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Homological and Bloch invariants for $\mathbb{Q}$-rank one spaces and flag structures

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Abstract. We use group homology to define invariants in algebraic K-theory and in an analogue of the Bloch group for $\mathbb{Q}$-rank one lattices and for some other geometric structures. We also show that the Bloch invariants of CR structures and of flag structures can be recovered by a fundamental class construction.

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Contents

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1. Introduction

Fundamental class constructions and Bloch invariants are by now a classical theme in the topological study of hyperbolic 3-manifolds, going back to the work of Dupont and Sah on scissors congruences and more recently the work of Neumann, Yang and Zickert on Bloch invariants. (See also the recent preprints [13] and [19].) For a hyperbolic 3-manifold $M = \Gamma\backslash\mathbb{H}^3$ with $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ one can on the one hand consider its $\mathrm{PSL}(2,\mathbb{C})$-fundamental class $[M]_{\mathrm{PSL}(2,\mathbb{C})}$, that is the image of the fundamental class $[M] \in H_3(M) \cong H_3(\Gamma)$ in $H_3(\mathrm{PSL}(2,\mathbb{C}))$, and on the other hand one can use ideal triangulations or more generally degree one ideal triangulations to define an invariant $\beta(M)$ in the Bloch group $\mathcal{B}(\mathbb{C})$. In [33] it was shown that one can recover the volume and the Chern–Simons invariant mod $\mathbb{Q}$ from $\beta(M)$. (In later work Neumann constructed an invariant in an extended Bloch group, from which one can recover the Chern–Simons invariant mod $\mathbb{Z}$.)

The approach via ideal triangulations is better suited for doing practical calculations, see for example [33]. On the other hand the fundamental class approach is useful for theoretical considerations, e.g. to study the behaviour of hyperbolic volume under cut and paste in [29]. By the Bloch–Wigner Theorem (proved in more generality by Dupont and Sah in [14, Appendix A]) there is an isomorphism

$$H_3(\mathrm{PSL}(2,\mathbb{C}),\mathbb{Z})/\mathrm{torsion} \cong \mathcal{B}(\mathbb{C}),$$

and this isomorphism sends $[M]_{\mathrm{PSL}(2,\mathbb{C})}$ to $\beta(M)$. (One may pictorially think of a triangulation whose vertices are moved to infinity to produce an ideal triangulation. In some weak sense this picture can be made precise, see [28].) In particular the Bloch invariant is determined by the $\mathrm{PSL}(2,\mathbb{C})$-fundamental class.

The construction of the Bloch invariant was generalized to higher-dimensional hyperbolic manifolds in [33, Section 8]. On the other hand Goncharov [21, Section 2] generalised the fundamental class construction to get – associated to an odd-dimensional hyperbolic manifold M^{2k-1} and a spinor representation $\mathrm{SO}(2k-1,1) \to \mathrm{GL}(n,\mathbb{C})$ – an element in $H_{2k-1}(\mathrm{GL}(n,\mathbb{C}))$ such that application of the Borel class recovers (a fixed multiple of) the volume. In [21, Section 3] he also used ideal triangulations to construct an extension $m(M^{2k-1}) \in \mathrm{Ext}^1_{\mathcal{M}_{\mathbb{Q}}}(Q(0);Q(k))$ in the category of mixed Tate motives over $\mathbb{Q}$ and thus an element in $K_{2k-1}(\overline{\mathbb{Q}})\otimes\mathbb{Q}$ according to Beilinson's description of K-theory of fields. (In degree 3 one has $K_3^{\mathrm{ind}}(\mathbb{C})\otimes\mathbb{Q} = \mathcal{B}(\mathbb{C})\otimes\mathbb{Q}$ and one recovers the Bloch invariant from this K-theoretic approach.)

In [27] the third-named author generalized Goncharov's fundamental class construction to finite-volume locally symmetric spaces of noncompact type $M = \Gamma\backslash G/K$ which are either closed or of $\mathbb{R}$-rank one. To each representation $\rho\colon G \to \mathrm{SL}(n,\mathbb{C})$ the construction yields an element in $H_*(\mathrm{SL}(n,\overline{\mathbb{Q}}))$ or, after a suitable projection, an element in

$$PH_*(\mathrm{GL}(\overline{\mathbb{Q}})) \cong K_*(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$$

such that application of the Borel class (to either the element in $H_*(\mathrm{SL}(n,\overline{\mathbb{Q}}))$ or its projection in $PH_*(\mathrm{GL}(\overline{\mathbb{Q}}))$) yields a multiple $c_\rho \operatorname{Vol}(M)$ of the volume $\operatorname{Vol}(M)$. The factor c_ρ depends only on ρ, in particular one can recover the volume if $c_\rho \neq 0$. Moreover, [27, Section 3] provides a complete list of fundamentals representations $\rho\colon G \to \mathrm{SL}(n,\mathbb{C})$ with $c_\rho \neq 0$. In the noncompact case, the only $\mathbb{R}$-rank one examples with $c_\rho \neq 0$ were (odd-dimensional) real-hyperbolic manifolds. On the other hand, for higher $\mathbb{R}$-rank there are many examples with $c_\rho \neq 0$.

In this paper we further generalize Goncharov's construction to $\mathbb{Q}$-rank one locally symmetric spaces. That means, for a $\mathbb{Q}$-rank one locally symmetric space of noncompact type $M = \Gamma\backslash G/K$ we construct elements

$$\bar{\gamma}(M) \in H_*(\mathrm{SL}(n,\overline{\mathbb{Q}}))$$

and

$$\gamma(M) \in K_*(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$$

such that application of the Borel class yields again $c_\rho \operatorname{Vol}(M)$. Compare Proposition 7.1 and Theorem 7.2 for the precise statements. Thus one can get many non-compact non-hyperbolic examples with nontrivial invariants. (Note that the universal cover of a $\mathbb{Q}$-rank one locally symmetric space can be a symmetric space of higher $\mathbb{R}$-rank. For example, Hilbert modular varieties are examples of $\mathbb{Q}$-rank one locally symmetric spaces with higher rank. In general, an irreducible non-uniform lattice in a semisimple algebraic Lie group without compact factor has $\mathbb{Q}$-rank one if and only if it does not contain a subgroup isomorphic to a finite index subgroup of $SL(3,\mathbb{Z})$ or $SO(2,3)_{\mathbb{Z}}$, see [34, Section 9H]. We refer the reader to [5, Part III] for more information about $\mathbb{Q}$-rank one locally symmetric spaces.) The arguments needed for the $\mathbb{Q}$-rank one case are an adaptation of those used in the $\mathbb{R}$-rank one case. The main difference between these cases is that in the $\mathbb{R}$-rank one case a horoball's boundary at infinity consists of only one point and therefore[2] a straightening map $\widehat{C}_*^{x_0}(M) \to \widehat{C}_*^{\mathrm{str},x_0}(M)$ is well-defined and moreover one has an isomorphism $C_*^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}}) \cong \widehat{C}_*^{\mathrm{str},x_0}(M)$. Both is not true for

[2] The notions $B\Gamma^{\mathrm{comp}}$, $\widehat{C}_*^{x_0}(M)$ and $\widehat{C}_*^{\mathrm{str},x_0}(M)$ are defined in Section 5.2, Definition 5.3 and Definition 6.2, respectively.

the higher $\mathbb{R}$-rank case, but in Section 6 (using Section 5) we show that nonetheless one has an isomorphism $H_d(M, \partial M) \cong H_d^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}})$ and therefore can define our invariant from the relative fundamental class.

The fundamental class approach is most natural for closed manifolds, whereas the Bloch group approach is most natural for cusped manifolds. The relative fundamental class unifies the two approaches.

The relative fundamental class construction shall be useful for deriving general results about the relation of topology and volume. For practical computations however the Bloch group approach appears to be more feasible, not only in 3-dimensional hyperbolic geometry [33] but also in the study of CR-structures [17] or flag structures [1].

In the 3-dimensional hyperbolic case, Neumann–Yang constructed the Bloch invariant [33, Definition 2.5] in the so-called pre-Bloch group $\mathcal{P}(\mathbb{C})$ and then proved in [33, Theorem 6] that it actually belongs to the Bloch group $\mathcal{B}(\mathbb{C}) \subset \mathcal{P}(\mathbb{C})$. The pre-Bloch group $\mathcal{P}(\mathbb{C})$ satisfies a natural isomorphism

$$\mathcal{P}(\mathbb{C}) \cong H_3(C_*(\partial_\infty G/K) \otimes_{\mathbb{Z}G} \mathbb{Z})$$

for $G/K = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2) = \mathbb{H}^3$. Thus it is natural to define a Bloch invariant of higher-dimensional locally symmetric spaces $M^d = \Gamma\backslash G/K$ (and representations $\rho\colon G \to \mathrm{SL}(n,\mathbb{C})$) as an element in

$$H_d(C_*(\partial_\infty \mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n)) \otimes_{\mathbb{Z}\,\mathrm{SL}(n,\mathbb{C})} \mathbb{Z}).$$

In [28] this was done for $\mathbb{R}$-rank one symmetric spaces and it was shown that the Bloch invariant is the image of the Goncharov invariant $\bar{\gamma}(M)$ under a naturally defined evaluation homomorphism which generalizes the homomorphism from the Bloch–Wigner Theorem. The construction of the generalized Bloch invariant uses proper ideal fundamental cycles since the existence of ideal triangulations is unclear in general. The proof of well-definedness of the Bloch invariant (i.e. independence from the chosen proper ideal fundamental cycle, [28, Lemma 3.4.1]) was building on the equality $H_d(C_*(\partial_\infty G/K) \otimes_{\mathbb{Z}\Gamma} \mathbb{Z}) \cong \mathbb{Z}$ for lattices $\Gamma \subset G$, which in the $\mathbb{R}$-rank one case can be proved by an immediate generalization of the results of Neumann–Yang (who proved this equality in [33] for hyperbolic 3-space). However it is unclear how to generalize this argument to the higher rank case. Therefore we avoid this point in Section 8 by directly defining the Bloch invariant

$$\beta(M) \in H_d(C_*(\partial_\infty \mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n)) \otimes_{\mathbb{Z}\,\mathrm{SL}(n,\mathbb{C})} \mathbb{Z})$$

for locally symmetric spaces (either closed or of $\mathbb{Q}$-rank one) as the image of the Goncharov invariant $\bar{\gamma}(M)$ under the evaluation homomorphism. As far as we know, this is the first construction of invariants in algebraic K-theory and in the Bloch group for higher rank locally symmetric spaces. As is well known, higher rank symmetric spaces have different geometry from $\mathbb{R}$-rank one symmetric spaces. For instance, the Tits metric on the ideal boundary of a symmetric space has completely different properties depending on its $\mathbb{R}$-rank (See [15, Chapter 3]). Such difference makes it difficult to extend the construction of the Bloch invariant to higher rank locally symmetric spaces. However, it turns out that $\mathbb{Q}$-rank one locally symmetric spaces it is possible to generalize the fundamental class construction.

In Section 9 we consider a similar construction for convex projective manifolds. We hope to further exploit this in other papers.

In Section 10 we prove that also the Bloch invariants of CR structures (as defined by Falbel and Wang in [17]) and of flag structures (as defined by Bergeron, Falbel, and Guilloux in [1]) can be recovered from a fundamental class construction. (Compare also [13] and [19].) We apply this to prove that these Bloch invariants are preserved under certain cut-and-paste operations.

2. Basics on Group (co)homology

2.1. Group homology. For a topological group G, let G^δ denote the group with discrete topology. Let BG^δ denote the simplicial set whose k-simplices are k-tuples $(g_1, \ldots, g_k)$ with the boundary operator ∂:

$$\partial(g_1,\ldots,g_k) = (g_2,\ldots,g_k) + \sum_{i=1}^{k-1}(-1)^i(g_1,\ldots,g_ig_{i+1},\ldots,g_k) + (-1)^k(g_1,\ldots,g_{k-1}).$$

It forms a chain complex $C_*^{\mathrm{simp}}(BG^\delta)$ whose homology with a coefficient ring R yields the group homology

$$H_*(G,R) = H_*^{\mathrm{simp}}(BG^\delta, R) = H_*^{\mathrm{simp}}(C_*(BG^\delta \otimes_{\mathbb{Z}} R), \partial \otimes 1).$$

Throughout the paper, BG will be understood as BG^δ.

Let M be a Riemannian manifold of nonpositive sectional curvature and $x_0 \in M$, a lift $\tilde{x}_0 \in \widetilde{M}$ of x_0 be fixed. Any *ordered* tuple of vertices in $\widetilde{M}$ determines a unique straight simplex. A singular simplex $\sigma \in C_*(M)$ is straight if some (hence any) lift $\tilde{\sigma} \in C_*(\widetilde{M})$ is straight. Let $C_*^{\mathrm{str},x_0}(M)$ be the chain complex of straight simplices with all vertices x_0.

Set $\Gamma = \pi_1(M, x_0)$. Then there are two canonical homomorphisms

$$\Psi\colon C_*^{\text{simp}}(B\Gamma) \longrightarrow C_*^{\text{str},x_0}(M)$$

defined by

$$\Psi(g_1, \dots, g_k) = \pi(\text{str}(\tilde{x}_0, g_1\tilde{x}_0, g_1 g_2 \tilde{x}_0, \dots, g_1 \cdots g_k \tilde{x}_0))$$

and

$$\Phi\colon C_*^{\text{str},x_0}(M) \longrightarrow C_*^{\text{simp}}(B\Gamma)$$

defined by

$$\Phi(\sigma) = ([\sigma|_{\zeta_1}], \dots, [\sigma|_{\zeta_k}])$$

where $e_0, \dots, e_k$ are the vertices of the standard simplex Δ^k, ζ_i is the standard sub-1-simplex with

$$\partial \zeta_i = e_i - e_{i-1},$$

each $[\sigma|_{\zeta_i}] \in \pi_1(M, x_0) = \Gamma$ is the homotopy class of $\sigma|_{\zeta_i}$ and $\pi\colon \widetilde{M} \to M$ is the universal covering map of M. It is easy to show that Ψ and Φ are chain isomorphisms inverse to each other.

The inclusion

$$i\colon C_*^{\text{str},x_0}(M) \subset C_*(M)$$

and the straightening ([27, Section 2.1], which is the composition of some homotopy moving all vertices of a chain into x_0 with the usual straightening operator)

$$\text{str}\colon C_*(M) \longrightarrow C_*^{\text{str},x_0}(M)$$

are chain homotopy inverses. Hence we have a chain homotopy equivalence, called the *Eilenberg–MacLane map*

$$\text{EM}\colon C_*^{\text{simp}}(B\Gamma) \longrightarrow C_*(M).$$

We will frequently use the induced isomorphism

$$\text{EM}_*^{-1} = \Phi_* \circ \text{str}_*\colon H_*(M, \mathbb{Z}) \longrightarrow H_*^{\text{simp}}(B\Gamma, \mathbb{Z}).$$

The geometric realization $|B\Gamma|$ is a $K(\Gamma, 1)$, thus there is a classifying map

$$h^M\colon M \longrightarrow |B\Gamma|$$

which induces an isomorphism on π_1 level. The inclusion map of simplices

$$i\colon C_*^{\text{simp}}(B\Gamma) \longrightarrow C_*(|B\Gamma|),$$

induces an isomorphism

$$i_*\colon H_*^{\mathrm{simp}}(B\Gamma,\mathbb{Z}) \longrightarrow H_*(|B\Gamma|,\mathbb{Z})$$

such that

$$h_*^M = i_* \circ \mathrm{EM}_*^{-1}$$

if M is aspherical and of the homotopy type of a CW complex (which is always the case for Riemannian manifolds of nonpositive sectional curvature).

For a commutative ring $A \subset \mathbb{C}$ with unit, let

$$\mathrm{GL}(A) = \bigcup_{n=1}^{\infty} \mathrm{GL}(n,A)$$

be the increasing union, and $|B\,\mathrm{GL}(A)|$ its classifying space as above.

Let

$$\rho\colon \Gamma \longrightarrow \mathrm{GL}(A)$$

be a representation. This induces

$$B\rho\colon B\Gamma \longrightarrow B\,\mathrm{GL}(A)$$

and

$$|B\rho|\colon |B\Gamma| \longrightarrow |B\,\mathrm{GL}(A)|.$$

The composition

$$H_*(M,\mathbb{Q}) \xrightarrow{\mathrm{EM}_*^{-1}} H_*^{\mathrm{simp}}(B\Gamma,\mathbb{Q}) \xrightarrow{(B\rho)_*} H_*^{\mathrm{simp}}(B\,\mathrm{GL}(A),\mathbb{Q})$$

induces a map

$$(H\rho)_*\colon H_*(M,\mathbb{Q}) \longrightarrow H_*^{\mathrm{simp}}(B\,\mathrm{GL}(A),\mathbb{Q}).$$

If M is a closed, oriented and connected d-dimensional manifold, $(H\rho)_d[M]$ will play an important role for us where $[M]$ is the fundamental class in $H_d(M,\mathbb{Q}) \cong \mathbb{Q}$.

2.2. Volume class and Borel class. Let G be a noncompact semisimple connected Lie group. Let $X = G/K$ be the associated symmetric space of dimension d with a maximal compact subgroup K of G. Let us denote by $H_c^*(G,\mathbb{R})$ the continuous cohomology of G. The comparison map

$$\mathrm{comp}\colon H_c^*(G,\mathbb{R}) \longrightarrow H^*_{\mathrm{simp}}(BG,\mathbb{R})$$

is defined by the cochain map

$$\mathrm{comp}(f)(g_1,\ldots,g_k) = f(1, g_1, g_1g_2, \ldots, g_1g_2\cdots g_k)$$

for a G-invariant cochain $f\colon G^{k+1} \to \mathbb{R}$. Fix a point $x \in X$. The volume class $v_d \in H_c^d(G,\mathbb{R})$ is defined by the cocycle

$$v_d(g_0,\ldots,g_d) = \mathrm{algvol}(\mathrm{str}(g_0x,\ldots,g_dx)) := \int_{\mathrm{str}(g_0x,\ldots,g_dx)} \mathrm{dvol}_X,$$

where dvol_X is the G-invariant Riemannian volume form on X, and str is the geodesic straightening of a simplex with those vertices. (That is $\mathrm{str}(g_0x,\ldots,g_dx)$ is the unique straight simplex with the given *ordered* set of vertices.) Under the comparison map

$$\mathrm{comp}(v_d)(g_1,\ldots,g_d) = \mathrm{algvol}(\mathrm{str}(x, g_1x, \ldots, g_1\cdots g_dx)).$$

Then it is not difficult to show that if $N = \Gamma\backslash X$ is a closed locally symmetric space, with $j\colon \Gamma \to G$ the inclusion, then (see [23][Lemma 3.1])

$$\mathrm{Vol}(N) = \langle \mathrm{comp}(v_d), Bj \circ \mathrm{EM}_d^{-1}[N]\rangle.$$

For a detailed proof of this, see [27, Theorem 1].

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebra of G and K respectively. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition, then the Lie algebra of the compact dual G_u of G is

$$\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}.$$

Note that the relative Lie algebra cohomology $H^*(\mathfrak{g},\mathfrak{k})$ is the cohomology of the complex of G-invariant differential forms on G/K and there is the van Est isomorphism

$$\mathfrak{J}\colon H_c^*(G,\mathbb{R}) \longrightarrow H^*(\mathfrak{g},\mathfrak{k}).$$

There is a well-known ([9, Section 5]) construction of a Borel class

$$b_{2k-1} = \Phi_A(R(\mathrm{Tr}_k)) \in H^{2k-1}(\mathrm{U}(n),\mathbb{R}) = H^{2k-1}(\mathfrak{u}(n),\mathbb{R}),$$

see [27, Section 2.4.3] for the details of the construction. Since $(\mathrm{GL}(n,\mathbb{C}))_u = \mathrm{U}(n)\times\mathrm{U}(n)$ and via the van Est isomorphism

$$\begin{aligned} H_c^*(\mathrm{GL}(n,\mathbb{C}),\mathbb{R}) &= H^*(\mathfrak{gl}(n,\mathbb{C}),\mathfrak{u}(n)) \\ &= H^*(\mathfrak{u}(n)\oplus\mathfrak{u}(n),\mathfrak{u}(n)) \\ &= H^*(\mathfrak{u}(n),\mathbb{R}) \end{aligned}$$

we may consider

$$b_{2k-1} \in H_c^{2k-1}(\mathrm{GL}(n,\mathbb{C}),\mathbb{R}).$$

2.3. Volume for compact locally symmetric manifolds. It is a standard fact that for $d = \dim(G/K)$, $H_c^d(G,\mathbb{R}) \cong \mathbb{R}$ via Van Est isomorphism. If Γ is a uniform lattice in G, then

$$H^d(\Gamma\backslash G/K,\mathbb{R}) \cong H^d(\Gamma,\mathbb{R}) \cong H_c^d(G,\mathbb{R}) \cong \mathbb{R},$$

where the second isomorphism is explicitly realised by integrating Γ-invariant cochains over a compact fundamental domain, see the proof of [23, Lemma 3.1]. Hence the volume class v_d as defined in Section 2.2 can be viewed as a generator of $H^d(\Gamma,\mathbb{R})$. For the proof of the following proposition see [27, Theorem 2].

Proposition 2.1. *For a symmetric space G/K of noncompact type with odd dimension $d = 2m-1$, and a representation $\rho\colon \Gamma \to \mathrm{GL}(n,\mathbb{C})$ with a closed manifold $N = \Gamma\backslash G/K$, there exists a constant $c_\rho \in \mathbb{R}$ such that*

$$\langle \mathrm{comp}(b_d), (H\rho)_*[N]\rangle = c_\rho \operatorname{Vol}(N).$$

If $\rho\colon \Gamma \to \mathrm{GL}(n,\mathbb{C})$ factors over a representation $\rho_0\colon G \to \mathrm{GL}(n,\mathbb{C})$, then c_ρ depends only on ρ_0.

3. $\mathbb{Q}$-rank 1 locally symmetric spaces

In this section, we consider only $\mathbb{Q}$-rank 1 lattices $\Gamma \subset G$. We first collect some definitions and results about $\mathbb{Q}$-rank 1 lattices.

3.1. Arithmetic lattices. Let G be a noncompact, semisimple Lie group with trivial center and no compact factors. Then one may define arithmetic lattices in the following way.

Definition 3.1. A lattice Γ in G is called *arithmetic* if there are

(1) a semisimple algebraic group $\mathbf{G} \subset \mathrm{GL}(n,\mathbb{C})$ defined over $\mathbb{Q}$ and

(2) an isomorphism $\varphi\colon \mathbf{G}(\mathbb{R})^0 \to G$

such that $\varphi(\mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^0)$ and Γ are commensurable.

It is well-known due to Margulis [31] that all irreducible lattices in higher rank Lie groups are arithmetic. The $\mathbb{Q}$-rank of a semisimple algebraic group $\mathbf{G}$ is defined as the dimension of a maximal $\mathbb{Q}$-split torus of $\mathbf{G}$. For an arithmetic lattice Γ in G, $\mathbb{Q}$-rank(Γ) is defined by the $\mathbb{Q}$-rank of $\mathbf{G}$ where $\mathbf{G}$ is an algebraic group as in Definition 3.1.

A closed subgroup $\mathbf{P} \subset \mathbf{G}$ defined over $\mathbb{Q}$ is called *rational parabolic subgroup* if $\mathbf{P}$ contains a maximal, connected solvable subgroup of $\mathbf{G}$. For any rational parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, one obtains the *rational Langlands decomposition* of $P = \mathbf{P}(\mathbb{R})$:

$$P = N_{\mathbf{P}} \times A_{\mathbf{P}} \times M_{\mathbf{P}},$$

where $N_{\mathbf{P}}$ is the real locus of the unipotent radical $\mathbf{N_P}$ of $\mathbf{P}$, $A_{\mathbf{P}}$ is a stable lift of the identity component of the real locus of the maximal $\mathbb{Q}$-torus in the Levi quotient $\mathbf{P}/\mathbf{N_P}$ and $M_{\mathbf{P}}$ is a stable lift of the real locus of the complement of the maximal $\mathbb{Q}$-torus in $\mathbf{P}/\mathbf{N_P}$.

Let $X = G/K$ be the associated symmetric space of noncompact type with a maximal compact subgroup K of G. Write

$$X_{\mathbf{P}} = M_{\mathbf{P}}/(K \cap M_{\mathbf{P}}).$$

Let us denote by

$$\tau\colon M_{\mathbf{P}} \longrightarrow X_{\mathbf{P}}$$

the canonical projection. Fix a base point $x_0 \in X$ whose stabilizer group is K. Then we have an analytic diffeomorphism

$$\mu\colon N_{\mathbf{P}} \times A_{\mathbf{P}} \times X_{\mathbf{P}} \longrightarrow X, \quad (n, a, \tau(m)) \longmapsto nam \cdot x_0,$$

which is called the *rational horocyclic decomposition* of X. For more detail, see [5, Section III.2].

3.2. Precise reduction theory. Let $\mathfrak{g}$ and $\mathfrak{a}_{\mathbf{P}}$ denote the Lie algebras of the Lie groups G and $A_{\mathbf{P}}$ defined above. Then the adjoint action of $\mathfrak{a}_{\mathbf{P}}$ on $\mathfrak{g}$ gives a root space decomposition:

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \Phi(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha = \{Z \in \mathfrak{g} \mid \mathrm{ad}(A)(Z) = \alpha(A)Z \text{ for all } A \in \mathfrak{a}_{\mathbf{P}}\},$$

and $\Phi(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$ consists of those nontrivial characters α such that $\mathfrak{g}_\alpha \neq 0$. It is known that $\Phi(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$ is a root system. Fix an order on $\Phi(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$ and denote by $\Phi^+(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$ the corresponding set of positive roots. Define

$$\rho_{\mathbf{P}} = \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})} (\dim \mathfrak{g}_\alpha)\alpha.$$

Let $\Phi^{++}(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$ be the set of simple positive roots. Since we consider only $\mathbb{Q}$-rank 1 arithmetic lattices, we restrict ourselves from now on to the case

$$\mathbb{Q}\text{-rank}(\mathbf{G}) = 1.$$

Then the followings hold:

(1) All proper rational parabolic subgroups of **G** are minimal.

(2) For any proper rational parabolic subgroup **P** of **G**, $\dim A_{\mathbf{P}} = 1$.

(3) The set $\Phi^{++}(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$ of simple positive $\mathbb{Q}$-roots contains only a single element.

For any proper rational parabolic subgroup **P** of **G** and any $t > 1$, define

$$A_{\mathbf{P},t} = \{a \in A_{\mathbf{P}} \mid \alpha(a) > t\},$$

where α is the unique root in $\Phi^{++}(\mathfrak{g}, \mathfrak{a}_{\mathbf{P}})$. For bounded sets $U \subset N_{\mathbf{P}}$ and $V \subset X_{\mathbf{P}}$, the set

$$\mathcal{S}_{\mathbf{P},U,V,t} = U \times A_{\mathbf{P},t} \times V \subset N_{\mathbf{P}} \times A_{\mathbf{P}} \times X_{\mathbf{P}}$$

is identified with the subset $\mu(U \times A_{\mathbf{P},t} \times V)$ of $X = G/K$ by the horospherical decomposition of X and called a *Siegel set* in X associated with the rational parabolic subgroup **P**. Given a $\mathbb{Q}$-rank 1 lattice Γ in G, it is a well-known result due to A. Borel and Harish and Chandra that there are only finitely many Γ-conjugacy classes of rational parabolic subgroups. Recall the precise reduction theory in $\mathbb{Q}$-rank 1 case as follows (see [5, Proposition III.2.21]).

Theorem 3.2. *Let Γ be a $\mathbb{Q}$-rank 1 lattice in G. Let* **G** *denote a semisimple algebraic group defined over $\mathbb{Q}$ with $\mathbb{Q}$-rank$(\mathbf{G}) = 1$ as in Definition 3.1. Denote by $\mathbf{P}_1, \ldots, \mathbf{P}_s$ representatives of the Γ-conjugacy classes of all proper rational parabolic subgroups of* **G**. *Then there exist a bounded set Ω_0 in $\Gamma\backslash G/K$ and Siegel sets*

$$U_i \times A_{\mathbf{P}_i,t_i} \times V_i, \quad i = 1, \ldots, s,$$

in $X = G/K$ such that

(1) *each Siegel set $U_i \times A_{\mathbf{P}_i,t_i} \times V_i$ is mapped injectively into $\Gamma\backslash X$ under the projection $\pi : X \to \Gamma\backslash X$,*

(2) *the image of $U_i \times V_i$ in $(\Gamma \cap P_i)\backslash N_{\mathbf{P}_i} \times X_{\mathbf{P}_i}$ is compact,*

(3) *$\Gamma\backslash X$ admits the following disjoint decomposition*

$$\Gamma\backslash X = \Omega_0 \cup \coprod_{i=1}^{s} \pi(U_i \times A_{\mathbf{P}_i,t_i} \times V_i).$$

Geometrically $B_{\mathbf{P}}(t) = \mu(N_{\mathbf{P}} \times A_{\mathbf{P},t} \times X_{\mathbf{P}})$ is a horoball for any proper minimal rational parabolic subgroup **P** of **G**. Hence each $\mu(U_i \times A_{\mathbf{P}_i,t_i} \times V_i)$ is a fundamental domain of the cusp group $\Gamma_i = \Gamma \cap \mathbf{P}(\mathbb{R})$ acting on the horoball $B_{\mathbf{P}_i}(t_i)$ and each $\mu(U_i \times V_i)$ is a bounded domain in the horosphere that bounds the horoball $B_{\mathbf{P}_i}(t_i)$. Furthermore, each set $\pi(U_i \times A_{\mathbf{P}_i,t_i} \times V_i)$ corresponds to a cusp of the locally symmetric space $\Gamma\backslash X$. We refer the reader to [5] for more details.

3.3. Rational horocyclic coordinates. Let $\mathbf{P}$ be a proper minimal rational parabolic subgroup of $\mathbf{G}$ with $\mathbb{Q}$-rank$(\mathbf{G}) = 1$. The pullback $\mu^* g$ of the metric g on X to $N_{\mathbf{P}} \times A_{\mathbf{P}} \times X_{\mathbf{P}}$ is given by

$$ds^2_{(n,a,\tau(m))} = \sum_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{a}_{\mathbf{P}})} e^{-2\alpha(\log a)} h_\alpha \oplus da^2 \oplus d(\tau(m))^2,$$

where h_α is some metric on $\mathfrak{g}_\alpha$ that smoothly depends on $\tau(m)$ but is independent of a. Choosing orthonormal bases $\{N_1, \ldots, N_r\}$ of $\mathfrak{n}_{\mathbf{P}}$, $\{Z_1, \ldots, Z_l\}$ of some tangent space $T_{\tau(m)} X_{\mathbf{P}}$ and $A \in \mathfrak{a}_{\mathbf{P}}$ with $\|A\| = 1$, one can obtain *rational horocyclic coordinates* $\eta\colon N_{\mathbf{P}} \times A_{\mathbf{P}} \times X_{\mathbf{P}} \to \mathbb{R}^r \times \mathbb{R} \times \mathbb{R}^l$ defined by

$$\eta\Big(\exp\Big(\sum_{i=1}^r x_i N_i\Big), \exp(yA), \exp\Big(\sum_{i=1}^l z_i Z_i\Big)\Big) = (x_1, \ldots, x_r, y, z_1, \ldots, z_l).$$

We abbreviate $(x_1, \ldots, x_r, y, z_1, \ldots, z_l)$ as (x, y, z). Then the G-invariant Riemannian volume form dvol_X on $X \cong N_{\mathbf{P}} \times A_{\mathbf{P}} \times X_{\mathbf{P}}$ with respect to the rational horocyclic coordinates is given by

$$\mathrm{dvol}_X = h(x,z) \exp^{-2\|\rho_{\mathbf{P}}\| y} \, dx dy dz,$$

where $h(x,z)$ is a smooth function that is independent of y. See [4, Corollary 4.4].

Note that all proper rational minimal parabolic subgroups are conjugate under $\mathbf{G}(\mathbb{Q})$. Hence the respective root systems are canonically isomorphic [3] and moreover, one can conclude $\|\rho_{\mathbf{P}}\| = \|\rho_{\mathbf{P}'}\|$ for any two proper minimal rational parabolic subgroups $\mathbf{P}, \mathbf{P}'$ of $\mathbb{Q}$-rank 1 algebraic group $\mathbf{G}$.

4. Straight simplices

Let X be a simply connected complete Riemannian manifold with nonpositive sectional curvature and $\partial_\infty X$ be the ideal boundary of X. For $x_0, \ldots, x_k \in X$, the straight simplex $\mathrm{str}(x_0, \ldots, x_k)$ is defined inductively as follows: First, $\mathrm{str}(x_0)$ is the point $x_0 \in X$, and $\mathrm{str}(x_0, x_1)$ is the unique geodesic arc from x_1 to x_0. In general, $\mathrm{str}(x_0, \ldots, x_k)$ is the geodesic cone on $\mathrm{str}(x_0, \ldots, x_{k-1})$ with the top point x_k. Since there is the unique geodesic connecting two points in X, each ordered $(k+1)$-tuple $(x_0, \ldots, x_k)$ determines the unique straight simplex.

If the sectional curvature of X is strictly negative, one can define the notion of straight simplex in $X \cup \partial_\infty X$. For any ordered tuple $(u_0, \ldots, u_k) \in X \cup \partial_\infty X$, the straight simplex $\mathrm{str}(u_0, \ldots, u_k)$ is well defined as above. A straight simplex $\mathrm{str}(u_0, \ldots, u_k)$ is called an *ideal straight simplex* if at least one of $u_0, \ldots, u_k$ is in $\partial_\infty X$. In general, however, an ideal straight simplex is not well defined for a simply connected Riemannian manifold with nonpositive sectional curvature. For example, let X be a higher rank symmetric space and consider two points θ_1, θ_2 in $\partial_\infty X$ which cannot be connected by any geodesic in X. Then, one cannot define a straight simplex $\mathrm{str}(\theta_1, \theta_2)$ with ideal vertices θ_1, θ_2. However, in the particular case that $x_0, \dots, x_{k-1} \in X$ and $\theta \in \partial_\infty X$, we can define an ideal straight simplex $\mathrm{str}(x_0, \dots, x_{k-1}, \theta)$ as usual. This is because there is the unique geodesic from a point in X to a point in $\partial_\infty X$. Hence, the geodesic cone on $\mathrm{str}(x_0, \dots, x_{k-1})$ with the top point θ is well defined. We only need such kind of ideal straight simplex to construct our invariant in K-theory for a $\mathbb{Q}$-rank 1 locally symmetric space.

Setup. We will stick to the following notations from now on. Let G be a noncompact, semisimple Lie group with trivial center and no compact factors and $X = G/K$ be the associated symmetric space with a maximal compact subgroup K of G. Given a $\mathbb{Q}$-rank 1 arithmetic lattice Γ in G, we denote by $\mathbf{G}$ a $\mathbb{Q}$-rank 1 semisimple algebraic group defined over $\mathbb{Q}$ as in Definition 3.1. Let $\mathbf{P}_1, \dots, \mathbf{P}_s$ be the representatives of the Γ-conjugacy classes of all proper rational parabolic subgroups of $\mathbf{G}$. According to the precise reduction theory, we fix a fundamental domain $F \subset X$ as in Theorem 3.2 as follows:

$$F = \Omega_0 \cup \coprod_{i=1}^{s} U_i \times A_{\mathbf{P}_i, t_i} \times V_i$$

Each half-geodesic $A_{\mathbf{P}_i, t_i}$ uniquely determines a point in $\partial_\infty X$, denoted by c_i. Write $\Gamma_i = \Gamma \cap \mathbf{P}_i(\mathbb{R})$ and $N = \Gamma \backslash X$. Note that each Γ_i is the stabilizer of c_i in Γ. Since N is tame, N is homeomorphic to the interior of a compact manifold M with boundary. Let $\partial_1 M, \dots, \partial_s M$ be the connected components of the boundary ∂M of M. Then there is a one-to-one correspondence between $\Gamma_1, \dots, \Gamma_s$ and $\partial_1 M, \dots, \partial_s M$. Indeed, we can assume that each $\partial_i M$ is homeomorphic to the quotient space of a horosphere based at c_i by the action of Γ_i.

Lemma 4.1. *For any $c \in \{c_1, \dots, c_s\}$, the volume of the ideal straight simplex* $\mathrm{str}(x_0, \dots, x_{d-1}, c)$ *is finite for any $x_0, \dots, x_{d-1} \in X$.*

Proof. Let $\mathbf{P}$ be the proper minimal rational parabolic subgroup associated with c. Let $\varphi \colon \Delta^{d-1} \to X$ be a parametrization of $\mathrm{str}(x_0, \dots, x_{d-1})$. Choose a coordinate system $s = (s_1, \dots, s_{d-1})$ in Δ^{d-1}. In the rational horocyclic coordinates of

$$X = N_{\mathbf{P}} \times A_{\mathbf{P}} \times X_{\mathbf{P}} \cong \mathbb{R}^r \times \mathbb{R} \times \mathbb{R}^l,$$

we can write

$$\varphi(s) = (x(s), y(s), z(s)).$$

Define a map

$$\psi\colon \Delta^{d-1} \times [0,\infty) \longrightarrow X$$

by

$$\psi(s,t) = (x(s), y(s)+t, z(s)).$$

A line $x(s) \times \mathbb{R}^+ \times z(s)$ is a geodesic representing c for any $s \in \Delta^{d-1}$. Hence, it is easy to see that ψ is a parametrization of $\mathrm{str}(x_0,\dots,x_{d-1},c)$.

Denote by $G(w_1,\dots,w_k)$ the Gram determinant of $w_1,\dots,w_k \in \mathbb{R}^d$. It is a standard fact that $\sqrt{G(w_1,\dots,w_k)}$ is the k-dimensional volume of the parallelogram with edges $w_1,\dots,w_k$. We abbreviate $\frac{\partial\psi}{\partial s_1},\dots,\frac{\partial\psi}{\partial s_{d-1}}$ as $\frac{\partial\psi}{\partial s}$. Then

$$\begin{aligned}
&\psi^*\,\mathrm{dvol}_X(s,t)\\
&\quad= h(x(s),z(s))e^{-2\|\rho_{\mathbf{P}}\|(y(s)+t)}\sqrt{G(\frac{\partial\psi}{\partial s}(s,t),\frac{\partial\psi}{\partial t}(s,t))}\,ds_1\cdots ds_{d-1}dt\\
&\quad= h(x(s),z(s))e^{-2\|\rho_{\mathbf{P}}\|(y(s)+t)}\sqrt{G(\frac{\partial\varphi}{\partial s}(s),\frac{\partial}{\partial y}(\psi(s,t)))}\,ds_1\cdots ds_{d-1}dt\\
&\quad\le h(x(s),z(s))e^{-2\|\rho_{\mathbf{P}}\|(y(s)+t)}\sqrt{G(\frac{\partial\varphi}{\partial s}(s))}\,ds_1\cdots ds_{d-1}\,dt
\end{aligned}$$

The last inequality follows from

$$\left\|\frac{\partial}{\partial y}(\psi(s,t))\right\| = 1.$$

Hence, we have

$$\begin{aligned}
&\mathrm{Vol}(\mathrm{str}(x_0,\dots,x_{d-1},c))\\
&\quad= \int_{\Delta^{d-1}\times[0,\infty)}\psi^*\,\mathrm{dvol}_X\;ds_1\cdots ds_{d-1}dt\\
&\quad\le \int_{\Delta^{d-1}} h(x(s),z(s))e^{-2\|\rho_{\mathbf{P}}\|y(s)}\sqrt{G(\frac{\partial\varphi}{\partial s}(s))}\,ds_1\cdots ds_{d-1}\cdot\int_0^\infty e^{-2\|\rho_{\mathbf{P}}\|t}\,dt\\
&\quad= \mathrm{Vol}(\mathrm{str}(x_0,\dots,x_{d-1}))\cdot\frac{1}{2\|\rho_{\mathbf{P}}\|}\\
&\quad< \infty.
\end{aligned}$$

This completes the proof. □

5. Cuspidal completion

In this section, we will define the notion of disjoint cone for M and the cuspidal completion of the classifying space $B\Gamma$, following [27, Section 4.2.1].

As we mentioned before, one can identify each component $\partial_i M$ with the quotient $\Gamma_i \backslash H_i$ where H_i is the horosphere that bounds a horoball

$$B_i = N_{\mathbf{P}_i} \times A_{\mathbf{P}_i, t_i} \times X_{\mathbf{P}_i}.$$

Note that such $B_i's$ are disjoint in $\mathbb{Q}$-rank 1 case. Hence we have a homeomorphism of tuples

$$(M, \partial_1 M, \dots, \partial_s M) \longrightarrow \Big(\Gamma\backslash\Big(X - \bigcup_{i=1}^{s} \Gamma B_i\Big), \Gamma_1\backslash H_1, \dots, \Gamma_s\backslash H_s\Big).$$

5.1. Disjoint cone of topological spaces. For a topological space Y and subspaces $A_1, \dots, A_s$ one can define a disjoint cone

$$\operatorname{Dcone}\Big(\bigcup_{i=1}^{s} A_i \longrightarrow Y\Big)$$

by coning each A_i to a point c_i. In other words, $\operatorname{Dcone}(\bigcup_{i=1}^{s} A_i \to Y)$ is the space obtained by gluing Y and $\bigcup_{i=1}^{s} \operatorname{Cone}(A_i)$ along $\bigcup_{i=1}^{s} A_i$. The following lemma is an elementary exercise in algebraic topology. It applies in particular to smooth manifolds with boundary because these have the homotopy type of CW-complexes.

Lemma 5.1. *Let (Y, A) be a pair of finite CW-complexes with Y connected and $A = A_1 \cup \dots \cup A_s$ the union of its connected components. Then there is an isomorphism*

$$H_*(Y, A) \cong H_*\Big(\operatorname{Dcone}\Big(\bigcup_{i=1}^{s} A_i \longrightarrow Y\Big)\Big)$$

in degrees $* \geq 2$.

5.2. Disjoint cone of simplicial sets. For a simplicial set (S, ∂_S) and a symbol c, *the cone over S with the cone point c* is the quasisimplicial set $\operatorname{Cone}(S)$ whose k-simplicies are either k-simplices in S or cones over $(k-1)$-simplices in S with the cone point c. The boundary operator ∂ in $\operatorname{Cone}(S)$ is defined by $\partial\sigma = \partial_S\sigma$ and

$$\partial \operatorname{Cone}(\sigma) = \operatorname{Cone}(\partial_S\sigma) + (-1)^{\dim(\sigma)+1}\sigma$$

for $\sigma \in S$.

If $\{T_i \mid i \in I\}$ is a family of simplicial subsets of S indexed over a set I, then define the quasisimplicial set $\mathrm{Dcone}(\dot{\bigcup}_{i\in I} T_i \to S)$ as the pushout

$$\begin{array}{ccc} \dot{\bigcup}_{i\in I} T_i & \longrightarrow & S \\ \downarrow & & \downarrow \\ \dot{\bigcup}_{i\in I} \mathrm{Cone}(T_i) & \longrightarrow & \mathrm{Dcone}\Big(\dot{\bigcup}_{i\in I} T_i \longrightarrow S\Big). \end{array}$$

Recall that in Section 2.1 we defined the simplicial set BG for a group G. Now let us consider $X = G/K$ a symmetric space of noncompact type and $\Gamma \subset G$ a lattice. We define the *cuspidal completion* BG^{comp} of BG to be

$$\mathrm{Dcone}\Big(\dot{\bigcup_{c\in\partial_\infty X}} BG \longrightarrow BG\Big).$$

In addition, define the *cuspidal completion* $B\Gamma^{\mathrm{comp}}$ of $B\Gamma$ to be

$$\mathrm{Dcone}\Big(\dot{\bigcup_{i=1}^{s}} B\Gamma_i \longrightarrow B\Gamma\Big),$$

where Γ_i are parabolic groups. More precisely, $B\Gamma^{\mathrm{comp}}$ is the quasisimplicial set whose k-simplices σ are either of the form

$$\sigma = (\gamma_1, \ldots, \gamma_k)$$

with $\gamma_1, \ldots, \gamma_k \in \Gamma$ or for some $i \in \{1, \ldots, s\}$ of the form

$$\sigma = (p_1, \ldots, p_{k-1}, c_i)$$

with $p_1, \ldots, p_{k-1} \in \Gamma_i$.

Lemma 5.2. *Let M be a compact, connected, smooth, oriented manifold with boundary ∂M. Then there is an isomorphism*

$$H_*(M, \partial M) \cong H_*^{\mathrm{simp}}\Big(\mathrm{Dcone}\Big(\bigoplus_{i=1}^{s} C_*(\partial_i M) \longrightarrow C_*(M)\Big)\Big)$$

in degrees ≥ 2. In particular

$$H_d^{\mathrm{simp}}\Big(\mathrm{Dcone}\Big(\oplus_{i=1}^{s} C_*(\partial_i M) \longrightarrow C_*(M); \mathbb{R}\Big)\Big) \cong \mathbb{R}$$

if $d = \dim(M) \geq 2$.

Proof. For a simplicial set S, we denote by $|S|$ the geometric realisation of S. One can think of $C_*(\partial_i M), C_*(M)$ as simplicial sets. Note that we have a natural isomorphism between the simplicial homology of the simplicial set and the singular homology of its geometric realisation. Thus to derive Lemma 5.2 from Lemma 5.1 it is sufficient to provide an isomorphism

$$H_*\Big(\mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)\Big) \cong H_*\big(\big|\mathrm{Dcone}\big(\bigoplus_{i=1}^{s} C_*(\partial_i M) \longrightarrow C_*(M)\big)\big|\big).$$

There is a natural Mayer-Vietoris sequence for CW-complexes (see the remark after [6, Proposition A.5]), hence the canonical continuous map

$$\Big|\mathrm{Dcone}\Big(\bigoplus_{i=1}^{s} C_*(\partial_i M) \longrightarrow C_*(M)\Big)\Big| \longrightarrow \mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)$$

yields the following commutative diagram

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
H_*(|C_*(M)|) \oplus \bigoplus_{i=1}^{s} H_*(|\mathrm{Cone}(C_*(\partial_i M))|) & \longrightarrow & H_*(M) \oplus \bigoplus_{i=1}^{s} H_*(\mathrm{Cone}(\partial_i M)) \\
\downarrow & & \downarrow \\
H_*\Big(\Big|\mathrm{Dcone}\Big(\bigoplus_{i=1}^{s} C_*(\partial_i M) \longrightarrow C_*(M)\Big)\Big|\Big) & \longrightarrow & H_*\Big(\mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \to M\Big)\Big) \\
\downarrow & & \downarrow \\
\bigoplus_{i=1}^{s} H_{*-1}(|C_*(\partial_i M)|) & \longrightarrow & \bigoplus_{i=1}^{s} H_{*-1}(\partial_i M) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

We note that $|\operatorname{Cone}(C_*(\partial_i M))|$ and $\operatorname{Cone}(\partial_i M)$ are contractible, hence their homology vanishes in degrees ≥ 1. Moreover

$$H_*(|C_*(M)|) \longrightarrow H_*(M)$$

and

$$H_*(|C_*(\partial_i M)|) \longrightarrow H_*(\partial_i M)$$

are isomorphisms: this follows from [24, Theorem 2.27] together with the fact that $H_*(X)$ is by definition the same as $H_*^{\text{simp}}(C_*(X))$ for any topological space X.

Thus the five lemma implies the wanted isomorphism

$$H_*\Big(\Big|\operatorname{Dcone}\Big(\bigoplus_{i=1}^{s} C_*(\partial_i M) \longrightarrow C_*(M)\Big)\Big|\Big) \longrightarrow H_*\Big(\operatorname{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)\Big). \qquad \square$$

In the sequel we will consider the situation that two points x_i, x in

$$X := \operatorname{Dcone}\Big(\bigoplus_{i=1}^{s} C_*(\partial_i M) \longrightarrow C_*(M)\Big)$$

are connected by a 1-simplex e_i with

$$\partial e_i = x - x_i.$$

For a 1-simplex σ with both vertices in x_i we can define "conjugation with e_i" by

$$C_{e_i}(\sigma) := \overline{e_i} * \sigma * e_i.$$

In particular, for $x_i \in \partial_i M$ and if

$$\pi_1(\partial_i M, x_i) \longrightarrow \pi_1(M, x_i)$$

is injective, then C_{e_i} realizes an isomorphism of $\pi_1(\partial_i M, x_i)$ to a subgroup $\Gamma_i \subset \pi_1(M, x)$. For a 1-simplex σ with

$$\partial\sigma = c_i - x_i$$

we define

$$C_{e_i}(\sigma) := \overline{e_i} * \sigma.$$

Definition 5.3. Let $(M, \partial M)$ be a pair of topological spaces, $\partial_1 M, \cdots, \partial_s M$ be the path components of ∂M. Denote by

$$c_i \in \mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)$$

the vertex of $\mathrm{Cone}(\partial_i M)$ for $i = 1, \ldots, s$. Let $x \in M, x_1 \in \partial_1 M, \ldots, x_s \in \partial_s M$. For $i \in \{1, \ldots, s\}$ we define

$$\widehat{C}_*^{x_i}(\partial_i M) \subset \mathrm{Cone}(C_*(\partial_i M)) \subset C_*\Big(\mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)\Big)$$

to be the subcomplex freely generated by those simplices in $\mathrm{Cone}(\partial_i M)$ for which

- either all vertices are in x_i ,
- or all but the last vertex is in x_i and the last vertex is in c_i.

For $i = 1, \ldots, s$ fix a path e_i from x to x_i and the corresponding C_{e_i}. Define

$$\widehat{C}_*^{x}(M) \subset C_*\Big(\mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)\Big)$$

to be the subcomplex freely generated by those simplices σ for which

- either all vertices are in x,
- or for some $i \in \{1, \ldots, s\}$ there exists a simplex $\sigma' \subset \widehat{C}_*^{x_i}(\partial_i M)$ such that C_{e_i} maps the 1-skeleton of σ' to the 1-skeleton of σ (up to homotopy fixing the 0-skeleton).

We remark that in the last case the homotopy classes (rel. $\{0, 1\}$) of all edges between all but the last vertices belong to $\Gamma_i \subset \pi_1(M, x)$.

For the statement of the following lemma we will denote by

$$j_1\colon \widehat{C}_*^{x}(M) \longrightarrow C_*\Big(\mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)\Big)$$

and

$$j_2\colon \mathrm{Dcone}\Big(\bigoplus_{i=1}^{s} C_*(\partial_i M) \longrightarrow C_*(M)\Big) \longrightarrow C_*\Big(\mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)\Big)$$

the inclusions.

Lemma 5.4. *Let M be a compact, connected, smooth, oriented manifold with boundary ∂M. Let $x \in M$. Then there exists a chain map*

$$F\colon C_*\Big(\operatorname{Dcone}\Big(\bigoplus_{i=1}^{s} C_*(\partial_i M) \longrightarrow C_*(M)\Big)\Big) \to \widehat{C}^x_*(M)$$

such that $j_1 \circ F$ is chain homotopic to j_2.

Proof. To write out the claim of the theorem: we want to show that there exist sequences of chain maps

$$F_n\colon C_n(\operatorname{Dcone}(\oplus_{i=1}^{s} C_*(\partial_i M) \to C_*(M))) \to \widehat{C}^x_n(M)$$

and of chain homotopies

$$K_n\colon C_n\Big(\operatorname{Dcone}\Big(\bigoplus_{i=1}^{s} C_*(\partial_i M) \longrightarrow C_*(M)\Big)\Big)$$
$$\longrightarrow C_{n+1}\Big(\operatorname{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)\Big)$$

for $n = 0, 1, 2, \ldots$ such that

$$\partial K_n(\sigma) + K_{n-1}(\partial\sigma) = F_n(\sigma) - \sigma$$

for all $\sigma \in C_n(\operatorname{Dcone}(\oplus_{i=1}^{s} C_*(\partial_i M) \to C_*(M)))$.

We will use the procedure for dividing Δ^n into $(n+1)$-simplices which is described in [24, page 112]. So for each $n \in \mathbb{N}$ we let $v_{n,0}, \ldots, v_{n,n}$ and $w_{n,0}, \ldots, w_{n,n}$ be the vertices of $\Delta^n \times \{0\}$ and $\Delta^n \times \{1\}$, respectively, and for $0 \le j \le n$, we denote by

$$\kappa_{n,j}\colon \Delta^{n+1} \longrightarrow \Delta^n \times [0,1]$$

the affine $(n+1)$-simplex with vertices $v_0, \ldots, v_j, w_j, \ldots, w_n$. We will inductively prove a slightly stronger statement as above, namely we will show that for each n-simplex σ in $\operatorname{Dcone}(\oplus_{i=1}^{s} C_*(\partial_i M) \to C_*(M))$ one can define a continuous map

$$L_\sigma\colon \Delta^n \times [0,1] \longrightarrow \operatorname{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)$$

such that $K_n(\sigma)$ is given by

$$K_n(\sigma) = \sum_{j=0}^{n} L_\sigma \circ \kappa_{n,j}$$

(and of course that the so defined K_n satisfies the above properties).

Suppose $n = 0$. A 0-simplex σ in $\mathrm{Dcone}(\oplus_{i=1}^{s} C_*(\partial_i M) \to C_*(M))$ is either a 0-simplex in M or a cone point c_i.

If $\sigma = c_i$, then we define

$$F_0(c_i) = c_i$$

and $K_0(c_i)$ is the 1-simplex mapped constantly to c_i.

If the 0-simplex σ belongs to $M - \partial M$, then we define

$$F_0(\sigma) = x$$

and $K_0(\sigma)$ is some (arbitrarily chosen) 1-simplex in $M \subset \mathrm{Dcone}(\bigcup_{i=1}^{s} \partial_i M \to M)$ with

$$\partial_0 K_0(\sigma) = x$$

and

$$\partial_1 K_0(\sigma) = \sigma.$$

If the 0-simplex σ belongs to $\partial_i M$, then we first choose some 1-simplex e_σ in $\partial_i M$ with

$$\partial_0 e_\sigma = x_i, \quad \partial_1 e_\sigma = \sigma.$$

(If $\sigma = x_i$, we let e_σ be the constant 1-simplex.) Recall from Definition 2.3 that we have fixed a path e_i from x_i to x which yields the isomorphism between $\pi_1(\partial_i M, x_i)$ and Γ_i by conjugation. Define then $F_0(\sigma) = x$ and $K_0(\sigma)$ is the 1-simplex obtained as concatenation of e_σ and e_i. In particular $\partial_0 K_0(\sigma) = x$ and $\partial_1 K_0(\sigma) = \sigma$.

Suppose $n = 1$. A 1-simplex σ in $\mathrm{Dcone}(\oplus_{i=1}^{s} C_*(\partial_i M) \to C_*(M))$ is either a 1-simplex in M or the cone (with cone point c_i) over a 0-simplex in $\partial_i M$. We have defined $K_0(\partial_1\sigma)$ and $K_0(\partial_0\sigma)$. Inclusion $\partial\Delta^1 \to \Delta^1$ is a cofibration, hence we have a continuous map

$$L_\sigma \colon \Delta^1 \times [0,1] \longrightarrow \mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)$$

such that

$$L_\sigma(x, 0) = x, \quad \text{for } x \in \sigma,$$

and

$$L_\sigma(\partial_j \sigma, t) = K_0(\partial_j \sigma)(t) \quad \text{for } j = 0, 1.$$

Then define

$$K_1(\sigma) := L_\sigma \circ \kappa_{1,0} + L_\sigma \circ \kappa_{1,1}$$

and $F_1(\sigma)$ by

$$F_1(\sigma)(x) := L_\sigma(x, 1)$$

for $x \in \Delta^1$.

It is clear that

$$\partial K_1(\sigma) + K_0(\partial\sigma) = F_1(\sigma) - \sigma$$

and that

$$F_0(\partial\sigma) = \partial F_1(\sigma).$$

We still have to check that $F_1(\sigma) \in \widehat{C}_1^x(M)$. If σ is the cone over a simplex in $\partial_i M$, then $\partial_0\sigma = c_i$ and $\partial_1\sigma \in \partial_i M \subset M$, hence $F_0(\partial_0\sigma) = c_i$ and $F_0(\partial_1\sigma) = x$, thus $F_1(\sigma) \in \widehat{C}_1^x(M)$. If $\sigma \in C_1(M)$, then

$$\partial_j F_1(\sigma) = F_0(\partial_j\sigma) = x \quad \text{for } j = 0, 1,$$

thus $F_1(\sigma) \in \widehat{C}_1^x(M)$. Moreover (this will be needed in the next steps) if $\sigma \in C_1(\partial_i M)$, then the homotopy class (rel. $\{0, 1\}$) of $F_1(\sigma)$ belongs to $\Gamma_i \subset \pi_1(M, x)$. Indeed, $F_1(\sigma)$ is in the homotopy class (rel. $\{0, 1\}$) of

$$\overline{K_0(\partial_1\sigma)} * \sigma * K_0(\partial_0\sigma) = \overline{e_i} * \overline{e_{\partial_1\sigma}} * \sigma * e_{\partial_0\sigma} * e_i,$$

where the bar means the 1-simplex with opposite orientation. Now $\overline{e_{\partial_1\sigma}} * \sigma * e_{\partial_0\sigma}$ represents an element in $\pi_1(\partial_i M, x_i)$ and by assumption conjugation with e_i provides to isomorphism to Γ_i, hence $F_1(\sigma)$ represents an element in Γ_i.

We now proceed to prove the theorem by induction. Assume that F_k and K_k have been defined for $k \leq n$. We will assume as part of the inductive hypothesis (and prove as part of the induction claim) that $F_n(\sigma)$ has all vertices in x if $\sigma \in C_*(M)$ and that $F_n(\sigma)$ has its last vertex in c_i if σ has its last vertex in c_i. (This is satisfied for $n \leq 1$ by the above construction.)

Let

$$\sigma\colon \Delta^{n+1} \longrightarrow \mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)$$

be an $(n+1)$-simplex in $\mathrm{Dcone}(\oplus_{i=1}^{s} C_*(\partial_i M) \to C_*(M))$. By the inductive hypothesis we have for $j = 0, \ldots, n+1$ a continuous map

$$L_{\partial_j\sigma}\colon \Delta^n \times [0,1] \longrightarrow \mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)$$

such that $K_n(\partial_j\sigma)$ is given by

$$K_n(\partial_j\sigma) = \sum_{l=0}^{n} L_{\partial_j\sigma} \circ \kappa_{n,l}.$$

(In particular $L_{\partial_j\sigma}(x,0) = x$ for $x \in \partial_j\Delta^n$.) Since the inclusion $\partial\Delta^n \to \Delta^n$ is a cofibration by [6, VII. Corollary 1.4] we have a continuous map

$$L_\sigma \colon \Delta^{n+1} \times [0,1] \longrightarrow \mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)$$

such that $L_\sigma|_{\Delta^{n+1}\times\{0\}}$ agrees with σ (after the obvious identification of Δ^{n+1} with $\Delta^{n+1}\times\{0\}$) and for $j = 0,\ldots,n+1$ $L_\sigma|_{\partial_j\Delta^{n+1}\times[0,1]}$ agrees with $L_{\partial_j\sigma}$. Then define

$$K_{n+1}(\sigma) := \sum_{j=0}^{n+1} L_\sigma \circ \kappa_{n+1,j}$$

and

$$F_{n+1}(\sigma) := L \circ \tau_{n+1},$$

where

$$\tau_{n+1} \colon \Delta^{n+1} \longrightarrow \Delta^{n+1} \times [0,1]$$

is defined by

$$\tau_{n+1}(x) = (x,1).$$

It is clear by construction that

$$\partial K_{n+1}(\sigma) + K_n(\partial\sigma) = F_{n+1}(\sigma) - \sigma$$

and that

$$\partial F_{n+1}(\sigma) = F_n(\partial\sigma).$$

We have to check that $F_{n+1}(\sigma) \in \widehat{C}^x_{n+1}(M)$. If σ is an $(n+1)$-simplex in M, then all $\partial_j\sigma$ are n-simplices in M, hence by induction all vertices of all $F_n(\partial_j)$ are in x. Because of $\partial_j F_{n+1}(\sigma) = F_n(\partial_j\sigma)$ this implies that all vertices of $F_{n+1}(\sigma)$ are in x, hence $F_{n+1}(\sigma) \in \widehat{C}^x_{n+1}(M)$.

If σ is the cone (with cone point c_i) over an n-simplex $\tau = \partial_n\sigma$, then we have by inductive hypothesis that $F_n(\partial_{n+1}\sigma)$ has all its vertices in x and moreover that all $F_n(\partial_j\sigma)$ with $0 \le j \le n$ have their last vertex in c_i. Because of

$$\partial_j F_{n+1}(\sigma) = F_n(\partial_j\sigma)$$

this implies that $F_{n+1}(\sigma)$ has its last vertex in c_i and the remaining vertices in x. Moreover, if $n+1=2$, then $\partial_2\sigma \in C_1(\partial_i M)$ and it follows (from the construction for $n=1$) that the homotopy class (rel. $\{0,1\}$) of $\partial_2 F_2(\sigma) = F_1(\partial_2\sigma)$ belongs to $\Gamma_i \subset \pi_1(M,x)$. If $n+1 \geq 3$, then, since each edge of σ is an edge of some $\partial_j\sigma$ and since $F_n(\partial_j\sigma) \in \widehat{C}_n^x(M)$, it follows that the homotopy classes (rel. $\{0,1\}$) of all edges between all but the last vertices belong to $\Gamma_i \subset \pi_1(M,x)$. Thus $F_{n+1}(\sigma) \in \widehat{C}_{n+1}^x(M)$. □

Corollary 5.5. *Let M be a compact, connected, smooth, oriented, aspherical manifold with aspherical π_1-injective boundary $\partial M = \partial_1 M \cup \ldots \cup \partial_s M$. Let $x \in M$. Assume $\Gamma_i \cap \Gamma_j = 0$ for $i \neq j$, where $\Gamma_i \subset \pi_1(M,x)$ for $i = 1,\ldots,s$ is defined by Definition 5.3. Then the chain map*

$$F : \mathrm{Dcone}\left(\bigoplus_{i=1}^{s} C_*(\partial_i M\right) \longrightarrow C_*(M)) \longrightarrow \widehat{C}_*^x(M)$$

induces an isomorphism of homology groups.

Proof. Lemma 5.4 implies that $j_{1*}F_* = j_{2*}$ and Lemma 5.2 implies that j_{2*} is an isomorphism. Hence F_* is injective. It remains to show that F_* is surjective, i.e., that every cycle in $\widehat{C}_*^x(M)$ is homologous to some cycle of the form F_*z with z a cycle in $\mathrm{Dcone}(\oplus_{i=1}^s C_*(\partial_i M) \to C_*(M))$.

Let $\sum_{j=1}^r a_j\sigma_j \in \widehat{C}_*^x(M)$ be a cycle. Let

$$J^{\deg} = \{j : \sigma_j \text{ has an edge representing } 0 \in \pi_1(M,x)\}.$$

The same argument as in the proof of [27, Lemma 5.15] shows that $\sum_{j\in J^{\deg}} a_j\sigma_j$ is a 0-homologous cycle, thus $\sum_{j\notin J^{\deg}} a_j\sigma_j$ is homologous to $\sum_{j=1}^r a_j\sigma_j$. We can and will therefore without loss of generality assume that no σ_j has an edge representing $0 \in \pi_1(M,x)$.

Let $c_1,\ldots,c_s$ be the cone points and for $i \in \{1,\ldots,s\}$ let

$$J_i = \{j : \sigma_j \text{ has its last vertex in } c_i\}.$$

We note that for $i \neq l$ a simplex in J_i can not have a face in common with a simplex in J_l. Indeed such a face would have edges representing elements in $\Gamma_i \subset \pi_1(M,x)$ and $\Gamma_l \subset \pi_1(M,x)$ which is impossible because of $\Gamma_i \cap \Gamma_l = \emptyset$.

Now let K be the simplicial complex defined as a union

$$K = \Delta_1 \cup \ldots \cup \Delta_s$$

of homeomorphic images of the d-dimensional standard simplex with identifications $\partial_i \Delta_j = \partial_k \Delta_l$ if and only if $\partial_i \sigma_j = \partial_k \sigma_l$. Let

$$\sigma\colon K \longrightarrow \mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)$$

be defined by

$$\sigma|_{\Delta_j} = \sigma_j,$$

where the homeomorphism from Δ_j to the standard simplex is understood. By construction,

$$\sum_{j=1}^{r} a_j \sigma_j = \sigma_*\Big[\sum_{j=1}^{r} a_j \Delta_j\Big].$$

We will now homotope σ such that its image becomes a chain in the complex $\mathrm{Dcone}(\oplus_{i=1}^{s} C_*(\partial_i M) \to C_*(M))$. First, if $j \in J_i$, then we homotope all but the last vertex of Δ_j from x to x_i along the path e_i from Definition 5.3. Since for $i \neq l$ simplices in J_i and J_l have no face in common this can be done simultaneously for all Δ_j with $j \in J_1 \cup \ldots \cup J_s$. By successive application of the cofibration property this homotopy can be extended to all of K. For $j \in J_i$ it follows from the definition of Γ_i in Definition 5.3 that after this homotopy the edges of Δ_j opposite to the cone point are all mapped to loops at x_i homotopic rel. $\{0, 1\}$ into $\partial_i M$. We may thus (using again the cofibration property to successively extend the homotopy from the 1-skeleton to K) further homotope σ to have all these edges in $\partial_i M$, and the remaining edges of Δ_i mapped to $\mathrm{Cone}(\partial_i M)$. Finally, since M and $\partial_i M$ are aspherical we have $\pi_{*\geq 2}(M, \partial M) = 0$, thus we can successively further homotope σ such that

- for $j \in J_i$ all higher-dimensional subsimplices and finally Δ_j are mapped to $\partial_i M$ (if they don't contain the cone point) or to $\mathrm{Cone}(\partial_i M)$ (if they do contain the cone point),
- for $j \notin J_1 \cup \ldots \cup J_s$ all higher-dimensional subsimplices and finally Δ_j are mapped to M.

Thus we obtain a cycle c in $\mathrm{Dcone}(\oplus_{i=1}^{s} C_*(\partial_i M) \to C_*(M))$. By construction $F_* c$ is homotopic, hence homologous, to $\sum_{j=1}^{r} a_j \sigma_j$. □

Corollary 5.6. *Under the assumptions of Corollary 5.5 we have*

$$H_d(\widehat{C}_*^{x}(M); \mathbb{R}) \cong \mathbb{R}$$

for $d = \dim(M) \geq 2$ *and* $x \in M$.

6. Eilenberg–MacLane map

Recall the homeomorphism of tuples as we describe in Section 5,

$$(M, \partial_1 M, \dots, \partial_s M) \longrightarrow \Big(\Gamma\backslash\Big(X - \bigcup_{i=1}^{s} \Gamma B_i\Big), \Gamma_1\backslash H_1, \dots, \Gamma_s\backslash H_s\Big).$$

Let c_i denote the cone point of $\mathrm{Cone}(\partial_i M)$. Identifying each $\mathrm{Cone}(\partial_i M) - c_i$ with $\Gamma_i\backslash B_i$, we have a homeomorphism

$$\Gamma\backslash X \longrightarrow \mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big) - \{c_1, \dots, c_s\}$$

extending the homeomorphism of tuples above. Composition of the universal covering $X \to \Gamma\backslash X$ with this homeomorphism yields a covering map

$$X \longrightarrow \mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big) - \{c_1, \dots, c_s\}.$$

Then we finally have a projection map

$$\pi : X \cup \bigcup_{i=1}^{s} \Gamma\partial_\infty B_i \longrightarrow \mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)$$

such that

$$\pi|_X : X \longrightarrow \mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big) - \{c_1, \dots, c_s\}$$

is a covering,

$$\pi|_{\Gamma B_i} : \Gamma B_i \longrightarrow \mathrm{Cone}(\partial_i M) - C_i$$

is a covering with deck group Γ and π maps $\Gamma\partial_\infty B_i$ to c_i for $i = 1, \dots, s$. Due to this projection map, we can define the notion of (ideal) straight simplex in $\mathrm{Dcone}(\bigcup_{i=1}^{s} \partial_i M \to M)$.

Definition 6.1. We say that a k-simplex σ in $\mathrm{Dcone}(\bigcup_{i=1}^{s} \partial_i M \to M)$ is *straight* if σ is of the form

$$\pi(\mathrm{str}(u_0, \dots, u_k))$$

for $u_0, \dots, u_k \in X \cup \bigcup_{i=1}^{s} \Gamma\partial_\infty B_i$.

Remark. In the $\mathbb{R}$-rank 1 case, every $\partial_\infty B_i$ consists of a point in $\partial_\infty X$ and for any ordered pair $(u_0, \dots, u_k) \in X \cup \bigcup_{i=1}^{s} \Gamma \partial_\infty B_i$, $\mathrm{str}(u_0, \dots, u_k)$ is well defined. In contrast, in the higher rank case, each $\partial_\infty B_i$ is not a point. More precisely, if B_i is any horoball centered at z_i, then

$$\partial_\infty B_i = \Big\{ w \in \partial_\infty X \;\Big|\; \mathrm{Td}(z_i, w) \le \frac{1}{2}\pi \Big\},$$

where Td is the Tits metric on $\partial_\infty X$ (see [25]). Furthermore, $\mathrm{str}(u_0, \dots, u_k)$ may not be defined for some ordered pair $(u_0, \dots, u_k)$.

We denote

$$\widehat{C}_*(M) := C_*\Big(\mathrm{DCone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)\Big).$$

Recall from Definition 5.3 that we choose base points $x_0, x_1, \dots, x_s$ of M, $\partial_1 M, \dots, \partial_s M$ respectively and identify $\pi_1(\partial_i M, x_i)$ with a subgroup Γ_i of $\pi_1(M, x_0)$ by choosing a path connecting x_0 and x_i for $i = 1, \dots, s$.

The assumptions of Corollary 5.6 are satisfied for $\mathbb{Q}$-rank 1 spaces, thus we have

$$\begin{aligned} H_d(\widehat{C}_*^{x_0}(M), \mathbb{Q}) &= H_d(\widehat{C}_*(M), \mathbb{Q}) \\ &= H_d(M, \partial M, \mathbb{Q}) \\ &= \mathbb{Q}. \end{aligned}$$

Definition 6.2. We define a chain complex

$$\widehat{C}_*^{\mathrm{str}, x_0}(M) := \mathbb{Z}[\{\sigma \in \widehat{C}_*^{x_0}(M) \mid \sigma \text{ is straight}\}],$$

and for the s-tuple $(c_1, \dots, c_s)$ with $c_i \in \partial_\infty B_i$ (see the setup in Section 4), we define the subcomplex $\widehat{C}_*^{\mathrm{str}, x_0, c}(M)$ of $\widehat{C}_*^{\mathrm{str}, x_0}(M)$ freely generated by those simplices that are either of the form

$$\sigma = \pi(\mathrm{str}(\tilde{x}_0, \gamma_1 \tilde{x}_0, \dots, \gamma_1 \cdots \gamma_k \tilde{x}_0))$$

where $\tilde{x}_0$ is a lift of x_0 and $\gamma_1, \dots, \gamma_k \in \Gamma$ or of the form

$$\sigma = \pi(\mathrm{str}(\tilde{x}_0, p_1 \tilde{x}_0, \dots, p_1 \cdots p_{k-1} \tilde{x}_0, c_i))$$

for $p_1, \dots, p_{k-1} \in \Gamma_i$ and $i \in \{1, \dots, s\}$.

Lemma 6.3. *The following hold.*

(*a*) *There is an isomorphism of chain complexes*

$$\Phi\colon \widehat{C}_*^{\mathrm{str},x_0,c}(M) \longrightarrow C_*^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}}).$$

(*b*) *The inclusion*

$$\widehat{C}_*^{\mathrm{str},x_0,c}(M) \longrightarrow \widehat{C}_*(M)$$

induces an isomorphism

$$H_d(\widehat{C}_*^{\mathrm{str},x_0,c}(M), \mathbb{Q}) \longrightarrow H_d\Big(\mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big), \mathbb{Q}\Big).$$

(*c*) *The composition of* Φ^{-1} *with the inclusion*

$$\widehat{C}_*^{\mathrm{str},x_0,c}(M) \longrightarrow \widehat{C}_*(M)$$

induces an isomorphism

$$\mathrm{EM}_d : H_d^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}}, \mathbb{Q}) \longrightarrow H_d\Big(\mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big), \mathbb{Q}\Big).$$

Proof. The proof of (a) is exactly the same as the proof of [27, Lemma 8a]. To prove (b) note that one can extend the domain of Φ (see Section 2.1) to $\widehat{C}_*^{x_0}(M)$ in the following way. Let σ be a k-simplex in $\widehat{C}_*^{x_0}(M)$ and $e_0, \dots, e_k$ be the vertices of the standard simplex Δ^k. For $i = 1, \dots, k$, let ζ_i be the standard sub-1-simplex with

$$\partial \zeta_i = e_i - e_{i-1}.$$

Define

$$\Phi(\sigma) = ([\sigma|_{\zeta_1}], \dots, [\sigma|_{\zeta_k}]),$$

where each $[\sigma|_{\zeta_i}] \in \Gamma = \pi_1(M, x_0)$ is the homotopy class of $\sigma|_{\zeta_i}$ if all vertices of σ are in x_0 and

$$\Phi(\sigma) = ([\sigma|_{\zeta_1}], \dots, [\sigma|_{\zeta_{k-1}}], c_i),$$

if the last vertex of σ is c_i. Then consider the composition of maps as follows:

$$\widehat{C}_*^{\mathrm{str},x_0,c}(M) \xrightarrow{\ i\ } \widehat{C}_*^{x_0}(M) \xrightarrow{\ \Phi\ } C_*^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}}) \xrightarrow{\ \Psi\ } \widehat{C}_*^{\mathrm{str},x_0,c}(M)\ .$$

Obviously, $\Psi \circ \Phi \circ i = \mathrm{id}$ and thus we have an injective homomorphism

$$i_*\colon H_d(\widehat{C}_*^{\mathrm{str},x_0,c}(M),\mathbb{Q}) \longrightarrow H_d(\widehat{C}_*^{x_0}(M),\mathbb{Q}).$$

From Corollary 5.6 we obtain

$$H_d(\widehat{C}_*^{x_0}(M),\mathbb{Q}) \cong \mathbb{Q},$$

thus $H_d(\widehat{C}_*^{\mathrm{str},x_0,c}(M),\mathbb{Q})$ as a vector space over $\mathbb{Q}$ is at most one dimensional. On the other hand Lemma 6.4 below shows that $\mathrm{EM}_d^{-1}[M,\partial M]$ is a nontrivial element in

$$H_d(B\Gamma^{\mathrm{comp}},\mathbb{R}) \cong H_d(\widehat{C}_*^{\mathrm{str},x_0,c}(M),\mathbb{R}),$$

because evaluation of some cocycle is not zero. Hence, i_* is actually an isomorphism. Furthermore, considering that by Corollary 5.5 the inclusion

$$\widehat{C}_*^{x_0}(M) \longrightarrow \widehat{C}_*(M)$$

is a homology equivalence, one can conclude that the inclusion

$$\widehat{C}_*^{\mathrm{str},x_0,c}(M) \longrightarrow \widehat{C}_*(M)$$

induces an isomorphism

$$H_d(\widehat{C}_*^{\mathrm{str},x_0,c}(M),\mathbb{Q}) \longrightarrow H_d\Big(\mathrm{Dcone}\Big(\bigcup_{i=1}^{s}\partial_i M \longrightarrow M\Big),\mathbb{Q}\Big).$$

Finally, (c) follows from (a) and (b). □

Remark. In the $\mathbb{R}$-rank 1 case, the geodesic straightening map

$$\mathrm{str}_*\colon \widehat{C}_*^{x_0}(M) \longrightarrow \widehat{C}_*^{\mathrm{str},x_0}(M)$$

that is a left-inverse to the inclusion $\widehat{C}_*^{\mathrm{str},x_0}(M) \subset \widehat{C}_*^{x_0}(M)$ is well defined and $C_*^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}})$ is isomorphic to $\widehat{C}_*^{\mathrm{str},x_0}(M) = \widehat{C}_*^{\mathrm{str},x_0,c}(M)$. (See [27]). However, in the higher rank case, the straightening map $\mathrm{str}_*\colon \widehat{C}_*^{x_0}(M) \to \widehat{C}_*^{\mathrm{str},x_0}(M)$ is not well defined as we mentioned in Section 4. Hence $C_*^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}})$ is not isomorphic to $\widehat{C}_*^{\mathrm{str},x_0}(M)$ but is isomorphic to its subcomplex $\widehat{C}_*^{\mathrm{str},x_0,c}(M)$. Despite such differences with $\mathbb{R}$-rank 1 case, we obtained the homology class

$$\mathrm{EM}_d^{-1}[M,\partial M] \in H_d^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}},\mathbb{Q})$$

as in the $\mathbb{R}$-rank 1 case. This will enable us in Section 7 to define an invariant in K-theory for a $\mathbb{Q}$-rank 1 locally symmetric space.

Recall that the volume cocycle $\mathrm{comp}(\nu_d) = c\nu_d \in C^d_{\mathrm{simp}}(BG)$ is defined by

$$c\nu_d(g_1,\ldots,g_d) = \int_{\mathrm{str}(\tilde{x}_0, g_1\tilde{x}_0,\ldots,g_1\cdots g_d\tilde{x}_0)} \mathrm{dvol}_X,$$

where dvol_X is the G-invariant Riemannian volume form on $X = G/K$. If X is a $\mathbb{R}$-rank 1 symmetric space, one can extend this volume cocycle $c\nu_d$ to a cocycle $\overline{c\nu}_d \in C^d_{\mathrm{simp}}(BG^{\mathrm{comp}})$. In the higher rank case, one cannot obtain an extended volume cocycle in $C^d_{\mathrm{simp}}(BG^{\mathrm{comp}})$. However, we can extend the volume cocycle in $C^d_{\mathrm{simp}}(B\Gamma)$ to at least a cocycle in $C^d_{\mathrm{simp}}(B\Gamma^{\mathrm{comp}})$. Define a cocycle $\overline{c\nu}_d \in C^d_{\mathrm{simp}}(B\Gamma^{\mathrm{comp}})$ as follows. For a d-simplex $(\gamma_1,\ldots,\gamma_d)$ with $\gamma_1,\ldots,\gamma_d \in \Gamma$, define

$$\overline{c\nu}_d(\gamma_1,\ldots,\gamma_d) = c\nu_d(\gamma_1,\ldots,\gamma_d).$$

For a d-simplex $(p_1,\ldots,p_{d-1},c_i)$ with $p_1,\ldots,p_{d-1} \in \Gamma_i$ and $i \in \{1,\ldots,s\}$, define

$$\overline{c\nu}_d(p_1,\ldots,p_{d-1},c_i) = \int_{\mathrm{str}(\tilde{x}_0, p_1\tilde{x}_0,\ldots,p_1\cdots p_{d-1}\tilde{x}_0, c_i)} \mathrm{dvol}_X\,.$$

It follows from Lemma 4.1 that $\overline{c\nu}_d$ is well defined. An application of Stokes' Theorem (to compact submanifolds with boundary of every ideal simplex exactly as in the proof of Lemma 6.4 below) shows that $\overline{c\nu}_d$ is a cocycle in $C^d_{\mathrm{simp}}(B\Gamma^{\mathrm{comp}})$. Hence, the cocycle $\overline{c\nu}_d$ determines a cohomology class in $H^d_{\mathrm{simp}}(B\Gamma^{\mathrm{comp}})$, denoted by $\bar{v}_d$.

Lemma 6.4. *If $N = \mathrm{int}(M)$ is a $\mathbb{Q}$-rank 1 locally symmetric space of dimension at least 3, then*

$$\langle \bar{v}_d, \mathrm{EM}_d^{-1}[M,\partial M]\rangle = \mathrm{Vol}(N).$$

Proof. Let z be a relative fundamental cycle in $C_d(M,\partial M)$ representing the relative fundamental class $[M,\partial M]$. We think of M as a submanifold of N via the homeomorphism of tuples

$$(M,\partial_1 M,\ldots,\partial_s M) \longrightarrow \Big(\Gamma\backslash\Big(X - \bigcup_{i=1}^{s}\Gamma B_i\Big), \Gamma_1\backslash H_1,\ldots,\Gamma_s\backslash H_s\Big)$$

and z as a chain in $C_d(N)$.

Since ∂z represents $[\partial M]$, we can write

$$\partial z = \partial_1 z + \cdots + \partial_s z,$$

where $\partial_i z$ is a cycle representing $[\partial_i M]$ for $i = 1, \ldots, s$. Make z a chain z^0 in $\widehat{C}_d^{x_0}(M)$ via the chain homotopy $\widehat{C}_*(M) \to \widehat{C}_*^{x_0}(M)$ in the proof of [27, Lemma 8]. Note that z^0 is obtained by adding several 1-dimensional paths to z and hence

$$\text{algvol}(z^0) = \text{algvol}(z) = \text{Vol}(M).$$

Now, consider a geodesic cone $\text{Cone}_g(\partial_i z)$ over $\partial_i z$ with the top point c_i for $i = 1, \ldots, s$. Due to dim $A_{\mathbf{P}_i} = 1$, it is not difficult to see that

$$\text{algvol}(\text{Cone}_g(\partial_i z)) = (-1)^{d+1}\,\text{Vol}(\Gamma_i \backslash B_i).$$

Since $\text{Cone}_g(\partial_i z^0)$ is obtained by adding two dimensional objects of N to the cone $\text{Cone}_g(\partial_i z)$, its algebraic volume is not changed, that is,

$$\text{algvol}(\text{Cone}_g(\partial_i z^0)) = \text{algvol}(\text{Cone}_g(\partial_i z)) = (-1)^{d+1}\,\text{Vol}(\Gamma_i \backslash B_i).$$

Now, define

$$c(z^0) = z^0 + (-1)^{d+1}\,\text{Cone}_g(\partial z^0).$$

Then it can be checked that $c(z^0)$ is a cycle in $\widehat{C}_d^{x_0}(M)$ by

$$\begin{aligned}
\partial c(z^0) &= \partial z^0 + (-1)^{d+1} \partial\, \text{Cone}_g(\partial z^0) \\
&= \partial z^0 + (-1)^{d+1}\,\text{Cone}_g(\partial\partial z^0) + (-1)^{d+1}(-1)^d \partial z^0 \\
&= 0.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\text{algvol}(c(z^0)) &= \text{Vol}(M) + \sum_{i=1}^{s} \text{Vol}(\Gamma_i \backslash B_i) \\
&= \text{Vol}(M) + \text{Vol}(N - M) \\
&= \text{Vol}(N).
\end{aligned}$$

Note that we can straighten $c(z^0)$ because all vertices of every ideal simplex in $c(z^0)$ are in x_0 except for the last vertex with c_i for some $i \in \{1, \ldots, s\}$. Furthermore, $\text{str}(c(z^0))$ is in $\widehat{C}_d^{\text{str}, x_0, c}(M)$ and represents $\Psi \circ \text{EM}_d^{-1}[M, \partial M]$. To prove the lemma, it is sufficient to show that

$$\text{algvol}(\text{str}(c(z^0))) = \text{Vol}(N).$$

Let

$$H_0, H_1\colon K \longrightarrow \operatorname{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)$$

be the simplicial maps realising the cycles

$$c(z^0) + \operatorname{Cone}_g(\partial z^0) \quad \text{and} \quad \operatorname{str}(c(z^0) + \operatorname{Cone}_g(\partial z^0)),$$

respectively. (See the construction in the proof of Corollary 5.5.) Let

$$H\colon K \times [0,1] \longrightarrow \operatorname{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)$$

be the straight line homotopy between

$$c(z^0) + \operatorname{Cone}_g(\partial z^0) \quad \text{and} \quad \operatorname{str}(c(z^0) + \operatorname{Cone}_g(\partial z^0)),$$

such that

$$H_0 = H(.,0) \quad \text{and} \quad H_1 = H(.,1).$$

The homotopy H yields a chain homotopy

$$L_*\colon \widehat{C}_*^{\operatorname{str},x_0,c}(M) \longrightarrow \widehat{C}_{*+1}^{\operatorname{str},x_0,c}(M)$$

from the straightening map str to the identity. (See the construction in the proof of Lemma 5.4.) This satisfies

$$\partial L_k + L_{k-1}\partial = \operatorname{str} - \operatorname{id}.$$

Then

$$\begin{aligned}
&\operatorname{str}(c(z^0)) - c(z^0)\\
&\quad = \operatorname{str}(z^0) - z^0 + (-1)^{d+1}(\operatorname{Cone}_g(\operatorname{str}(\partial z^0)) - \operatorname{Cone}_g(\partial z^0))\\
&\quad = \partial L_d(z^0) + L_{d-1}(\partial z^0) + (-1)^{d+1}\operatorname{Cone}_g(\partial L_{d-1}(\partial z^0) + L_{d-2}(\partial\partial z^0))\\
&\quad = \partial L_d(z^0) + L_{d-1}(\partial z^0) + (-1)^{d+1}\partial\operatorname{Cone}_g(L_{d-1}(\partial z^0)) - L_{d-1}(\partial z^0)\\
&\quad = \partial(L_d(z^0) + (-1)^{d+1}\operatorname{Cone}_g(L_{d-1}(\partial z^0)))
\end{aligned}$$

In order to conclude $\operatorname{algvol}(\operatorname{str}(c(z^0))) = \operatorname{algvol}(c(z^0))$ we want to apply Stokes' Theorem to show that the integral of the volume form over

$$\partial(L_d(z^0) + (-1)^{d+1}\operatorname{Cone}_g(L_{d-1}(\partial z^0)))$$

vanishes. It is clear that the integral of the closed form dvol over $\partial L_d(z^0)$ vanishes and thus it remains to look at simplices in $\partial\,\mathrm{Cone}_g(L_{d-1}(\partial z^0))$.

Since the volume form is defined on the complement of the cone points we can apply Stokes' Theorem to compact submanifolds with boundary of every (ideal) simplex in Let $\mathcal{H}_{ik}$ be a sequence of horospheres converging towards c_i for $k \to \infty$. In the following we will call simplices in $\mathrm{Cone}_g(\partial_i z)$ *proper ideal* simplices if they have a vertex in c_i, i.e. if they are not contained in $\partial_i z$. For a proper ideal simplex in $\mathrm{Cone}_g(\partial_i z)$ its edges are either edges of a simplex in $\partial_i z$ or otherwise they are geodesics ending in c_i, which therefore are transverse to the horospheres $\mathcal{H}_{ik}$. Moreover all higher-dimensional proper ideal simplices in $\mathrm{Cone}_g(\partial_i z^0)$ and their straightenings are a union of geodesic lines ending in c_i. In particular all proper ideal simplices occurring in $\mathrm{Cone}_g(\partial_i z^0)$ and $\mathrm{str}(\mathrm{Cone}_g(\partial_i z^0))$ are transverse to the $\mathcal{H}_{ik}$'s.

The Relative Transversality Theorem (see [22]) yields that any map

$$H\colon K \times [0,1] \longrightarrow \mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)$$

whose restriction to $K \times \{0,1\}$ is transverse to $\bigcup_{i.k} \mathcal{H}_{ik}$ can be homotoped (by an arbitrarily small homotopy, keeping $K \times \{0,1\}$ fixed) to a map which is transverse on all of $K \times [0,1]$. The homotopy can be chosen to fix subsimplices which are already transverse. Thus we can homotope (keeping $c(z^0)$ and $\mathrm{str}(c(z^0))$ as well as $L_d(z^0)$ fixed) the simplices in $\mathrm{Cone}_g(L_{d-1}(\partial z^0))$ to be transverse to the $\mathcal{H}_{ik}$'s.

Then, for any $(d+1)$-dimensional simplex

$$\kappa\colon \Delta^{d+1} \longrightarrow \mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big) \cong N \cup \{\text{cusps}\}$$

occurring in $\mathrm{Cone}_g(L_{d-1}(\partial z^0))$, and for each $k \in \mathbb{N}$, we conclude from transversality that $\kappa^{-1}(\mathcal{H}_{ik})$ is a d-dimensional submanifold K_{ik} bounding a $(d+1)$-dimensional submanifold $\Omega_{ik} \subset \Delta^{d+1}$ which does not contain the preimage of c_i. Thus we can apply Stokes' Theorem to Ω_{ik} and obtain

$$\int_{\partial\Delta^{d+1}\cap\Omega_{ik}} H^*\,\mathrm{dvol}_X + \int_{K_{ik}} H^*\,\mathrm{dvol}_X = \int_{\Omega_{ik}} dH^*\,\mathrm{dvol}_X = 0$$

because $H^*\,\mathrm{dvol}_X$ is a closed form. We note that H maps the d-dimensional submanifold K_{ik} to the $(d-1)$-dimensional submanifold $\mathcal{H}_{ik} \subset N$ and therefore

$$\int_{K_{ik}} H^*\,\mathrm{dvol}_X = 0.$$

Thus

$$\int_{\partial\Delta^{d+1}\cap\Omega_{ik}} H^*\,\mathrm{dvol}_X = 0$$

for all $k \in \mathbb{N}$. Then, since the countable union $\bigcup_{k\in\mathbb{N}} \partial\Delta^{d+1} \cap \Omega_{ik}$ equals $\Delta^{d+1} \setminus \kappa^{-1}(c_i)$ and since dvol_X is defined to be zero on c_i, we conclude that

$$\int_{\partial\Delta^{d+1}} H^*\,\mathrm{dvol}_X = 0.$$

Summing up over all

$$\kappa\colon \Delta^{d+1} \longrightarrow \mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i M \longrightarrow M\Big)$$

occurring in $\mathrm{Cone}_g(L_{d-1}(\partial z^0))$ we obtain

$$\int_{\partial\,\mathrm{Cone}_g(L_{d-1}(\partial z^0))} \mathrm{dvol}_X = 0.$$

Then, we have

$$\mathrm{algvol}(\mathrm{str}(c(z^0))) = \int_{\mathrm{str}(c(z^0))} \mathrm{dvol}_X = \int_{c(z^0)} \mathrm{dvol}_X = \mathrm{Vol}(N),$$

which completes the proof. □

Assume that d is odd. Let b_d be the Borel class in $H_c^d(\mathrm{SL}(n,\mathbb{C}),\mathbb{R})$. (It is obtained by restriction of the Borel class $b_d \in H_c^d(\mathrm{GL}(n,\mathbb{C}),\mathbb{R})$ defined in Section 2.2.) According to the Van Est isomorphism, a representative β_d of b_d is given by

$$\beta_d(g_0,\dots,g_d) = \int_{\mathrm{str}(g_0\tilde{o},\dots,g_d\tilde{o})} \mathrm{dbol}$$

where dbol is an $\mathrm{SL}(n,\mathbb{C})$-invariant differential d-form on $\mathrm{SL}(n,\mathbb{C})/\,\mathrm{SU}(n)$ and $\tilde{o}$ is a point in $\mathrm{SL}(n,\mathbb{C})/\,\mathrm{SU}(n)$. Recall that a comparison map

$$\mathrm{comp}\colon C_c^*(\mathrm{SL}(n,\mathbb{C}),\mathbb{R}) \longrightarrow C^*_{\mathrm{simp}}(B\,\mathrm{SL}(n,\mathbb{C}),\mathbb{R})$$

is defined by

$$\mathrm{comp}(f)(g_1,\dots,g_k) = f(1,g_1,g_1g_2,\dots,g_1\cdots g_k).$$

As in [27, Section 4.2.3] define $B\,\mathrm{SL}(n,\mathbb{C})^{\mathrm{fb}}_d$ as the set consisting of d-simplices in $B\,\mathrm{SL}(n,\mathbb{C})^{\mathrm{comp}}_d$ which are either d-simplices in $B\,\mathrm{SL}(n,\mathbb{C})_d$ or of the form

$$(p_1,\dots,p_{d-1},c)$$

satisfying

$$\left|\int_{\mathrm{str}(\tilde{o},p_1\tilde{o},\dots,p_1\cdots p_{d-1}\tilde{o},c)} \mathrm{dbol}\right| < \infty.$$

We define $B\,\mathrm{SL}(n,\mathbb{C})^{\mathrm{fb}}$ to be the quasisimplicial set generated by $B\,\mathrm{SL}(n,\mathbb{C})^{\mathrm{fb}}_d$ under face maps. Then consider a cocycle

$$\overline{c\beta}_d\colon C^{\mathrm{simp}}_d(B\,\mathrm{SL}(n,\mathbb{C})^{\mathrm{fb}},\mathbb{R}) \longrightarrow \mathbb{R}$$

defined by

$$\overline{c\beta}_d(g_1,\dots,g_d) = \int_{\mathrm{str}(\tilde{o},g_1\tilde{o},\dots,g_1\cdots g_{d-1}\tilde{o})} \mathrm{dbol}$$

for $(g_1,\dots,g_d) \in B\,\mathrm{SL}(n,\mathbb{C})^{\mathrm{fb}}_d$, and

$$\overline{c\beta}_d(p_1,\dots,p_{d-1},c) = \int_{\mathrm{str}(\tilde{o},p_1\tilde{o},\dots,p_1\cdots p_{d-1}\tilde{o},c)} \mathrm{dbol}$$

for $(p_1,\dots,p_{d-1},c) \in B\,\mathrm{SL}(n,\mathbb{C})^{\mathrm{fb}}_d$. By the construction of $\overline{c\beta}_d$, it is obvious that

$$(\mathrm{Bi})^*(\overline{c\beta}_d) = \mathrm{comp}(\beta_d)$$

and hence, $(\mathrm{Bi})^*(\overline{c\beta}_d)$ is a cocycle representing $\mathrm{comp}(b_d)$, where

$$\mathrm{Bi}\colon B\,\mathrm{SL}(n,\mathbb{C}) \longrightarrow B\,\mathrm{SL}(n,\mathbb{C})^{\mathrm{fb}}$$

is the natural inclusion map. The following lemma can be shown by the arguments in the proof of [27, Lemma 7].

Lemma 6.5. *Let*

$$\rho\colon (G,K) \longrightarrow (\mathrm{SL}(n,\mathbb{C}),\mathrm{SU}(n))$$

be a representation. Let

$$j : \Gamma \longrightarrow G$$

be the natural inclusion map. Then we have

$$(B\rho\circ Bj)_*(C^{\mathrm{simp}}_d(B\Gamma^{\mathrm{comp}})) \subset C^{\mathrm{simp}}_d(B\,\mathrm{SL}(n,\mathbb{C})^{\mathrm{fb}}),$$

where

$$B\rho\colon BG^{\mathrm{comp}} \longrightarrow B\,\mathrm{SL}(n,\mathbb{C})^{\mathrm{comp}} \quad \textit{and} \quad Bj : B\Gamma^{\mathrm{comp}} \longrightarrow BG^{\mathrm{comp}}$$

are the induced maps from ρ and j respectively. Furthermore, there exists a constant $c_\rho \in \mathbb{R}$ such that $(B\rho\circ Bj)^*(\overline{c\beta}_d)$ represents $c_\rho \bar{v}_d$.

7. Invariants in group homology and K-theory

In this section we state the construction of $\gamma(N) \in K_d(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$ from [27], which with the results proven so far can now be extended to a $\mathbb{Q}$-rank 1 locally symmetric space $N = \Gamma\backslash G/K$. The following proposition has exactly the same proof as [27, Proposition 1].

Proposition 7.1. *Let $\Gamma \subset \mathbf{G}(\overline{\mathbb{Q}})$ be a $\mathbb{Q}$-rank 1 lattice. Let $N = \Gamma\backslash G/K$ be the interior of the manifold with boundary M. Let*

$$\rho\colon \mathbf{G}(\overline{\mathbb{Q}}) \longrightarrow \mathrm{SL}(n, \overline{\mathbb{Q}})$$

be a representation. Suppose that $\rho(\Gamma_i)$ is unipotent for all $i \in \{1, \dots, s\}$. Then

$$(B_\rho \circ B_j)_* \,\mathrm{EM}_d^{-1}[M, \partial M] \in H_d^{\mathrm{simp}}(B\,\mathrm{SL}(n, \overline{\mathbb{Q}}))^{\mathrm{fb}}, \mathbb{Q})$$

has a preimage

$$\bar{\gamma}(N) \in H_d^{\mathrm{simp}}(B\,\mathrm{SL}(n, \overline{\mathbb{Q}}), \mathbb{Q}),$$

where

$$j\colon \Gamma \longrightarrow \mathbf{G}(\overline{\mathbb{Q}})$$

is the natural inclusion map.

Let A be a subring with unit of the ring of complex numbers. One can regard an element in $H_*^{\mathrm{simp}}(B\,\mathrm{SL}(A), \mathbb{Q})$ as an element in $H_*(|B\,\mathrm{SL}(A)|^+, \mathbb{Q})$ due to the canonical identifications

$$H_*^{\mathrm{simp}}(B\,\mathrm{SL}(A), \mathbb{Q}) \cong H_*(|B\,\mathrm{SL}(A)|, \mathbb{Q}) \cong H_*(|B\,\mathrm{SL}(A)|^+, \mathbb{Q}).$$

By the Milnor-Moore theorem, the Hurewicz homomorphism

$$K_*(A) = \pi_*(|B\,\mathrm{GL}(A)|^+) \longrightarrow H_*(|B\,\mathrm{GL}(A)|^+, \mathbb{Z})$$

gives, after tensoring with $\mathbb{Q}$, an injective homomorphism

$$I_*\colon K_*(A) \otimes \mathbb{Q} = \pi_*(|B\,\mathrm{GL}(A)|^+) \otimes \mathbb{Q} \longrightarrow H_*(|B\,\mathrm{GL}(A)|^+, \mathbb{Q}).$$

Its image consists of primitive elements, denoted by $PH_*(|B\,\mathrm{GL}(A)|^+, \mathbb{Q})$.

By Quillen, inclusion $|B\,\mathrm{GL}(A)| \subset |B\,\mathrm{GL}(A)|^+$ induces an isomorphism

$$Q_*\colon PH_*(|B\,\mathrm{GL}(A)|, \mathbb{Q}) \longrightarrow PH_*(|B\,\mathrm{GL}(A)|^+, \mathbb{Q}) \cong K_*(A) \otimes \mathbb{Q}.$$

Once the projection

$$\mathrm{pr}_*\colon H_*^{\mathrm{simp}}(B\,\mathrm{GL}(A),\mathbb{Q}) \longrightarrow PH_*^{\mathrm{simp}}(B\,\mathrm{GL}(A),\mathbb{Q}) \cong PH_*(|B\,\mathrm{GL}(A)|,\mathbb{Q})$$

is fixed, we can define an element $I_*^{-1} \circ Q_* \circ \mathrm{pr}_*(\alpha) \in K_*(A) \otimes \mathbb{Q}$ for each $\alpha \in H_*^{\mathrm{simp}}(B\,\mathrm{GL}(A),\mathbb{Q})$. For $h \in K_{2m-1}(A) \otimes \mathbb{Q}$, define

$$\langle b_{2m-1}, h\rangle = \langle \mathrm{comp}(b_{2m-1}), Q_{2m-1}^{-1} \circ I_{2m-1}(h)\rangle.$$

In particular, when $A = \overline{\mathbb{Q}}$, note that by [27, Corollary 2] the projection pr_* can be chosen such that

$$\langle \mathrm{comp}(b_{2m-1}), pr_{2m-1}(\alpha)\rangle = \langle \mathrm{comp}(b_{2m-1}), \alpha\rangle$$

for all $m \in \mathbb{N}$. We refer the reader to [27, Section 2.5] for more details about this.

By a proof analogous to [27, Theorem 4] we obtain the following theorem.

Theorem 7.2. *Let G/K be a symmetric space of noncompact type with odd dimension d and $N = \Gamma\backslash G/K$ be a $\mathbb{Q}$-rank 1 locally symmetric space. Let*

$$\rho\colon G \longrightarrow \mathrm{GL}(n,\mathbb{C})$$

be a representation with $\rho_c^* b_d \neq 0$. Suppose that $\rho(\Gamma_i)$ is unipotent for all $i \in \{1,\dots,s\}$. Then there is an element

$$\gamma(N) \in K_d(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$$

such that the application of the Borel class b_d yields

$$\langle b_d, \gamma(N)\rangle = c_\rho \,\mathrm{Vol}(N)$$

for some constant $c_\rho \neq 0$.

Due to $\langle b_d, \gamma(N)\rangle = c_\rho \,\mathrm{Vol}(N) \neq 0$, it can be checked that $\gamma(N)$ is a nontrivial element in $K_d(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$ if $\rho^* b_d \neq 0$. Kuessner [27, Theorem 3] characterizes a complete list of irreducible symmetric spaces G/K of noncompact type and fundamental representation $\rho\colon G \to \mathrm{GL}(n,\mathbb{C})$ with $\rho^* b_{2m-1} \neq 0$ for $d = 2m-1 = \dim(G/K)$. In the noncompact case, he only gets invariants for hyperbolic manifolds since there exist no such fundamental representations for the other $\mathbb{R}$-rank 1 semisimple Lie groups. However, in the list, there are a lot of fundamental representations for higher rank semisimple Lie groups. Theorem 7.2 enables us to get invariants for $\mathbb{Q}$-rank 1 locally symmetric spaces, including hyperbolic manifolds, by using the fundamental representations in the list.

8. Relation to classical Bloch group

In [33], Neumann and Yang constructed an invariant of finite volume hyperbolic 3-manifolds which lie in the Bloch group $\mathcal{B}(\mathbb{C})$. In [28] their construction was generalised to $\mathbb{R}$-rank 1-spaces. Here we choose another approach (via the fundamental class) to give an invariant of a $\mathbb{Q}$-rank 1 locally symmetric space, which coincides with the classical Bloch invariant of cusped hyperbolic 3-manifolds.

Let X be a symmetric space of noncompact type and $C_k(\partial_\infty X)$ the free abelian group generated by $(k+1)$-tuples of points of $\partial_\infty X$ modulo the relations

(1) $(\theta_0, \ldots, \theta_k) = \operatorname{sign}(\tau)(\theta_{\tau(0)}, \ldots, \theta_{\tau(k)})$ for any permutation τ,

(2) $(\theta_0, \ldots, \theta_k) = 0$ whenever $\theta_i = \theta_j$ for some $i \neq j$,

(3) $\partial(\theta_0, \ldots, \theta_k) = \sum_{i=0}^{k} (-1)^i (\theta_0, \ldots, \hat{\theta}_i, \ldots, \theta_k)$

As in [28] the *generalized pre-Bloch group of X* is

$$\mathcal{P}_*(X) = H_*(C_*(\partial_\infty X) \otimes_{\mathbb{Z}G} \mathbb{Z}, \partial \otimes_{\mathbb{Z}G} id)$$

and the *generalized pre-Bloch group of* $\mathbb{C}$ as

$$\mathcal{P}_*^n(\mathbb{C}) = \mathcal{P}_*(\mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n)).$$

Note that $\mathcal{P}_3(H^3_{\mathbb{R}}) = \mathcal{P}^2_3(\mathbb{C})$ is the classical pre-Bloch group $\mathcal{P}(\mathbb{C})$. (Frequently the pre-Bloch group is defined by using the complex of nondegenerate tuples $C_*^{\mathrm{nd}}(\partial_\infty X)$, that is, tuples consisting of pairwise distinct elements. Since $\partial_\infty X$ has infinite cardinality, barycentric subdivision as in [20, Proposition 5.4] shows that the inclusion

$$C_*^{\mathrm{nd}}(\partial_\infty X) \otimes_{\mathbb{Z}G} \mathbb{Z} \longrightarrow C_*(\partial_\infty X) \otimes_{\mathbb{Z}G} \mathbb{Z}$$

yields an isomorphism of homology groups.)

Remark. One may wonder what happens if condition (1) is omitted. Let $\widehat{C}_*(\partial_\infty X)$ be the chain complex analogously defined without condition (1) and

$$\pi : \widehat{C}_*(\partial_\infty X) \longrightarrow C_*(\partial_\infty X)$$

the projection, then (because of condition (ii)) each element of $C_n(\partial_\infty X)$ has $(n+1)!$ preimages and we may define a right-inverse to the projection π by sending each $c \in C_n(\partial_\infty X)$ to the formal sum of its preimages divided by $(n+1)!$. One checks that this defines a chain map. Thus one obtains an injection

$$H_*(C_*(\partial_\infty X) \otimes_{\mathbb{Z}G} \mathbb{Z}) \longrightarrow H_*(\widehat{C}_*(\partial_\infty X) \otimes_{\mathbb{Z}G} \mathbb{Z}).$$

Therefore, each G-equivariant cocycle on $\widehat{C}_*(\partial_\infty X)$ also defines a G-equivariant cocycle on $C_*(\partial_\infty X)$. (One may think of this new cocycle as taking the signed average of the evaluations of the old cocycle on the $(n+1)!$ simplices obtained by permuting the vertices.) In particular, for $\mathbb{R}$-rank one spaces, the algebraic volume algvol yields a well-defined cocycle on $C_*(\partial_\infty X) \otimes_{\mathbb{Z}G} \mathbb{Z}$. (This was not made explicit in [28].)

If $\rho\colon G \to \mathrm{SL}(n,\mathbb{C})$ is a nontrivial (hence reductive) representation, then it induces a smooth map

$$X = G/K \longrightarrow \mathrm{SL}(n,\mathbb{C})/\,\mathrm{SU}(n)$$

and its extension to the ideal boundary

$$\rho_\infty\colon \partial_\infty X \longrightarrow \partial_\infty(\mathrm{SL}(n,\mathbb{C})/\,\mathrm{SU}(n)).$$

Using this one can define a generalized Bloch invariant of a $\mathbb{Q}$-rank 1 locally symmetric space $N = \Gamma\backslash G/K$ as follows. Fix a point $c_0 \in \partial_\infty X$ and define

$$\mathrm{ev}_{\Gamma,c_0,c_1,\dots,c_s}\colon C_*(B\Gamma^{\mathrm{comp}}) \longrightarrow C_*(\partial_\infty X) \otimes_{\mathbb{Z}G} \mathbb{Z}$$

on generators by

$$\mathrm{ev}_{\Gamma,c_0,c_1,\dots,c_s}(\gamma_1,\cdots,\gamma_k) = (c_0,\gamma_1 c_0,\cdots,\gamma_1\cdots\gamma_k c_0)\otimes 1$$

for $\gamma_1,\dots,\gamma_k \in \Gamma$, and

$$\mathrm{ev}_{\Gamma,c_0,c_1,\dots,c_s}(p_1,\dots,p_{k-1},c_i) = (c_0,p_1c_0,\dots p_1\dots p_{k-1}c_0,c_i)\otimes 1$$

for $p_1,\dots,p_{k-1} \in \Gamma_i$. It is straightforward to check that $\mathrm{ev}_{\Gamma,c_0,c_1,\dots,c_s}$ extends linearly to a chain map and thus, it induces a homomorphism

$$(\mathrm{ev}_{\Gamma,c_0,c_1,\dots,c_s})_*\colon H_*^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}},\mathbb{Z}) \longrightarrow \mathcal{P}_*(X).$$

In addition, one can define a map

$$\rho_\infty\colon C_*(\partial_\infty X)\otimes_{\mathbb{Z}G}\mathbb{Z} \longrightarrow C_*(\partial_\infty(\mathrm{SL}(n,\mathbb{C})/\,\mathrm{SU}(n)))\otimes_{\mathbb{Z}\,\mathrm{SL}(n,\mathbb{C})}\mathbb{Z}$$

defined by

$$\rho_\infty((\theta_0,\dots,\theta_k)\otimes 1) = (\rho_\infty(\theta_0),\dots,\rho_\infty(\theta_k))\otimes 1.$$

Hence, it yields a homomorphism

$$(\rho_\infty)_*\colon \mathcal{P}_*(X) \longrightarrow \mathcal{P}_*^n(\mathbb{C}).$$

To obtain an element of $\mathcal{P}^n_*(\mathbb{C})$ for a $\mathbb{Q}$-rank 1 locally symmetric space

$$N = \Gamma\backslash G/K,$$

we need an integer homology class in $H_d^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}}, \mathbb{Z})$. Note that

$$\mathrm{EM}_d^{-1}[M, \partial M] \in H_d^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}}, \mathbb{Q})$$

is a rational homology class. However, it can be easily checked that we can obtain an integer homology class $\mathrm{EM}_d^{-1}[M, \partial M]_{\mathbb{Z}} \in H_d^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}}, \mathbb{Z})$ from the relative fundamental class $[M, \partial M]_{\mathbb{Z}} \in H^d(M, \partial M, \mathbb{Z})$ with integer coefficients as follows. Given a relative fundamental cycle z with integer coefficients,

$$c(z^0) = z^0 + (-1)^{d+1}\,\mathrm{Cone}_g(\partial z^0)$$

defined in the proof of Lemma 6.4 is also a cycle with integer coefficients. For another relative fundamental cycle $\bar{z}$ with integer coefficients, there are chains $w \in C_{d+1}(M, \mathbb{Z})$ and $y \in C_d(\partial M, \mathbb{Z})$ with all vertices in x_0 and such that

$$z^0 - \bar{z}^0 = \partial w^0 + y^0.$$

Then

$$\begin{aligned} c(z^0) - c(\bar{z}^0) &= z^0 - \bar{z}^0 + (-1)^{d+1}(\mathrm{Cone}_g(\partial z^0) - \mathrm{Cone}_g(\partial \bar{z}^0)) \\ &= \partial w^0 + y^0 + (-1)^{d+1}\,\mathrm{Cone}_g(\partial y^0) \\ &= \partial w^0 + y^0 + (-1)^{d+1}(\partial\,\mathrm{Cone}_g(y^0) + (-1)^d y^0) \\ &= \partial(w^0 + (-1)^{d+1}\,\mathrm{Cone}_g(y^0)). \end{aligned}$$

It is obvious that $\Phi(w^0 + (-1)^{d+1}\,\mathrm{Cone}_g(y^0))$ is a chain in $C_{d+1}^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}}, \mathbb{Z})$. Hence $c(z^0)$ determines a homology class in $H_d^{\mathrm{simp}}(B\Gamma^{\mathrm{comp}}, \mathbb{Z})$ independent of the choice of relative fundamental cycle z, denoted by $\mathrm{EM}_d^{-1}[M, \partial M]_{\mathbb{Z}}$.

Definition 8.1. For a $\mathbb{Q}$-rank 1 locally symmetric space N of dimension d, define an element $\beta_\rho(N)$ in the generalized pre-Bloch group $\mathcal{P}^n_d(\mathbb{C})$ by

$$\beta_\rho(N) := (\rho_\infty)_d \circ (\mathrm{ev}_{\Gamma, c_0, c_1, \ldots, c_s})_d \circ \mathrm{EM}_d^{-1}[M, \partial M]_{\mathbb{Z}}.$$

Suppose that every $\rho(\Gamma_i)$ is unipotent. Then the proof of Proposition 7.1 works for $(B_\rho \circ B_j)_* \,\mathrm{EM}_d^{-1}[M, \partial M]_{\mathbb{Z}} \in H_d^{\mathrm{simp}}(B\,\mathrm{SL}(n, \overline{\mathbb{Q}})^{\mathrm{fb}}, \mathbb{Z})$. Thus, it has a preimage $\bar{\gamma}(N)_{\mathbb{Z}} \in H_d^{\mathrm{simp}}(B\,\mathrm{SL}(n, \overline{\mathbb{Q}}), \mathbb{Z})$. In the case that N is a $\mathbb{R}$-rank 1 locally symmetric space, Kuessner [28] showed that

$$(\mathrm{ev}_{\mathrm{SL}(n,\mathbb{C})})_d(\bar{\gamma}(N)_{\mathbb{Z}}) = \beta_\rho(N)$$

where the evaluation map

$$\mathrm{ev}_{\mathrm{SL}(n,\mathbb{C})} \colon C_*^{\mathrm{simp}}(B\,\mathrm{SL}(n,\mathbb{C}),\mathbb{Z}) \longrightarrow C_*(\partial_\infty(\mathrm{SL}(n,\mathbb{C})/\,\mathrm{SU}(n)) \otimes_{\mathbb{Z}\,\mathrm{SL}(n,\mathbb{C})} \mathbb{Z}$$

is defined on generators by

$$ev(g_1,\dots,g_k) = (c_0, g_1c_0,\dots,g_1\dots g_kc_0) \otimes 1.$$

In the same way, this holds for $\mathbb{Q}$-rank 1 locally symmetric spaces. Specially it recovers the classical Bloch invariant of cusped hyperbolic 3-manifolds. For this reason we will call $\beta_\rho(N)$ the *generalized Bloch invariant* for either a compact locally symmetric manifold or a finite volume $\mathbb{Q}$-rank 1 locally symmetric space N.

One advantage of our approach is that one can define the Bloch invariant without the notion of degree one ideal triangulation. Neumann and Yang used the fact that cusped hyperbolic 3-manifolds admit a degree one ideal triangulation to define the classical Bloch invariant of cusped hyperbolic 3-manifolds. Since it is not known whether general locally symmetric spaces admit such a triangulation, it seems to be difficult to extend the definition of the classical Bloch invariant from hyperbolic 3-manifolds to general locally symmetric spaces in the same way that Neumann and Yang constructed it. However, we here use only the relative fundamental cycle $[M,\partial M]_{\mathbb{Z}}$ to define $\beta_\rho(N)$ and moreover, this agrees with the classical Bloch invariant for cusped hyperbolic 3-manifolds [28]. Hence our approach makes it possible to generalize the Bloch invariant for cusped hyperbolic 3-manifolds without the existence of a degree one ideal triangulation. Furthermore, even if one defines an invariant by using a degree one ideal triangulation of N, the invariant should agree with $\beta_\rho(N)$. (This is shown for $\mathbb{R}$-rank 1 spaces in [28, Theorem 4.0.2].)

9. Bloch group for convex projective manifolds

Given a strictly convex real projective manifold N, up to taking a double cover, there is a holonomy map

$$\rho\colon \pi_1(N) = \Gamma \longrightarrow \mathrm{SL}(n,\mathbb{R})$$

where Γ acts on a strictly convex projective domain Ω equipped with a Hilbert metric. If Ω is conic, then the image of ρ is in $\mathrm{SO}(n-1,1)$ and the manifold is a real hyperbolic manifold.

Let's assume that $\dim(N) = 3$. For a given hyperbolic representation

$$\rho_0 \colon \Gamma \longrightarrow \mathrm{SL}(2,\mathbb{C}) \subset \mathrm{SL}(4,\mathbb{R}) \subset \mathrm{SL}(4,\mathbb{C}),$$

assume there is a deformation of ρ_0 to convex projective structures. The inclusion

$$i \colon \mathrm{SL}(2,\mathbb{C}) = \mathrm{SO}(3,1) \subset \mathrm{SL}(4,\mathbb{R}) \subset \mathrm{SL}(4,\mathbb{C})$$

is not the standard one but by [27, Corollary 5] we have $i^* b_3 \neq 0$.

There is a canonical invariant called Bloch invariant developed by Dupont-Sah and others. We want to generalize this notion to projective manifolds.

When Γ acts on the strictly convex domain $\Omega \subset \mathbb{R}^3$, in general $\mathrm{Aut}(\Omega)$ does not have a Lie group structure and we do not have a volume form naturally induced from the Lie group. On the other hands, Ω has a Finsler metric, called Hilbert metric invariant under $\mathrm{Aut}(\Omega)$. Ω with a Hilbert metric behaves like a hyperbolic space, hence one can define a straight simplex. Fix a base point $x \in \Omega$. The volume class $v_d \in H^d(\Gamma,\mathbb{R})$ is defined by the cocycle

$$\nu_d(\gamma_0,\ldots,\gamma_d) = \int_{\mathrm{str}(\gamma_0 x,\ldots,\gamma_d x)} \mathrm{dvol}_\Omega, \tag{1}$$

where dvol_Ω is a signed Finsler volume form on Ω, and str is a geodesic straightening. One can check that it is a cocycle since $\mathrm{str}(\gamma_0 x,\ldots,\gamma_d x)$ is a top dimensional simplex. We have the following lemma similar to Lemma 4.1.

Lemma 9.1. *The volume of the ideal straight simplex* $\mathrm{str}(x_0,\ldots,x_{d-1},c)$ *is finite for any* $x_0,\ldots,x_{d-1} \in \Omega$ *and* c *is a cuspidal point.*

Proof. This follows from the Proposition 11.2 of [11]. □

This lemma allows us to carry out the similar constructions as in previous sections. Hence we can define a cocycle $\bar{\nu}_d \in C^d_{\mathrm{simp}}(B\Gamma^{\mathrm{comp}})$ extending $\mathrm{comp}(\nu_d)$. Let $\bar{v}_d$ denote the element represented by $\bar{\nu}_d$ in $H^d_{\mathrm{simp}}(B\Gamma^{\mathrm{comp}},\mathbb{R})$. By Lemma 6.3,

$$H_d(B\Gamma^{\mathrm{comp}},\mathbb{R}) = H_d(M,\partial M,\mathbb{R}) = \mathbb{R}$$

where M is a compact manifold with boundary whose interior is homeomorphic to N. It is known that for a cusped hyperbolic 3-manifold, there exists a degree one ideal triangulation. We can use the same ideal triangulation to obtain a triangulation of M by Hilbert metric ideal tetrahedra to obtain

$$\langle \bar{v}_d, \mathrm{EM}_d^{-1}[M,\partial M]\rangle = \mathrm{Vol}_{\mathrm{Finsler}}(N).$$

For the Borel class $b_3 \in H_c^3(\mathrm{GL}(\mathbb{C}), \mathbb{R})$, it is known that $\rho^* b_3 \neq 0$ for any nontrivial representation $\rho\colon \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}(\mathbb{C})$. Then for any finite volume hyperbolic manifold $N = \Gamma \backslash H^3_{\mathbb{R}}$, the induced representation

$$\rho_0 \colon \Gamma \longrightarrow \mathrm{GL}(\mathbb{C})$$

gives rise to a nontrivial Borel class

$$\rho_0^* b_3 = c_{\rho_0} \bar{v}_{\rho_0}$$

by Lemma 6.5. Since a strictly convex projective structure $\rho\colon \Gamma \to \mathrm{SL}(4, \mathbb{R})$ lies in the same component containing ρ_0 in the character variety $\chi(\Gamma, \mathrm{SL}(4, \mathbb{C}))$, $(H\rho)_*[M, \partial M]$ is nontrivial, indeed equal to $(H\rho_0)_*[M, \partial M] \in H_3(B\,\mathrm{GL}(\mathbb{C})^\delta, \mathbb{Z})$. Hence

$$\begin{aligned}
\langle b_3, (B\rho)_* \circ \mathrm{EM}^{-1}[M, \partial M] \rangle &= \langle b_3, (B\rho_0)_* \circ \mathrm{EM}^{-1}[M, \partial M] \rangle \\
&= \langle \rho_0^* b_3, \mathrm{EM}^{-1}[M, \partial M] \rangle \\
&= c_{\rho_0} \operatorname{Vol}_{\mathrm{hyp}}(M) \\
&= c_\rho \operatorname{Vol}_{\mathrm{Finsler}}(M).
\end{aligned}$$

Since

$$H^d_{\mathrm{simp}}(B\Gamma^{\mathrm{comp}}, \mathbb{R}) = H_d(B\Gamma^{\mathrm{comp}}, \mathbb{R})^* = \mathbb{R}$$

we get

$$\rho^* b_3 = c_\rho \bar{v}_\rho.$$

If $\sum a_i \tau_i$ is a proper ideal fundamental cycle (i.e. each simplex is a proper ideal straight simplex) of M, the sum of the cross-ratios

$$\beta(M) = \sum a_i [\mathrm{cr}(\tau_i)] \in \mathcal{P}_d(\Omega) := H_3(C_*(\partial\Omega)_\Gamma)$$

defines a generalized Neumann–Yang invariant. Note that if Ω is not conic, the cross-ratio is not a complex number as in $H^3_{\mathbb{R}}$. It is a question how to interpret this invariant in terms of a volume and the Chern–Simon invariant.

10. Bloch invariant in SU(2, 1) and SL(3, $\mathbb{C}$)

10.1. Falbel-Wang invariant. CR structures on 3-manifolds correspond to discrete representations of their fundamental group into SU(2, 1). For example, Falbel constructed a discrete representation

$$\rho\colon \pi_1(S^3 \setminus K) \longrightarrow \mathrm{SU}(2,1) \subset \mathrm{SL}(3, \mathbb{C})$$

which is faithful and parabolic on the torus boundary where K is a figure 8 knot. In [17], Falbel and Wang constructed a Bloch invariant for such representations. Here we prove that their invariant can be computed from $(B\rho)_* \operatorname{EM}^{-1}[M, \partial M]$.

Following [1, Section 3.8] we identify the complex hyperbolic space

$$H^2_{\mathbb{C}} = \mathrm{SU}(2,1)/S(U(2) \times U(1))$$

with $\pi(V_-) \subset \mathbb{C}P^2$, where

$$V_- := \{(x, y, z) \in \mathbb{C}^3 \colon x\bar{z} + y\bar{y} + z\bar{x} < 0\}$$

and

$$\pi \colon \mathbb{C}^3 - \{0\} \longrightarrow \mathbb{C}P^2$$

is the canonical projection. The ideal boundary $\partial_\infty H^2_{\mathbb{C}} \simeq S^3$ is then identified with $\pi(V_0)$, where

$$V_0 := \{(x, y, z) \in \mathbb{C}^3 \colon x\bar{z} + y\bar{y} + z\bar{x} = 0\}.$$

The following construction involves an identification of $\mathbb{C}P^1$ with the set of complex lines through a given point in $\mathbb{C}P^2$. There is some arbitrariness in choosing such an identification, a specific choice is given in [17, Section 2.5] with a derived explicit formula in [17, Definition 2.24].

Definition 10.1 (Falbel–Wang construction, [17, Section 2.5]). a) Using the canonical identification $\partial_\infty H^3_{\mathbb{R}} = \mathbb{C}P^1$ we define a homomorphism

$$FW_{01} \colon C_3^{\mathrm{nd}}(\partial_\infty H^2_{\mathbb{C}}) \longrightarrow C_3^{\mathrm{nd}}(\partial_\infty H^3_{\mathbb{R}})$$

on generators (p_0, p_1, p_2, p_3) of $C_3^{\mathrm{nd}}(\partial_\infty H^2_{\mathbb{C}})$ by

$$FW_{01}(p_0, p_1, p_2, p_3) = (t_0, t_1, t_2, t_3),$$

where we define $t_0 \in \mathbb{C}P^1 = \partial_\infty H^3_{\mathbb{R}}$ to be the complex line through p_0 tangent to $\partial_\infty H^2_{\mathbb{C}} \subset \mathbb{C}P^2$ and for $i = 1, 2, 3$ we define $t_i \in \mathbb{C}P^1 = \partial_\infty H^3_{\mathbb{R}}$ to be the complex line in $\mathbb{C}P^2$ passing through p_0 and p_i.

b) For $a \neq b \in \{0, 1, 2, 3\}$ there are unique $k, l \in \{0, 1, 2, 3\}$ such that the ordered set (a, b, k, l) is an even permutation of $(0, 1, 2, 3)$ and we define

$$FW_{ab} \colon C_3^{\mathrm{nd}}(\partial_\infty H^2_{\mathbb{C}}) \longrightarrow C_3^{\mathrm{nd}}(\partial_\infty H^3_{\mathbb{R}})$$

on generators (p_0, p_1, p_2, p_3) of $C_3^{\mathrm{nd}}(\partial_\infty H^2_{\mathbb{C}})$ by

$$FW_{ab}(p_0, p_1, p_2, p_3) = FW_{01}(p_a, p_b, p_k, p_l).$$

We will use the abbreviation

$$FW := FW_{01} + FW_{10} + FW_{23} + FW_{32} \colon C_3^{\mathrm{nd}}(\partial_\infty H^2_{\mathbb{C}}) \longrightarrow C_3^{\mathrm{nd}}(\partial_\infty H^3_{\mathbb{R}}).$$

Recall that the cross ratio

$$X \colon \mathcal{P}_3^{\mathrm{nd}}(H^3_{\mathbb{R}}) \longrightarrow \mathcal{P}(\mathbb{C})$$

is well defined and yields an isomorphism between $\mathcal{P}_3^{\mathrm{nd}}(H^3_{\mathbb{R}})$ and $\mathcal{P}(\mathbb{C})$. It is proved in [17, Lemma 3.2] that $X(FW(\partial u)) = 0 \in \mathcal{P}(\mathbb{C})$ for all $u \in C_4^{\mathrm{nd}}(\partial_\infty H^2_{\mathbb{C}})_G$. (Following [16, Theorem 5.2], which shows that

$$X(FW(\partial C_4^{\mathrm{nd}}(\partial_\infty H^2_{\mathbb{C}})_G)) \subset \mathbb{Z}[\mathbb{C} - \{0,1\}]$$

is in the subgroup generated by the 5-term relations.) Therefore FW induces a well-defined map

$$FW \colon H_3(C_*^{\mathrm{nd}}(\partial_\infty H^2_{\mathbb{C}})_G) \longrightarrow H_3(C_*^{\mathrm{nd}}(\partial_\infty H^3_{\mathbb{R}})_G) \cong \mathcal{P}(\mathbb{C}).$$

Together with the isomorphism $H_3(C_*^{\mathrm{nd}}(\partial_\infty H^2_{\mathbb{C}})_G) = H_3(C_*(\partial_\infty H^2_{\mathbb{C}})_G)$, given by barycentric subdivision, we obtain a well-defined map

$$FW \colon H_3(C_*(\partial_\infty H^2_{\mathbb{C}})_G) \longrightarrow \mathcal{P}(\mathbb{C}).$$

One may think of the FW_{ab} as maps which send (SU(2, 1)-orbits of) ideal simplices in $H^2_{\mathbb{C}}$ to (SO(3, 1)-orbits of) ideal simplices in $H^3_{\mathbb{R}}$. For degenerate simplices the FW_{ab} are given by performing barycentric subdivision (to produce nondegenerate simplices) and applying the maps to the simplices of the subdivision.

If M is a hyperbolic 3-manifold and $\rho \colon \pi_1 M \to \mathrm{SU}(2,1)$ is a reductive representation (that means $\rho(\pi_1 M) \subset \mathrm{SU}(2,1)$ is a reductive subgroup), then by [12] and [30] we obtain a ρ-equivariant developing map

$$D \colon H^3_{\mathbb{R}} = \widetilde{M} \longrightarrow H^2_{\mathbb{C}}$$

and by [8] a measurable boundary map

$$\partial_\infty D \colon \partial_\infty H^3_{\mathbb{R}} \longrightarrow \partial_\infty H^2_{\mathbb{C}}.$$

In particular, if T is an ideal simplex in M (that is a $\pi_1 M$-orbit of some ideal simplex $\widetilde{T}$ with vertices $v_0, v_1, v_2, v_3 \in \partial_\infty H^3_{\mathbb{R}}$), then we can define $D(T)$ to be the $\pi_1 M$-orbit of the (possibly degenerate) ideal simplex in $H^2_{\mathbb{C}}$, whose vertices are $\partial_\infty D(v_i)$ for $i = 0, 1, 2, 3$.

Definition 10.2 (Falbel–Wang invariant). Let

$$M = \bigcup_{i=1}^{r} T_i$$

be an ideal triangulation of a hyperbolic 3-manifold and

$$\rho\colon \pi_1 M \longrightarrow \mathrm{SU}(2,1)$$

be a reductive representation; then the invariant

$$\beta_{\mathrm{FW}}(M) \in \mathcal{B}(\mathbb{C}) \subset \mathcal{P}(\mathbb{C})$$

is defined by

$$\beta_{\mathrm{FW}}(M) := \sum_{i=1}^{r} X(FW(\mathrm{cr}(D(T_i)))).$$

(It is proved in [17, Theorem 1.1] that $\beta_{\mathrm{FW}}(M)$ lies in $\mathcal{B}(\mathbb{C})$ and does not depend on the ideal triangulation. The latter fact will also follow from our Lemma 10.3.)

Lemma 10.3. *Let M be a finite-volume hyperbolic 3-manifold and*

$$\rho\colon \pi_1 M \longrightarrow \mathrm{SU}(2,1)$$

a reductive representation. Then

$$\beta_{\mathrm{FW}}(M) = X(FW((\mathrm{ev})_*((B\rho)_*(\mathrm{EM}^{-1}[M,\partial M])))).$$

Proof. Let

$$x_0 \in M,$$

$$c_0 \in \partial_\infty \widetilde{M} = \partial_\infty H^3_{\mathbb{R}}$$

and

$$b_0 := \partial_\infty D(c_0) \in \partial_\infty H^2_{\mathbb{C}}.$$

Lemma 3.3.4 in [28] constructs a chain map

$$\hat{C}\colon \hat{C}^{\mathrm{str},x_0}_*(M) \longrightarrow \hat{C}^{\mathrm{str},c_0}_*(M).$$

Let

$$G = \mathrm{SU}(2,1),$$

$$K = \mathrm{S}(\mathrm{U}(2)\times \mathrm{U}(1)),$$

$$\Gamma = \pi_1 M,$$

and

$$\Gamma_i = \pi_1 \partial_i M$$

for the path components $\partial_i M$ of ∂M, c_i the cusps associated to Γ_i and

$$b_i := \partial_\infty D(c_i).$$

We use the commutative diagram

$$\begin{array}{ccc}
C_*(BG^{\mathrm{comp}}) & \xrightarrow{\mathrm{ev}_{b_0}} & C_*(\partial_\infty G/K)_G \\
\uparrow{\scriptstyle B\rho} & & \uparrow{\scriptstyle \partial_\infty D} \\
C_*(B\Gamma^{\mathrm{comp}}) & \xrightarrow{\mathrm{ev}_{c_0}} & C_*(\partial_\infty \widetilde{M})_\Gamma \\
\uparrow{\scriptstyle \hat{\Phi}} & & \uparrow{\scriptstyle cr} \\
\hat{C}_*^{\mathrm{str},x_0}(M) & \xrightarrow{\hat{C}} & \hat{C}_*^{\mathrm{str},c_0}(M) \\
\uparrow{\scriptstyle \mathrm{str}} & & \\
C_*(M \cup \{\Gamma c_1, \ldots, \Gamma c_s\}) & & \\
\uparrow{\scriptstyle \simeq} & & \\
Z_*(\bar{M}, \partial \bar{M}) \longrightarrow C_*\Big(\mathrm{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i \bar{M} \longrightarrow \bar{M}\Big)\Big), & &
\end{array}$$

whose derivation (except for the first square) is explained in the proof of [28, Theorem 4.0.2]. In the first square we have

$$\mathrm{ev}_{b_0}(g_1, \ldots, g_n) = (b_0, g_1 b_0, \ldots, g_1 \ldots g_n b_0) \otimes 1$$

for $(g_1, \ldots, g_n) \in BG$ and

$$\mathrm{ev}_{b_0}(g_1, \ldots, g_{n-1}, b) = (b_0, g_1 b_0, \ldots, g_1 \ldots g_{n-1} b_0, b) \otimes 1$$

for $b \in \partial_\infty G/K$, $(g_1, \ldots, g_{n-1}, b) \in \mathrm{Cone}_b(BG)$, similarly for ev_{c_0}.

The diagram shows that

$$(\mathrm{ev}_{b_0})_*((B\rho)_*(\mathrm{EM}^{-1}[M, \partial M]))$$

is represented by

$$\partial_\infty D(\mathrm{cr}(\hat{C}(\mathrm{str}(z + \mathrm{Cone}(\partial z)))))$$

whenever $z \in Z_*(\bar{M}, \partial \bar{M})$ is a relative fundamental cycle.

If $z \in Z_*(\bar{M}, \partial \bar{M})$ is a relative fundamental cycle, then

$$\operatorname{str}(z + \operatorname{Cone}(\partial z)) \in \hat{C}_*^{\operatorname{str}, x_0}(M)$$

is an ideal fundamental cycle in the sense of [28, Definition 3.1.4]. By [28, Lemma 3.3.4] this implies that $\hat{C}(\operatorname{str}(z + \operatorname{Cone}(\partial z)))$ is an ideal fundamental cycle.

On the other hand, let $M = \bigcup_{i=1}^r T_i$ be an ideal triangulation. Let

$$p_0^i, p_1^i, p_2^i, p_3^i \in \partial_\infty \widetilde{M} = \partial_\infty H_{\mathbb{R}}^3$$

be the ideal vertices of T_i. Then

$$FW(\partial_\infty D(p_0^i)), \quad FW(\partial_\infty D(p_1^i)), \quad FW(\partial_\infty D(p_2^i)), \quad FW(\partial_\infty D(p_3^i))$$

are the ideal vertices of $FW(D(T_i))$. By definition we have

$$\beta_{\mathrm{FW}}(M) = \sum_{i=1}^r X(FW(D(T_i))).$$

We have proved in the proof of [28, Lemma 3.4.1] that the homology classes of $\operatorname{cr}(\hat{C}(\operatorname{str}(z + \operatorname{Cone}(\partial z))))$ and of $\sum_{i=1}^r \operatorname{cr}(T_i)$ are the same in $H_*(C_*(\partial_\infty \widetilde{M})_\Gamma)$. Thus there is some $w \in C_*(\partial_\infty \widetilde{M})_\Gamma$ with

$$\partial w = \operatorname{cr}(\hat{C}(\operatorname{str}(z + \operatorname{Cone}(\partial z)))) - \sum_{i=1}^r \operatorname{cr}(T_i).$$

$\partial_\infty D$ is a chain map by construction. By the remark before Definition 10.2 (following [16, Theorem 5.2]), we have that

$$X(FW(\partial u)) = 0 \in \mathcal{B}(\mathbb{C}), \quad \text{for all } u \in C_4(\partial_\infty H_{\mathbb{C}}^2)_G.$$

Hence we obtain

$$\begin{aligned} &X(FW(\partial_\infty D(\operatorname{cr}(\hat{C}(\operatorname{str}(z + \operatorname{Cone}(\partial z))))))) - \sum_{i=1}^r X(FW(\partial_\infty D(\operatorname{cr}(T_i)))) \\ &\quad = X(FW(\partial \partial_\infty D(w))) \\ &\quad = 0. \end{aligned}$$

Hence the cycle

$$\sum_{i=1}^r X(FW(\partial_\infty D(\operatorname{cr}(T_i))))$$

represents the homology class

$$X(FW(H(\operatorname{ev}_{b_0})(H(\rho)(\mathrm{EM}^{-1}[M, \partial M])))).$$

Because of $\partial_\infty D(\operatorname{cr}(T_i)) = \operatorname{cr}(D(T_i))$ this implies the claim. □

For Corollary 10.4 and Corollary 10.8 we will consider the situation that a 3-manifold M^τ is obtained from another 3-manifold M by cutting along some π_1-injective surface $\Sigma \subset M$ and regluing via $\tau\colon \Sigma \to \Sigma$. If in this situation for a representation $\rho\colon \pi_1 M \to G$ we have some $A \in G$ with $\rho(\tau_* h) = A\rho(h)A^{-1}$ for all $h \in \pi_1\Sigma$, then we get an induced representation $\rho^\tau\colon \pi_1 M^\tau \to G$ by a standard application of the Seifert–van Kampen Theorem as in [29, Section 2]. This representation ρ^τ will be used in the statements of Corollary 10.4 and Corollary 10.8. We say that the representation is parabolics-preserving if it sends $\pi_1\partial M$ to parabolic elements. (It is easy to see from the explicit description in the proof of [29, Proposition 3.1] that ρ^τ is reductive and parabolics-preserving if ρ is.)

Corollary 10.4. *Let M be a compact, orientable 3-manifold, $\Sigma \subset M$ a properly embedded, incompressible, boundary-incompressible, 2-sided surface,*

$$\tau\colon \Sigma \longrightarrow \Sigma$$

an orientation-preserving diffeomorphism of finite order and M^τ the manifold obtained by cutting M along Σ and regluing via τ.

If M and M^τ are hyperbolic and if the reductive, parabolics-preserving representation

$$\rho\colon \pi_1 M \longrightarrow \mathrm{SU}(2,1)$$

satisfies

$$\rho(\tau_*\sigma) = A\rho(\sigma)A^{-1} \quad \text{for some } A \in \mathrm{SU}(2,1) \text{ and all } \sigma \in \pi_1\Sigma,$$

then

$$\beta_{\mathrm{FW}}(M) \otimes 1 = \beta_{\mathrm{FW}}(M^\tau) \otimes 1 \in \mathcal{B}(\mathbb{C}) \otimes \mathbb{Q}$$

with respect to the representations ρ and ρ^τ.

Proof. The proof is essentially the same as the one of Corollary 10.8 below, which in turn is essentially the same as that for [29, Theorem 1]. Therefore we omit the proof at this point and just mention that literally the same argument (just replacing $\mathrm{SL}(3,\mathbb{C})$ by $\mathrm{SU}(2,1)$) as given below in the proof of Corollary 10.8 shows that

$$(B\rho)_*(\mathrm{EM}^{-1}[M,\partial M]_{\mathbb{Q}}) = (B\rho^\tau)_*(\mathrm{EM}^{-1}[M^\tau,\partial M^\tau]_{\mathbb{Q}})$$

and in view of Lemma 10.3 this implies $\beta_{\mathrm{FW}}(M) \otimes 1 = \beta_{\mathrm{FW}}(M^\tau) \otimes 1$. □

10.2. Tetrahedra of flags

Definition 10.5 ([1, Section 2]). Let

$$\mathcal{F}l(\mathbb{C}) = \{([x],[f]) \in P(\mathbb{C}^3) \times P(\mathbb{C}^{3*})\colon f(x) = 0\}$$

where $P(V)$ denotes the projectivization of V, be the configuration space of flags and let

$$C_*^r(\mathcal{F}l(\mathbb{C})) \subset C_*(\mathcal{F}l(\mathbb{C}))$$

be the subcomplex generated by generic configurations

$$(([x_0],[f_0]),\dots,([x_n],[f_n]))$$

(see [1]), that is, those with the x_i in general position and with $f_j(x_i) \neq 0$ for $i \neq j$. For

$$T = (([x_0],[f_0]),([x_1],[f_1]),([x_2],[f_2]),([x_3],[f_3])) \in C_3^r(\mathcal{F}l(\mathbb{C}))$$

and $a \neq b \in \{0,1,2,3\}$ we define $z_{ab} \in \mathbb{C}$ as follows: choose $k,l \in \{0,1,2,3\}$ such that (a,b,k,l) is a positive permutation of $(0,1,2,3)$ and let

$$z_{ab} := \frac{f_a(x_k)\det(x_a,x_b,x_l)}{f_a(x_l)\det(x_a,x_b,x_k)}.$$

Then define

$$\beta\colon C_3^r(\mathcal{F}l) \longrightarrow \mathcal{P}(\mathbb{C})$$

by

$$\beta(T) := [z_{01}] + [z_{10}] + [z_{23}] + [z_{32}].$$

Let

$$H_3(\mathcal{F}l) := H_3(C_*(\mathcal{F}l(\mathbb{C}))_G) \stackrel{\sim}{=} H_3(C_*^r(\mathcal{F}l(\mathbb{C})_G))$$

for the canonical action of

$$G := \mathrm{SL}(3,\mathbb{C})$$

on $\mathcal{F}l(\mathbb{C})$. (The isomorphism is again by barycentric subdivision.) Then [1, Proposition 3.3] implies that β yields a well-defined map

$$\beta_*\colon H_3(\mathcal{F}l) \longrightarrow \mathcal{P}(\mathbb{C}).$$

Moreover, if $M = \Gamma \backslash H^3_{\mathbb{R}}$ is a hyperbolic 3-manifold and

$$h\colon \mathbb{C}P^1 \longrightarrow \mathcal{F}l(\mathbb{C})$$

a map equivariant with respect to *some* homomorphism

$$\Gamma \longrightarrow \mathrm{SL}(3, \mathbb{C}),$$

then one obtains a well-defined chain map

$$h_*\colon C_*(\mathbb{C}P^1)_\Gamma \longrightarrow C_*(\mathcal{F}l(\mathbb{C}))_{\mathrm{SL}(3,\mathbb{C})}.$$

The following definition is due to Bergeron, Falbel, and Guilloux ([1]).

Definition 10.6. If $M = \bigcup_{i=1}^r T_i$ is an ideal triangulation of a hyperbolic 3-manifold, $\rho\colon \pi_1 M \to \mathrm{SL}(3, \mathbb{C})$ a representation and

$$h\colon \mathbb{C}P^1 \longrightarrow \mathcal{F}l(\mathbb{C})$$

a ρ-equivariant map, then define

$$\beta_h(M) := \sum_{i=1}^{r} \beta_*(h_*(P_0^i, P_1^i, P_2^i, P_3^i)) \in \mathcal{P}(\mathbb{C}),$$

where $P_0^i, P_1^i, P_2^i, P_3^i$ are the vertices of T_i.

We remark (compare [7, Proposition 10.79]) that $\mathcal{F}l(\mathbb{C})$ corresponds to the set $\mathrm{SL}(3, \mathbb{C})/P$ of Weyl chambers in $\partial_\infty(\mathrm{SL}(3, \mathbb{C})/\mathrm{SU}(3))$ where P is a minimal parabolic subgroup. If $\rho\colon \pi_1 M \to \mathrm{SL}(3, \mathbb{C})$ is a reductive representation, then there exists a ρ-equivariant harmonic map

$$H^3_{\mathbb{R}} = \widetilde{M} \longrightarrow \mathrm{SL}(3, \mathbb{C})/\mathrm{SU}(3)$$

(see [12],[30]). But it is not easy to prove the existence of a ρ-equivariant boundary map

$$\mathbb{C}P^1 \longrightarrow \partial_\infty(\mathrm{SL}(3, \mathbb{C})/\mathrm{SU}(3)).$$

It should be easier to show the existence of the boundary map

$$h\colon \mathbb{C}P^1 \longrightarrow \mathrm{SL}(3, \mathbb{C})/P = \mathcal{F}l(\mathbb{C})$$

instead. We will not deal with that issue in this paper, we always assume the existence of such a map.

Relation to hyperbolic Bloch invariant, [1, Section 3.7]. If M is an orientable hyperbolic manifold, then by Culler's Theorem its monodromy representation $\Gamma \to \mathrm{PSL}(2,\mathbb{C})$ lifts to $\mathrm{SL}(2,\mathbb{C})$. Composition with the (unique) irreducible representation $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(3,\mathbb{C})$ yields representations $\rho\colon \Gamma \to \mathrm{SL}(3,\mathbb{C})$. In this case there is a canonically (independent of Γ) defined ρ-equivariant map $\mathbb{C}P^1 \to \mathcal{F}l(\mathbb{C})$ as follows.

Recall that the irreducible 3-dimensional representation of $\mathrm{SL}(2,\mathbb{C})$ can be defined as follows. Consider the $\mathbb{C}$-vector space of complex homogeneous polynomials of degree 2 in two variables. This is a 3-dimensional vector space V generated by x^2, xy and y^2. $\mathrm{SL}(2,\mathbb{C})$ acts by

$$(AP)(x,y) := P(A^{-1}(x,y)).$$

We may consider its projectivization $P(V)$ and the projectivization of the dual space $P(V^*)$, whose elements we will write as homogeneous column vectors. Then a ρ-equivariant map

$$h\colon \mathbb{C}P^1 \longrightarrow \mathcal{F}l(\mathbb{C}) \subset P(V) \times P(V^*)$$

is given by

$$h([x,y]) := \Big([x^2, xy, y^2], \Big[\frac{1}{2}y^2, -xy, \frac{1}{2}x^2\Big]^T\Big).$$

([1] gives an apparently different construction which however – after computation – this yields the same map h.) It turns out that the so-defined $\beta_h(M)$ coincides with 4 times the usual Bloch invariant $\beta(M)$ of the hyperbolic 3-manifold M. Indeed, if $T = (P_0, P_1, P_2, P_3) \in C_3(\mathbb{C}P^1)$ is an ideal simplex of cross ratio t, then explicit computation shows that the simplex

$$h(T) = (h(P_0), h(P_1), h(P_2), h(P_3)) \in C_3(\mathcal{F}l)$$

satisfies

$$z_{01}(h(T)) = z_{10}(h(T)) = z_{23}(h(T)) = z_{32}(h(T)) = t.$$

Relation to CR Bloch invariant, [1, Section 3.8]. If D is the developing map of a reductive representation $\pi_1 M \to \mathrm{SU}(2,1)$ and h is the composition of $\partial_\infty D$ with the map $S^3 \to \mathcal{F}l(\mathbb{C})$ given in [1, Section 3.8], then $\beta_h(M)$ coincides with $\beta_{\mathrm{FW}}(M)$.

Lemma 10.7. *Let M be a finite-volume hyperbolic 3-manifold, and let*

$$h\colon \mathbb{C}P^1 \longrightarrow \mathcal{F}l(\mathbb{C})$$

equivariant with respect to some homomorphism

$$\pi_1 M \longrightarrow \mathrm{SL}(3,\mathbb{C}).$$

Then

$$\beta_h(M) = \beta_* h_* \,\mathrm{ev}_* \,\mathrm{EM}^{-1}[M, \partial M].$$

Proof. Let $c_0 \in \partial_\infty H^3_{\mathbb{R}}$. The commutative diagram in the proof of Lemma 10.3 shows that $(\mathrm{ev}_{c_0})_* \,\mathrm{EM}^{-1}[M, \partial M]$ is represented by $\mathrm{cr}(\hat{C}(\mathrm{str}(z + \mathrm{Cone}(\partial z))))$ whenever $z \in Z_*(\bar{M}, \partial \bar{M})$ is a relative fundamental cycle.

In the proof of Lemma 10.3 we have seen that the homology classes of $\mathrm{cr}(\hat{C}(\mathrm{str}(z + \mathrm{Cone}(\partial z))))$ and of $\sum_{i=1}^r \mathrm{cr}(T_i)$ are the same in $H_*(C_*(\partial_\infty \widetilde{M})_\Gamma)$, whenever $M = \bigcup_{i=1}^r T_i$ is an ideal triangulation. Thus we deduce that there is some $w \in C_*(\partial_\infty \widetilde{M})_\Gamma$ with

$$\partial w = \mathrm{cr}(\hat{C}(\mathrm{str}(z + \mathrm{Cone}(\partial z)))) - \sum_{i=1}^r \mathrm{cr}(T_i).$$

h_* is a chain map by construction. Moreover, by [1, Proposition 3.3] (following from [16, Theorem 5.2]) we have that β maps boundaries to zero, thus

$$\beta(\partial h_*(w)) = 0,$$

which implies

$$\beta(h_*(\mathrm{cr}(\hat{C}(\mathrm{str}(z + \mathrm{Cone}(\partial z)))))) - \sum_{i=1}^r \beta(h_*(\mathrm{cr}(T_i))) = \beta(\partial h_*(w)) = 0.$$

Thus the cycle $\sum_{i=1}^r \beta(h_*(\mathrm{cr}(T_i)))$ represents the homology class

$$\beta_* h_* (\mathrm{ev}_{c_0})_* \,\mathrm{EM}^{-1}[M, \partial M],$$

which implies the claim.

□

For the following corollary we will use the notations introduced before in Corollary 10.4. The following corollary applies for example when M^τ is a (generalized) mutation of M and $\rho\colon \pi_1 M \to \mathrm{SL}(3,\mathbb{C})$ is the composition of the inclusion $\pi_1 M \subset \mathrm{SL}(2,\mathbb{C})$ with some representation $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(3,\mathbb{C})$. In this case ρ^τ is obtained from the composition of the inclusion $\pi_1 M^\tau \subset \mathrm{SL}(2,\mathbb{C})$ (see [29, Section 2]) with the same representation $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(3,\mathbb{C})$ and we have $h = h^\tau$.

Corollary 10.8. *Let M be a compact, orientable 3-manifold, $\Sigma \subset M$ a properly embedded, incompressible, boundary-incompressible, 2-sided surface,*

$$\tau \colon \Sigma \longrightarrow \Sigma$$

an orientation-preserving diffeomorphism of finite order and M^τ the manifold obtained by cutting M along Σ and regluing via τ.

If M and M^τ are hyperbolic and if the parabolics-preserving representation

$$\rho \colon \pi_1 M \longrightarrow \mathrm{SL}(3, \mathbb{C})$$

satisfies

$$\rho(\tau_* \sigma) = A \rho(\sigma) A^{-1}$$

for some $A \in \mathrm{SL}(3, \mathbb{C})$ and all $\sigma \in \pi_1 \Sigma$, then

$$\beta_h(M) \otimes 1 = \beta_{h^\tau}(M^\tau) \otimes 1 \in \mathcal{B}(\mathbb{C}) \otimes \mathbb{Q}$$

when h and h^τ are ρ- resp. ρ^τ-equivariant maps from $\partial_\infty H^3_{\mathbb{R}}$ to $\mathcal{F}l$.

Proof. The proof is essentially the same as for [29, Theorem 1]. Since Σ is a 2-sided, properly embedded surface, it has a neighborhood $N \simeq \Sigma \times [0,1]$ in M, and a neighborhood $N^\tau \simeq \Sigma \times [0,1]$ in M^τ. The complements $M - \mathrm{int}(N)$ and $M^\tau - \mathrm{int}(N^\tau)$ are diffeomorphic and we let X be the union of M and M^τ along this identification of $M - \mathrm{int}(N)$ and $M^\tau - \mathrm{int}(N^\tau)$. The union of N and N^τ yields a copy of the mapping torus T^τ in X. We have

$$i_{M*}[M, \partial M] - i_{M^\tau *}[M^\tau, \partial M^\tau] = i_{T^\tau *}[T^\tau, \partial T^\tau] \in H_3(X, \partial X, \mathbb{Z}). \tag{2}$$

The made assumption implies that ρ and ρ^τ extend to a representation

$$\rho_X : \pi_1 X \longrightarrow \mathrm{SL}(3, \mathbb{C}).$$

As in the proof of [29, Theorem 1] we have a finite cyclic covering $\widehat{X} \to X$ such that $\widehat{X}$ contains a copy of $\Sigma \times S^1$ finitely covering $T^\tau \subset X$. Let $\widehat{M}, \widehat{M}^\tau \subset \widehat{X}$ be the preimages of M and M^τ. Application of the transfer map yields

$$i_{\widehat{M}*}[\widehat{M}, \partial \widehat{M}] - i_{\widehat{M}^\tau *}[\widehat{M}^\tau, \partial \widehat{M}^\tau] = i_{\Sigma \times \mathbf{S}^1 *}[\Sigma \times \mathbf{S}^1, \partial \Sigma \times \mathbf{S}^1]. \tag{3}$$

Again as in the proof of [29, Theorem 1] we obtain a representation

$$\rho_{\widehat{X}} \colon \pi_1 \widehat{X} \longrightarrow \mathrm{SL}(3,\mathbb{C})$$

and – because the lift $\widehat{X}$ is chosen such that $\rho_{\widehat{X}}(\pi_1 \partial \widehat{X})$ consists of parabolics – a continuous map

$$B\rho_{\widehat{X}} \colon (B\pi_1 \widehat{X})^{\mathrm{comp}} \longrightarrow B\,\mathrm{SL}(3,\mathbb{C})^{\mathrm{comp}}.$$

The classifying map $\Psi_{\widehat{X}} \colon \widehat{X} \to |B\pi_1 \widehat{X}|$ extends to

$$\Psi_{\widehat{X}} \colon \operatorname{Dcone}\Big(\bigcup_{i=1}^{s} \partial_i \widehat{X} \longrightarrow \widehat{X}\Big) \longrightarrow |(B\pi_1 \widehat{X})^{\mathrm{comp}}|$$

and the same argument as in [29] shows that $(|B\rho_{\widehat{X}}|\Psi_{\widehat{X}} i_{\Sigma\times \mathbf{S}^1})_*$ factors over $H_3(\Sigma,\partial\Sigma)=0$ and is therefore 0. Thus Equation 3 implies

$$\begin{aligned}(|B\rho_{\widehat{X}}|\Psi_{\widehat{X}} i_{\widehat{M}})_*[\widehat{M},\partial\widehat{M}] = ((|B\rho_{\widehat{X}}|\Psi_{\widehat{X}} i_{\widehat{M}^\tau})_*[\widehat{M}^\tau,\partial\widehat{M}^\tau]\\ \in H_3(|B\,\mathrm{SL}(3,\mathbb{C})^{\mathrm{comp}}|).\end{aligned}$$

Again following the same argument from [29] we conclude

$$(B\rho)_*(\mathrm{EM}^{-1}[M,\partial M]_{\mathbb{Q}}) = (B\rho^\tau)_*(\mathrm{EM}^{-1}[M^\tau,\partial M^\tau]_{\mathbb{Q}}).$$

The ρ-equivariance of h implies that

$$h \circ \mathrm{ev}_{\mathrm{SL}(2,\mathbb{C})} = \mathrm{ev}_{\mathrm{SL}(3,\mathbb{C})} \circ \rho,$$

hence

$$h_*\,\mathrm{ev}_* = \mathrm{ev}_*(B\rho)_*$$

and thus

$$\begin{aligned}\beta_h(M)\otimes 1 &= \beta_* h_*\,\mathrm{ev}_*\,\mathrm{EM}^{-1}[M,\partial M]_{\mathbb{Q}}\\ &= \beta_*\,\mathrm{ev}_*(B\rho)_*(\mathrm{EM}^{-1}[M,\partial M]_{\mathbb{Q}})\\ &= \beta_*\,\mathrm{ev}_*(B\rho^\tau)_*(\mathrm{EM}^{-1}[M^\tau,\partial M^\tau]_{\mathbb{Q}})\\ &= \beta_* h_*\,\mathrm{ev}_*\,\mathrm{EM}^{-1}[M^\tau,\partial M^\tau]_{\mathbb{Q}}\\ &= \beta_h(M^\tau)\otimes 1\end{aligned}$$

in view of Lemma 10.7. □

The so-called Bloch regulator map

$$\rho\colon \mathcal{B}(\mathbb{C}) \longrightarrow \mathbb{C}/\mathbb{Q}$$

is known to send the Bloch invariant $\beta(M)$ of hyperbolic 3-manifolds to

$$\frac{i}{2\pi^2}(\mathrm{Vol}(M) + i\,\mathrm{CS}(M)) \mod \mathbb{Q},$$

as was proved in [33, Theorem 1.3]. In other words, the imaginary part of $\rho(\beta(M))$ determines the volume (and the real part determines the Chern–Simons invariant mod $\mathbb{Q}$). Thus it is natural to define the volume of flag structures as (a multiple of) the imaginary part of $\rho(\beta_h(M))$. Bergeron, Falbel, and Guilloux in fact define in [1, Section 3.6] the volume of a flag structure to be $\frac{2\pi^2}{4}\,\mathrm{Im}(\rho(\beta_h(M)))$. The analogously defined volume of CR structures is always zero by [17, Theorem 3.12] but the volume of flag structures is a nontrivial and potentially interesting invariant. Corollary 10.4 of course implies its invariance under the cut-and-paste operation described in the statement of the corollary.

References

[1] N. Bergeron, E. Falbel, and A. Guilloux, Tetrahedra of flags, volume and homology of SL(3). *Geom. Topol.* **18** (2014), no. 4, 1911–1971. Zbl 06356603 MR 3268771

[2] A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compactes. *Ann. of Math.* (2) **57** (1953), 115–207. Zbl 0052.40001 MR 0051508

[3] A. Borel, *Introduction aux groupes arithmétiques.* Publications de l'Institut de Mathématique de l'Université de Strabourg, XV. Actualités Scientifiques et Industrielles, 1341. Hermann, Paris, 1969. Zbl 0186.33202 MR 0244260

[4] A. Borel, Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup.* (4) **7** (1974), 235–272. Zbl 0316.57026 MR 0387496

[5] A. Borel and L. Ji, *Compactifications of symmetric and locally symmetric spaces.* Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 2006. Zbl 1100.22001 MR 2189882

[6] G. E. Bredon, *Topology and geometry.* Graduate Texts in Mathematics, 139. Springer, New York, 1993. Zbl 0791.55001 MR 1224675

[7] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature.* Grundlehren der Mathematischen Wissenschaften, 319. Springer, Berlin, 1999. Zbl 0988.53001 MR 1744486

[8] M. Burger and S. Mozes, CAT(-1)-spaces, divergence groups and their commensurators. *J. Amer. Math. Soc.* **9** (1996), no. 1, 57–93. Zbl 0847.22004 MR 1325797

[9] J. I. Burgos, *The regulators of Beilinson and Borel.* CRM Monograph Series, 15. American Mathematical Society, Providence, R.I., 2002. Zbl 0994.19003 MR 1869655

[10] H. Cartan, La transgession dans un groupe de Lie et dans un espace fibré principal. *Colloque de topologie* (*epaces fibrés*). Masson et Cie, 1950, 57–71.

[11] D. Cooper, D. Long, and S. Tillmann, On convex projective manifolds and cusps. *Adv. Math.* **277** (2015), 181–251. Zbl 3336086 MR 06431971

[12] K. Corlette, Flat G-bundles with canonical metrics. *J. Differential Geom.* **28** (1988), no. 3, 361–382. Zbl 0676.58007 MR 0965220

[13] T. Dimofte, M. Gabella, and A. Goncharov, K-decompositions and 3d gauge theories. Preprint 2013. arXiv:1301.0192 [hep-th]

[14] J. L. Dupont and H. Sah, *Scissors congruences.* II. *J. Pure Appl. Algebra* **25** (1982), no. 2, 159–195. Zbl 0496.52004 MR 0662760

[15] P. B. Eberlein, *Geometry of nonpositively curved manifolds.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1996. Zbl 0883.53003 MR 1441541

[16] E. Falbel, A volume function for spherical CR tetrahedra. *Q. J. Math.* **62** (2011), no. 2, 397–415. Zbl 1238.32026 MR 2805210

[17] E. Falbel and Q. Wang, A combinatorial invariant for spherical CR structures. *Asian J. Math.* **17** (2013), no. 3, 391–422. Zbl 1296.57013 MR 3119793

[18] W. Fulton and J. Harris, *Representation theory.* A first course. Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer, New York, 1991. Zbl 0744.22001 MR 1153249

[19] S. Garoufalidis, M. Goerner, and C. Zickert, Gluing equations for $\mathrm{PGL}(n,\mathbb{C})$-representations of 3-manifolds. *Algebr. Geom. Topol.* **15** (2015), no. 1, 565–622. Zbl 06425414 MR 3325748

[20] S. Garoufalidis, D. Thurston, and C. Zickert, The complex volume of $\mathrm{SL}(n,\mathbb{C})$-representations of 3-manifolds. *Duke Math. J.* **164** (2015), no. 11, 2099–2160. MR 3385130

[21] A. Goncharov, Volumes of hyperbolic manifolds and mixed Tate motives. *J. Amer. Math. Soc.* **12** (1999), no. 2, 569–618. Zbl 0919.11080 MR 1649192

[22] V. Guillemin and A. Pollack, *Differential topology.* Prentice-Hall, Englewood Cliffs, N.J., 1974. Zbl 0361.57001 MR 0348781

[23] T. Hartnick and A. Ott. Surjectivity of the comparison map in bounded cohomology for Hermitian Lie groups. *Int. Math. Res. Not. IMRN* **2012** (2012), no. 9, 2068–2093. Zbl 1246.22013 MR 2920824

[24] A. Hatcher, *Algebraic topology.* Cambridge University Press, Cambridge, 2002. Zbl 1044.55001 MR 1867354

[25] T. Hattori, Geometric limit sets of higher rank lattices. *Proc. London Math. Soc.* (3) **90** (2005), no. 3, 689–710. Zbl 1077.22015 MR 2137827

[26] Th. Kuessner, Generalizations of Agol's inequality and nonexistence of tight laminations. *Pacific J. Math.* **251** (2011), no. 1, 109–172. Zbl 1221.57037 MR 2794617

[27] Th. Kuessner, Locally symmetric spaces and K-theory of number fields. *Algebr. Geom. Topol.* **12** (2012), no. 1, 155–213. Zbl 1271.57062 MR 2916273

[28] Th. Kuessner, Group homology and ideal fundamental cycles. *Topology Proc.* **40** (2012), 239–258. Zbl 1261.57025 MR 2854083

[29] Th. Kuessner, Mutation and recombination for hyperbolic 3-manifolds. *J. Gökova Geom. Topol. GGT* **5** (2011), 20–30. Zbl 06182833 MR 2872549

[30] F. Labourie, Existence d'applications harmoniques tordues à valeurs dans les variétés à courbure négative. *Proc. Amer. Math. Soc.* **111** (1991), no. 3, 877–882. Zbl 0783.58016 MR 1049845

[31] G. A. Margulis, *Discrete subgroups of semisimple Lie groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 17. Springer, Berlin etc., 1991. Zbl 0732.22008 MR 1090825

[32] W. Neumann, Realizing arithmetic invariants of hyperbolic 3-manifolds. In A. Champanerkar, O. Dasbach, E. Kalfagianni, I. Kofman, W. Neumann and N. Stoltzfus (eds.), *Interactions between hyperbolic geometry, quantum topology and number theory.* Contemporary Mathematics, 541. American Mathematical Society, Providence, R.I., 2011. 233–246. Zbl 1237.57013 MR 2796636

[33] W. Neumann and J. Yang, Bloch invariants of hyperbolic 3-manifolds. *Duke Math. J.* **96** (1999), no. 1, 29–59. Zbl 0943.57008 MR 1663915

[34] D. Morris, Introduction to arithmetic groups. Deductive Press, 2015. Zbl 06444012 http://members.shaw.ca/deductivepress/IntroArithGrps-FINAL.pdf

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