description of the dynamics of G, as stated in Theorem 1.4 below.

Theorem 1.4. *Let $G \subset \mathrm{Homeo}_+([0,1])$ be a group without linked fixed points, and $\{a,b\}$ be a pair of successive fixed points. Then*

- *for every $g \in G$, either*
$$g([a,b]) = [a,b]$$
or
$$g((a,b)) \cap (a,b) = \emptyset.$$
We denote by $G_{[a,b]}$ the stabilizer of $[a,b]$;
- *there is a morphism*
$$\tau_{a,b} \colon G_{[a,b]} \longrightarrow \mathbb{R}$$
whose kernel is precisely the set of $g \in G_{[a,b]}$ having fixed points in (a,b), and which is positive at $g \in G_{[a,b]}$ if and only if $g(x) - x > 0$ on (a,b);
- *the union of the minimal sets of the action of $G_{[a,b]}$ on (a,b) is a non-empty closed subset $\Lambda_{[a,b]}$ on which the action of $G_{[a,b]}$ is semi-conjugated to the group of translations $\tau_{a,b}(G_{[a,b]})$;*
- *the elements of the kernel of $\tau_{a,b}$ induce the identity map on $\Lambda_{a,b}$.*

The morphism $\tau_{a,b}$ is unique up to multiplication by a positive number.

The morphism $\tau_{a,b}$ is called a *relative translation number*.

To describe the dynamics of G, we are led to consider:

- the nested configuration of the intervals of successive fixed points;
- for each interval (a, b) of successive fixed points, the relative translation number $\tau_{a,b}$ and the set $\Lambda_{a,b}$, union of the minimal sets in (a, b). Each connected component I of $(a, b) \setminus \Lambda_{a,b}$ is
  - either a wandering interval,
  - or an interval of successive fixed points,
  - or else it may be the union of an increasing sequence of intervals of successive fixed points.

  In the first case, we can stop the study. In both last cases, we consider the restriction to I of the stabilizer G_I: it is a group without linked fixed points, so we may proceed the study.

1.3. Completion of a group without linked fixed points. One difficulty for classifying the groups C^1-close to the identity is that each element $g \in G$ may have infinitely many pairs of successive fixed points. To bypass this difficulty, we enrich the group G so that, for every $g \in G$ and any pair $\{a, b\}$ of successive fixed points of g, the group G contains the diffeomorphism $g_{a,b}$ which coincides with g on $[a, b]$ and with the identity out of $[a, b]$. The diffeomorphism $g_{a,b}$ has a unique pair of successive fixed points. Such a diffeomorphism will be called *simple*. Every element $g \in G$ can be seen as an infinite product of the simple elements $g_{a,b}$, for all the pairs $\{a, b\}$ of successive fixed points of g.

More precisely, given $g, h \in \mathrm{Homeo}_+([0, 1])$, we say that *$h$ is induced by g* if g and h coincide on the support of h. We say that a group $G \subset \mathrm{Homeo}_+([0, 1])$ without linked fixed points is *complete* if it contains any homeomorphism h induced by an element $g \in G$.

Corollary 5.10 shows that every group C^1-close to the identity is a subgroup of a complete group C^1-close to the identity. Analogous results hold for groups of homeomorphisms without linked fixed points, or for groups of diffeomorphisms without hyperbolic fixed point: Proposition 5.3 associates to each group G without linked fixed points its *completion* $\widetilde{G}$ which is the smallest complete group without linked fixed points containing G. The families of intervals of successive fixed points of $\widetilde{G}$ and G are the same, and Corollary 5.6 states that the morphisms of translation numbers associated to each interval of successive fixed points also coincide for G and $\widetilde{G}$.

1.3.1. Totally rational groups and topological basis. Let us present here a problem, coming from an unsuccessful attempt of us to classify groups C^1-close to the identity.

We say that a group G without linked fixed points is *totally rational* if, for any pair of successive fixed points $\{a,b\}$, the image of the translation number $\tau_{a,b}(G_{[a,b]})$ is a cyclic (monogene) group (i.e. a group of the form $\alpha\mathbb{Z}$). Proposition 5.3 and Corollary 5.6 imply that the completion of a totally rational group is totally rational, which allows us to consider complete totally rational groups.

Given a complete totally rational group G, let us call a *topological basis* any family $\{f_i\}_{i\in\mathcal{I}}$ of elements $f_i \in G$ satisfying the following conditions.

- For every $i \in \mathcal{I}$, f_i has a unique pair $\{a_i, b_i\}$ of successive fixed points; thus $[a_i, b_i]$ is the support of f_i.
- For every $i \in \mathcal{I}$, let τ_i be the relative translation number associated to (a_i, b_i). As G is totally rational, its image is a cyclic group. We require that $\tau_i(f_i)$ is a generator of the image of τ_i.
- For every pair $\{a,b\}$ of successive fixed points of G, there is a unique $i \in \mathcal{I}$ such that $\{a_i, b_i\}$ and $\{a,b\}$ are in the same G-orbit.
- For every pair $\{a,b\}$ of successive fixed points of G, there is an element g of the subgroup $\langle f_i, i \in \mathcal{I}\rangle$ generated by the f_i such that $\{a,b\}$ is a pair of successive fixed points of g.

The family of intervals of successive fixed points is countable (Proposition 4.9), so that any countable basis is countable. We can get rather easily the three first items of the definition: just choose one interval in each G-orbit of intervals of successive fixed points and for this interval choose a generator of the corresponding translation number. Thus, the difficulty is the last item. We have not been able to solve it[1], and the following natural question remains open:

[1] Consider the group G generated by a family $g_i \in \mathrm{Diff}^1_+([0,1])$, $i \in \mathbb{N}$, such that the support of g_i consists of one interval $[a_i, b_i]$ contained in a *fundamental domain* of g_{i+1} (i.e. a compact segment of the form $[x, g_{i+1}(x)]$). Now consider the family $f_i = g_{i+1} g_i g_{i+1}^{-1}$, $i \in \mathbb{N}$. This family satisfies the three first hypotheses, but not the last one: indeed, the f_i's, $i \in \mathbb{N}$, have pairwise disjoint supports $[c_i, d_i] = [g_{i+1}(a_i), g_{i+1}(b_i)]$. Any pair $\{a_i, b_i\}$ is a pair of successive fixed points of g_i, $i \in \mathbb{N}$, but is not the image by an element $f \in \langle f_i, i \in \mathbb{N}\rangle$ of a pair $\{c_j, d_j\}$, $j \in \mathbb{N}$.

Question 5. *Given a complete totally rational group* $G \subset \mathrm{Homeo}_+([0,1])$ *without linked fixed points, does* G *admit a topological basis*?

If the answer is negative, same question with the more restrictive assumption that $G \subset \mathrm{Diff}^1_+([0,1])$ *is* C^1*-close to the identity.*

Our intuition is that every element $g \in G$ is determined by its intervals of successive fixed points and, on each of them, by the value of the corresponding translation number, which provides coordinates in the topological basis. However, even assuming the existence of a topological basis, there remain many issues before making our intuition rigorous. In particular:

Question 6. *Let* $G \subset \mathrm{Homeo}_+([0,1])$ *be without linked fixed points, and assume that* G *admits a topological basis* $\{f_i\}_{i\in\mathcal{I}}$. *Is* G *contained in the* C^0*-closure of the group generated by the* f_i*'s*?

Same question, in the C^1*-topology, with the more restrictive assumption that* $G \subset \mathrm{Diff}^1_+([0,1])$ *is* C^1*-close to the identity.*

1.4. Realization results

1.4.1. Invariance of $\mathcal{C}^1_{\mathrm{id}}$ by some extensions. Let us first present two results, enlightening some invariance of the class $\mathcal{C}^1_{\mathrm{id}}$ by some natural extensions.

Theorem 1.5. *Let* $G_n \subset \mathrm{Diff}^1_+([0,1])$, $n \in \mathbb{N}$, *be an increasing sequence of subgroups:*

$$G_n \subset G_{n+1}, \quad \text{for all } n \in \mathbb{N}.$$

Assume that every G_n *is finitely generated and is* C^1*-close to the identity. Then* $G = \bigcup_{n\in\mathbb{N}} G_n \subset \mathrm{Diff}^1_+([0,1])$ *is* C^1*-close to the identity.*

We don't know if Theorem 1.5 holds for uncountable groups G_n.

The technical heart of this paper consists in proving:

Theorem 1.6. *Consider a segment* $I \subset (0,1)$. *Let* $G \subset \mathrm{Diff}^1_+(I)$ *be a group* C^1*-close to the identity and* $f \in \mathrm{Diff}^1_+([0,1])$ *be a diffeomorphism such that*

$$f(I) \cap I = \emptyset$$

(*in other words,* I *is contained in the interior of a fundamental domain of* f). *Then the group* $\langle f, G\rangle$ *generated by* f *and* G *is* C^1*-close to the identity.*

Actually, we will prove in Theorem 8.1 a slightly stronger version, where I is not assumed to be contained in the interior of a fundamental domain of f, but only contained in a fundamental domain. This weaker hypothesis requires extra technical conditions.

Remark 1.4. Under the hypotheses of Theorem 1.6, any conjugates $f^i G f^{-i}$ and $f^j G f^{-j}$, $i \neq j$, have disjoint supports and therefore commute. Actually, the group $\langle f, G\rangle$ generated by f and G is isomorphic to

$$\langle f, G\rangle = \Big(\bigoplus_{\mathbb{Z}} G\Big) \rtimes \mathbb{Z},$$

where the factor $\mathbb{Z}$ is generated by f and acts on $\big(\bigoplus_{\mathbb{Z}} G\big)$ by conjugacy as a shift of the G factors (see Lemma 5.14).

Definition 1.5. The group $\big(\bigoplus_{\mathbb{Z}} G\big) \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on $\big(\bigoplus_{\mathbb{Z}} G\big)$ by conjugacy as a shift of the G factors, has been considered in the same context in [9, Chapitre 5, exercice 1.7 vi]. It is called the *wreath-product of G by $\mathbb{Z}$*, and denoted by

$$G \wr \mathbb{Z}.$$

1.4.2. Elementary groups and fundamental systems. We now define a class of subgroups $G \subset \mathrm{Diff}^1([0,1])$, called *elementary groups*, whose topological dynamics implies that they are C^1-close to the identity: *every group G' topologically conjugated to G is C^1-close to the identity*.

Definition 1.6. A collection $(f_n)_{n\in\mathbb{N}}$, $f_n \in \mathrm{Diff}^1(\mathbb{R})$ is called a *fundamental system* if, for every $n \in \mathbb{N}$, $\mathbb{R} \setminus \mathrm{Fix}(f_n)$ consists of a unique connected component with compact closure S_n (called the support of f_n), and if for every $n \in \mathbb{N}$ there is a fundamental domain I_n of f_n such that, for every $i, j \in \mathbb{N}$ we have the following property:

- either $S_i \subset I_j$,
- or $S_j \subset I_i$,
- or else f_i and f_j have supports with disjoint interiors

$$\mathring{S}_i \cap \mathring{S}_j = \varnothing.$$

A group $G \subset \mathrm{Diff}^1_+([0,1])$ is said to be an *elementary group* if it is generated by a fundamental system supported in $[0,1]$.

Remark 1.7. Every group $G' \subset \mathrm{Diff}^1_+([0,1])$ topologically conjugated to an elementary group is an elementary group.

Our main result is:

Theorem 1.7. *Every elementary group $G \subset \mathrm{Diff}^1_+([0,1])$ is C^1-close to the identity.*

Theorem 1.7 is obtained as a consequence of Theorem 8.1 (the stronger version of Theorem 1.6 above) and Theorem 1.5: by Theorem 1.5 one only needs to consider groups generated by a finite fundamental system, and for these groups, Theorem 8.1 enables us to argue by induction on the cardinal of the fundamental system.

The groups contained in an elementary group are very specific. In particular, we can prove:

Proposition 1.8. *Every finitely generated group contained in an elementary group is solvable.*

1.5. Examples and counter examples. An easy solvable (non nilpotent) group is the Baumslag–Solitar group whose presentation is

$$B(1,n) = \langle a, b \mid aba^{-1} = b^n \rangle,$$

where n is an integer such that $|n| > 1$. This group has a very natural affine action on $\mathbb{R}$ (a acts as the homothety of ratio n and b is a translation) and on the circle (by identifying the affine group to a subgroup of $PSL(2,\mathbb{R})$), and therefore on the segment $[0,1]$ (by opening the fixed point ∞ of the circle). In particular, $B(1,n)$ has analytic actions on $[0,1]$. These analytic actions have been classified in [2]. All these actions have linked pairs of successive fixed points, and therefore are not C^1-close to the identity. On the other hand, $B(1,n)$ admits C^0-actions without linked fixed points, but [10] shows that these C^0-actions cannot be C^1 (see also [4]).

Indeed, [12] shows the following theorem, which holds in any dimension:

Theorem 1.8 (A. McCarthy). *For every n, $|n| > 1$, and any compact manifold M, there is a C^1-neighborhood U_n of the identity in $\mathrm{Diff}^1(M)$ such that every morphism $\rho\colon B(1,n) \to \mathrm{Diff}^1(M)$ with $\rho(a) \in U_n$ and $\rho(b) \in U_n$ satisfies*

$$\rho(b) = \mathrm{id}\,.$$

In particular, no subgroup of $\mathcal{C}^1_{\mathrm{id}}$ is isomorphic to $B(1,n)$. Actually, [12] considers a more general class of groups called *abelian-by-cyclic*, and which are characterized as follows:

$$G_A = \langle a, b_1, \ldots, b_k \mid b_i b_j = b_j b_i \text{ for all } i, j, \\ f g_i f^{-1} = g_1^{a_{i,1}} \ldots g_k^{a_{i,k}} \text{ for all } i \rangle,$$

where $A = (a_{i,j}) \in GL(k, \mathbb{R})$ has integer entries. [12] shows that, if A has no eigenvalue of modulus 1, then there is no faithful action on compact manifolds such that the generators belong to a neighborhood U_A of the identity in $\mathrm{Diff}^1(M)$. In particular, no subgroup in $\mathcal{C}^1_{\mathrm{id}}$ is isomorphic to G_A. Actually, [4] showed recently that, for every faithful C^1-action of the group G_A on $[0,1]$, the diffeomorphism associated to a admits a hyperbolic fixed point whose derivative is an eigenvalue of the matrix A. Consequently, no group in $\mathcal{C}^1_{\mathrm{nonhyp}}$ is isomorphic to G_A.

This shows that there are algebraic obstructions for a group admitting faithful actions on $[0,1]$ to be C^1-close to identity. It is natural to ask if $\mathcal{C}^1_{\mathrm{id}}$ contains free groups. The answer is positive as stated below:

Theorem 1.9. *There is a group $G \subset \mathrm{Diff}^1([0,1])$, C^1-close to the identity (and totally rational), such that G is isomorphic to the free group $\mathbb{F}_2$.*

(See also [1] which proves that the free group $\mathbb{F}_2$ admits a faithful discrete representation into $\mathrm{Diff}^1_+([0,1])$.)

The proof of Theorem 1.9 consists in building a group

$$G_\omega = \langle f_\omega, g_\omega \rangle \subset \mathrm{Diff}^1_+([0,1]),$$

for all the reduced words ω, so that G_ω is C^1-close to the identity and the pair $\{f_\omega, g_\omega\}$ does not satisfy the relation ω. Then, we glue together the groups G_ω in such a way that their supports are pairwise disjoint. Theorem 1.6 is our main tool for building the groups G_ω.

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2. Isotopies versus sequences of conjugacy to the identity

2.1. The cohomological equation and the proof of Theorem 1.1. The aim of the section is the proof (suggested by A. Navas) of Theorem 1.1. Let us first start with the following observation:

Lemma 2.1. *Consider* $f \in \mathrm{Diff}^1_+([0,1])$ *and a sequence* $\{h_n\}_{n\in\mathbb{N}}$ *with* $h_n \in \mathrm{Diff}^1_+([0,1])$. *Then*

$$(h_n f h_n^{-1} \xrightarrow{C^1} \mathrm{id}) \iff (\log Dh_n(f(x)) - \log Dh_n(x) \xrightarrow{\text{unif}} -\log Df(x)).$$

Proof. Just notice that the right term means that $\log D(h_n f h_n^{-1}(h(x)))$ converges uniformly to 0, that is, $D(h_n f h_n^{-1})$ converges uniformly to 1. This implies that $h_n f h_n^{-1}$ is C^1-close to an isometry of $[0,1]$, that is, to the identity map. □

A straightforward calculation implies:

Corollary 2.2. *Assume that*

$$\psi_n \colon [0,1] \longrightarrow \mathbb{R}$$

is a sequence of continuous maps satisfying

$$\psi_n(f(x)) - \psi_n(x) \xrightarrow{\text{unif}} -\log Df(x).$$

Then

$$h_n \colon [0,1] \longrightarrow \mathbb{R}$$

defined by

$$h_n(x) = \frac{\displaystyle\int_0^x e^{\psi_n(t)}\,dt}{\displaystyle\int_0^1 e^{\psi_n(t)}\,dt}$$

is a sequence of diffeomorphisms of $[0,1]$ *such that*

$$h_n f h_n^{-1} \xrightarrow{C^1} \mathrm{id}\,.$$

For any continuous map

$$\psi : [0,1] \longrightarrow \mathbb{R},$$

we will denote by $h_\psi \in \mathrm{Diff}^1_+([0,1])$ the diffeomorphism defined as above, that is

$$h_\psi(x) = \frac{\displaystyle\int_0^x e^{\psi(t)}dt}{\displaystyle\int_0^1 e^{\psi(t)}dt}.$$

Notice that

- $\psi \mapsto h_\psi$ is a continuous map from $\mathcal{C}^0([0,1],\mathbb{R})$ to $\mathrm{Diff}^1_+([0,1])$;
- $h_{\log Dg} = g$ for every $g \in \mathrm{Diff}^1_+([0,1])$.

Consequently, finding diffeomorphisms h_n conjugating f C^1-close to the identity is equivalent to find approximate solutions for the cohomological equation. The advantage of this approach is that the cohomological equation is a linear equation. As a consequence, convex sums of approximate solutions are still approximate solutions.

Lemma 2.3. *Assume that $\psi_n : [0,1] \to \mathbb{R}$ is a sequence of continuous maps satisfying*

$$\psi_n(f(x)) - \psi_n(x) \xrightarrow{\text{unif}} -\log Df(x).$$

Let $\psi_t, t \in [0,+\infty)$ be defined as follows:

- *if $t = n \in \mathbb{N}$, then $\psi_t = \psi_n$;*
- *if $t \in (n, n+1)$, then $\psi_t(x) = (n+1-t)\psi_n(x) + (t-n)\psi_{n+1}(x)$.*

Then $\{h_t = h_{\psi_t}\}_{t\in[0,+\infty)}$ is a continuous path of diffeomorphisms satisfying

$$h_t f h_t^{-1} \xrightarrow[t\to+\infty]{C^1} \mathrm{id}.$$

Proof of Theorem 1.1. Let G be a group and assume that h_n is a sequence of diffeomorphisms such that

$$h_n g h_n^{-1} \xrightarrow{C^1} \mathrm{id}, \quad \text{for all } g \in G.$$

Let ψ_n denote $\log Dh$ and define $\psi_t, t \in [0,+\infty)$, as in Lemma 2.3 as convex sums of the ψ_n. One denotes $h_t = h_{\psi_t}$. Notice that, for $t = n \in \mathbb{N}$, one has $h_t = h_n$, so that the notation is coherent. Now, for every $g \in G$, one has

$$h_t g h_t^{-1} \xrightarrow[t\to+\infty]{C^1} \mathrm{id}. \qquad \square$$

2.2. Increasing union of groups C^1-close to the identity: proof of Theorem 1.5.

Proof of Theorem 1.5. Let $\{G_n\}_{n\in\mathbb{N}}$ be an increasing sequence of finitely generated groups C^1-close to the identity. For every $n \in \mathbb{N}$, let k_n be an integer such that $\mathcal{S}_n = \{g_{1,n}, \dots, g_{k_n,n}\}$ is a system of generators of G_n.

One denotes
$$\mathcal{E}_n = \bigcup_{i\le n} \mathcal{S}_i.$$
As G_n is C^1-close to the identity, there is a sequence $h_{n,i} \in \mathrm{Diff}^1_+([0,1])$ such that
$$h_{n,i} g h_{n,i}^{-1} \xrightarrow[i\to\infty]{C^1} \mathrm{id} \quad \text{for all } g \in G_n.$$
Fix a sequence $\varepsilon_n > 0$ such that $\varepsilon_n \xrightarrow[n\to\infty]{} 0$. For every n, there is $i(n)$ so that
$$\|h_{n,i(n)} g h_{n,i(n)}^{-1} - \mathrm{id}\,\|_1 < \varepsilon_n, \quad \text{for all } g \in \mathcal{E}_n.$$

As a straightforward consequence, for every $g \in G = \bigcup_{n\in\mathbb{N}} G_n$ one has
$$h_{n,i(n)} g h_{n,i(n)}^{-1} \xrightarrow[n\to\infty]{C^1} \mathrm{id}\,.$$
According to Definition 1.1 (which is coherent according to Theorem 1.1), this is equivalent to the fact that G is C^1-close to the identity, ending the proof. □

As any countable group is an increasing union of finitely generated groups, we easily deduce:

Corollary 2.4. *If $\{G_n\}_{n\in\mathbb{N}}$ is an increasing sequence of countable groups and if $G_n \in \mathcal{C}^1_{\mathrm{id}}$ for all n, then*
$$G = \bigcup_{n\in\mathbb{N}} G_n \in \mathcal{C}^1_{\mathrm{id}}.$$

3. Relative translation numbers for groups C^1-close to the identity

The aim of this section is to give a direct proof of Theorem 1.4 in the case of groups of diffeomorphisms C^1-close to the identity. This proof was essentially done (and never written) by the first author with S. Crovisier and A. Wilkinson in 2003. Section 4.2 presents the (later) proof due to A. Navas, with completely different arguments, for groups of homeomorphisms without linked fixed points.

Let us restate Theorem 1.4 in the settings of groups $G \in \mathcal{C}^1_{\mathrm{id}}$:

Theorem 3.1. *Let $G \subset \mathrm{Diff}^1_+([0,1])$ be a subgroup C^1-close to the identity, f in G, and I a connected component of $[0,1] \setminus \mathrm{Fix}(f)$. Assume that $(f(x)-x) > 0$ for $x \in I$.*

- *For every $g \in G$, either $g(I) = I$ or $g(I) \cap I = \emptyset$.*
- *Let us denote $G_I = \{g \in G \mid g(I) = I\}$ the stabilizer of I. There is a unique group morphism $\tau_{f,I} : G_I \to \mathbb{R}$ with the following properties:*
  - *the kernel $Ker(\tau_{f,I})$ is precisely the set of elements $g \in G$ having a fixed point in I,*
  $$(g \in G_I \text{ and } \tau_{f,I}(g) = 0) \iff \mathrm{Fix}(g) \cap I \neq \emptyset;$$
  - *$\tau_{f,I}$ is increasing: given any $g, h \in G_I$, if there exists $x \in I$ such that $g(x) \geq h(x)$, then $\tau_{f,I}(g) \geq \tau_{f,I}(h)$;*
  - *$\tau_{f,I}(f) = 1$.*

Theorem 3.1 is the purpose of this Section 3.

3.1. Background on diffeomorphisms C^1-close to the identity and compositions.

According to [3, Lemme 4.3.B-1], we have:

Lemma 3.1. *Let M be a compact manifold. For any $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists $\delta > 0$ such that, for any $f \in \mathrm{Diff}^1(M)$ whose C^1-distance to the identity is less than δ, for every $x \in M$, for any y with $\|y - x\| < n\|f(x) - x\|$, one has*

$$\|(f(y) - y) - (f(x) - x)\| < \varepsilon\|f(x) - x\|.$$

As a consequence, [3] shows:

Lemma 3.2. [3, Lemme 4.3.D-1] *Let M be a compact manifold. For any $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists $\delta > 0$ such that, for any $f_1, \dots, f_n \in \mathrm{Diff}^1(M)$ whose C^1-distance to the identity is less than δ, for every $x \in M$, one has*

$$\left\|(f_n \dots f_1(x) - x) - \sum_i (f_i(x) - x)\right\| < \varepsilon \sup_i \|f_i(x) - x\|.$$

3.2. Translation numbers. In this section, $G \subset \mathrm{Diff}^1_+([0,1])$ denotes a group C^1-close to the identity. Thus, there is a C^1-continuous path $h_t \in \mathrm{Diff}^1_+([0,1])$, $t \in [0,1)$, such that $h_t g h_t^{-1}$ tends to id as $t \to 0$ for the C^1-topology, for every $g \in G$. For every element g of G, we will denote $g_t = h_t g h_t^{-1}$. For every point $x \in [0,1]$, we denote $x_t = h_t(x)$.

As a direct consequence of Lemma 3.2, we get:

Lemma 3.3. *Let $g \in G$ and $x \in [0,1]$ such that $g(x) \neq x$. Then, for every $n \in \mathbb{Z}$, one has*

$$\lim_{t\to 1} \frac{g_t^n(x_t) - x_t}{g_t(x_t) - x_t} = n$$

We deduce:

Corollary 3.4. *Consider $f, g \in G$ and $x \in [0,1]$ such that $f(x) > x$. Assume that there are $n > 0$ and $m \in \mathbb{Z}$ such that*

$$g^n(x) \in [f^m(x), f^{m+1}(x)].$$

Then, for every t close enough to 1, one has

$$\frac{g_t(x_t) - x_t}{f_t(x_t) - x_t} \in \left[\frac{m-1}{n}, \frac{m+2}{n}\right].$$

Analogous statement holds in the case $f(x) < x$ or $n < 0$.

Proof. Obviously, the statement is satisfied if $g(x) = x$. One assumes now that $g(x) \neq x$. The conjugacy by h_t preserves the order. Thus, by assumption, one has

$$\frac{g_t^n(x_t) - x_t}{f_t^m(x_t) - x_t} \in \left[1, \frac{f_t^{m+1}(x_t) - x_t}{f_t^m(x_t) - x_t}\right].$$

For every t we define α_t by

$$\frac{g_t^n(x_t) - x_t}{f_t^m(x_t) - x_t} = \alpha_t \frac{n}{m} \frac{g_t(x_t) - x_t}{f_t(x_t) - x_t}.$$

Thus

$$\alpha_t \frac{g_t(x_t) - x_t}{f_t(x_t) - x_t} \in \left[\frac{m}{n}, \frac{m}{n} \frac{f_t^{m+1}(x_t) - x_t}{f_t^m(x_t) - x_t}\right].$$

Lemma 3.3 implies that

$$\alpha_t \xrightarrow[t\to 1]{} 1 \quad \text{and} \quad \frac{m}{n}\frac{f_t^{m+1}(x_t)-x_t}{f_t^m(x_t)-x_t} \xrightarrow[t\to 1]{} \frac{m+1}{n}.$$

One concludes by choosing t large enough so that

$$\left[\alpha_t^{-1}\frac{m}{n}, \alpha_t^{-1}\frac{m+1}{n}\right] \subset \left(\frac{m-1}{n}, \frac{m+2}{n}\right). \qquad \square$$

Conversely, we have:

Corollary 3.5. *Consider $f, g \in G$ and $x \in [0,1]$ such that $f(x) > x$ and assume there are $n, m \in \mathbb{Z}$, $n > 0$, and a sequence $t_i \to 1$, $i \in \mathbb{N}$ such that*

$$\frac{g_{t_i}(x_{t_i}) - x_{t_i}}{f_{t_i}(x_{t_i}) - x_{t_i}} \in \left[\frac{m}{n}, \frac{m+1}{n}\right].$$

Then

$$g^n(x) \in [f^{m-1}(x), f^{m+2}(x)]$$

Analogous statement holds in the case where $f(x) < x$ or $n < 0$. The proof of Corollary 3.5 follows from the same estimates as Corollary 3.4 and is left to the reader.

Corollary 3.6. *Consider $f \in G$ and $x \in [0,1]$ such that $f(x) \neq x$.*

(1) *For any $g \in G$, the ratio*

$$\frac{g_t(x_t) - x_t}{f_t(x_t) - x_t}$$

converges as $t \to 1$. Its limit will be denoted by

$$\tau_f(g, x) \in \mathbb{R} \cup \{-\infty, +\infty\}.$$

(2) $\tau_{f^{-1}}(g, x) = \tau_f(g^{-1}, x) = -\tau_f(g, x)$ *(with the convention $-\pm\infty = \mp\infty$).*

(3) $\tau_f(g, x) \in \{-\infty, +\infty\}$ *if and only if f has a fixed point in $(x, g(x))$ or $(g(x), x)$ (according to the sign of $g(x) - x$).*

(4) *Let us denote $G_{x,f} = \{g \in G, \tau_f(g, x) \in \mathbb{R}\}$. Then $G_{x,f}$ is a subgroup of G containing f and $\tau_f(g, x)\colon G_{x,f} \to \mathbb{R}$ is a group morphism sending f to 1.*

(5) (a) *If $\tau_f(g, x) \in \mathbb{R}^*$, then*

$$\tau_g(f, x) = \frac{1}{\tau_f(g, x)}.$$

(b) $\tau_f(g, x) \in \{-\infty, +\infty\} \iff \tau_g(f, x) = 0.$

(6) *If* $\tau_f(g,x) \in \mathbb{R}^*$, *then for every* $h \in G$ *one has*

- $\tau_f(h,x) \in \{-\infty,+\infty\} \iff \tau_g(h,x) \in \{-\infty,+\infty\}$;
- $\tau_f(h,x) \in \mathbb{R} \implies \tau_g(h,x) = \tau_g(f,x)\tau_f(h,x)$.

(7) $\tau_f(g,x) = 0 \iff g$ *has a fixed point in* $[x, f(x)]$.

Proof. First notice that the sign of $\tau_f(g,x)$ is determined by the sign of $(f(x)-x)(g(x)-x)$.

For the first item, assume that there is a sequence t_i such that $\frac{g_{t_i}(x_{t_i})-x_{t_i}}{f_{t_i}(x_{t_i})-x_{t_i}}$ is bounded. Hence, up to consider a subsequence, this ratio converges to some $\tau \in \mathbb{R}$. Then Corollary 3.5 implies that any rational estimate of τ

$$\frac{m}{n} < \tau < \frac{m+1}{n}$$

leads to a dynamical estimate of $g^n(x)$. Now Corollary 3.4 implies that $\frac{g_t(x_t)-x_t}{f_t(x_t)-x_t}$ belongs to $[\frac{m-2}{n}, \frac{m+3}{n}]$ for any t close enough to 1. By considering more precise rational estimates, one easily deduces that $\frac{g_t(x_t)-x_t}{f_t(x_t)-x_t}$ converges to τ. This concludes the proof of item 1.

Item 2 is a straightforward consequence of Lemma 3.3. Item 3 is a straightforward consequence of Corollaries 3.4 and 3.5 and of item 1. Assuming $g(x) > x$ and $f(x) > x$, notice that if f has no fixed point in $[x, g(x)]$, then there is $n > 0$ with $f^n(x) > g(x)$, and therefore Corollary 3.4 implies that $\frac{g_t(x_t)-x_t}{f_t(x_t)-x_t}$ is bounded by $n+1$ for any t close to 1.

Item 4 is a straightforward consequence of Lemma 3.2. Items 5 and 6 are easy consequences of the definition.

Finally, Item 7 is a straightforward consequence of Items 3 and 5. □

Lemma 3.7. *Consider* $f \in G$ *and* $x \notin \mathrm{Fix}(f)$. *Let* I *be the connected component of* $[0,1] \setminus \mathrm{Fix}(f)$ *containing* x.

- *for every* $g \in G$ *and any* $y \in I$, *one has*

$$\tau_f(g,x) = \tau_f(g,y).$$

  We will denote this quantity by $\tau_{f,I}(g)$.

- $\tau_{f,I}(g) \in \mathbb{R} \iff g(I) = I$; *that is,* g *belongs to the stabilizer* G_I *of* I.
- $\tau_{f,I} : G_I \to \mathbb{R}$ *is a group homomorphism sending* f *to* 1.
- *If* $\tau_{f,I}(g) \in \{-\infty,+\infty\}$ *then* $g(I)$ *and* I *are disjoint.*

Proof. If $\tau_f(g,x) = 0$, then Corollary 3.6 states that g has a fixed point in the interior of I. One easily deduces that $\tau_f(g,y) = 0$ for every $y \in I$. Therefore, Corollary 3.6 implies that $[y, f(y)) \cap \mathrm{Fix}(g) \neq \emptyset$ for every $y \in I$. This implies in particular that the endpoints of I are fixed points of g, so that $g \in G_I$.

Assume now that $\tau_f(g,x) \in \mathbb{R}^*$. Then g has no fixed point in I. Let J be the connected component of x in $[0,1] \setminus \mathrm{Fix}(g)$. We saw that $\tau_g(f,x) = \frac{1}{\tau_f(g,x)} \in \mathbb{R}^*$, so f has no fixed point in J. One concludes that $I = J$, implying that $g \in G_I$.

Furthermore, Lemma 3.1 implies that, when t tends to 1, the difference $g_t(y_t) - y_t$ (resp. $f_t(y_t) - y_t$) is almost constant on an interval of the form $[g_t^{-k}(x_t), g_t^k(x_t)]$ (resp. $[f_t^{-k}(x_t), f_t^k(x_t)]$), with $|k|$ arbitrarily large, the error term being arbitrarily small compared to $g_t(x_t) - x_t$ (resp. $f(x_t) - x_t$). As, by hypothesis, $g_t(x_t) - x_t$ and $f_t(x_t) - x_t$ remain in bounded ratio as $t \to 1$, one gets that the error term is arbitrarily small compared with both $g_t(x_t) - x_t$ and $f(x_t) - x_t$. As a consequence, one gets that $\tau_f(g,y) = \lim_{t \to 1} \frac{g_t(y_t) - y_t}{f_t(y_t) - y_t}$ is locally constant, and hence is constant on I.

Finally, assume that $\tau_f(g,x) \in \{-\infty, +\infty\}$. This is equivalent to the condition $\tau_g(f,x) = 0$. Let J denote the connected component of x in $[0,1] \setminus \mathrm{Fix}(g)$. We saw that each fundamental domain of g in J contains a fixed point of f. Notice that the extremities of J are disjoint from I, whereas at least one of the extremities of I is contained in J. One easily concludes that $I \subset J$.

Now, one denotes $(a,b) = I$. Up to replace g by g^{-1}, one can assume that $g(x) - x > 0$ so that $g(y) - y > 0$ for every $y \in J$. Now, for every small $\varepsilon > 0$, $[a+\varepsilon, g(a+\varepsilon)]$ contains a fixed point of f. As there is no fixed point of f between a and b, one deduces that $g(a+\varepsilon) \geq b$. As a consequence, $g(I) \cap I = \emptyset$, completing the proof. □

We end the construction of the relative translation number $\tau_{f,I}$ and the proof of Theorem 3.1 by showing

Lemma 3.8. *Consider $f \in G$ and x such that $f(x) > x$. Let I be the connected component of x in $[0,1] \setminus \mathrm{Fix}(f)$. Then, for g, g' in the stabilizer G_I, if there exists $y \in I$ such that $g(y) \geq g'(y)$, then $\tau_{f,I}(g) \geq \tau_{f,I}(g')$.*

Proof. Just notice that

$$\frac{g_t(y_t) - y_t}{f_t(y_t) - y_t} \geq \frac{g'_t(y_t) - y_t}{f_t(y_t) - y_t}$$

for every t. □

3.3. Groups of diffeomorphisms without linked fixed points

Remark 3.9. A group $G \subset \mathrm{Homeo}_+([0,1])$ is without linked fixed points if and only if, given any $f, g \in G$, given (a,b) and (c,d) connected components of $[0,1] \setminus \mathrm{Fix}(f)$ and $[0,1] \setminus \mathrm{Fix}(g)$ respectively, then either

$$(a,b) \cap (c,d) = \emptyset,$$

or

$$(a,b) = (c,d),$$

or

$$[a,b] \subset (c,d,)$$

or

$$(a,b) \supset [c,d].$$

The following straightforward lemma gives another formulation of being without linked fixed points:

Lemma 3.10. *A group $G \subset \mathrm{Homeo}_+([0,1])$ is without linked fixed points if and only if, given $\{a,b\}$ and $\{c,d\}$ two pairs of successive fixed points of $f \in G$ and $g \in G$ respectively, then either*

$$\{c,d\} \cap (a,b) = \emptyset,$$

or

$$[c,d] \subset (a,b).$$

Theorem 3.2. *Let $G \subset \mathrm{Diff}^1_+([0,1])$ be a group C^1-close to the identity. Then G is without linked fixed points.*

Proof. Let $f, g \in G$, and I and J be connected components of $[0,1] \setminus \mathrm{Fix}(f)$ and $[0,1] \setminus \mathrm{Fix}(g)$ respectively. Assume that one endpoint of J belongs to I. This implies that $\tau_{f,I}(g) = 0$. Therefore, according to Corollary 3.6, g has fixed points in every fundamental domains of f in I. This implies that J is contained in the interior of I. If no endpoint of I or J belongs to the other interval, then either $I = J$, or $I \cap J = \emptyset$. □

4. Groups without linked fixed points

4.1. Groups without linked fixed points and groups without crossing. In [13], Navas defines the notion of *groups without crossing* as follows:

Definition 4.1. One says that a group $G \subset \mathrm{Homeo}_+([0,1])$ of homeomorphisms is *without crossing* if, given any $f, g \in G$, and any connected component (a,b) of $[0,1] \setminus \mathrm{Fix}(f)$, one has:

$$\{g(a), g(b)\} \cap (a,b) = \emptyset.$$

These two notions are equivalent:

Lemma 4.2. *A group $G \subset \mathrm{Homeo}_+([0,1])$ is without crossing if and only if it has no linked fixed points.*

Proof. Assume first that G admits a crossing. Thus there is $f \in G$; a, b successive fixed points of f, and $g \in G$ such that $g(b) \in (a,b)$ (or $g(a) \in (a,b)$, but this case is analogous).

- First assume that g has a fixed point in $[a,b)$. Let c, d be the successive fixed points of g such that $b \in (c,d)$. Then $b \in (c,d)$ but $[a,b] \not\subseteq (c,d)$, so G has linked fixed points.
- Assume now that g has no fixed points in $[a,b)$. Then gfg^{-1} admits $c = g(a) < a$ and $d = g(b) \in (a,b)$ as successive fixed points, so $\{a,b\}$ and$\{c,d\}$ are linked pairs of successive fixed points.

Conversely, assume that $\{a,b\}$ and $\{c,d\}$ are linked pairs of fixed points of $f \in G$ and $g \in G$ respectively. Up to reverse the orientation or to exchange the roles of f and g, one may assume that $d \in (a,b)$ and $c \notin (a,b)$. Up to exchange f with f^{-1}, one may assume that $f(d) < d$. Therefore $c \le a < f(d) < d$, so $f(d) \in (c,d)$: G admits a crossing. □

4.2. Dynamics of groups without crossing. In this section, $G \subset \mathrm{Homeo}_+([0,1])$ is a group without crossing (or equivalently, without linked fixed points). We fix an element $f \in G$ and a connected component I of $[0,1]\setminus\mathrm{Fix}(f)$. As G is without linked fixed point, for any $g \in G$:

- either $g(I) \cap I = \emptyset$,
- or $g(I) = I$, that is: g belongs to the stabilizer G_I of I.

In this section, we consider the dynamics of G_I in restriction to I.

As f has no fixed point (by assumption) on I, every orbit $G_I(y)$ where $y \in I$ intersects a (compact) fundamental domain $[x, f(x)]$ of f. We easily deduce (using Zorn lemma) that:

Lemma 4.3. *The action of G_I on I admits minimal closed sets.*

Furthermore, the following classical result explains what are the possibilities:

Lemma 4.4. *Let $H \subset \mathrm{Homeo}_+(\mathbb{R})$ be a group and assume that there is $h \in H$ without fixed point. Then H admits minimal sets, the union of the minimal sets is closed, and*

- *either there is a unique minimal set which is either $\mathbb{R}$ or the product $\mathcal{C} \times \mathbb{Z}$ where $\mathcal{C}$ is a Cantor set,*
- *or there is $h \in H$ without fixed point such that every minimal of G is exactly one orbit of h.*

Let us denote by $U \subset I$ the union of the open sets $I \setminus \mathrm{Fix}(g)$, for $g \in G$ with $\mathrm{Fix}(g) \cap I \neq \emptyset$. That is,

$$U = \bigcup_{\{g \in G, \mathrm{Fix}(g) \cap I \neq \emptyset\}} I \setminus \mathrm{Fix}(g).$$

U is an open set as union of open sets. As there are no linked fixed points, each connected component of U is the union of an increasing sequence of connected components of $I \setminus \mathrm{Fix}(g_n)$. As a consequence, each connected component of U is contained in a fundamental domain $[x, f(x)]$, where $x \in I$.

Remark 4.5. If g has a fixed point x in I, then it has another fixed point in $(x, f(x)]$: otherwise, the next fixed point of g would be such that (x,y) contains $f(x)$ and thus G would have a crossing.

Thus, g has fixed points in every fundamental domain of f in I.

We deduce that

Lemma 4.6. *$\Lambda = I \setminus U$ is a nonempty closed subset invariant by G_I. Furthermore, for every $g \in G_I$ with fixed points in I, the restriction of g to Λ is the identity map*:

$$\mathrm{Fix}(g) \cap I \neq \emptyset \iff g|_\Lambda = \mathrm{id}_\Lambda .$$

Proof. Let J be a component of U. If g is the identity on J, then the extremities of J are fixed points of g. Assume now that $x \in J$ is not a fixed point of g and consider the pair $\{a, b\}$ of successive fixed points of g around x. By construction of U, $(a, b) \subset J$. One deduces easily that the extremities of J are fixed points of g. Thus, g is the identity map on the boundary ∂U. Finally, every point $x \in I \setminus \mathrm{Fix}(g)$ is contained in a connected component of U, so that g is the identity map on Λ, as announced. □

Let G_I^0 be the set of elements $g \in G$ such that $\mathrm{Fix}(g) \cap I \neq \emptyset$. We deduce that

Lemma 4.7. *G_I^0 is a normal subgroup of G_I. Furthermore, $\Gamma = G_I / G_I^0$ induces a group of homeomorphisms of Λ whose action is free* (*the nontrivial elements have no fixed points*).

As a consequence we get:

Lemma 4.8. *Assume that the action of G_I on I has a unique minimal set $\mathcal{M}$ which is either I or $\mathcal{C} \times \mathbb{Z}$. Then, there is an increasing continuous map from $\mathcal{M}$ to $\mathbb{R}$ which induces a semi-conjugacy of the action of Γ on $\mathcal{M}$ with a dense group of translation of $\mathbb{R}$.*

Otherwise, Γ is cyclic (*monogenous*)*, and its action on Λ is conjugated to a translation.*

Proof. If the minimal set is I itself, this implies that the elements of G_I have no fixed point in I. Therefore, Hölder Theorem implies that the action is conjugated to a dense translations group.

Assume now that the action of G_I on I has a unique minimal set $\mathcal{M}$ homeomorphic to $\mathcal{C} \times \mathbb{Z}$. By collapsing each closure of a connected component of the complement of the minimal on a point, one defines a projection of I to an interval, which is surjective on $\mathcal{M}$. The action passes to the quotient and defines a free minimal action on the quotient interval: this quotient action is therefore conjugated to a dense translations group.

When a minimal set $\mathcal{M}_0$ is the orbit of an element $g \in G$, every minimal set is an orbit of g; the union of the minimal sets is a closed subset K on which the dynamics is generated by g. In this case, the minimal sets are precisely the orbits of g which are invariant by the whole group. If the orbit of x by g is not a minimal set for the action, there is $h \in G_I$ such that $x \neq h(x)$ but x and $h(x)$ are in the same fundamental domain for g, and hence belong to the same connected component of $I \setminus K$. This implies that h leaves invariant this component, hence has fixed points in I. This shows that x belongs to the open set U. In other words, we have shown that $\Lambda = K$. As g is a generator of the action on K, one deduces that g is a generator of Γ, ending the proof. □

As a consequence, we proved Theorem 1.4 which provides for groups of homeomorphisms without crossing (or without linked fixed points) the same dynamical description as for group of diffeomorphisms C^1-close to the identity, given by Theorem 3.1.

4.3. Countable family of intervals

Proposition 4.9. *Let $G \subset \mathrm{Homeo}_+([0,1])$ be a group without crossing. Then, the family of pairs of successive fixed points of elements of G is at most countable.*

Proof. Let G be a group without crossing and let $\mathcal{P}_n$ be the set of pairs of successive fixed points $\{a,b\}$ such that $|a-b| \geq \frac{1}{n}$. To prove the lemma, it is enough to prove that $\mathcal{P}_n$ is at most countable for every $n \in \mathbb{N} \setminus \{0\}$.

Let $\{a,b\} \in \mathcal{P}_n$ be such a pair and $g \in G$ such that $\{a,b\}$ are successive fixed points of g. Up to replace g by g^{-1}, we can assume that $g(x)-x > 0$ for $x \in (a,b)$.

Let $\varepsilon > 0$ be such that $0 < g(x) - x < \frac{1}{n}$ for $x \in (a, a+\varepsilon)$ and denote $c = g^{-1}(a+\varepsilon)$.

Claim 1. *Any pair $\{p,q\} \in \mathcal{P}_n$ is disjoint from $(a,c]$.*

Proof. Assume that $\{p,q\}$ meets $(a,c]$. As G is without crossing and as $\{a,b\}$ is a pair of successive fixed points of g, the pair $\{p,q\}$ is contained in a fundamental domain of g, that is in $[p, g(p)]$. However, the choice of c implies that $|p-q| < \frac{1}{n}$, contradicting the fact that $\{p,q\} \in \mathcal{P}_n$. □

Now, to every pair $\{a,b\}$, one can associate the largest open interval $J_{a,b} = (a,d)$ such that (a,d) is disjoint from any pair $\{p,q\} \in \mathcal{P}_n$. In other words, $d = \inf\{p > a \mid \text{there exists } q \text{ such that } \{p,q\} \in \mathcal{P}_n\}$. The claim asserts that this open interval $J_{a,b}$ is not empty (that is, $d > a$).

By construction, if $\{a_1,b_1\}, \{a_2,b_2\}$ are two distinct pairs in $\mathcal{P}_n$, then $J_{a_1,b_1} \cap J_{a_2,b_2} = \emptyset$. Now, any family of disjoint open intervals is countable, concluding. □

4.4. A characterization of crossing, and entropy. In the next section we will show that groups of diffeomorphisms admitting a crossing have hyperbolic fixed points. The main step of the proof is the next lemma, which provides a dynamical characterization of the existence of crossings:

Lemma 4.10. *Consider $G \subset \mathrm{Homeo}_+([0,1])$. Assume that G admits crossings. Then there are $h_1, h_2 \in G$ and a segment $I \subset [0,1]$ such that $h_1(I)$ and $h_2(I)$ are disjoint segments contained in I.*

Thus (see [8]), the topological entropy of the semi-group generated by h_1^{-1} and h_2^{-1} is $\log 2$.

Proof. According to Lemma 3.10 (up to reverse the orientation, and to exchange f for g), there are $f, g \in G$ and successive fixed points $\{a, b\}$ and $\{c, d\}$ of f, g respectively, such that $b \in (c, d)$ and such that $(a, b) \not\subseteq (c, d)$, that is $a < c < b < d$. Up to exchange f for f^{-1} and g with g^{-1}, one may assume that $f(x) - x > 0$ on (a, b) and $g(x) - x > 0$ on (c, d).

Consider $x_0 \in (a, c)$, and fix $I = [x_0, b]$. Then, for $n > 0$ large, $f^n(I)$ is a small segment in I arbitrarily close to b. Then, for positive m, $g^{-m} f^n(I)$ form an infinite collection of disjoint segments contained in $[c, b)$, and hence contained in the interior of I. □

4.5. Groups without hyperbolic fixed point. Let us recall that a group $G \subset \mathrm{Diff}^1_+([0,1])$ is *without hyperbolic fixed point* if, for every $g \in G$, one has $Dg(x) = 1$ for every $x \in \mathrm{Fix}(g)$. The aim of this section is to prove Theorem 1.3, that is, *if G is without hyperbolic fixed point, then it is without crossing.*

We present here a proof of A. Navas. Let us start by stating two lemmas.

Lemma 4.11. *Let I be a segment and $f, g \colon I \to I$ be diffeomorphisms onto their images, and such that $f(I) \cap g(I) = \emptyset$. Then there is an infinite sequence ω_i, $i \in \mathbb{N}$, $\omega_i \in \{f, g\}$ such that*

$$\limsup \frac{1}{n} \log \ell(\omega_{n-1}\omega_{n-2}\cdots\omega_0(I)) < 0,$$

where ℓ denotes the length.

The proof of Lemma 4.11 is postponed to the end of the section.

Lemma 4.12. *Let $f, g \in \mathrm{Diff}^1_+([0,1])$ be two C^1-diffeomorphisms and assume there is a segment I for which there is an infinite word ω_i, $i \in \mathbb{N}$, $\omega_i \in \{f, g\}$ such that*

$$\limsup \frac{1}{n} \log \ell(\omega_{n-1}\omega_{n-2}\cdots\omega_0(I)) < 0$$

Then, for every $t \in I$, one has

$$\limsup \frac{1}{n} \log |D((\omega_{n-1}\omega_{n-2}\cdots\omega_0)(x)| = \limsup \frac{1}{n} \log \ell(\omega_{n-1}\omega_{n-2}\cdots\omega_0(I)) < 0$$

Proof. This lemma looks like a distortion control, which does not exist in the C^1-setting. However, the proof is an easy consequence of the uniform continuity of Df and Dg: the length of $\omega_{n-1}\omega_{n-2}\cdots\omega_0(I)$ tends to 0, by assumption.

Therefore,

$$\log|D(\omega_n)(\omega_{n-1}\omega_{n-2}\cdots\omega_0(x)| - \log\frac{\ell(\omega_n\omega_{n-1}\cdots\omega_0(I))}{\ell(\omega_{n-1}\omega_{n-2}\cdots\omega_0(I))} \xrightarrow[n\to\infty]{\text{unif}} 0.$$

One concludes by noticing that

$$\frac{1}{n}\log|D\left((\omega_{n-1}\omega_{n-2}\cdots\omega_0\right)(x)| = \frac{1}{n}\sum_0^{n-1}\log|(D\omega_i)(\omega_{i-1}\cdots\omega_0(x))|,$$

and

$$\frac{1}{n}\log\ell(\omega_{n-1}\omega_{n-2}\cdots\omega_0(I)) = \frac{1}{n}\Big(\log\ell(I) + \sum_1^{n-1}\log\frac{\ell(\omega_i\cdots\omega_0(I))}{\ell(\omega_{i-1}\cdots\omega_0(I))}\Big). \qquad \square$$

Before giving the proof of Lemma 4.11, let us conclude the proof of Theorem 1.3.

Proof of Theorem 1.3 . Let $G \subset \mathrm{Diff}^1_+([0,1])$ be a group with a crossing. According to Lemma 4.10, there are $f, g \in G$ and a segment $I \subset [0,1]$ such that $f(I)$ and $g(I)$ are disjoint segments contained in I.

According to Lemma 4.11, there is an infinite word $\omega_i \in \{f, g\}$ such that the size of the segments $\omega_{n-1}\cdots\omega_0(I)$ decreases exponentially. Then, Lemma 4.12 implies that, for every $x \in I$, one has $|D\omega_{n-1}\cdots\omega_0(x)| < 1$. As we have $\omega_{n-1}\cdots\omega_0(I) \subset I$, there is a fixed point in I and this fixed point has derivative < 1, so that it is hyperbolic.

This implies that G is not without hyperbolic fixed point, concluding the proof. $\square$

Proof of Lemma 4.11. For any $n \in \mathbb{N}$, let $\Omega_n = \{f, g\}^n$ be the set of words $(\omega_i)_{i\in\{0,\dots,n-1\}}$ of length n, with letters $\omega_i \in \{f, g\}$. In particular the cardinal of Ω_n is

$$\#\Omega_n = 2^n.$$

As the intervals $\omega_{n-1}\dots\omega_0(I)$, for $(\omega_i)_{i\in\{0,\dots,n-1\}} \in \Omega_n$, are pairwise disjoints, one expects that generally the length is not much more than $\frac{1}{2^n}\ell(I)$.

Let us denote

$$B_n = \Big\{(\omega_i)_{i\in\{0,\dots,n-1\}} \in \Omega_n | \ell(\omega_{n-1}\dots\omega_0(I)) \geq \ell(I)\cdot\Big(\frac{2}{3}\Big)^n\Big\}.$$

From a simple calculation, it follows that

$$\#B_n \leq \Big(\frac{3}{2}\Big)^n.$$

Thus

$$\frac{\#B_n}{\#\Omega_n} \leq \Big(\frac{3}{4}\Big)^n.$$

Choose $0 < \varepsilon < 1$ and $T > 0$ such that $\sum_{i=1}^{\infty}\big(\frac{3}{4}\big)^{iT} < \varepsilon$ and let us denote

$$\mathcal{B}_n^T = \{(\omega_i)_{i\in\{0,\dots,nT-1\}} \in \Omega_{nT} \mid \text{there exists } j > 0 \text{ such that } (\omega_i) \in B_{jT}\}.$$

A simple calculation shows

$$\frac{\#\mathcal{B}_n^T}{\#\Omega_{nT}} \leq \sum_{i=1}^{n-1}\Big(\frac{3}{4}\Big)^{iT}.$$

Let $\Omega_\infty = \{f, g\}^{\mathbb{N}}$ be the set of infinite words in letters f, g. It is a Cantor set. Consider

$$\mathcal{G}_n^T = \{(\omega_i)_{i\in\mathbb{N}} \in \Omega_\infty \mid (\omega_i)_{i\in\{0,\dots,nT-1\}} \notin \mathcal{B}_n^T\}$$

and

$$\mathcal{G}_\infty^T = \{(\omega_i)_{i\in\mathbb{N}} \in \Omega_\infty \mid n > 0, (\omega_i)_{i\in\{0,\dots,nT-1\}} \notin \mathcal{B}_n^T\}$$

Then $\mathcal{G}_n^T$ is a decreasing sequence of compact subsets of Ω_∞ and

$$\mathcal{G}_\infty^T = \bigcap_{n>0}\mathcal{G}_n^T.$$

The fact that

$$\frac{\#\mathcal{B}_n^T}{\#\Omega_{nT}} < 1$$

implies that $\mathcal{G}_n^T$ is not empty. One deduces that $\mathcal{G}_\infty^T$ is not empty (as a decreasing sequence of nonempty compact sets).

One concludes by noticing that, for every word $(\omega_i)_{i\in\mathbb{N}} \in G_\infty^T$ one has

$$\limsup \frac{1}{n}\log \ell(\omega_{n-1}\cdots\omega_0)(I) \leq \log\Big(\frac{2}{3}\Big). \qquad \square$$

Remark 4.13. The proof above gives much more: if one endows Ω_∞ with the measure whose weight on each cylinder of length n is $\frac{1}{2^n}$ then, for almost every word (ω_i) in Ω_∞, the exponential rate of decreasing of the length is upper bounded by $-\log 2$ (this is proved in [5]).

5. Completion of a group without crossing

5.1. Complete groups without crossing. Consider a $g \in \mathrm{Homeo}_+([0,1])$. We say that a homeomorphism $h \in \mathrm{Homeo}_+([0,1])$ is *induced* by g if, for every $x \in [0,1]$, one has

$$h(x) \in \{x, g(x)\}.$$

In other words, h is obtained from g by replacing g by the identity map on the union of some connected components of $[0,1] \setminus \mathrm{Fix}(g)$. More precisely, we can easily check:

Lemma 5.1. *Given $g \in \mathrm{Homeo}_+([0,1])$, a homeomorphism h is induced by g if and only if there is a family $\mathcal{I}$ of connected components of $[0,1] \setminus \mathrm{Fix}(g)$ such that h is the map $g_{\mathcal{I}}$ defined as follows*:

$$g_{\mathcal{I}}(x) = \begin{cases} g(x) & \text{if } x \in \bigcup_{I \in \mathcal{I}} I, \\ x & \text{otherwise.} \end{cases}$$

Definition 5.2. Let $G \subset \mathrm{Homeo}_+([0,1])$ be a group without crossing. We say that G is *complete* if, for any $g \in G$, any homeomorphism induced by g belongs to G.

The aim of this section is to show:

Proposition 5.3. *Any group G without crossing is contained in a complete group without crossing.*

Let $G \subset \mathrm{Homeo}_+([0,1])$ be a group without crossing. We will denote by $I(G) \subset \mathrm{Homeo}_+([0,1])$ the group generated by all the elements h induced by elements $g \in G$.

Remark 5.4. For every $h \in I(G)$ and $x \in [0,1]$, there is $g \in G$ with $h(x) = g(x)$. In other words, $I(G)$ and G have the same orbits.

Lemma 5.5. *Assume that G is without crossing. If $\{a, b\}$ are successive fixed points for some $h \in I(G)$, then $\{a, b\}$ are successive fixed points for some $g \in G$.*

Proof. One proves the lemma by contradiction. Assume that there is a pair of successive fixed points $\{a, b\}$ for an element $h \in I(G)$, which is not a pair of successive fixed points for G. So, h can be written as $h = h_n \cdots h_1$ where h_i is induced by $g_i \in G$. One chooses the pair $\{a, b\}$ so that n is as small as possible. Fix a point $x \in (a, b)$.

For every $i \in \{1, \ldots, n\}$, let a_i be the largest fixed point of g_i less than or equal to x, and b_i the smallest fixed point of g_i larger than or equal to x. If $g_i(x) \neq x$, then $\{a_i, b_i\}$ are successive fixed points of g_i (otherwise, $a_i = b_i = x$). As $h(x) \neq x$, there is at least one index i for which $g_i(x) \neq x$.

As G is without crossing, the intervals (a_i, b_i) are totally ordered for the inclusion. The union $I = \bigcup_{i=1}^{n} (a_i, b_i)$ is invariant by all the g_i. Indeed, either I is an interval of successive fixed points of g_i, or g_i has a fixed point in I, and then belongs to the stabilizer of I. One deduces that I is fixed by all the h_i's. As a consequence, one gets $(a, b) \subset I$.

Notice that there is $j \in \{1, \ldots, n\}$ such that $I = (a_j, b_j)$. Let $i_1, \ldots, i_k$ be the set of indices such that $I = (a_{i_j}, b_{i_j})$. If there is i such that $h_i = \mathrm{id}$ on I, then n was not the minimal number. As the indices i_j have been chosen so that I is a component of $[0, 1] \setminus \mathrm{Fix}(g_{i_j})$, this implies that $h_{i_j} = g_{i_j}$ on I. Thus, one gets the same interval $\{a, b\}$ if we substitute the h_{i_j}'s by g_{i_j}'s. Thus, one will now assume that $h_{i_j} = g_{i_j}$.

One considers now the group generated by the restriction of the g_i's to I. It is a group without crossing and the restrictions of the h_i's to I are induced by the g_i's. One considers a minimal set $\mathcal{M} \subset I$ of the action of the group generated by the g_i's on I: every g_i, $i \notin \{i_1, \ldots, i_k\}$, has fixed points in I. According to Lemma 4.6, these g_i's induce the identity map on $\mathcal{M}$. Therefore the same happens for h_i, $i \notin \{i_1, \ldots, i_k\}$. So the action of h on $\mathcal{M}$ is the same as the one of $g = g_n \cdots g_1$.

Consider now the translation number τ relative to g_{i_1} on I. Then $\tau(g)$ is the sum of the $\tau(g_{i_j})$'s.

First assume that $\tau(g) \neq 0$. Thus the orbits of g on the minimal $\mathcal{M}$ go from one endpoint of I to the other, and so does the orbit of h on $\mathcal{M}$. In particular, $I = (a, b)$, so that $\{a, b\}$ is a pair of successive fixed points of g_{i_1}, contradicting the definition of $\{a, b\}$.

Thus $\tau(g) = 0$. This implies $k > 1$. Now one will use the fact that, if $f \in G$ and h is induced by $g \in G$, then fhf^{-1} is induced by fgf^{-1} in G. By using inductively the elementary fact $f(f^{-1}hf) = hf$, one can rewrite the word

$$h_n \dots h_{n_k+1} g_{n_k} h_{n_k-1} \dots h_{n_1+1} g_{n_1} h_{n_1-1} \dots h_1 = g_{n_k} \dots g_{n_1} \tilde{h}_{n-k} \cdots \tilde{h}_1,$$

where the $\tilde{h}_i$'s are induced by elements of G. However, $\tilde{g} = g_{n_k} \dots g_{n_1}$ belongs to G. Thus, one can rewrite this word as $\tilde{g}\tilde{h}_{n-k} \cdots \tilde{h}_1$, which has only $n-k+1 < n$ letters. This contradicts the fact that n was chosen as realizing the minimum. □

Corollary 5.6. *Assume that the group G is without crossing. Then $I(G)$ is without crossing.*

Let $\{a,b\}$ be a pair of successive fixed points, for some $f \in G$. Then, the image of the translation number associated to $\{a,b\}$ is the same for G and $I(G)$:

$$\tau_{f,[a,b]}(G) = \tau_{f,[a,b]}(I(G)) \subset \mathbb{R}$$

Proof. Groups without crossing are the groups without linked fixed points. This property is a property of the set of pairs of successive fixed points. According to Lemma 5.5, the pairs of successive fixed points are the same for G and for $I(G)$, concluding.

Let now $\{a,b\}$ be a pair of successive fixed points for some $g \in G$. Consider the stabilizer $I(G)_{[a,b]}$. As $I(G)$ is without crossing, the translation number relative to g can be extended to $I(G)_{(a,b)}$. Furthermore, the action of $G_{[a,b]}$ on (a,b) admits a minimal set $\mathcal{M}$. For every $x \in [0,1]$ and every $h \in I(G)$, there exists $g \in G$ such that $h(x) = g(x)$. This implies that

Claim 2. *The minimal set $\mathcal{M}$ is invariant under the action of $I(G)_{(a,b)}$.*

Proof. If $h \in I(G)$ and $x \in \mathcal{M}$, and if $h(x) \in (a,b)$, then there is $g \in G$ with $g(x) = h(x)$. Then $g(x) \in \mathcal{M}$ (because $\mathcal{M}$ is invariant by $G_{(a,b)}$). One concludes that $h(x) \in \mathcal{M}$. Thus the minimal set $\mathcal{M}$ is invariant by $I(G)_{(a,b)}$. □

Now the action of $I(G)_{(a,b)}/Ker(\tau_{g,(a,b)})$ is a free action. For every $h \in I(G)$ and $x \in \mathcal{M}$, there is $g \in G$ with $g(x) = h(x)$. This implies that g and h coincide on $\mathcal{M}$, and thus $\tau_{f,(a,b)}(h) = \tau_{f,(a,b)}(g)$, concluding. □

We don't know if, in general, $I(G)$ is a complete group, that is, if $I(I(G)) = I(G)$. For this reason, let us denote $I^n(G)$ defined as $I^{n+1}(G) = I(I^n(G))$. The sequence $I^n(G)$ is an increasing sequence of groups. We denote

$$I^\infty(G) = \bigcup_{n \in \mathbb{N}} I^n(G).$$

Next Lemma ends the proof of Proposition 5.3.

Lemma 5.7. *For every $G \subset \mathrm{Homeo}_+([0,1])$ without crossing, $I^\infty(G)$ is a complete group without crossing. Furthermore,*

- *the orbits of $I^\infty(G)$ and of G are equal*;
- *any pair of successive fixed points $\{a,b\}$ of $I^\infty(G)$ is a pair of successive fixed points of G*;
- *for any pair $\{a,b\}$ of successive fixed points of some $f \in G$, the images $\tau_{f,[a,b]}(G)$ and $\tau_{f,[a,b]}(I^\infty(G))$ are equal*;
- *any complete group without crossing containing G contains $I^\infty(G)$.*

Proof. The unique non-trivial point is that $I^\infty(G)$ is complete. For that, it is enough to show that, if $g \in I^\infty(G)$, then every homeomorphism h induced from g also belongs to $I^\infty(G)$. Notice that there is n so that $g \in I^n(G)$; thus $h \in I^{n+1}(G)$, concluding. □

The group $I^\infty(G)$ is called the *completion* of G.

5.2. Completion of groups C^1-close to the identity or without hyperbolic fixed points. The aim of this section is to prove that the completion of groups C^1-close to the identity or without hyperbolic fixed points are respectively C^1-close to the identity or without hyperbolic fixed points. Notice that:

Remark 5.8. For any sequence $\mathcal{H} = \{h_n, n \in \mathbb{N}\}$ of diffeomorphisms of $[0,1]$, let $G_{\mathcal{H}}$ be the set of diffeomorphisms g such that

$$h_n g h_n^{-1} \xrightarrow[n\to\infty]{C^1} \mathrm{id}\,.$$

Then $G_{\mathcal{H}}$ is a group C^1-close to the identity.

Lemma 5.9. *Consider a group G, C^1-close to the identity, and $\{h_n\}_{n\in\mathbb{N}}$ a sequence of diffeomorphisms such that*

$$h_n g h_n^{-1} \xrightarrow[n\to\infty]{C^1} \mathrm{id} \quad \text{for every } g \in G.$$

Consider an element $g \in G$, and a family $\mathcal{I}$ of connected components of $[0,1] \setminus \mathrm{Fix}(g)$.

Then, the induced map $g_{\mathcal{I}}$ (equal to g on the components in $\mathcal{I}$ and equal to id *out of these components), is a C^1-diffeomorphism of* $[0,1]$. *Furthermore,*

$$h_n g_{\mathcal{I}} h_n^{-1} \xrightarrow[n\to\infty]{C^1} \mathrm{id}\,.$$

In other words, $g_{\mathcal{I}} \in G_{\mathcal{H}}$.

Proof. First notice that g has no hyperbolic fixed point: the derivative of g is 1 at each endpoint of the components of $\mathcal{J}$. One deduces that $g_{\mathcal{J}}$ is a diffeomorphism.

Then, $h_n g_{\mathcal{J}} h_n^{-1}$ is induced by $h_n g h_n^{-1}$. Therefore its C^1-distance to the identity is smaller than the distance between $h_n g h_n^{-1}$ and the identity. □

Corollary 5.10. *Consider a group G, C^1-close to the identity, and a sequence $\mathcal{H} = \{h_n\}_{n\in\mathbb{N}}$ of diffeomorphisms such that*

$$h_n g h_n^{-1} \xrightarrow[n\to\infty]{C^1} \mathrm{id} \quad \text{for every } g \in G.$$

Then the completion $I^\infty(G)$ is contained in $G_{\mathcal{H}}$. In particular, $I^\infty(G)$ is C^1-close to the identity.

Lemma 5.11. *If $G \subset \mathrm{Diff}^1_+([0,1])$ is a group without hyperbolic fixed point, then the completion $I^\infty(G)$ is contained in $\mathrm{Diff}^1_+([0,1])$ and is without hyperbolic fixed point.*

Proof. It is enough to show that $I(G)$ is a group of diffeomorphisms without hyperbolic fixed point.

As we saw previously, since the elements $g \in G$ are diffeomorphism without hyperbolic fixed point, every induced map is a diffeomorphism. It remains to show that $I(G)$ is without hyperbolic fixed point.

Assume that a is a hyperbolic fixed point of an element $h \in I(G)$. Then a is an isolated fixed point of h. Let b be the next fixed point, so that $\{a, b\}$ is a pair of successive fixed points of h. According to Lemma 5.5, there exists $g \in G$ such that $\{a, b\}$ are successive fixed points of g. Thus $\tau_{g,(a,b)}(h)$ is well defined and finite. However, the derivative $g'(a)$ is 1 because G is without hyperbolic fixed point. One easily deduces that $h'(a) = 1$ (otherwise,$\tau_{g,(a,b)}(h)$ would be infinite), contradicting the hypothesis. □

5.3. Algebraic presentation: specific subgroups. The finitely generated groups C^1-close to the identity may be complicated. However, the complete groups without crossing admit special subgroups with a simple presentation.

An element in G is called *simple* if $[0,1] \setminus \mathrm{Fix}(g)$ consists of a unique interval whose closure is the support of g and is denoted by $\mathrm{Supp}(g)$.

Remark 5.12. (1) Let $G \subset \mathrm{Homeo}_+([0,1])$ be a complete group without crossing. Each $g \in G$ is the limit, for the C^0-topology, of a product of induced simple elements g_I, where I covers the set of connected components of $[0,1] \setminus \mathrm{Fix}(g)$.

(2) Let $G \subset \mathrm{Diff}^1_+([0,1])$ be a complete group without hyperbolic fixed point. Each $g \in G$ is the limit, for the C^1-topology, of a product of the induced simple elements g_I, where I covers the set of connected components of $[0,1] \setminus \mathrm{Fix}(g)$.

Let $G \subset \mathrm{Homeo}_+([0,1])$ be a group without crossing. According to Theorem 1.4, given any pair f, g of simple elements of G, we have one of the following possibilities:

- either the supports $\mathrm{Supp}(f)$ and $\mathrm{Supp}(g)$ have disjoint interiors,
- or the supports are equal,
- or the support of one of the diffeomorphisms is contained in a fundamental domain of the other.

The group generated by f and g depends essentially on these 3 configurations and admits a simple presentation if $\mathrm{Supp}(f) \neq \mathrm{Supp}(g)$.

Proposition 5.13. *Let G be a group without crossing, and let g_1 and g_2 be two simple elements of G.*

(1) *If the interior* $(\mathrm{Supp}(g_1) \overset{\circ}{\cap} \mathrm{Supp}(g_2))$ *is empty, then g_1 and g_2 commute*:

$$\langle g_1, g_2 \rangle = \mathbb{Z}^2.$$

(2) *If* $\mathrm{Supp}(g_2)$ *is contained in the interior* $\overset{\circ}{\mathrm{Supp}}(g_1)$*, then the group $\langle g_1, g_2\rangle$ admits as unique relation that the conjugates $g_1^i g_2 g_1^{-i}$, $i \in \mathbb{Z}$, pairwise commute. More precisely,*

$$\langle g_1, g_2 \rangle = \Big(\bigoplus_{\mathbb{Z}} \mathbb{Z}\Big) \rtimes \mathbb{Z},$$

where $\mathbb{Z}$ acts by conjugacy on $\big(\bigoplus_{\mathbb{Z}} \mathbb{Z}\big)$ as a shift of the $\mathbb{Z}$ factors. In other words, $\langle g_1, g_2\rangle$ is the wreath-product:

$$\langle g_1, g_2 \rangle = \mathbb{Z} \wr \mathbb{Z}.$$

Proof. One just needs to prove the second point. If $\mathrm{Supp}(g) \subset \overset{\circ}{\mathrm{Supp}}(f)$, then the images of $\mathrm{Supp}(g)$ by f^i are pairwise disjoint. This proves that the $g_i = f^i g f^{-i}$, $i \in \mathbb{Z}$, pairwise commute. This allows to define a morphism

$$\varphi\colon \Big(\bigoplus_{\mathbb{Z}} \mathbb{Z}\Big) \rtimes \mathbb{Z} = < a, b | [a^i b a^{-i}, a^j b a^{-j}], i, j \in \mathbb{Z}\} \to \langle f, g\rangle,$$

such that $\varphi(a) = f$ and $\varphi(b) = g$.

It remains to show that φ is injective. Note that every element of the group can be written as $a^i b_{i_1}^{\beta_1} \dots b_{i_k}^{\beta_k}$ with pairwise distinct $b_{i_k} = a^{i_k} b a^{-i_k}$. The image is $f^i g_{i_1}^{\beta_1} \dots g_{i_k}^{\beta_k}$. The translation number relative to f is i, so that $f^i g_{i_1}^{\beta_1} \dots g_{i_k}^{\beta_k}$ is not the identity unless $i = 0$. In that case, the element is $g_{i_1}^{\beta_1} \dots g_{i_k}^{\beta_k}$, which is the identity only if all the β_i vanish, ending the proof. □

The proposition admits a straightforward generalization as follows:

Lemma 5.14. (1) *Let I, J be two segments with disjoint interiors. Consider subgroups $H, K \subset \mathrm{Homeo}_+([0,1])$, respectively supported on I and J. Then the group generated by H and K is isomorphic to $H \oplus K$.*

(2) *Consider $f \in G$ and I a fundamental domain of the morphism f. Let $H \subset \mathrm{Homeo}_+([0,1])$ be a subgroup of homeomorphisms supported in I. Then the group generated by H and f is*

$$\langle H, f \rangle = H \wr \mathbb{Z}.$$

6. Elementary groups

In this section, we establish under what conditions on a family $\mathcal{S} \subset \mathrm{Homeo}_+([0,1])$ the generated group is without crossing and admits $\mathcal{S}$ as a topological basis.

6.1. Fundamental systems and elementary groups. Recall that a morphism $f \in \mathrm{Homeo}_+([0,1])$ is called *simple* if $[0,1] \setminus \mathrm{Fix}(f)$ has a unique connected component (whose closure is the support $\mathrm{Supp}(f)$). A simple homeomorphism f is called *positive* if $f(x) \geq x$ for all $x \in [0,1]$.

Let $\mathrm{Int}([0,1])$ denote the set of segments of $[0,1]$.

Definition 6.1. Consider a family

$$\mathcal{S} = \{f, S_f, I_f\} \subset \mathrm{Homeo}_+([0,1]) \times \mathrm{Int}([0,1]) \times \mathrm{Int}([0,1]).$$

We say that $\mathcal{S}$ is a *fundamental system* if

- for any $(f, S_f, I_f) \in \mathcal{S}$, f is simple positive, $S_f = \mathrm{Supp}(f)$, and $I_f \subset \mathrm{Supp}(f)$ is a fundamental domain of f;
- for any distinct $(f, S_f, I_f) \neq (g, S_g, I_g) \in \mathcal{S}$,
  - either S_f and S_g have disjoint interiors,
  - or $S_f \subset I_g$,
  - or $S_g \subset I_f$.

The aim of this section is to prove:

Proposition 6.2. *Let $\mathcal{S}$ be a fundamental system. Then $G \subset \mathrm{Homeo}_+([0,1])$ generated by the elements of $\mathcal{S}$ is without crossing, and totally rational.*

Lemma 6.3. *Let $\mathcal{S}$ be a fundamental system and $G \subset \mathrm{Homeo}_+([0,1])$ be the group generated by $\mathcal{S}$. Assume that $\{a,b\}$ is a pair of successive fixed points of an element of G. Then there are $f \in G$ and $(g, S_g, I_g) \in \mathcal{S}$ such that $[a,b] = f(S_g)$.*

Proof. Assume that it is not true, and consider a pair $\{a,b\}$ of successive fixed points which are not in the orbit of some S_g. Let $h = f_n^{\pm 1} \cdots f_1^{\pm 1}$, $f_i \in \mathcal{S}$, having $\{a,b\}$ as a pair of successive fixed points. One chooses $\{a,b\}$ and h so that n is minimal for these properties. Notice that (a,b) is not disjoint from all the supports S_{f_i}, otherwise h would be the identity on $[a,b]$, contradicting the definition of $\{a,b\}$.

Claim 3. *Consider $x \in (a,b)$ such that $h(x) \neq x$. The supports S_{f_j} containing x are totally ordered by inclusion, by definition of a fundamental system. Let i such that S_{f_i} is the largest support S_{f_j}, $j = 1 \dots n$, containing x. Then S_{f_i} contains $[a,b]$.*

Proof. For every j, one has $f_j(S_{f_i}) = S_{f_i}$, because the support of f_j is either disjoint from S_{f_i}, or contained in it. Therefore the endpoints of S_{f_i} are fixed points of all the f_j, hence of h. The definition of successive fixed points of h implies that $[a,b] \subset S_{f_i}$, concluding. □

One deduces

Claim 4. *$S_{f_j} \subset S_{f_i}$, for every j.*

Proof. If S_{f_j} is not contained in S_{f_i}, then f_j is the identity map on S_{f_i}. Furthermore, S_{f_i} is invariant under all the f_k's, and contains $[a,b]$. One deduces that f_i is the identity map on the orbit of $[a,b]$ by the subgroup generated by the f_k's, $k = 1 \dots n$. Thus $\{a,b\}$ is still a pair of successive fixed points of the word obtained by deleting the letter f_j. This contradicts the minimality of n. □

Now one splits $\{1,\dots,n\} = A \coprod B$ where A is the set of indices j such that $f_j = f_i$, and B the set of the other indices, so that, for $j \in B$, one has

$$S_{f_j} \subset I_{f_i}.$$

Every element f_j, $j \in B$, acts as the identity on the orbit of ∂I_{f_i} by f_i. One deduces:

Claim 5. *Let α denote the sum of the coefficients ± 1 of the f_k's, $k \in A$. Then $\alpha = 0$.*

Proof. Notice that $h = f_i^{\alpha}$ on the orbit of ∂I_{f_i}. If $\alpha \neq 0$, this implies that h has no fixed point on S_{f_i}. Thus $(a,b) = S_{f_i}$, contradicting the definition of $\{a,b\}$. □

Now, one can rewrite h as a product of conjugates of f_j, $j \in B$ by some power $f_i^{\beta_j}$. In other words, there are $\beta_j \in \mathbb{Z}$, $j \in B$, such that

$$h = \prod_{j \in B} f_i^{\beta_j} f_j f_i^{-\beta_j}.$$

Each of these conjugates $f_i^{\beta_j} f_j f_i^{-\beta_j}$ is supported in $f_i^{\beta_j}(S_j)$, which is contained in the fundamental domain $f_i^{\beta_j}(I_{f_i})$.

If all the β_j's are not equal, then the interiors of the supports of the conjugates corresponding to different β_j's are disjoint, hence these conjugates commute. Furthermore, $\{a,b\}$ is contained in one of these intervals. One does not change the pair of successive fixed points by deleting the terms corresponding to the other β_j's. Here again, if the β_j's are not constant, one gets a smaller word, contradicting the minimality of n.

One can now assume that all the β_j's are equal to some β. So, h is the conjugate by f_i^{β} of the product of the f_j's, $j \in B$. So, $\{a,b\}$ is the image by f_i^{β} of a pair $\{c,d\}$ of successive fixed points of the product of the f_j, $j \in B$. As the cardinal of B is strictly smaller than n, the minimality of n implies that $\{c,d\}$ is the image by an element of G of one of the S_g's, $g \in \mathcal{S}$. One deduces the same property for $\{a,b\} = f_i^{\beta}(\{c,d\})$, leading to a contradiction with the definition of $\{a,b\}$. □

If the group G admits a crossing, this means by definition that there is a pair of successive fixed points $\{a,b\}$ and an element $g \in G$ with $g(\{a,b\}) \cap (a,b) \neq \emptyset$. According to Lemma 6.3, one can assume that (a,b) is some S_f with $f \in \mathcal{S}$. One concludes the proof of Proposition 6.2 by showing:

Lemma 6.4. *Let $\mathcal{S}$ be a fundamental system and $G \subset \mathrm{Homeo}_+([0,1])$ be the group generated by the elements of $\mathcal{S}$. Given any $f \in \mathcal{S}$ and any $g \in G$, either $g(S_f)$ and S_f have disjoint interior, or $g(S_f) = S_f$.*

Proof. One proves it by contradiction. Assume that it is not true and consider $g = g_n^{\alpha_n} \dots g_1^{\alpha_1}$, $g_i \in \mathcal{S}$, and $f \in \mathcal{S}$ such that

- $g_{i+1} \neq g_i$ for every $i \in \{1, \dots n-1\}$,
- $g(S_f) \neq S_f$,
- and $g(S_f)$ intersects the interior of S_f.

One chooses g, f so that $\sum_{i=1}^{n} |\alpha_i|$ is minimal for these properties.

If $S_{g_1} \subset S_f$ or if S_{g_1} and S_f have disjoint interiors, then $g_1(S_f) = S_f$. In this case, one may delete g_1 contradicting the minimality of n. Thus, $S_f \subset I_{g_1}$ by definition of a fundamental system. So, $g_1^{\alpha_1}(S_f) \subset g_1^{\alpha_1}(I_{g_1})$ and its interior is disjoint from I_{g_1} but is contained in S_{g_1}.

If $S_{g_1} \not\subset I_{g_2}$, as $g_2 \neq g_1$, one gets that either the interior of S_{g_2} is disjoint from S_{g_1}, or $S_{g_2} \subset I_{g_1}$. In both cases, g_2 is the identity map on $g_1^{\alpha_1}(S_f)$. Thus, one may delete g_2, contradicting the minimality of the word. So $S_{g_1} \subset I_{g_2}$ and $g_2^{\alpha_2} g_1^{\alpha_1}(S_f) \subset g_2^{\alpha_2}(I_{g_2}) \subset S_{g_2}$. In particular, $g_2^{\alpha_2} g_1^{\alpha_1}(S_f)$ is disjoint from S_f.

An easy induction proves that $S_{g_i} \subset I_{g_{i+1}}$ and $g_{i+1}^{\alpha_i+1} \cdots g_2^{\alpha_2} g_1^{\alpha_1}(S_f)$ is disjoint from S_f for every i, concluding. □

Definition 6.5. A group $G \subset \mathrm{Homeo}_+([0,1])$ is called an *elementary group* if it is generated by a fundamental system.

Remark 6.6. If H is topologically conjugate to an elementary group G, then H is an elementary group.

6.2. The topological dynamics and the topology of the fundamental systems. The next proposition explains that an elementary group generated by a finite fundamental system is topologically determined by the topological configuration of the intervals (S_f, I_f) of the fundamental system.

Proposition 6.7. *Let $\mathcal{S}$ and Σ be two finite fundamental systems and let G and Γ be the groups respectively generated by $\mathcal{S}$ and Σ. Assume that there is a homeomorphism $h\colon [0,1] \to [0,1]$ and a bijection*

$$\varphi\colon \mathcal{S} \to \Sigma, \quad \varphi(f, S_f, I_f) = (\varphi(f), S_{\varphi(f)}, I_{\varphi(f)}),$$

such that for every f,

$$S_{\varphi(f)} = h(S_f) \quad \text{and} \quad I_{\varphi(f)} = h(I_f).$$

Then there is a homeomorphism $\tilde{h}\colon [0,1] \to [0,1]$ conjugating G to Γ:

$$\Gamma = \{\tilde{h} g \tilde{h}^{-1}, g \in G\}.$$

More precisely, for every $(f, S_f, I_f) \in \mathcal{S}$,

$$\tilde{h} f \tilde{h}^{-1} = \varphi(f).$$

Proof. One proves it by induction on the cardinals of $\mathcal{S}$ and Σ. If the cardinal is 1, one just notices that any two positive simple homeomorphisms of [0, 1] are topologically conjugate.

One assumes now that the statement has been proved for any fundamental system of cardinal less than or equal to n, and one considers fundamental systems $\mathcal{S}$ and Σ of cardinal $n+1$. Write $\mathcal{S} = \{(f, S_f, I_f)\} \cup \widetilde{\mathcal{S}}$, where S_f is a maximal interval in the (nested) family of supports, and $\Sigma = \{(\phi, S_\phi, I_\phi)\} \cup \widetilde{\Sigma}$, where $\phi = \varphi(f)$. One considers h_1 such that $h_1 g h_1^{-1} = \varphi(g)$ for $g \in \widetilde{\mathcal{S}}$. Notice that $h_1(S_g) = h(S_g) = S_{\varphi(g)}$ for $g \in \widetilde{\mathcal{S}}$. As a consequence, there is a homeomorphism h_2 which coincides with h_1 on the union $S_{\widetilde{\mathcal{S}}}$ of the supports S_g, $g \in \widetilde{\mathcal{S}}$, and with h out of this union. Notice that $S_{\widetilde{\mathcal{S}}} \cap S_f \subset I_f$. Therefore $h_2(I_f) = h(I_f) = I_\phi$.

Thus there is a unique homeomorphisms $h_3\colon S_f \to S_\phi$ conjugating f with ϕ and coinciding with h_2 on I_f. The announced homeomorphism $\tilde{h}$ is the homeomorphism which coincides with h_3 on S_f, and with h_2 out of S_f. ☐

6.3. Elementary groups of diffeomorphisms. If the homeomorphisms of a fundamental system are diffeomorphisms, the corresponding elementary group will be a group of diffeomorphisms.

Proposition 6.8. *Any elementary group $G \subset \mathrm{Diff}^1_+([0,1])$ is without hyperbolic fixed point.*

Proof. Let us consider a fundamental system $\mathcal{S} = \{(f, S, f, I_f)\}$, where $f \in \mathrm{Diff}^1_+([0,1])$. Notice that every diffeomorphism f in $\mathcal{S}$ is, by definition of a fundamental system, a simple diffeomorphism. Therefore, it has no hyperbolic fixed point.

Let G be the group generated by $\mathcal{S}$. Every $g \in G$ can be written as

$$g = f_n^{\alpha_n} \dots f_1^{\alpha_1} \in G, \quad f_i \in \mathcal{S},\ f_{i+1} \neq f_i, \alpha_i \in \mathbb{Z}$$

(this presentation of g may be not unique). One argues by contradiction and assumes that there is $g \in G$ having a hyperbolic fixed point. One chooses g such that $\sum |\alpha_i|$ is the minimal number with this property. A hyperbolic fixed point is isolated, so that it belongs to a pair of successive fixed points $\{a, b\}$ (let us assume it is a).

One considers f_i such that the support S_{f_i} is the largest of the supports S_{f_j} containing a, b. Then S_{f_i} is invariant by all the f_j's. If some S_{f_j} is not contained in S_{f_i}, one may delete the letter f_j, contradicting the minimality of $\sum |\alpha_i|$.

One considers the sum α of the α_j's for which $f_j = f_i$ (recall that S_{f_i} is the largest support). If $\alpha \neq 0$, then g coincides with f_i^α on the orbit of $\partial(I_{f_i})$. One deduces that $\{a, b\} = \partial S_{f_i}$, and that the derivative of g at each end point is the same as the one of f_i^α, which is 1 (because f_i is simple).

So $\alpha = 0$. This allows to rewrite g as the product of conjugates of the $f_j \neq f_i$ by powers $f_i^{\beta_j}$. These diffeomorphisms are supported on the $f_i^{\beta_j}(I_{f_i})$'s, which have disjoint interiors. Exactly as in the proof of Lemma 6.3, one deduces that, if the β_j's are not all equal, then one may delete some of the $f_j^{\alpha_j}$'s, contradicting the minimality of $\sum |\alpha_i|$.

So the β_j's are all equal to some β and one gets that g is the conjugate by f_i^β of the product of the $f_j^{\alpha_j}$'s, $f_j \neq f_i$. However, having a hyperbolic fixed point is invariant by conjugacy, so that one deduces that the product of the $f_j^{\alpha_j}$'s, $f_j \neq f_i$ has an hyperbolic fixed point. This contradicts again, the minimality of $\sum |\alpha_i|$. □

One of the main results of this paper consists in proving that every elementary group of diffeomorphisms of $[0, 1]$ is actually C^1-close to the identity. This will be the aim of Section 9.

7. Free group

The aim of this section is to prove Theorem 1.9 (assuming Theorem 1.6). Let us recall its statement:

Theorem 7.1. *There exists a subgroup $G \subset \mathrm{Diff}^1([0,1])$, C^1-close to the identity, and isomorphic to the free group $\mathbb{F}_2$.*

Before expounding its proof, let us notice that the elementary groups do not contain any free group $\mathbb{F}_2$.

7.1. Finitely generated subgroups of elementary groups are solvable: proof of Proposition 1.8.

In contrast, we notice that elementary groups do not contain non-cyclic free groups:

Proposition 7.1. *Any elementary group G generated by a finite fundamental system $\mathcal{S}$ is solvable, and the associated solvability length is bounded by the cardinal of $\mathcal{S}$.*

Let us recall that a group is *solvable* if the sequence

$$G_1 = [G, G], \dots, G_{n+1} = [G_n, G_n]$$

stratifies: there exists k such that

$$G_k = \{1\}.$$

The infimum of all such k's is the *solvability length* $\ell(G)$ of the solvable group G.

As a direct corollary, we get:

Corollary 7.2. *For any fundamental system $\mathcal{G}$, the group G generated by $\mathcal{G}$ does not contain any non-cyclic free group.*

Proof of Corollary 7.2. Assume that there is a subgroup $\langle f, g\rangle \subset G$ isomorphic to $\mathbb{F}_2$. Notice that f and g are written as finite words in the generators of G in $\mathcal{G}$. Thus the group $\langle f, g\rangle$ is a subgroup of a group generated by a finite fundamental system. This group is solvable, hence does not contain any subgroup isomorphic to $\mathbb{F}_2$, contradicting the hypothesis. □

Notice that our argument above proved Proposition 1.8.

We start the proof of Proposition 7.1 by showing:

Lemma 7.3. *Fix and integer k. Assume that $\{G_i\}_{i\in\mathbb{N}}$ is a sequence of solvable groups of solvability length $\ell(G_i) \leq k$. Then the direct sum*

$$G = \bigoplus_{i\in\mathbb{N}} G_i$$

is solvable, and the solvability length associated is $\ell(G) \leq k$.

Proof. Just notice that $[G, G] = \bigoplus_{i\in\mathbb{N}}[G_i, G_i]$. □

Proof of Proposition 7.1. One presents a proof by induction on the cardinal of $\mathcal{S}$. If this cardinal is 1, the group is a cyclic abelian group, hence is solvable of solvability length 1. One assumes now that Proposition 7.1 is proved for fundamental systems of cardinal less or equal to n. Let $\mathcal{S} = \{(f_i, S_i, I_i), i \in \{1, \dots, n+1\}\}$ be a fundamental system. Up to re-index the f_i's, one may assume that S_{n+1} is maximal among the S_i's for the inclusion. Let $A \subset \{1, \dots, n\}$ denote the set of indices i for which the interiors of S_i and S_{n+1} are disjoint, and $B = \{1, \dots, n\} \setminus A$ the set of indices j for which S_j is contained in I_{n+1}.

First assume that A is not empty. Then $\mathcal{S}_A = \{(f_i, S_i, I_i), i \in A\}$ and $\mathcal{S}' = \{(f_j, S_j, I_j), j \in B \cup \{n+1\}\}$ are fundamental systems with cardinal $\leq n$. Let G_A and G' be the elementary groups respectively generated by $\mathcal{S}_A$ and $\mathcal{S}'$. They are solvable groups of length bounded by n. Therefore, $G = G_A \oplus G'$ is solvable, of length bounded by n, according to Lemma 7.3, which concludes the proof in this case.

One assumes now that A is empty, so that $B = \{1, \dots, n\}$. Let G_B denote the group generated by the fundamental system $\mathcal{S}_B = \{(f_j, S_j, I_j), j \in B\}$. One easily shows that $[G, G]$ is contained in $\bigoplus_{i \in \mathbb{Z}} G_{B,i}$, where $G_{B,i} = f_{n+1}^i G_B f_{n+1}^{-i}$. These groups are solvable, of solvability length $\ell(G_{B_i}) \leq n$, by our induction hypothesis, so that G is solvable, of solvability length $\ell(G) \leq n+1$, according to Lemma 7.3. □

7.2. Proof of Theorem 1.9. Let $\mathcal{A} = \{a_i, i \in \mathbb{N}\}$ be a countable set, called alphabet. We say that a word $\omega = \{\omega_i\}$ in n letters of the alphabet $\mathcal{A}$ is *universal* (*among the groups in* $\mathcal{C}^1_{\mathrm{id}}$) if $\omega(f_1, \dots, f_n) = \mathrm{id}$ for any $f_1, \dots, f_n \in \mathrm{Diff}^1_+([0,1])$ for which the group $\langle f_1, \dots, f_n \rangle$ is C^1-close to the identity.

The length of such a word is n. The word is reduced if $\omega_{i+1}\omega_i \neq 1$, and cyclically reduced if furthermore $\omega_n \omega_1 \neq 1$.

We will prove:

Proposition 7.4. *There is no universal reduced non-trivial word.*

Let us deduce Theorem 1.9 from Proposition 7.4:

Proof. Assume that there is no universal reduced word. In particular, there is no universal word in 2 letters. Therefore, for any word ω in 2 letters, there is a pair f_ω, g_ω such that the group $\langle f_\omega, g_\omega \rangle$ is C^1-close to the identity and $\omega(f_\omega, g_\omega) \neq \mathrm{id}$.

One fixes a sequence $I_\omega \subset [0,1]$ of pairwise disjoint segments. For every ω, one chooses diffeomorphisms $\tilde{f}_\omega, \tilde{g}_\omega$ supported on I_ω, smoothly conjugated to (f_ω, g_ω), and such that $\tilde{f}_\omega, \tilde{g}_\omega$ tend uniformly to the identity when the length of ω tends to ∞ (that is possible because $\langle f_\omega, g_\omega \rangle$ is C^1-close to the identity). One defines f and g as being $\tilde{f}_\omega$,$\tilde{g}_\omega$ on I_ω and the identity map out of the union of the I_ω's. One easily checks that f and g are homeomorphisms. Now, f and g are diffeomorphisms because $\tilde{f}_\omega, \tilde{g}_\omega$ tend uniformly to the identity.

Finally, the group $\langle f, g \rangle$ is C^1-close to the identity and $\omega(f, g) \neq \mathrm{id}$ for every reduced word ω. □

7.3. No universal relation: proof of Proposition 7.4. Notice that, if ω is a universal word, then for each letter $a_i \in \mathcal{A}$, the sum of the coefficients in a_i of the ω_i's vanishes. Otherwise, $\omega(\mathrm{id}, \dots, \mathrm{id}, f, \mathrm{id}, \dots, \mathrm{id})$ would be different from id, for $f \neq \mathrm{id}$ at the i^{th} position, contradicting the universality of the relation ω.

Consider a letter appearing in ω, say ω_1. Then, one can write ω as a product of conjugates of the other letters by powers of the letter corresponding to ω_1. One replaces each $\omega_1^j a_i \omega_1^{-j}$ with a new letter denoted by $b_{i,j}$. One gets a reduced word $\varphi(\omega)$ of length less or equal than $n-2$ (in the alphabet $\mathcal{B} = \{b_{i,j}\}$). Notice that a reduced word of length ≤ 2 cannot be universal. One concludes the proof of Proposition 7.4 and therefore of Theorem 1.9 by proving

Lemma 7.5. *If ω is universal, then $\varphi(\omega)$ is universal.*

Proof. Assume that $\varphi(\omega)$ is not universal. Therefore, there is an interval I and $h_{i,j} \in \mathrm{Diff}^1(I)$ such that $\varphi(\omega)(h_{i,j}) \neq \mathrm{id}$, and the group generated by the $h_{i,j}$ is C^1-close to the identity. One considers $f \in \mathrm{Diff}^1_+([0,1])$, without hyperbolic fixed point, and such that I is contained in the interior of a fundamental domain of f. According to Theorem 1.6, as the group generated by the $h_{i,j}$'s is C^1-close to the identity, the group generated by f and the $h_{i,j}$'s is still C^1-close to the identity.

One denotes by g_i the diffeomorphism which coincides with $f^{-j} h_{i,j} f^j$ on $f^{-j}(I)$, for every j for which $h_{i,j}$ is defined, and with the identity out of the $f^{-j}(I)$'s. The group generated by f and the g_i's is C^1-close to the identity (it is the group generated by f and the $h_{i,j}$'s). Now, $\omega(f, \{g_i\})$ is a diffeomorphism which coincides with $\varphi(\omega)(h_{i,j})$ in restriction to I, hence is not the identity map, contradicting the hypothesis on ω. ☐

8. Group extensions in the class $\mathcal{C}^1_{\mathrm{id}}$

The aim of this section is to prove Theorem 1.6: given a group $G \subset \mathrm{Diff}^1_+([0,1])$, C^1-close to the identity and supported in the interior of a fundamental domain of a diffeomorphism $f \in \mathrm{Diff}^1_+([0,1])$, without hyperbolic fixed point, the group $\langle G, f\rangle$ is C^1-close to the identity. We will prove here a slightly stronger version which will be used in the proof of Theorem 1.7.

Theorem 8.1. *Let $f \in \mathrm{Diff}^1_+([0,1])$ be a diffeomorphism without hyperbolic fixed points and $G \subset \mathrm{Diff}^1_+([0,1])$ be a group supported on a fundamental domain $[x_0, f(x_0)]$. Assume that there is a C^1-continuous path $h_t \in \mathrm{Diff}^1_+([0,1])$, $t \in [0,1)$, such that*

- *for every* $g \in G$,

$$h_t g h_t^{-1} \xrightarrow{C^1} \mathrm{id}$$

  and

- h_t *is supported on* $[x_0, f(x_0)]$.

Then the group $\langle f, G\rangle$ *generated by* f *and* G *is* C^1*-close to the identity.*

More precisely, there is a C^1*-continuous path* $H_t \in \mathrm{Diff}^1_+([0,1])$, $t \in [0,1)$, *such that*

- *for every* $g \in \langle f, G\rangle$,

$$H_t g H_t^{-1} \xrightarrow{C^1} \mathrm{id}$$

  and

- H_t *is supported on the support of* f *and* $DH_t(0) = DH_t(1) = 1$.

Notice that Theorem 1.6 follows directly from Theorem 8.1: if the support of G is contained in the interior of the fundamental domain $(x_0, f(x_0))$, then, given $h_t^0 \in \mathrm{Diff}^1(\mathrm{Supp}(G))$, $t \in [0,1)$ realizing an isotopy by conjugacy of G to the identity, we easily build another isotopy by conjugacy $h_t \in \mathrm{Diff}^1([0,1])$, supported on $[x_0, f(x_0]$.

Theorem 8.1 is the main technical result of this paper. The proof is the aim of the whole section. Let us first present a sketch of proof.

8.1. Sketch of proof. The proof uses strongly arguments in [6] which builds explicit conjugacies from a given diffeomorphism to another in a given neighborhood of the identity. Here, we will need to come back to the construction of [6] to get a simultaneous conjugacy. For this reason it will be sometimes practical to use the following notation:

Notation 1. Given $f, g \in \mathrm{Diff}^1_+([0,1])$ and $h_t \in \mathrm{Diff}^1_+([0,1])$, $t \in [0,1)$ a C^1-continuous path of diffeomorphisms, the notation

$$f \underset{h_t}{\rightsquigarrow} g$$

means that $(h_t f h_t^{-1})_{t\in[0,1)}$ is an *isotopy by conjugacy from* f *to* g, that is

$$h_t f h_t^{-1} \xrightarrow[t\to 1]{C^1} g.$$

We will write $f \rightsquigarrow g$ if there exists an isotopy by conjugacy from f to g.

Sketch of proof of Theorem 8.1. Let G be a group C^1-close to the identity, supported in a fundamental domain $[a,b]$ of $f \in \mathrm{Diff}^1_+([0,1])$. Let $(h_t)_{t\in[0,1)}$ be a continuous path of C^1-diffeomorphisms realizing an isotopy by conjugacy from G to id and such that h_t has derivative equal to 1 at a and b. One will extend these diffeomorphisms $\{h_t\}_{t\in[0,1)}$ to diffeomorphisms $\tilde{h}_t$ of $[0,1]$ in such a way that

$$h_t f h_t^{-1} \xrightarrow{C^1} f$$

(Lemma 8.3). In other words, for $t \to 1$, the extensions $\tilde{h}_t$ almost commute with f. In this way, the impact induced on f by the isotopy by conjugacy from G to id will be slight.

Let $(\varphi_t)_{t\in[0,1)}$ be a continuous path of C^1-diffeomorphisms for which

$$\varphi_t f \varphi_t^{-1} \xrightarrow[t\to 1]{C^1} \mathrm{id}$$

(the existence of $(\varphi_t)_{t\in[0,1)}$ is given by [6]). Assume that one can choose φ_t so that, furthermore, φ_t is affine on $[a,b]$ for all $t \in [0,1)$. Under this assumption, the conjugacy by $(\varphi_t)_{t\in[0,1)}$ will not affect the C^1-distance of $h_t g h_t^{-1}$ to the identity for $g \in G$, and one can compose the two isotopies, by first conjugating by $\tilde{h}_t$ and then by φ_t:

$$g \underset{\varphi_t\circ\tilde{h}_t}{\rightsquigarrow} \mathrm{id}, \quad \text{for all } g \in \langle G, f\rangle.$$

Indeed, one will ensure the existence of such φ_t affine on the support of G when the support of G is contained in the interior of a fundamental domain of f. When the support of G is precisely one fundamental domain, one will weaken slightly this assumption, ensuring that the logarithm of the derivative of φ_t is equicontinuous on $[a,b]$ (Lemma 8.4). This will ensure that conjugating by φ_t will have a bounded effect on the C^1-distance of $h_t g h_t^{-1}$ to the identity for $g \in G$. □

As explained in this sketch of proof, the two main tools for Theorem 1.6 are Lemmas 8.3 and 8.4 stated in next sections.

8.2. Background from [6]. We rewrite [6, Proposition 9] in a form which will be more convenient here.

If h_0 is a diffeomorphism supported in a fundamental domain of the diffeomorphism f, we get a homeomorphism h commuting with f by defining the restriction $h_n = h|_{I_n}$, $I_n = f^n(I_0)$ as $h_n = f_{n-1} h_{n-1} f_{n-1}^{-1}$, where f_n is the restriction $f|_{I_n}$. The homeomorphism h will be a diffeomorphism if and only if h_n tends to the identity map in the C^1 topology as $n \to \pm\infty$. That is not the case in general. However, if we only want that f and h *almost* commute, then we are allowed to modify slightly the induction process, and we can do it without affecting this convergence to the identity. That was the aim of [6]. In Propositions 8.1 and 8.2 below, we renormalized the intervals I_n so that f_n appears as diffeomorphisms of $[0, 1]$.

Proposition 8.1. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of diffeomorphisms of $[0, 1]$ such that $(f_n)_{n\in\mathbb{N}}$ converges to the identity in the C^1-topology when n tends to infinity.Let h_0 be a diffeomorphism of $[0, 1]$ such that, $Dh_0(0) = 1 = Dh_0(1)$. Fix $\varepsilon > 0$.*

Then there exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ of C^1-diffeomorphisms of $[0, 1]$ such that

- $D\psi_n(0) = 1 = D\psi_n(1)$ *for all* $n \in \mathbb{N}$;
- $\|f_n \circ \psi_n - f_n\|_1 < \varepsilon$ *for all* $n \in \mathbb{N}$;
- *the sequence $(h_n)_{n\in\mathbb{N}}$ of C^1-diffeomorphisms of $[0, 1]$, defined by induction by h_0 and*

$$h_n = f_{n-1}\psi_{n-1}h_{n-1}f_{n-1}^{-1}$$

  if $n \in \mathbb{N}^*$, satisfies

$$\textit{there exists } N > 0 \textit{ such that } h_n = \mathrm{id} \textit{ for all } n \geq N.$$

For Theorem 9.1 we will need the version with parameters, also due to [6], of this proposition. Let us state it below:

Proposition 8.2. *Let $(f_{t,n})_{(t,n)\in[0,1)\times\mathbb{N}}$ be a collection of diffeomorphisms such that*

- *for all n, $(f_{t,n})_{t\in[0,1)}$ is a C^1-continuous path in $\mathrm{Diff}^1_+([0, 1])$ such that both $Df_{t,n}(0)$ and $Df_{t,n}(1)$ do not depend on $t \in [0, 1)$;*
- *for all $t \in [0, 1)$, $(f_{t,n})_{n\in\mathbb{N}}$ converges to the identity in the C^1-topology, when n tends to infinity.*

Let $(h_{t,0})_{t\in[0,1)}$ be a C^1-continuous path of diffeomorphisms of $[0, 1]$ such that, for all $t \in [0, 1)$, one has $Dh_{t,0}(0) = 1 = Dh_{t,0}(1)$. Let $(\varepsilon_t)_{t\in[0,1)}$ be a continuous path of strictly positive real numbers.

Then there exists a collection $(\psi_{t,n})_{(t,n)\in[0,1)\times\mathbb{N}}$ *of* C^1*-diffeomorphisms of* $[0,1]$ *such that*

- $D\psi_{t,n}(0) = 1 = D\psi_{t,n}(1)$ *for all* $(t,n) \in [0,1) \times \mathbb{N}$;
- $\| f_{t,n} \circ \psi_{t,n} - f_{t,n} \|_1 < \varepsilon_t$ *for all* $(t,n) \in [0,1) \times \mathbb{N}$;
- *if* $(h_{t,n})_{n\in\mathbb{N},t\in[0,1)}$ *is the collection of* C^1*-diffeomorphisms of* $[0,1]$ *defined by induction from* $h_{t,0}$ *by*

  $$h_{t,n} = f_{t,n-1}\psi_{t,n-1}h_{t,n-1}f_{t,n-1}^{-1} \quad \text{if } n \in \mathbb{N}^*,$$

  then, for every $t \in [0,1)$*, there is* $N_t > 0$*, increasing with* $t \in [0,1)$*, such that*

  $$h_{t,n} = f_{t,n-1}\psi_{t,n-1}h_{t,n-1}f_{t,n-1}^{-1} = \mathrm{id} \quad \text{for all } n \geq N_t;$$

- $(\psi_{t,n})_{t\in[0,1)}$ *is* C^1*-continuous for all* $n \in \mathbb{N}$;
- $(h_{t,n})_{t\in[0,1)}$ *is* C^1*-continuous for all* $n \in \mathbb{N}$.

8.3. Diffeomorphisms almost commuting with f and prescribed in a fundamental domain.

Lemma 8.3. *Consider* $f \in \mathrm{Diff}^1_+([0,1])$*,* $I = [x, f(x)]$*, a fundamental domain of* f*,* (a,b) *the connected component of* $[0,1] \setminus \mathrm{Fix}(f)$ *containing* x*, and* $(h_t)_{t\in[0,1)}$*, a* C^1*-continuous path of* C^1*-diffeomorphisms of* $[0,1]$ *supported on* I*. Then, for all continuous path* $(\varepsilon_t)_{t\in[0,1)}$ *of strictly positive real numbers, there exists a* C^1*-continuous path* $(\tilde{h}_t)_{t\in[0,1)}$ *of* C^1*-diffeomorphisms of* $[0,1]$ *such that, for all* $t \in [0,1)$*,*

- $\tilde{h}_t$ *coincides with the identity map on a neighborhood of* a *and* b;
- *the support of* $\tilde{h}_t$ *is contained in the orbit of the support of* h_t *by* f:

  $$\mathrm{Supp}(\tilde{h}_t) \subset \bigcup_{n\in\mathbb{Z}} f^n(\mathrm{Supp}(h_t));$$

- $\tilde{h}_t$ *coincides with* h_t *on the fundamental domain* I:

  $$\tilde{h}_t|_I = h_t;$$

- $\|\tilde{h}_t f \tilde{h}_t^{-1} - f\|_1 < \varepsilon_t$.

The proof of Lemma 8.3 consists in pushing h_t by f in the iterates $f^n(I)$ of the fundamental domain. In each of these fundamental domains, we apply a small perturbation so that the diffeomorphism obtained in $f^n(I)$ becomes closer to id.

Proof of Lemma 8.3. Let $(\varepsilon_t)_{t\in[0;1)}$ be a continuous path of strictly positive real numbers converging to 0. One denotes by $(f_n)_{n\in\mathbb{N}}$ the sequence of C^1-diffeomorphisms of $[0,1]$ defined by: for all $n \in \mathbb{N}$, f_n is the normalization of the diffeomorphism $f|_{[f^n(x);f^{n+1}(x)]}$, that is : f_n is obtained by conjugating $f|_{[f^n(x),f^{n+1}(x)]}$ by the affine maps from $[f^n(x), f^{n+1}(x)]$ and $[f^{n+1}(x), f^{n+2}(x)]$to $[0,1]$. Notice that, as f is C^1, the sequence $(f_n)_{n\in\mathbb{N}}$ converges to id when n tends to ∞, with respect to the C^1-topology. One considers then $h_{t,0}$ as being the normalization of $h_t|_{[x,f(x)]}$ on the interval $[0,1]$. In particular, the equality $Dh_{t,0}(0) = 1 = Dh_{t,0}(1)$ is satisfied.

Proposition 8.2 asserts that there exists a collection $(\psi_{t,n})_{t,n\in[0,1)\times\mathbb{N}}$ of diffeomorphisms of $[0,1]$ such that

- $(\psi_{t,n})_{t\in[0,1)}$ is a C^1-continuous path for all $n \in \mathbb{N}$;
- for all $(t,n) \in [0,1)\times\mathbb{N}$, $D\psi_{t,n}(0) = 1 = D\psi_{t,n}(1)$;
- for all $(t,n) \in [0,1)\times\mathbb{N}$,
$$\|f_n \circ \psi_{t,n} - f_n\|_1 < \varepsilon_t;$$
- for all $t \in [0,1)$, the sequence of diffeomorphisms of $[0,1]$ defined by $h_{t,0}$ and
$$h_{t,n} = f_{n-1}\psi_{t,n-1}h_{t,n-1}f_{n-1}^{-1}, \quad \text{for all } n \in \mathbb{N}^*,$$
is stationary, equal to id for all $n \in \mathbb{N}$ large enough;
- for all $n \in \mathbb{N}$, the path $(h_{t,n})_{t\in[0,1)}$ is a C^1-continuous path.

We get a similar result and a similar C^1-continuous collection $(h_{t,n})_{(t,n)\in[0,1)\times(-\mathbb{N})}$ by considering the negative iterates of f.

Consider now the C^1-continuous path $(h_t)_{t\in[0,1)}$ of C^1-diffeomorphisms of $[0,1]$ defined by

- the restriction $h_t|_{[f^n(x),f^{n+1}(x)]}$ is conjugated to $h_{t,n}$ by the affine map from $[f^n(x), f^{n+1}(x)]$ to $[0,1]$, for all $n \in \mathbb{Z}$,
- $h_t = \text{id}$ on the complement of $\bigcup_{n\in\mathbb{Z}} f^n([x,f(x)])$.

By straightforward calculations using that $\|f_n \circ \psi_{t,n} - f_n\|_1 < \varepsilon_t$, one gets
$$\|D(h_{t,n+1}f_nh_{t,n}^{-1}) - Df_n\|_1 < \varepsilon_t,$$
from which follows $\|h_{t,n+1}f_nh_{t,n}^{-1} - f_n\|_1 < \varepsilon_t$, thus $\|h_tfh_t^{-1} - f\|_1 < \varepsilon_t$, concluding the proof. □

8.4. Conjugacy to the identity prescribed in a fundamental domain. The aim of this section is to prove the following lemma.

Lemma 8.4. *Let $f \in \mathrm{Diff}^1_+([0,1])$ be a diffeomorphism without hyperbolic fixed point, and let $[a,b]$, $b = f(a)$, be a fundamental domain of f. There exists a C^1-continuous path $(\alpha_t)_{t\in[0,1)}$ of C^1-diffeomorphisms of $[0,1]$ such that*

- $D\alpha_t(0) = 1 = D\alpha_t(1)$ *for all* $t \in [0,1)$;
- $(\alpha_t)_{t\in[0,1)}$ *has equicontinuous* Log-*derivative on* $[a,b]$*: for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that, for all* $t \in [0,1)$ *and* $x, y \in [a,b]$ *satisfying* $|x-y| \le \delta$*,*
$$|\log D\alpha_t(x) - \log D\alpha_t(y)| < \varepsilon;$$
- $f \underset{\alpha_t}{\rightsquigarrow} \mathrm{id}$.

The proof is a variation on the proof of the following main result in [6]:

Theorem 8.2 ([6]). *Given $f \in \mathrm{Diff}^1_+([0,1])$ without fixed point in $(0,1)$, given any continuous paths $0 < a_t < b_t < 1$, $t \in [0,1)$, given any C^1-continuous path $(g_t)_{t\in[0,1)}$, where $g_t \in \mathrm{Diff}^1_+([0,1])$ is a diffeomorphism without fixed point in $(0,1)$ which coincides with f on $[0,a_t]$ and on $[b_t,1]$, there is a C^1-continuous path $h_t \in \mathrm{Diff}^1_+([0,1])$, $t \in [0,1)$ such that, for every t, h_t coincides with the identity on a neighborhood of 0 and of 1, and the C^1-distance $\|h_t f h_t^{-1} - g_t\|_1$ tends to 0 as $t \to 1$.*

Let us sketch the proof of [6], so that we will explain the modification we need here.

Sketch of proof of [6]. As f and g_t coincide on $[0,a_t]$, there is a unique diffeomorphism $\hat{h}_t$ of $[0,1)$ which is the identity map in a neighborhood of 0, and which conjugates the restriction $f|_{[0,1)}$ to $g_t|_{[0,1)}$. [6] chooses h_t so that it coincides with $\hat{h}_t$ out of an arbitrarily small neighborhood of 1. The idea is that, in a neighborhood of 1, f and g_t coincide so that $\hat{h}_t$ commute with f. One concludes as in the proof of Lemma 8.3: by using Proposition 8.2, one can modify $\hat{h}_t$ slowly in the successive fundamental domains of f in order to get a diffeomorphism h_t coinciding with $\hat{h}_t$ out of a small neighborhood of 1, with the identity map in a smaller neighborhood of 1 and almost commuting with f on $[b_t,1]$. ☐

Let us now modify slightly the proof of [6]. Consider points $x_t, y_t \in [a_t, b_t]$ varying continuously with $t \in [0,1)$. Let

$$\varphi_t\colon [x_t, f(x_t)] \longrightarrow [y_t, g_t(y_t)], \quad t \in [0,1),$$

be a C^1-continuous path of diffeomorphisms satisfying

$$D\varphi_t(f(x_t))Df(x_t) = Dg_t(y_t)D\varphi_t(x_t).$$

Then there is a unique C^1-diffeomorphism

$$\tilde{h}_t\colon (0,1) \longrightarrow (0,1)$$

conjugating f to g_t and coinciding with φ_t on $[x_t, f(x_t)]$. As before, since f and g_t coincide on $[0, a_t]$ and $[b_t, 1]$, we get that $\tilde{h}_t$ commutes with f in a neighborhood of 0 and of 1. We conclude as before: we can modify $\tilde{h}_t$ slowly in the successive fundamental domains of f and f^{-1} in order to get a diffeomorphism h_t coinciding with $\tilde{h}_t$ out of a small neighborhood of 0 and of 1, with the identity map in a smaller neighborhoods of 0 and 1 and almost commuting with f on $[0, a_t]$ and on $[b_t, 1]$.

Summarizing, this proves:

Theorem 8.3. [6]

- *Given* $f \in \mathrm{Diff}^1_+([0,1])$ *without fixed points in* $(0,1)$ *such that* $f(x) - x > 0$ *on* $(0,1)$*,*
- *given any continuous paths* $0 < a_t < b_t < 1$*,* $t \in [0,1)$*,*
- *given any continuous paths* $x_t, y_t \in [a_t, b_t]$*,* $t \in [0,1)$*,*
- *given any* C^1*-continuous path* $(g_t)_{t\in[0,1)}$*, where* $g_t \in \mathrm{Diff}^1_+([0,1])$ *is a diffeomorphism without fixed point in* $(0,1)$ *such that* g_t *coincides with* f *on* $[0, a_t]$ *and on* $[b_t, 1]$*,*
- *given any* C^1*-continuous path*

$$\varphi_t\colon [x_t, f(x_t)] \longrightarrow [y_t, g_t(y_t)]$$

such that

$$D\varphi_t(f(x_t)Df(x_t) = Dg_t(y_t)D\varphi_t(x_t),$$

there is a C^1-continuous path $h_t \in \mathrm{Diff}^1_+([0,1])$, $t \in [0,1)$, such that

- *for every t, the diffeomorphism h_t coincides with the identity on a neighborhood of 0 and of 1,*
- *h_t coincides with φ_t on $[x_t, f(x_t)]$,*
- *the C^1-distance $\|h_t f h_t^{-1} - g_t\|_1$ tends to 0 as $t \to 1$.*

According to Theorem 8.3, Lemma 8.4 is now a direct consequence of the following lemma:

Lemma 8.5. *Let $f \in \mathrm{Diff}^1_+([0,1])$ without hyperbolic fixed point, without fixed point in $(0,1)$ such that $f(x) - x > 0$ on $(0,1)$. Let $[a,b]$, $b = f(a)$, be a fundamental domain of f.*

- *There is a C^1-continuous path g_t, $t \in [0,1)$, $g_t \in \mathrm{Diff}^1_+([0,1])$ such that*
  - *g_t is without fixed point in $(0,1)$,*
  - *there are continuous paths $0 < a_t < b_t < 1$ such that g_t coincides with f on $[0,a_t] \cup [b_t,1]$,*
  - *$g_t \xrightarrow[t\to 1]{C^1} \mathrm{id}$;*
- *there is a C^1-continuous path of diffeomorphisms*

$$\varphi_t\colon [a, f(a)] \longrightarrow [a, g_t(a)]$$

  such that

$$D\varphi_t(f(a)Df(a) = Dg_t(a)D\varphi_t(a);$$

- *$(\varphi_t)_{t\in[0,1)}$ has equicontinuous* Log-*derivative on $[a, f(a)]$.*

Proof of Lemma 8.4. Lemma 8.5 and Theorem 8.3 imply that there exists h_t such that $h_t f h_t^{-1}$ is C^1-asymptotic to the isotopy g_t, which tends to the identity. Furthermore, h_t coincides with φ_t on $[a, f(a)]$, hence has equicontinuous Log-derivative on the fundamental domain $[a, f(a)]$, ending the proof. □

Lemma 8.5 announces the existence of two objects: the path

$$g_t \xrightarrow[t\to 1]{C^1} \mathrm{id},$$

and the path φ_t with equicontinous Log-derivative. This suggests a natural splitting of the proof in two easy observations.

Lemma 8.6. *Consider* $f \in \mathrm{Diff}^1_+([0,1])$ *without hyperbolic fixed point, without fixed point in* $(0,1)$ *so that* $f(x) - x > 0$ *on* $(0,1)$*, and a fundamental domain* $[c, d = f(c)]$ *of* f*. Let* $c < d_t \leq d$ *be a continuous path such that* $d_t \to c$ *as* $t \to 1$ *and* $\alpha_t > 0$ *be a continuous path with* $\alpha_t \to 1$ *as* $t \to 1$*.*

There is a C^1*-continuous path of diffeomorphisms* $g_t \in \mathrm{Diff}^1_+([0,1])$*, there are continuous paths* $0 < a_t < b_t < 1$ *such that*

- $g_t(x) > x$ *for* $x \in (0,1)$*,*
- g_t *coincides with* f *on* $[0, a_t] \cup [b_t, 1]$*,*
- $g_t(c) = d_t$*,*
- $Dg_t(c) = \alpha_t$*,*
- $g_t \xrightarrow[t\to 1]{C^1} \mathrm{id}$.

Hint of proof. In other words, g_t is an isotopy from g_0 to the identity map, which does not require to create any fixed point, whose image and derivative may be prescribed at a point c, and satisfying that g_t coincides with f on a small neighborhood of 0 and 1. This is possible because we require that the image $d_t = g_t(c)$ tends to c, that the derivative $\alpha_t = Dg_t(c)$ tends to 1, and because $Df(0) = Df(1) = 1$, so that f is arbitrarily C^1-close to the identity map on a sufficiently small neighborhoods of 0 and 1. □

Lemma 8.7. *Consider* $f \in \mathrm{Diff}^1_+([0,1])$ *without hyperbolic fixed point, without fixed point in* $(0,1)$*, such that* $f(x) - x > 0$ *on* $(0,1)$*. Let* $[c,d]$*,* $d = f(c)$ *be a fundamental domain of* f*. Then there exists a* C^1*-continuous path of diffeomorphisms* $\varphi_t \colon [c,d] \to [c, \varphi_t(d)]$*,* $t \in [0,1)$*, such that*

- $\varphi_t(d) \xrightarrow[\mathrm{t}\to 1]{} c$*;*
- *if we set*

  $$\alpha_t = \frac{Df(c) \cdot D\varphi_t(d)}{D\varphi_t(c)},$$

  then

  $$\alpha_t \xrightarrow[\mathrm{t}\to 1]{} 1;$$

- $\{\log(D\varphi_t)\}_{t\in[0,1)}$ *is equicontinuous.*

Proof. Notice that adding a constant to a function does not change the equicontinuity properties. As a consequence, one can compose each φ_t by some affine map without changing the equicontinuity of the family $\log(D\varphi_t)$. Furthermore, composing by an affine map does not change the ratio $\frac{Df(c)\cdot D\varphi_t(d)}{D\varphi_t(c)}$. In other words, the first item is for free.

Now, one chooses some φ_{t_0} so that $\frac{Df(c)\cdot D\varphi_t(d)}{D\varphi_t(c)} = 1$, and, for $t > t_0$, one chooses φ_t as being the composition of φ_{t_0} by some affine map. □

8.5. Conjugacy by an equicontinous Log-derivative map.

Lemma 8.8. *Let $(\alpha_t)_{t\in[0,1)}$ be a C^1-continuous path of C^1-diffeomorphisms of* $[0,1]$ *with equicontinuous Log-derivative: $\{\log D\alpha_t\}_{t\in[0,1)}$ is equicontinuous. For every $\eta > 0$, there is $\varepsilon > 0$ such that, for all $g \in \mathrm{Diff}^1_+([0,1])$ satisfying $\|g - \mathrm{id}\|_1 < \varepsilon$,*

$$\|\alpha_t g \alpha_t^{-1} - \mathrm{id}\|_1 < \eta \quad \textit{for all } t \in [0,1).$$

In particular, if $(g_t)_{t\in[0,1)}$ is a path of diffeomorphisms converging to id *when t tends to* 1, *then*

$$\alpha_t g_t \alpha_t^{-1} \xrightarrow[t\to 1]{C^1} \mathrm{id}\,.$$

Proof. Consider $x \in [0,1]$ and $y = \alpha_t^{-1}(x)$. Then

$$D(\alpha_t g \alpha_t^{-1})(x) = \frac{D\alpha_t(g(y))}{D\alpha_t(y)} \cdot Dg(y).$$

By assumption, $|Dg(y) - 1| < \varepsilon$. Therefore, it is enough to check that

$$\log\Big(\frac{D\alpha_t(g(y))}{D\alpha_t(y)}\Big) = \log D\alpha_t(g(y)) - \log D\alpha_t(y)$$

is uniformly bounded with respect to ε, and that this bound tends to 0 as $\varepsilon \to 0$. Notice that $|g(y) - y| < \epsilon$. Thus, the equicontinuity of $\log D\alpha$ provides the uniform bound of $\log(\frac{D\alpha_t(g(y))}{D\alpha_t(y)})$ as a function of ε. □

8.6. Isotopy to the identity by conjugacy and perturbations.

Definition 8.9. Let $\varepsilon_t > 0$ and $\eta_t > 0$, $t \in [0,1)$, be continuous paths such that $\varepsilon_t \xrightarrow[t\to 1]{} 0$ and $\eta_t \xrightarrow[t\to 1]{} 0$. A C^1-continuous path $(\psi_t)_{t\in[0,1)}$, $\psi_t \in \mathrm{Diff}^1_+([0,1])$ is *an* $(\varepsilon_t)_{t\in[0,1)}$*-robust isotopy by conjugacy of speed* $(\eta_t)_{t\in[0,1)}$ *from f to* id if, for all continuous path $(g_t)_{t\in[0,1)}$ satisfying $\|g_t - f\|_1 < \varepsilon_t$, one has

$$\|\psi_t g_t \psi_t^{-1} - \mathrm{id}\|_1 < \eta_t.$$

Lemma 8.10. *Let f, $(\varphi_t)_{t\in[0,1)}$ be C^1-diffeomorphisms of* $\mathbb{R}$ *such that*

$$\|\varphi_t f \varphi_t^{-1} - \mathrm{id}\,\|_1 \underset{t\to 1}{\longrightarrow} 0, \quad \text{and} \quad \varphi_0 = \mathrm{id}\,.$$

For all continuous path $(\varepsilon_t)_{t\in[0,1)}$ of strictly positive real numbers converging to 0*, there exist a continuous path $(\eta_t)_{t\in[0,1)}$ of strictly positive real numbers converging to* 0 *and a continuous map*

$$r\colon [0,1) \longrightarrow [0,1),$$

satisfying

$$r(0) = 0, \quad \text{and} \quad r(t) \underset{t\to 1}{\longrightarrow} 1,$$

such that, $(\psi_t = \varphi_{r(t)})_{t\in[0,1)}$ is an $(\varepsilon_t)_{t\in[0,1)}$-robust isotopy by conjugacy of speed $(\eta_t)_{t\in[0,1)}$ from f to id.

We split the proof in two lemmas. The first one just states that any isotopy by conjugacy is ε_t-robust, if we choose $\varepsilon_t > 0$ small enough.

Lemma 8.11. *Consider $f \in \mathrm{Diff}^1_+([0,1])$ and a C^1-continuous path $(\varphi_t)_{t\in[0,1)}$ of $\mathcal{C}^1$-diffeomorphisms of* $[0,1]$ *and* $\varphi_0 = \mathrm{id}$*, such that*

$$\|\varphi_t f \varphi_t^{-1} - \mathrm{id}\,\|_1 \underset{t\to 1}{\longrightarrow} 0.$$

Denote

$$\mu_t = 2 \cdot \|\varphi_t f \varphi_t^{-1} - \mathrm{id}\,\|_1.$$

There exists a continuous path $\nu_t > 0$ such that

$$\|\varphi_t g_t \varphi_t^{-1} - \mathrm{id}\,\|_1 < \mu_t$$

for all continuous path $(g_t)_{t\in[0,1)}$ satisfying $\|g_t - f\|_1 < \nu_t$.In other words, the isotopy by conjugacy φ_t is ν_t-robust of speed μ_t.

Sketch of proof. For every $t \in [0,1)$, one needs to bound $\frac{|D\varphi_t(g(x)) - D\varphi_t(f(x)|}{D\varphi_t(x)}$, for $|g(x) - f(x)| < \nu_t$, uniformly in $x \in [0,1]$, by a constant $\tilde{\mu}_t$ depending in a simple continuous way on μ_t. As $D\varphi_t$ is bounded on $[0,1]$, one essentially needs to bound (uniformly in x) $D\varphi_t(g(x)) - D\varphi_t(f(x))$, for $|g(x) - f(x)| < \nu_t$. In other words, ν_t depends strongly on the continuity modulus δ_t of $D\varphi_t$ for the constant $\hat{\mu}_t$, where $\hat{\mu}_t = \tilde{\mu}_t \cdot \max_{x\in[0,1]} |D\varphi_t(x)|$.

$$|x - y| < \delta_t \implies D\varphi_t(x) - D\varphi_t(y) < \tilde{\mu}_t \cdot \max_{x\in[0,1]} |D\varphi_t(x)|$$

The unique difficulty is to choose $0 < \nu_t < \delta_t$ depending continuously on $t \in [0, 1)$. This is possible because the modulus of continuity of a continuous function (on a compact metric space) associated to a given constant depends lower-semi-continuously on the function. One concludes by noticing that, given a strictly positive lower-semi-continuous map $\delta_t \colon [0, 1) \to \mathbb{R}$, there is a strictly positive function $0 < \nu_t < \delta_t$. □

To get Lemma 8.10 from Lemma 8.11, one just needs to apply the following simple observation:

Lemma 8.12. *Let* $\varepsilon_t > 0$ *and* $\nu_t > 0$, $t \in [0, 1)$, *be continuous paths such that* $\varepsilon_t \underset{t\to 1}{\longrightarrow} 0$. *Then there is a continuous map*

$$r \colon [0, 1) \longrightarrow [0, 1),$$

such that

$$r(0) = 0 \quad \textit{and} \quad r(t) \underset{t\to 1}{\longrightarrow} 1,$$

and there is $0 \leq t_0 < 1$ *such that, for every* $t_0 \leq t < 1$,

$$\nu_{r(t)} > \varepsilon_t.$$

Proof of Lemma 8.10. Choose $\mu_t, \nu_t > 0$ given by Lemma 8.11, and $r(t)$ and t_0 given by Lemma 8.12. Then, for all continuous path $(g_t)_{t\in[0,1)}$ satisfying

$$\|g_t - f\|_1 < \varepsilon_t, \quad \text{for every } t \geq t_0,$$

one has

$$\|\varphi_{r(t)} g_t \varphi_{r(t)}^{-1} - \mathrm{id}\,\|_1 < \mu_{r(t)}.$$

Notice that $t \mapsto \mu_{r(t)}$ is continuous and tends to 0 as $t \to 1$.

In other words, the choice $\eta_t = \mu_{r(t)}$ is convenient for $t \geq t_0$. One extends such η_t for $t \in [0, t_0]$ by a simple compactness argument. More precisely, one chooses η_t, $t \in [0, 1)$ so that:

- for every $t \in [0, 1)$, $\eta_t \geq \mu_{r(t)}$;
- $\eta_t = \mu_{r(t)}$ for every t close enough to 1;
- for every $t \in [0, t_0)$,

$$\eta_t \geq \max_{x\in[0,1]} Df(x) + \max \varepsilon_t + \max_{t\in[0,t_0],x\in[0,1]} D\varphi_{r(t)}(x) + \max_{t\in[0,t_0],x\in[0,1]} D\varphi_{r(t)}^{-1}(x);$$

- $t \mapsto \eta_t$ is continuous.

For this choice of η, $\varphi_{r(t)}$ is ε_t-robust with speed η_t, concluding the proof. □

8.7. Group extensions in the class $\mathcal{C}^1_{\mathrm{id}}$: proof of Theorem 1.6. We are now ready to prove Theorem 1.6:

Proof of Theorem 1.6. Let f be a C^1-diffeomorphism of $[0,1]$, without hyperbolic fixed point and $I = [x, f(x)]$ a fundamental domain of f. Let $G \subset \mathrm{Diff}^1([0,1])$ be a group of diffeomorphisms whose supports are included in I. Assume that G is C^1-close to id; more precisely, one assumes that there is a C^1-continuous path of diffeomorphisms h_t, $t \in [0,1)$, supported on I, which realizes an isotopy by conjugacy from the elements of G to the identity. One will prove that the group $\langle G, f\rangle$, generated by f and the elements of G, is C^1-close to the identity and admits an isotopy by conjugacy to the identity H_t, $t \in [0,1)$, such that $D(H_t)(0) = D(H_t(1)) = 1$. Actually, H_t will coincide with the identity in small neighborhoods of 0 and 1.

One begins by extending the path $(h_t)_{t\in[0,1)}$ to $[0,1]$ by Lemma 8.3, in such a way that $\|h_t f (h_t)^{-1} - f\|_1 < \varepsilon_t$, where $(\varepsilon_t)_{t\in[0,1)}$ is some continuous path of strictly positive real numbers converging to 0, and that h_t coincides with the identity on a neighborhood of 0 and 1. As explained in Section 8.4, and from Lemma 8.10, one can choose an $(\varepsilon_t)_{t\in[0,1)}$-robust isotopy $(\alpha_t)_{t\in[0,1)}$ from f to id which has equicontinuous Log-derivative, and so that α_t coincides with the identity map on small neighborhoods of 0 and 1. Then, by definition of an $(\varepsilon_t)_{t\in[0,1)}$-robust isotopy, one has:

$$\|\alpha_t h_t f h_t^{-1} \alpha_t^{-1} - \mathrm{id}\,\|_1 \underset{\mathrm{t}\to 1}{\longrightarrow} 0,$$

and, from Lemma 8.8,

$$\|\alpha_t h_t g h_t^{-1} \alpha_t^{-1} - \mathrm{id}\,\|_1 \underset{\mathrm{t}\to 1}{\longrightarrow} 0, \quad \text{for all } g \in G.$$

Thus $H_t = \alpha_t h_t$ is the announced isotopy by conjugacy from $\langle f, G\rangle$ to the identity. □

9. Isotopy to the identity of groups generated by a fundamental system

The aim of this section is to prove Theorem 1.7: any group G of diffeomorphisms of $[0,1]$ generated by a fundamental system is C^1-close to the identity. We will prove a slightly stronger version: the diffeomorphisms f_n are not assumed to be simple.

Theorem 9.1. *Let $(f_n)_{n\in\mathbb{N}}$ be a collection of C^1-diffeomorphisms of $\mathbb{R}$ with compact support S_n and without hyperbolic fixed point and, for each $n \in \mathbb{N}$, let I_n be a given fundamental domain of f_n such that for all $i < n$,*

- *either $S_n \subset I_i$,*
- *or $S_i \subset I_n$,*
- *or $\mathring{S}_n \cap \mathring{S}_i = \varnothing$.*

Then the group $\langle f_n, n \in \mathbb{N}\rangle$ generated by $(f_n)_{n\in\mathbb{N}}$ is isotopic by conjugacy to the identity.

Let f_n, $n \in \mathbb{N}$, be a collection of diffeomorphisms satisfying the hypotheses of Theorem 9.1 and let us denote by G the group generated by the f_n's. Therefore, G is the increasing union of the groups $G_n = \langle f_0, \dots, f_n\rangle$. According to Theorem 1.5, if all the G_n's are C^1-close to the identity, then G is C^1-close to the identity.

Therefore, Theorem 9.1 is a straightforward consequence of Theorem 1.5 with the following finite version of Theorem 9.1:

Theorem 9.2. *Let $N > 0$ be an integer and $(f_n)_{n\in\{0,\dots,N\}}$ be a collection of C^1-diffeomorphisms of $[0, 1]$ without hyperbolic fixed point, with compact supports S_n and, for each $n \in \{0, \dots, N\}$, let I_n be a given fundamental domain of f_n such that, for all $i < n$,*

- *either $S_n \subset I_i$,*
- *or $S_i \subset I_n$,*
- *or $\mathring{S}_n \cap \mathring{S}_i = \varnothing$.*

Then the group $\langle f_0, \dots, f_N\rangle$ is C^1-close to the identity. More precisely, there is a C^1-continuous family $\{h_t\}_{t\in[0,1)}$, $h_t \in \mathrm{Diff}^1_+([0, 1])$, supported on $\bigcup_0^N S_n$, such that $Dh_t(0) = Dh_t(1) = 1$ and

$$f_i \underset{h_t}{\rightsquigarrow} \mathrm{id}, \quad \textit{for all } i \in \{0, \dots, N\}.$$

9.1. Proof of Theorem 9.2. One proves Theorem 9.2 by induction on N. For $N = 0$, this is precisely the main result of [6]. Assume now that Theorem 9.2 is proved for $N \geq 0$; one will prove it for $N + 1$.

Let $f_0, \dots, f_{N+1}$ be a collection of diffeomorphisms satisfying the hypotheses of Theorem 9.2. Either the supports S_i, S_j of f_i, f_j, $i \neq j$, have disjoint interiors, or one of them is included in a fundamental domain of the other. Let $\mathcal{I} \subset \{0, \dots, N+1\}$ be the set of indices for which S_i is maximal for the inclusion.

First assume that $\mathcal{I}$ contains more than 1 element. Then, for every $i \in \mathcal{I}$ the collection $\{f_j, S_j \subset S_i\}$ satisfies the hypotheses of Theorem 9.2 and contains strictly less elements than $N + 1$. Therefore, the induction hypothesis provides continuous paths $h^i_t, t \in [0, 1)$, supported on S_i, realizing an isotopy from all the f_j's with $S_j \subset S_i$ to the identity, and so that the derivatives at 0 and 1 are equal to 1. One defines the announced family h_t as coinciding with h^i_t on S_i, $i \in \mathcal{I}$.

Assume now that $\mathcal{I}$ contains a unique element. Up to change indices, one may assume that $\mathcal{I} = \{N + 1\}$. Thus, the group $G_N = \langle f_0, \ldots, f_N \rangle$ is supported in the fundamental domain I_{n+1} of f_{N+1}. By the induction hypothesis, there is a C^1-continuous path h^N_t, $t \in [0, 1)$, supported on I_{N+1} and realizing an isotopy by conjugacy from the elements of G_N to the identity. Thus, G_N and f_{N+1} satisfy the hypotheses of Theorem 8.1, which provides the announced path h^{N+1}_t, concluding the proof.

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