Boundary values, random walks, and ℓ^p-cohomology in degree one

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Abstract. The vanishing of reduced ℓ^2-cohomology for amenable groups can be traced to the work of Cheeger and Gromov in [10]. The subject matter here is reduced ℓ^p-cohomology for $p \in]1, \infty[$, particularly its vanishing. Results for the triviality of $\underline{\ell^p H}^1(G)$ are obtained, for example: when $p \in]1, 2]$ and G is amenable; when $p \in]1, \infty[$ and G is Liouville (*e.g.* of intermediate growth).

This is done by answering a question of Pansu in [34, §1.9] for graphs satisfying certain isoperimetric profile. Namely, the triviality of the reduced ℓ^p-cohomology is equivalent to the absence of non-constant harmonic functions with gradient in ℓ^q (q depends on the profile). In particular, one reduces questions of non-linear analysis (p-harmonic functions) to linear ones (harmonic functions with a very restrictive growth condition).

Contents

1. Introduction

A graph $\Gamma = (X, E)$ is defined by X, its set of vertices, and E, its set of edges. All graphs will be assumed to be of bounded valency. The set of edges will be thought of as a subset of $X \times X$. The subject matter is the reduced ℓ^p-cohomology in degree one of the graph Γ. This is the quotient

$$\underline{\ell^p H}^1(\Gamma) := \mathsf{D}^p(\Gamma)/\overline{\ell^p(X) + \mathbb{K}}^{\mathsf{D}^p},$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is the field where our functions take values (this choice plays no role). See subsection §2.1 for more details. The main goal of this paper is to give partial answers to a question (dating back at least to Gromov [20, §8.A_1.(A_2), p.226]):

Question 1.1. *Let G be an amenable group, is it true that for one* (*and hence all*) *Cayley graph Γ and all $1 < p < \infty$, $\underline{\ell^p H}^1(\Gamma) = 0$?*

The original question concerns cohomology in all degrees. The case $p = 1$ is slightly singular and the case $p = \infty$ is trivially false (see appendix A). For $p = 2$, the positive answer is a famous result of Cheeger and Gromov [10] (see also Lück's book [29]). The results presented here will give a positive answer for all $p \in]1, 2]$ and covers (for all p) many cases.

The basic idea relies on a standard argument which shows that a function which (essentially) only takes one value at infinity has trivial cohomology class. With a few more efforts, this determines $\underline{\ell^1 H}^1$ via boundary values on the end (see appendix A). The idea is to define a "boundary value" of $g \in \mathsf{D}^p(\Gamma)$ on another ideal boundary, namely the Poisson boundary. This boundary is made up by harmonic functions, so a natural candidate for this boundary value is $\lim_{n \to \infty} P^{(n)}g$ where P is the random walk operator.

The convergence of this limit can be obtained as a consequence of return probability (or heat kernels) estimates. To see that the behaviour of the boundary value still says something about the behaviour of g at infinity, a transport problem (between a Dirac measure and the time n distribution of a simple random walk) has to be studied. Some hypothesis on the isoperimetric profile will be necessary. For $F \subset X$ a subset of the vertices, let ∂F be the edges between F and F^c. Let $d \in \mathbb{R}_{\geq 1}$. Then, a graph Γ has

IS$_d$ if there is a $\kappa > 0$ such that for all finite $F \subset X, |F|^{(d-1)/d} \leq \kappa |\partial F|$;

IS$_\omega$ if there is a $\kappa > 0$ such that for all finite $F \subset X, |F| \qquad \leq \kappa |\partial F|$.

Quasi-homogeneous graphs with a certain (uniformly bounded below) volume growth in n^d will satisfy these isoperimetric profiles, see Woess' book [48, (4.18) Theorem]. The Cayley graph of a group G does not satisfy IS_ω if and only if G is amenable. It satisfies IS_d *for all d* if and only if G is not virtually nilpotent. The upcoming result will apply best to groups which are not virtually nilpotent. See §2.1 or [48, §14] for more details.

Theorem 1.2. *Let Γ be a graph satisfying IS_d and let $1 \leq q \leq p < d/2$. Then*

- *the natural quotient*

$$\underline{\ell^q H}^1(\Gamma) \longrightarrow \underline{\ell^p H}^1(\Gamma)$$

  is an injection;

- *if Γ has no non-constant bounded harmonic functions whose gradient is in $\ell^p(E)$ then*

$$\underline{\ell^q H}^1(\Gamma) = \{0\}, \quad \text{for all } q < \frac{pd}{p + 2d};$$

- *if Γ has a non-constant (*bounded or not*) harmonic functions whose gradient is in $\ell^p(E)$ then*

$$\underline{\ell^p H}^1(\Gamma) \neq \{0\}.$$

More precisely, a map from $\mathsf{D}^p(\Gamma)$ to harmonic functions modulo constants on Γ is exhibited, and it is shown does not depends on the representative of the reduced ℓ^p-cohomology class ($1 \leq p < \infty$). This maps sends bounded functions to bounded functions. To establish vanishing of reduced ℓ^p-cohomology in degree 1, it is sufficient to consider only bounded functions (for $1 < p < \infty$), thanks to a lemma of Holopainen and Soardi [22, Lemma 4.4] (see Lemma 2.1 for a simple proof).

Theorem 1.2 almost answers a question of Pansu [34, Question 6 in §1.9]: if Γ has IS_d for all d, is the existence of a non-constant harmonic form whose gradient is in $\ell^p(E)$ equivalent to non-trivial reduced ℓ^p-cohomology in degree 1? Theorem 1.2 shows this holds if one allows to lose some regularity (q is bigger than p). This theorem is the compilation of Corollaries 3.10, 3.11 and 3.14.

Note that though the current methods are not sufficient to show the existence of a harmonic function with finite ℓ^p gradient in each reduced cohomology class, it is an easy consequence of the methods that if such a function exists (and the isoperimetric profile is nice enough), then it is unique up to a constant, see Remark 4.2.

Recall that all groups of subexponential growth are Liouville (see Avez [2]), *i.e.* the Poisson boundary associated to the simple random walk on the Cayley graph is trivial.

Corollary. *If G is a group of growth at least polynomial of degree d and one of its Cayley graphs has trivial Poisson boundary (for the simple random walk, **i.e.** G is Liouville), then $\underline{\ell^p H}^1(\Gamma) = \{0\}$ for any $1 \le p < d/2$.*

In particular, groups of intermediate growth and $\mathbb{Z}_2 \wr \mathbb{Z}^2$ have trivial reduced ℓ^p-cohomology in degree 1, for any $p \in [1, \infty[$.

Note it is unknown whether being Liouville is an invariant of quasi-isometry for Cayley graphs (it is not even known whether it is possible for a group to have a non-Liouville Cayley graph and a Liouville one). But, for the current purposes, it actually suffices that G has a Cayley graph quasi-isometric to a Liouville graph in order to have that its ℓ^p-cohomology vanish for all $p \in]1, \infty[$.

Using [16, Theorem 1.3] and Theorem 1.2 one can show that [amenable] lamp-lighters on $\mathbb{Z}^d$ (*e.g.* $\mathbb{Z}_2 \wr \mathbb{Z}^d$) have harmonic functions with gradient in ℓ^p. See also [17] for more results on semi-direct products $G = N \rtimes H$ where N is not finitely generated but G is.

The reduced ℓ^2-cohomology is trivial by Cheeger and Gromov [10] (and the ℓ^p-cohomology is trivial for any $p \in]1, \infty[$ for groups which *do not* have IS_d for some d, see §2.2 below), using Theorem 1.2 one gets a positive answer to Question 1.1:

Corollary. ***Any*** *finitely generated amenable groups has trivial reduced ℓ^p-cohomology (in degree one) for all $p \in]1, 2]$. Virtually-$\mathbb{Z}$ groups are the only amenable groups with non-trivial reduced ℓ^1-cohomology in degree 1.*

The Poisson boundary is not an invariant of quasi-isometry (see, for example, T. Lyons' examples [30]). However, the following corollary, which was known for $p = 2$ (trivially), may now be extended:

Corollary. *If Γ is a graph satisfying IS_d and $\underline{\ell^p H}^1(\Gamma) \neq \{0\}$ for some $1 \le p < d/2$, then any graph quasi-isometric to Γ has non-trivial Poisson boundary.*

Indeed, if Γ satisfies IS_d and given that $p < d/2$, then inside the Poisson boundary lives the image (by boundary values) of the non-trivial reduced ℓ^p-cohomology class representable by bounded functions; denote this image $\mathcal{P}_p$. Furthermore, if

$$D = \sup_{\Gamma \text{ has } \mathrm{IS}_d} d \in [1, \infty],$$

then

$$\mathcal{P} = \bigcup_{p < D/2} \mathcal{P}_p$$

is a part of the Poisson boundary which will persist under quasi-isometry. One could also try to thicken $\mathcal{P}$ by considering more generic Banach spaces (*e.g.* Orlicz spaces).

The second result is but a consequence of the first.

Theorem 1.3. *If Γ has IS_ω with constant κ, then the cohomology is always reduced (i.e. $\underline{\ell^p H}^1(\Gamma) = \ell^p H^1(\Gamma)$). Furthermore, if $n = \lceil \kappa^{-1} \rceil$ and $\Gamma^{[n]}$ is the n-fuzz of Γ (the graph obtained from Γ by adding edges between all points at distance n in Γ), then there is a spanning tree T in $\Gamma^{[n]}$ so that the non-trivial cohomology classes are exactly those non-trivial cohomology class in T which also belong to $\mathsf{D}^p(\Gamma^{[n]})$.*

The equality between reduced and unreduced cohomology was already known. For Cayley graphs, see Guichardet [21, Corollaire 1] or Martin and Valette [31, Corollary 2.4]): a group is amenable if and only if $\underline{\ell^p H}^1(G) \neq \ell^p H^1(G)$, for some, and hence any, $p \in]1, \infty[$. For more general graphs, it is implicit at least in Lohoué [28].

In the author's mind, the interest of this results lies in the following idea: to compute the ℓ^p cohomology of a graph with positive isoperimetric constant, one needs only to run through the list of boundary values for a spanning tree, and look which one are in D^p of the initial graph. Of course, even if boundary values of the tree are somehow much more reasonable to compute (either by the methods of Bourdon and Pajot in [9] or as the harmonic functions associated to the random walk), this is probably not directly usable unless the spanning tree produced by Benjamini and Schramm in [5] may be made explicit.

Some results on $\ell^{p,q}$-cohomology are presented in §4.2.

Using Theorem 1.2, it is possible to get a vanishing result for groups with normal subgroups:

Theorem 1.4. *Let $p \in [1, \infty[$. Assume G is a finitely generated group, $N \lhd G$ is finitely generated as a group and the growth of N is at least polynomial of degree $> 2p$. Assume further that G/N is infinite and $\underline{\ell^p H}^1(\Gamma_N) = \{0\}$, where Γ_N is some Cayley graph of N. Then for all Cayley graphs Γ_G of G, $\underline{\ell^p H}^1(\Gamma_G) = 0$. If further N is non-amenable then the statement is true in unreduced cohomology.*

The case where N is non-amenable was already done in Bourdon, Martin and Valette [8, Theorem 1,1)]. Bourdon in [7, paragraph **4)** in §1.6] has given a very nice example showing sharpness of the previous statement: there is a group Γ with $\underline{\ell^p H}^1(\Gamma) \neq \{0\}$ for $p > 2$ and an exact sequence $1 \to N \to \Gamma \to \mathbb{Z} \to 1$ where N has IS_ω and $\underline{\ell^p H}^1(N) \neq \{0\}$ for $p > 2$.

All of this (as well as [16] and [17]) seems to support a positive answer to 1.1. Here is a probably easier question which should shed more light on this topic:

Question 1.5. *Is there an amenable group G so that its Cayley graph has non-constant harmonic functions with gradient in c_0?*

A negative answer would give a positive answer to Question 1.1. Note that this condition is much more restrictive than asking for harmonic functions of, say, sublinear growth. In fact, an answer to this question for G solvable would already be interesting. Indeed, as pointed out by G. Kozma, some wreath products have harmonic functions of sublinear growth.

Another tempting path to a positive answer to Question 1.1, comes from Corollary 4.1:

Question 1.6. *Given a Cayley graph of an amenable group of exponential growth, what is the largest $d \in \mathbb{R}_{\geq 1}$ so that there is connected spanning subgraph which is Liouville and satisfies IS_d?*

In fact, it would be sufficient to know what is the largest d so that, for any pair of geodesic rays, there is connected subgraph [not necessarily spanning] containing these rays and which is Liouville and satisfies IS_d.

Lastly, it seems very plausible that the inequalities $p < d/2$ could be changed to inequalities of the form $p < d$ by looking at more carefully defined transport plans. It is easy to see that this is true in some simple graphs.

2. Definitions and further discussions

2.1. Preliminaries. Recall that the conventions are that a graph $\Gamma = (X, E)$ is defined by X, its set of vertices, and E, its set of edges. All graphs will be assumed to be of bounded valency. The set of edges will be thought of as a subset of $X \times X$. The set of edges will be assumed symmetric (*i.e.* $(x, y) \in E \Rightarrow (y, x) \in E$). Functions will take value in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Functions on E will often be anti-symmetric (*i.e.* $f(x, y) = -f(y, x)$). This said $\ell^p(X)$ is the Banach space of functions on the vertices which are p-summable, while $\ell^p(E)$ will be the subspace of functions on the edges which are p-summable.

The gradient

$$\nabla \colon \mathbb{K}^X \longrightarrow \mathbb{K}^E$$

is defined by

$$\nabla g(\gamma, \gamma') = g(\gamma') - g(\gamma).$$

Given a finitely generated group G and a finite set S, the Cayley graph $\mathrm{Cay}(G, S)$ is the graph whose vertices are the element of G and $(\gamma, \gamma') \in E$ if there exists $s \in S$ such that $s^{-1}\gamma = \gamma'$. This convention might be unusual from the point of view of random walks, but is much more convenient to write convolutions. In order for the resulting graph to have a symmetric edge set, S is always going to be symmetric (*i.e.* $s \in S \Rightarrow s^{-1} \in S$). Also, Cayley graphs are always going to be connected (*i.e.* S is generating). This said, it is worthwhile to observe that the gradient is made of $\{(\lambda_s - \mathrm{Id})g\}_{s \in S}$ where λ is the left-regular representation. As for the right-regular representation, it is a (injective) homomorphism from G into $\mathrm{Aut}(\mathrm{Cay}(G, S))$, the automorphism group of the Cayley graph.

The Banach space of p-Dirichlet functions is the space of functions f on X such that $\nabla f \in \ell^p(E)$. It will be denoted $\mathsf{D}^p(\Gamma)$. In order to introduce the $\mathsf{D}^p(\Gamma)$-norm on $\mathbb{K}^X$, it is necessary to choose a vertex, denoted e_Γ (in a Cayley graph, it is convenient to choose the neutral element). This said

$$\|f\|_{\mathsf{D}^p(\Gamma)}^p = \|\nabla f\|_{\ell^p(E)}^p + |f(e_\Gamma)|^p.$$

Lastly, p' will denote the Hölder conjugate exponent of p, *i.e.* $p' = p/(p-1)$ (with the usual convention that 1 and ∞ are conjugate).

The ℓ^p-cohomology in degree one of the graph Γ is the quotient

$$\ell^p H^1(\Gamma) := (\ell^p(E) \cap \nabla \mathbb{K}^X)/\nabla \ell^p(X).$$

This space is not always separated, and it is sometimes more convenient to look at the largest separated quotient, the reduced ℓ^p-cohomology,

$$\underline{\ell^p H}^1(\Gamma) := (\ell^p(E) \cap \nabla \mathbb{K}^X)/\overline{\nabla \ell^p(X)}^{\ell^p(E)}.$$

By taking the primitive of these gradients, one may also prefer to define this by

$$\underline{\ell^p H}^1(\Gamma) := \mathsf{D}^p(\Gamma)/\overline{\ell^p(X) + \mathbb{K}}^{\mathsf{D}^p}.$$

A common abuse of language/notation will happen when we say the reduced cohomology is equal to the non-reduced one: this means that the "natural" quotient map

$$\ell^p H^1(\Gamma) \longrightarrow \underline{\ell^p H}^1(\Gamma)$$

is injective.

When G is a finitely generated group, this is isomorphic to the cohomology of the left-regular representation on $\ell^p(G)$, see Puls' paper [37] or Martin and Valette [31]. Another important result is that ℓ^p-cohomology is an invariant of quasi-isometry:

Theorem. (*see Élek* [12, §3] *or Pansu* [35]) *If two graphs of bounded valency Γ and Γ' are quasi-isometric, then they have the same ℓ^p-cohomology* (*in all degrees, reduced or not*).

The result is actually much more powerful, in the sense that it holds for a large category of measure metric spaces (see above mentioned references). For shorter proofs in more specific situations see Puls [39, Lemma 6.1] or Bourdon and Pajot [9, Théorème 1.1]. A first useful consequence is that it is possible to work on graphs and obtain results about manifold (or *vice-versa*, when it is more convenient). A second corollary is that if G is a finitely generated group, the ℓ^p-cohomology of any two Cayley graphs are isomorphic. Consequently, one may speak of the ℓ^p-cohomology of a group without making reference to a Cayley graph.

Kanai has shown [26] that any Riemannian manifold with Ricci curvature and injectivity radius bounded from below is quasi-isometric to a graph (of bounded valency). Also, if M is a compact Riemannian manifold and $\widetilde{M}$ its universal covering, then $\widetilde{M}$ is quasi-isometric to the fundamental groups $\pi_1(M)$. Even if the language is that of graphs, there is always a corresponding result for Riemannian manifolds of bounded geometry (see Corollary 2.2 for a summary of the results expressed on Riemannian manifolds).

Before moving on to the results, it is important to say a bit more about isoperi-metric profiles (see *e.g.* Woess' book [48, (4.1) Definition]). For a set of vertices A let ∂A be the edges between A and A^c. Let $\mathfrak{F}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function. Then Γ (of bounded valency) satisfies the isoperimetric profile $\mathrm{IS}_{\mathfrak{F}}$ if there is a $\kappa > 0$ such that, for any non-empty finite set of vertices A,

$$\mathfrak{F}(|A|) \leq \kappa |\partial A|.$$

If $\mathfrak{F}(t) = t^{1-1/d}$ then this is called a *d-dimensional isoperimetric profile* (d need not be an integer); the short notation is IS_d. If $\mathfrak{F}(t) = t$ this is called a *strong isoperimetric profile* (or inequality); the short notation used here will be IS_ω. In the latter case, the constant κ is sometimes referred to as the isoperimetric con-stant. It is straightforward to see that $\mathrm{IS}_\omega \Rightarrow \mathrm{IS}_d$ for all d. The converse is false (whence the notation with ω rather than ∞). Obviously $d' \leq d$, then $\mathrm{IS}_d \Rightarrow \mathrm{IS}_{d'}$.

Under the convention that Cayley graphs are always connected, recall that G is a *non-amenable* group precisely when one (hence all) of its Cayley graph have a strong isoperimetric profile (Følner's classical result, [15]). To see that a Cayley graph of a group satisfy IS_d for all d if and only if the groups is not virtually nilpotent requires more effort (*e.g.* one needs Gromov's theorem on groups of polynomial growth [19]). See again [48, §14] for more details.

The constant κ in the various isoperimetric profiles is not an invariant of quasi-isometry. However (see again [48, (4.7) Theorem]), satisfying a d-dimensional or a strong isoperimetric profile is an invariant of quasi-isometry.

2.2. Discussion. Theorem 1.2 extends the result of Bourdon and Pajot [9, Théo-rème 1.1] from hyperbolic groups or spaces to those satisfying a d-dimensional isoperimetric profile, and extends the result of Lohoué [28] from graphs with IS_ω to graphs with IS_d for all d.

Tessera's [44, Theorem 2.2] showed that groups with CF also have vanishing of the reduced ℓ^p-cohomology (in fact, for any weakly mixing representation, not just the left-regular representation). Let B_n be the ball of radius n around the identity element in some Cayley graph. An amenable group is *CF* if there exists a sequence $F_n \subset B_n$ of [finite] sets and a constant $K > 0$ such that $\frac{|\partial F_n|}{|F_n|} \leq K/n$.

To position Theorem 1.2 with respect to Tessera's [44, Theorem 2.2], note that many groups with CF were already known to have trivial Poisson boundary, see Kaimanovich's [25, Theorem 3.3] for $F \wr \mathbb{Z}$ and [25, Corollary on p.23] for poly-cyclic groups. Groups with CF have compression exponent 1 (see Tessera [45, Theorems 9 and 10], so that using a bound of Austin, Naor and Peres from [1] on the speed exponent, one concludes that their Poisson boundary must be trivial.

For some other result implying the Liouville property, see also Saloff-Coste and Zheng [41] and [18].

However, $\mathbb{Z}_2 \wr \mathbb{Z}^2$ has a trivial Poisson boundary, but is not CF. This is a consequence of estimates of Erschler on the isoperimetric profile [13] of wreath products. These estimates are not compatible with the isoperimetric profile of CF groups computed by Tessera in [46]. It has become frequent to show that a group is amenable by showing that it is Liouville. For example, Bartholdi and Virág proved the Basilica group also has trivial Poisson boundary [3, Theorem 1]. Since CF groups of exponential growth have return probability $\approx e^{Kn^{-1/3}}$ (see [46]), one may see that many Liouville groups are not CF. For more examples, see the results of Revelle [40], Pittet and Saloff-Coste [36] and the references therein.

Let us also mention that groups of intermediate growth are most likely not CF (though the author ignores the existence of a direct argument). Any semi-direct product $N \rtimes H$ where N is finitely generated nilpotent and H is finitely generated and Liouville has trivial Poisson boundary by Kaimanovich [25, Theorem 3.2].

Theorem 1.2 does not cover (for all p) groups of polynomial growth. But for these groups, many (quite different) proofs of the vanishing of reduced ℓ^p-cohomology are available to the reader: groups of polynomial growth are quasi-isometric to nilpotent groups, these have infinitely many finite conjugacy class (in fact, infinite center) and see Kappos [24, Theorem 6.4] or [16, Theorem 3.2]; they are also polycyclic, hence CF, and see Tessera [44, Theorem 2.2] or [16, Theorem 1.3]; lastly they satisfy certain Poincaré inequalities and see Holopainen and Soardi [23, Corollary 1.10]. The first assertion requires to use that groups of polynomial growth are virtually nilpotent by Gromov's famous result [19].

Theorems 1.3 and/or 1.2 essentially unify many preceding notions of an ideal boundary which allows to compute the reduced ℓ^p-cohomology (in degree one). These boundaries are ℓ^p-corona (see Gromov [20, §8.C] and Élek [11]), the Bourdon and Pajot boundary for hyperbolic spaces (see [9]), the Floyd boundary (Puls, see [38]) and the p-harmonic boundary (Puls, see [39]). The advantage of the Poisson boundary is that it is better understood than most of the above (*e.g.* it possesses a linear structure). It is worthwhile to underline that, in non-amenable groups, a result of Karlsson [27] exhibits a strong link between Floyd and Poisson boundaries.

[16, Theorem 1.3] and [17] shows many wreath products also have trivial reduced ℓ^p-cohomology. This means that groups such as $H \wr \mathbb{Z}^k$ (where H, the "lamp state" group, is amenable and $k > 0$) have no [bounded or not] harmonic functions with gradient in ℓ^p (though they have many bounded harmonic func-

tions if $k > 2$). Actually, the Poisson boundary of these groups is fully described by Erschler in [14, Theorem 1] (under the further assumption that $k > 4$), and one may directly check that these do not have harmonic functions in D^p.

It is also possible to show that certain semi-direct products (where N is not finitely generated) have trivial reduced ℓ^p cohomology. See [17] for more details.

Also, using Erschler's result [14, Theorem 2], one may check that the free metabelian groups of rank ≥ 5 also do not have harmonic functions with ℓ^p gradient, and hence trivial reduced ℓ^p cohomology in degree 1. As a last note on this topic, Martin and Valette [31, Theorem.(iv)] shows that wreath products $H' \wr H$ where H' is *non-amenable* have trivial reduced ℓ^p-cohomology. See [17] for even more wreath products with trivial reduced ℓ^p-cohomology in degree 1.

In higher degree, there is no hope to extend Theorem 1.2 or 1.3. Pansu computed in [34, Théorème B] that, already in degree 2, some groups have non-trivial cohomology exactly in an interval.

Finally, let us sum up the results in the language of p-harmonic functions and on Riemannian manifolds. When $1 < p < \infty$, it is known (see Puls [37, §3] or Martin and Valette [31, §3]) that the existence of non-constant p-harmonic function (*i.e.* $h \in \mathsf{D}^p(\Gamma)$ with $\nabla^* \mu_{p,p'} \nabla h = 0$, where $\mu_{p,p'}$ is the Mazur map defined by $(\mu_{p,p'} f)(\gamma) = |f(\gamma)|^{p-2} f(\gamma)$) is equivalent to the non-vanishing of reduced cohomology in degree 1. In fact, up to a constant there is exactly one p-harmonic function in each reduced class. In this light, Theorem 1.2 is even more surprising, as one may replace solutions of a non-linear equation (the p-Laplacian) by solutions to a linear one (the Laplacian).

Using the fact that the reduced ℓ^p-cohomology of groups of polynomial growth is trivial, let's sum up the results for groups in terms of p-harmonic functions:

Corollary. *Let $p \in]1, \infty[$ and Γ be the Cayley graph of a finitely generated group G. Assume one of the following holds*

- *G is amenable and $p \in]1, 2]$;*

- *there exists $q \in [p, \infty]$ such that Γ has no non-constant bounded harmonic function whose gradient is in $\ell^q(E)$ (**e.g.** Γ is Liouville);*

- *there exists $q \in [p, d/2[$ such that there are no non-constant bounded q-harmonic functions on Γ;*

- *there exists $N \lhd G$ such that N is infinite, finitely generated, G/N is infinite, $\underline{\ell^p H}^1(N) = \{0\}$ and the growth of N is at least a polynomial of degree $d > 2p$.*

Then there are no non-constant p-harmonic functions on Γ.

Note that the corollary also holds

- when G has infinitely many finite conjugacy classes (*e.g. G has infinite centre*);

- when G is transport amenable (*e.g. G is of the form $H \wr \mathbb{Z}^k$ where H is amenable and $k > 0$*);

- when $G = N \rtimes H$ such that G is finitely generated, N is Abelian but not finitely generated, $\underline{\ell^p H}^1(H) = \{0\}$ and H has IS_d for some $d > 2p$;

thanks to [16, Theorem 1.3], [16, Theorem 3.2] and [17] (see also Holopainen and Soardi [23], Kappos [24], Martin and Valette [31] and Tessera [44]).

Using quasi-isometry to re-express the results on Riemannian manifolds, one obtains:

Corollary. *Let M be a Riemannian manifold* (*with Ricci curvature and injectivity radius bounded below*). *The degree one reduced ℓ^p-cohomology of M vanishes and* (*equivalently*) *there are no non-constant continuous p-harmonic functions on M*

- *if M is the universal cover of a compact Riemannian manifold M' with $\pi_1(M') =: G$ and p satisfying one of the conditions of Corollary* 2.2;

- *if M satisfies a d-dimensional isoperimetric profile with $d > 2p$ and M is Liouville*;

- *if M satisfies a d-dimensional isoperimetric profile, $\underline{\ell^q H}^1(M) = 0$ for some $q \in [p, \infty[$ and $d > 2q$.*

Furthermore, $\underline{\ell^1 H}^1(M) = \{0\}$ if and only if M has one end.

In particular, if M is the universal covering of M' with $\pi_1(M')$ amenable, then $\underline{\ell^p H}^1(M) = \{0\}$ for $p \in]1, 2]$.

Other known consequences of the triviality of the reduced ℓ^p-cohomology include the triviality of the p-capacity between finite sets and ∞ (see Yamasaki [49] and Puls [39, Corollary 2.3]) and existence of continuous translation invariant linear functionals on $\mathsf{D}^p(\Gamma)/\mathbb{K}$ (see [39, §8]). It also has implication on the possibility of realising the graph as a packing of spheres in $\mathbb{R}^k$ (see Benjamini and Schramm [6]).

Since Lemma 4.4 from Holopainen and Soardi [22] is quite important to the current methods and its proof relies on p-harmonic functions, the author feels he owes the reader a proof which does not require the use of p-harmonic functions.

Lemma 2.1 (Holopainen and Soardi [22], 1994). *Let $g \in \mathsf{D}^p(\Gamma)$ be such that $g \notin [0] \in \underline{\ell^p H}^1(\Gamma)$. For $t \in \mathbb{R}_{>0}$, let g_t be defined as*

$$g_t(x) = \begin{cases} g(x) & \text{if } |g(x)| < t, \\ t\,\dfrac{g(x)}{|g(x)|} & \text{if } |g(x)| \geq t. \end{cases}$$

Then there exists t_0 such that $g_t \notin [0]$, for any $t > t_0$. In particular, the reduced ℓ^p cohomology is trivial if and only if all bounded functions in $\mathsf{D}^p(\Gamma)$ have trivial classes.

Proof. Assume, without loss of generality that $g(o) = 0$ for some preferred vertex (*i.e.* root) $o \in X$. Since $\|\nabla g\|_{\ell^\infty(E)} \leq \|\nabla g\|_{\ell^p(E)} =: K$, given $x \in X$ and P a path from o to x,

$$|g(x)| = |g(x) - g(o)| = \sum_{e \in P: o \to x} \nabla g(e) \leq d(o, x) \|\nabla g\|_{\ell^p(E)}.$$

In particular, g_t is identical to g on $B_{t/K}$. Hence $\|g - g_t\|_{\mathsf{D}^p(\Gamma)} \leq \|\nabla g\|_{\ell^p(B^c_{t/K})}$, where $\ell^p(B^c_{t/K})$ denotes the ℓ^p-norm restricted to edges which are not inside $B_{t/K}$. Because $\nabla g \in \ell^p(E)$, $\|\nabla g\|_{\ell^p(B^c_{t/K})}$ tends to 0, as t tends to ∞.

Now if there is a infinite sequence t_n such that g_{t_n} are in $[0]$ and $t_n \to \infty$, then g_{t_n} is a sequence of functions in $[0]$ which tends (in D^p-norm) to g. This implies $g \in [0]$, a contradiction. Hence, for some t_0, $g_t \notin [0]$ given that $t > t_0$. $\square$

It seems worthwhile to note that this proof works for c_0 (though it is clearly false for ℓ^∞, see Proposition A.3), whereas the original proof requires p-harmonic functions (which are not defined in this extremal cases). Of course, one can check directly that $\mathsf{D}^1(\Gamma) \subset \ell^\infty(X)$ (see §A), so that even if the proof still works for $p = 1$, this is not much of a surprise.

3. Boundary values and simple random walks

Let $\Gamma = (X, E)$ be a graph. Let $\xi_x \in \ell^1(X)$ be a family (as x varies in X) of finitely supported, positive elements of ℓ^1-norm one. Define

$$\xi * g(x) = \int_X g(y) \mathrm{d}\xi_x(y) = \sum_{y \in X} g(y) \xi_x(y).$$

The main idea of this section will be to find a sequence $\xi_x^{(n)}$ of such families and suppose even that $\xi^{(n)} * g$ converges pointwise. It will be shown that elements of trivial class converge to a constant function. Using a good transport plan will allow to show the converse: if the limit is a constant function, the g takes essentially one value at infinity. A classical truncation argument ensures such functions are of trivial class.

Throughout the text, $[g]$ denote the class of g in $\underline{\ell^p H}^1(\Gamma)$. There should be no confusion as to which cohomology is considered.

Note that if there exists a $R \in \mathbb{Z}_{\geq 0}$ so that the support of ξ_x is contained in a ball of radius R at x, then $\xi * g \in [g]$. This seems to be decent motivation to look at such sequences. One could even be tempted to prove more than just pointwise convergence, namely, that there is convergence of $\xi^{(n)} * g$ in $\mathsf{D}^p(\Gamma)$. This second option is investigated in [16]. It gives better results for small groups (*i.e.* virtually nilpotent groups and some small wreath products), but is much more restrictive.

The first subsection treats a fairly generic situation and is more elaborated than strictly required. Existence of boundary values when $\xi_x^{(n)}$ is the time n distribution of a simple random walker starting at x can be significantly simplified. This can be done by looking at Remark 3.12.

3.1. Boundary values.

Here is the avatar of this viewpoint: if g does not take "enough distinct values" then its class is trivial. A function $h\colon X \to \mathbb{K}$ will be said *to tend to* $c \in \mathbb{K}$ as $|x| \to \infty$ if the following holds: there exists a $c \in \mathbb{K}$ such that for all $\varepsilon > 0$ the set $X \setminus h^{-1}(B_\varepsilon(c))$ is finite (where $B_\varepsilon(c) = \{k \in \mathbb{K} \mid |c-k| < \varepsilon\}$).

The following argument is well-known (it may probably be traced back to Strichartz [43], if not earlier) and motivates the introduction of boundary values. The proof is extremely similar to that of Lemma 2.1 (it also works for c_0 and ℓ^1, but not ℓ^∞).

Lemma 3.1. *Assume $h \in \mathsf{D}^p(\Gamma)$ is such that $h \to c$ as $|x| \to \infty$. Then h belongs to $\overline{\ell^p(X) + \mathbb{K}}^{\mathsf{D}^p}$ (so $[h] = 0$).*

Proof. First, one may assume $c = 0$ by changing h up to a constant. Then for all $\varepsilon > 0$, the truncated function h_ε defined by

$$h_\varepsilon(\gamma) = \begin{cases} \varepsilon h(\gamma)/|h(\gamma)| & \text{if } |h(\gamma)| > \varepsilon, \\ h(\gamma) & \text{otherwise,} \end{cases}$$

is distinct from h only on a finite set. Let $X_\varepsilon = h^{-1}(B_\varepsilon^c)$ be this finite set and $g_\varepsilon = h - h_\varepsilon$ be their difference (it is finitely supported, hence in $\ell^p(X)$ for any p).

Note that $[g_\varepsilon] = [0]$ and $\|h - g_\varepsilon\|_{\mathsf{D}^1(\Gamma)} = \|h_\varepsilon\|_{\mathsf{D}^1(\Gamma)}$. Furthermore,

$$
\nabla h_\varepsilon \text{ is } \begin{cases} \text{equal to } \nabla h & \text{on } E \cap (X_\varepsilon^c \times X_\varepsilon^c), \\ \text{smaller in } |\cdot| \text{ than } \nabla h & \text{on } \partial X_\varepsilon, \\ 0 & \text{on } E \cap (X_\varepsilon \times X_\varepsilon). \end{cases}
$$

But $E \cap (X_\varepsilon \times X_\varepsilon)$ increases, as $\varepsilon \to 0$, to the whole of E. The important conse-
quence is that the ℓ^p-norm of ∇h outside this set tends to 0. Thus $\|h_\varepsilon\|_{\mathsf{D}^p(\Gamma)} \to 0$
as $\varepsilon \to 0$, and consequently $h \in \overline{\ell^p(X)}^{\mathsf{D}^p(\Gamma)}$. $\square$

The aim here is to define a "boundary value" for functions so that the "value"
does not depend on p or the representative in the reduced cohomology class and
it is constant exactly when the hypothesis of Lemma 3.1 apply. In order to do so,
one must show some continuity in D^p-norm. This will be done by an old trick in
a new disguise: integration by parts under the cover of transportation problem.

For two finitely supported function f and g on a countable set Y, define the
pairing $\langle f \mid g \rangle_Y = \sum_{y \in Y} f(y)g(y)$. The subscript Y will often be dropped. This
allows to define the adjoint of the gradient ∇, denoted ∇^* and called divergence,
by $\langle f \mid \nabla g \rangle_E = \langle \nabla^* f \mid g \rangle_X$. More precisely, for $f : E \to \mathbb{K}$, one finds

$$
\nabla^* f(x) = \sum_{y \in N(x)} f(y, x) - \sum_{y \in N(x)} f(x, y)
$$

where $N(x)$ are the neighbours of x.

Definition 3.2. A *transport pattern from ξ to ϕ* (two finitely supported probability
measures) is a finitely supported function on the edges $\tau_{\xi,\phi}$ such that

$$
\nabla^* \tau_{\xi,\phi} = \xi - \phi.
$$

The name can be explained by the following simple fact: if f represents any
oriented path (with oriented edges counted with multiplicities if necessary) from
a vertex x to a vertex y then $\nabla^* f = \delta_y - \delta_x$. As such, $\tau_{\xi,\phi}$ can be seen as the
indication of how much mass is going to be transported through each edge in a
transport from ϕ to ξ.

Lemma 3.3. *Let ξ, ϕ be as above, $g \in \mathsf{D}^p(\Gamma)$ and $\tau_{\xi,\phi} \in \ell^{p'}(E)$ be a transport
pattern. Then*

$$
\left| \int g \, d\xi - \int g \, d\phi \right| \leq \|\nabla g\|_{\ell^p(E)} \|\tau_{\xi,\phi}\|_{\ell^{p'}(E)}.
$$

Proof. Simply note that

$$\int g\,\mathrm{d}\xi - \int g\,\mathrm{d}\phi = \langle g \mid \xi - \phi \rangle = \langle g \mid \nabla^* \tau_{\xi,\phi} \rangle = \langle \nabla g \mid \tau_{\xi,\phi} \rangle,$$

then conclude using Hölder's inequality.   □

As a consequence obtaining the following bound will become important:

$$\lim_{n\to\infty} \sup_{k\geq 0} \| \tau_{\xi_x^{(n)}, \xi_x^{(n+k)}} \|_{\ell^r(E)} = 0. \tag{3.4}$$

where $\xi_x^{(n)}$ is a sequence of (finitely supported probability) measures.

Definition 3.5. Let $\{\xi_x^{(n)}\}_{x\in X, n\in\mathbb{Z}_{\geq 0}}$ be (finitely supported probability) measures such that (3.4) holds for $r = p'$ and for all $x \in X$. If $g \in \mathsf{D}^p(\Gamma)$, then $\xi^{(n)} * g$ converges pointwise and the resulting pointwise limit is called the boundary value of g for $\xi^{(n)}$.

The above definition of boundary value is closer to that of Pansu [33] than Bourdon and Pajot [9]. With the choice of $\xi_x^{(n)}$ which will considered in a few paragraphs, these boundary values are harmonic functions.

An important point in the preceding definition is that, if the condition (3.4) holds for $r = p'$ it holds for $r = q' > p'$, and if $g \in \mathsf{D}^q(\Gamma)$ then $g \in \mathsf{D}^p(\Gamma)$. In other words, if this boundary value exists for p, it does not depend on p and exists for all $q < p$.

The choice of τ is very important in the computation of the norm. Even in $\mathbb{Z}^2$ and for $\xi_x^{(n)}$ the normalised characteristic function of the balls around x, some (too simple) choices of τ will not satisfy the required condition whereas others will.

3.2. Simple random walks. In order to start with the easiest setting, the measure $\xi_x^{(n)}$ will, in this subsection and subsection 3.3, be $P_x^{(n)}$ where $P_x^{(n)}(y)$ is the probability a simple random walker starting at x reaches y in n steps. With this vocabulary, $P^{(n)} * g(x)$ is the expected value of g after n steps of a simple random walk starting at x. When the boundary value exists, it is a harmonic functions (for the simple random walk); see also Remark 3.12 below.

The advantage of these function is that there is a natural (albeit very inefficient) choice for $\tau_{P_x^{(n)}, P_x^{(n+k)}}$. Namely, $\tau_{P_x^{(n)}, P_x^{(n+k)}}$ is given by continuing the random walk k steps. Let us dwell on an explicit realisation of this idea.

First, if ξ, ϕ and ψ are finitely supported probability measures, $\tau_{\xi,\phi}$ is a transport pattern from ϕ to ξ and $\tau_{\psi,\xi}$ is a transport pattern from ξ to ψ, then

$$\tau_{\psi,\phi} = \tau_{\xi,\phi} + \tau_{\psi,\xi}.$$

This follows trivially from the linearity of ∇^*. For our current purpose, it will be sufficient to define the transport plan $\tau_{P_x^{(n)},P_x^{(n+1)}}$, because one can then pick

$$\tau_{P_x^{(n)},P_x^{(n+k)}} = \sum_{i=n}^{n+k-1} \tau_{P_x^{(n)},P_x^{(n+1)}}.$$

Now let $P_{x,E}^{(i)}$ be the measure on the edges defined by

$$P_{x,E}^{(i)}(y,z) = \frac{1}{|N(y)|} P_x^{(i)}(y), \quad \text{for all } y \in X,\ z \in N(y).$$

The claim is that this is a candidate for $\tau_{P_x^{(i)},P_x^{(i+1)}}$. To verify this, fix a $y \in X$ and look at

$$\sum_{z \in N(y)} P_{x,E}^{(i)}(y,z) = P_x^{(i)}(y)$$

and

$$\sum_{z \in N(y)} P_{x,E}^{(i)}(z,y) = \sum_{z \in N(y)} \frac{1}{|N(z)|} P_x^{(i)}(z) = P_x^{(i+1)}(y).$$

The first equality follows from the definition of $P_{x,E}^{(i)}$, the second from the definition of a simple random walk. The difference of these sum is the value of $\nabla^* P_{x,E}^{(i)}$ and shows the claim.

This formula is particularly useful as one can give an upper bound in terms of more well-studied quantities:

$$\|\tau_{P_x^{(n)},P_x^{(n+k)}}\|_{\ell^{p'}(E)} \leq \sum_{i=n}^{n+k-1} \|P_x^{(i)}\|_{\ell^{p'}(X)}.$$

When all vertices have the same valency, one can take a slightly smaller constant in front of the sum on the right-hand side, but it is completely irrelevant for the present purpose. This proves:

Lemma 3.6. *If $\xi_x^{(n)} = P_x^{(n)}$ as above, and, for all $x \in X$, $\sum_{i=0}^{\infty} \|P_x^{(i)}\|_{\ell^{p'}(X)} < +\infty$, then any $g \in \mathsf{D}^p(\Gamma)$ admits a boundary value for $\xi^{(n)}$.*

In particular, the simple random walk is transient exactly when the condition of Lemma 3.6 holds for $p' = \infty$. (It never holds in the case $p' = 1$, but $p = \infty$ is also not of interest.)

Fortunately, there are very good estimates at hand for $\|P_x^{(i)}\|_{\ell^{p'}(X)}$, and some of them rely only on isoperimetric profiles (and in the case of a Cayley graph, the growth of the group). Indeed, if Γ has IS_d, then there exists $K > 0$ such that

$$P_x^{(n)}(y) \leq Kn^{-d/2}, \quad \text{for all } x, y \in X,$$

see Woess' book [48, (14.5) Corollary].

Corollary 3.7. *If Γ has IS_d, then boundary values of $g \in \mathsf{D}^p(\Gamma)$ exists for $\xi_x^{(n)} = P_x^{(n)}$ if $p < d/2$.*

Proof. Obviously $\|P_x^{(n)}\|_{\ell^1(X)} = 1$. By Hölder's inequality,

$$\|P_x^{(n)}\|_{\ell^{p'}(X)} \leq \|P_x^{(n)}\|_{\ell^1(X)}^{1/p'} \|P_x^{(n)}\|_{\ell^\infty(X)}^{1/p}.$$

Hence

$$\|P_x^{(n)}\|_{\ell^{p'}(X)} \leq K'n^{-d/2p}$$

uniformly in x, for some $K' > 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

This shows that there are plenty of graphs where boundary values may be defined. It remains to prove these boundary values have the desired properties.

Lemma 3.8. *Assume Γ has IS_d and $p < \frac{d}{2}$. If $g \in \overline{\ell^p(X)}^{\mathsf{D}^p(\Gamma)}$ then its boundary value is trivial.*

Proof. If $g \in \ell^p(G)$ then $P_x^{(n)}g$ tends to 0 (as $\ell^p \subset c_0$ and the mass of $P^{(n)}$ tends to 0 on finite sets). It remains to be checked that convergence in $\mathsf{D}^p(\Gamma)$ does not alter the boundary value. However, by Lemma 3.3, if $g - g_n$ tends to 0 in $\mathsf{D}^p(\Gamma)$ norm so does the difference of their boundary values. $\qquad\qquad\qquad\square$

In fact, since the boundary value of a sum is the sum of boundary values, this shows the boundary value does not depend on the representative of the reduced cohomology class, up to a constant function.

Lemma 3.9. *Assume $g \in \mathsf{D}^p(\Gamma)$, Γ has IS_d and $p < \frac{d}{2}$. If the boundary value of g for $P^{(n)}$ is a constant function, then $g \to c$ as $|x| \to \infty$ and (consequently) $[g] = 0 \in \underline{\ell^p H}^1(\Gamma)$.*

Proof. Up to changing g by a constant (this does not affect its cohomology class), one may assume that the limit takes value 0 everywhere. Fix a root $o \in X$ of the graph. For any $\varepsilon > 0$, let n_ε be such that $\|\nabla g\|_{\ell^p(E \setminus B_{n_\varepsilon}(o))} < \varepsilon$ and, uniformly in x,

$$\sum_{i \geq n_\varepsilon} \|P_x^{(i)}\|_{\ell^{p'}(E)} < \varepsilon.$$

Let $E[Y]$ denote the edges incident with $Y \subset X$ and $\langle . \mid . \rangle_{E'}$ restriction of the pairing to the set E'. If $x \notin B_{3n_\varepsilon}(o)$, then

$$
\begin{aligned}
|P^{(k)} * g(x) - g(x)| &= |\langle g \mid P_x^{(k)} - \delta_x \rangle| \\
&= |\langle g \mid \nabla^* \tau_{P_x^{(k)}, \delta_x} \rangle| \\
&= |\langle \nabla g \mid \tau_{P_x^{(k)}, \delta_x} \rangle| \\
&\leq |\langle \nabla g \mid \tau_{P_x^{(k)}, \delta_x} \rangle_{E[B_{n_\varepsilon}(x)]}| + |\langle \nabla g \mid \tau_{P_x^{(k)}, \delta_x} \rangle_{E[B_{n_\varepsilon}]^c}| \\
&\leq \|\nabla g\|_{\ell^p(E[B_{n_\varepsilon}(x)])} \|\tau_{P_x^{(k)}, \delta_x}\|_{\ell^{p'}(E)} \\
&\quad + \|\nabla g\|_{\ell^p(E)} \|\tau_{P_x^{(k)}, \delta_x}\|_{\ell^{p'}(E \setminus B_{n_\varepsilon}(x))} \\
&\leq \varepsilon \sum_{i \geq 0} \|P_x^{(i)}\|_{\ell^{p'}(E)} + \|\nabla g\|_{\ell^p(E)} \sum_{i \geq n_\varepsilon} \|P_x^{(i)}\|_{\ell^{p'}(E)} \\
&\leq \varepsilon \sum_{i \geq 0} \|P_x^{(i)}\|_{\ell^{p'}(E)} + \|\nabla g\|_{\ell^p(E)} \varepsilon \\
&\leq c\varepsilon,
\end{aligned}
$$

where c is a constant depending only on the constant in IS_d and the D^p-norm of g. Thus, letting $k \to \infty$, for all $x \notin B_{3n_\varepsilon}(e)$, $|g(x)| \leq c\varepsilon$. Thus, Lemma 3.1 may be applied to yield that $[g] = 0$. □

A trivial, but useful, remark, is that if g is bounded (*i.e.* in $\ell^\infty(X)$) then its boundary value is also bounded.

3.3. Inclusion and vanishing

Corollary 3.10. *Let Γ be a graph with IS_d and $1 \leq q \leq p < d/2$. Then the natural quotient $\underline{\ell^q H}^1(\Gamma) \to \underline{\ell^p H}^1(\Gamma)$ is an injection.*

Proof. Assume $[g] \neq [0] \in \underline{\ell^q H}^1(\Gamma)$. Then, by Lemma 3.9, its boundary value is not trivial. However, this boundary value does not depend on p and so, by Lemma 3.8, g is not trivial in $\underline{\ell^p H}^1(\Gamma)$. □

Finally, let us consider the case where the graph has is Liouville, *i.e.* there are no non-constant bounded harmonic functions. This means that, if $g \in \ell^\infty(X)$ and $P^{(n)} * g$ converges pointwise, then the limit is a constant function.

Corollary 3.11. *Let Γ be a Liouville graph with IS_d and $1 \le q \le p < d/2.$, then $\underline{\ell^p H}^1(\Gamma) = \{0\}$ for all $p \in [1, \infty[$.*

Proof. By the truncation lemma from Holopainen and Soardi (see Lemma 2.1), it suffices to show all bounded functions in $\mathsf{D}^p(\Gamma)$ are trivial. But if g is in $\ell^\infty(X)$ and the graph is Liouville, the boundary value of g for $P_x^{(n)}$ is constant. By Lemma 3.9, the conclusion follows. □

Remark 3.12. As P. Pansu pointed out to the author, a result of Lohoué [28] shows that in graphs satisfying IS_ω, there is a harmonic element in each (unreduced) class. This element is the boundary value above. To see this, first recall that the representative in the (reduced or unreduced) class of g exhibited by N. Lohoué is defined by $g + u$ where $u = \Delta^{-1}(-\Delta g)$, where $\Delta = \mathrm{Id} - R$ and R is the random walk (or averaging operator): $Rg = P^{(1)} * g$. Thanks to H. Kesten, graphs satisfying IS_ω are exactly those where $\|R\|_{\ell^2 \to \ell^2} < 1$. On the other hand, $\|R\|_{\ell^p \to \ell^p} \le 1$ for $p = 1$ or ∞. Thus $\|R\|_{\ell^p \to \ell^p} < 1$ for any $p \in]1, \infty[$ by Riesz-Thorin interpolation, and $\Delta^{-1} = \sum_{i \ge 0} R^i$ is bounded from $\ell^p(X)$ to itself. Next notice that

$$
\begin{aligned}
\tilde{g} - g &= \lim_{n \to \infty} P^{(n)} * g - g \\
&= \lim_{n \to \infty} R^n g - g \\
&= \sum_{i \ge 0} (R^{i+1} g - R^i g) \\
&= \sum_{i \ge 0} R^i (R - \mathrm{Id}) g \\
&= \Big(\sum_{i \ge 0} R^i \Big) * (-\Delta g),
\end{aligned}
$$

to conclude that $\tilde{g} = g + u$.

This argument may not apply in (many, if not all) Cayley graphs of amenable groups. Indeed, this would mean the harmonic function belongs to the same (unreduced!) cohomology class. But, in amenable groups, the reduced and unreduced ℓ^p-cohomologies are never equal (see Guichardet [21, Corollaire 1]). Thus, if the

reduced ℓ^p-cohomology is trivial (which is already known for many amenable groups), this would give a contradiction.

However, it is still possible to answer positively a weaker form of P. Pansu's question [34, Question 6 in §1.9] (*i.e.* is the absence of non-constant harmonic function whose gradient has finite ℓ^p-norm is equivalent to $\underline{\ell^p H}^1(\Gamma) = \{0\}$). Lemma 3.9 shows that if there is such a harmonic function and Γ has IS_d for $d > 2p$, then $\underline{\ell^p H}^1(\Gamma)$ is not trivial. Indeed, such a function would be its own boundary value, and being non-constant, it would be non-trivial in cohomology.

Let us now address the reverse implication.

Lemma 3.13. *Let Γ be a graph with IS_d and $1 \leq p < d/2$. For any $g \in \mathsf{D}^p(\Gamma)$, let $\tilde{g}$ be its boundary value for $P_x^{(n)}$. Then $\tilde{g}$ is in the same $\ell^q H^1(\Gamma)$ class as g for all $q > \frac{dp}{d-2p}$.*

Proof. As before, write

$$\tilde{g} - g = \lim_{n \to \infty} P^{(n)} * g - g$$
$$= \sum_{i \geq 0} (P^{(i+1)} * g - P^{(i)} * g)$$
$$= \sum_{i \geq 0} P^{(i)} * (P^{(1)} - \mathrm{Id}) * g.$$

Let

$$h = (P^{(1)} - \mathrm{Id}) * g = \Delta g \in \ell^p(X)$$

and remember $P^{(i)}$ are operators defined by a kernel. Using Young's inequality (see *e.g.* Sogge's book [42, Theorem 0.3.1]) for $r > p$ and $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$,

$$\|\tilde{g} - g\|_{\ell^r(X)} = \|(\sum_{i \geq 0} P^{(i)}) * h\|_{\ell^r(X)}$$
$$\leq \sup_{x \in X} \|\sum_{i \geq 0} P_x^{(i)}\|_{\ell^q(X)} \|h\|_{\ell^p(X)}$$
$$\leq 2D \sup_{x \in X} \|\sum_{i \geq 0} P_x^{(i)}\|_{\ell^q(X)} \|\nabla, g\|_{\ell^p(E)}$$

where D is the maximal valency of a vertex. Since

$$\sup_{x \in X} \|\sum_{i \geq 0} P^{(i)}\|_{\ell^q(X)} < +\infty \quad \text{for all } q' < d/2,$$

this means

$$\tilde{g} - g \in \ell^r(X) \quad \text{for all } r > \frac{dp}{d-2p},$$

and consequently that $\tilde{g}$ and g belong in the same (unreduced) ℓ^r-cohomology class. □

As mentioned before, to address the question of whether $\tilde{g}$ and g are in the same reduced ℓ^p-cohomology class, the author believes one would need to consider question similar to those of the transport problem from [16].

Corollary 3.14. *Let Γ be a graph with IS_d and $1 \leq p < d/2$. If there are no non-constant bounded harmonic functions whose gradient has finite $\ell^p(E)$ norm, then $\underline{\ell^q H}(\Gamma) = \{0\}$ for all $q < \frac{pd}{d+2p}$.*

*Conversely, if there is a non-constant harmonic functions whose gradient has finite $\ell^p(E)$ norm, then $\underline{\ell^p H}(\Gamma) \neq \{0\}$ and there is also a **bounded** non-constant harmonic functions whose gradient has finite $\ell^r(E)$ norm for $r > \frac{pd}{d-2p}$.*

Proof. Under this new hypothesis, one has that $\tilde{g}$ is constant (by Lemma 3.13), so Lemma 3.9 implies that $[g] = 0 \in \underline{\ell^p H}^1(\Gamma)$.

As mentioned above, the converse is a consequence of Lemma 3.9. Boundedness may be added thanks to the Holopainen and Soardi lemma (see Lemma 2.1), but upon taking boundary values again the gradient might loose regularity and be only in ℓ^r for some bigger r. □

Corollaries 3.10, 3.11 and 3.14 are particularly interesting for graphs satisfying IS_d for every $d \in \mathbb{Z}_{\geq 1}$. This will be developed in subsection 4.2.

4. Consequences

This section finishes the proof of theorems announced in the introduction, namely Theorems 1.3 and 1.4.

4.1. Graphs having IS_ω

Proof of Theorem 1.3. When the graph has IS_ω, the fact that the quotient is always separated (*i.e.* $\underline{\ell^q H}^1(\Gamma) = \ell^q H^1(\Gamma)$) is a consequence Remark 3.12.

Let $\Gamma =: \Gamma_1$. For any finite subset F of Γ_1, $|\partial^{\Gamma_1} F| \geq c|F|$ for some $c > 0$. So take $n \in \mathbb{N}$ such that $nc \geq 1$. Let Γ_k be graph obtained by adding, for all $\gamma \in \Gamma$, an edge between γ and all vertices at distance $\leq n$ from γ (*i.e.* the k-fuzz of Γ). Denote $\partial^{\Gamma_k} F$ to be the boundary of F in Γ_k and $B_k(F)$ to be the sets of vertices at distance $\leq k$ from F (F included). Then, for any finite subset F of Γ_n,

$$|\partial^{\Gamma_n} F| \geq |\partial^{\Gamma_1} B_n(F)| + |\partial^{\Gamma_{n-1}} F| \geq c|F| + |\partial^{\Gamma_{n-1}} F| \geq \ldots \geq nc|F| \geq |F|.$$

In other words, there is a graph Γ_n, quasi-isometric to Γ_1, satisfying a strong isoperimetric profile with constant ≥ 1. But by a result of Benjamini and Schramm [5, Theorem 1.2], there is then a spanning tree in Γ_n which is obtained by adding edges between disjoint copies of binary trees. Let T be this tree, and look at the simple random walk on this tree. The boundary value of g for this tree will be in the same class for $\ell^p H^1(\Gamma)$ as g (the difference being in $\ell^p(X)$, the class is preserved for any set of edges as long as the valency remains bounded).

Furthermore, the fact that the boundary value is constant depends only on the boundary value for the tree. Boundary values actually shows the existence of a commuting diagram of "natural" injections (where T is the spanning tree inside Γ_n):

$$
\begin{array}{ccc}
\ell^q H^1(\Gamma_n) & \longrightarrow & \ell^q H^1(T) \\
\downarrow & & \downarrow \\
\ell^p H^1(\Gamma_n) & \longrightarrow & \ell^p H^1(T).
\end{array}
$$

As a consequence, the only boundary values possible (in a graph Γ having IS_ω) are those of T. To know if a boundary value of the tree is an actual cohomology class in Γ, it remains to check that it belongs to $\mathsf{D}^p(\Gamma)$. $\qquad\square$

4.2. Further corollaries. One of the most important consequence of boundary values is that they really give an idea of how the function behaves at infinity. It is already nice that it does not depend on p (up to some value), but it in fact also does not depend so much on the graph. Recall that a spanning subgraph H of Γ is a graph on the same vertices but with some edges removed. Two things are easy to check. First, if a spanning subgraph H has IS_d, then so does the full graph Γ. Second, if $g \in \mathsf{D}^p(\Gamma)$ then $g \in \mathsf{D}^p(H)$.

However, $\underline{\ell^p H}^1(H) = \{0\}$ has, in general, no incidence on the triviality of $\underline{\ell^p H}^1(\Gamma)$. For example, there are many graphs with a spanning line (or half-line) which have non-trivial ℓ^p-cohomology.

Corollary 4.1. *Let H be a connected spanning subgraph of Γ which has IS_d (hence Γ also has IS_d) and assume $p < d/2$. Then if $g \in \mathsf{D}^p(\Gamma)$ (hence in $\mathsf{D}^p(H)$) is such that $[g] = 0 \in \underline{\ell^p H}^1(H)$ then $[g] = 0 \in \underline{\ell^p H}^1(\Gamma)$.*

In particular, if $\underline{\ell^p H}^1(\Gamma) \neq \{0\}$ then $\underline{\ell^p H}^1(H) \neq \{0\}$.

Proof. Indeed, if $[g] = 0 \in \underline{\ell^p H}^1(H)$ then, by Lemmas 3.8 and 3.9, then $g \to c$ as $|x| \to \infty$. This statement remains true in Γ (because H is spanning and connected), hence, by Lemma 3.1, the conclusion follows. $\qquad\square$

The main application of this idea is done in §4.3, *i.e.* the proof of Theorem 1.4.

Remark 4.2. Recall that there is a unique (up to a constant) p-harmonic function in each reduced ℓ^p-cohomology class. The existence of a harmonic function in reduced ℓ^p-cohomology class seems unclear (at least to the author). However, it is easy to see that, if it exists and $p < \frac{d}{2}$, it is unique (up to constants). Indeed, assume h_1 and h_2 are two harmonic functions in $\mathsf{D}^p(\Gamma)$ and Γ has IS_d for $d > 2p$. Then $h_1 - h_2$ is harmonic and of trivial class. By Lemma 3.9, this means that it is constant at infinity. But a harmonic function which is constant at infinity is also constant by the maximum principle. Hence, $h_1 - h_2$ is a constant function.

The upcoming corollary is just to make a cleaner statement in the case of groups of superpolynomial growth (the case of polynomial growth being well-understood, see §1 or §2.2). N. Varopoulos showed that superpolynomial growth of a group implies superpolynomial decay of $\|P^{(i)}\|_{\ell^\infty(G)}$ (*e.g.* see [47] or Woess' book [48, (14.5) Corollary, p.148]. In particular, Cayley graphs of groups of polynomial growth of degree d satisfy IS_d and Cayley graphs of groups of superpolynomial growth will satisfy IS_d for any $d \in \mathbb{R}_{\geq 1}$. In the latter case, Corollaries 3.10, 3.11, and 3.14 yield

Corollary 4.3. *Let G be a group of superpolynomial growth and Γ a Cayley graph. Let $1 \leq p < \infty$. Then there exists a map π from $\mathsf{D}^p(\Gamma)$ to the space of harmonic functions modulo constants such that*

(1) $\pi(g) = \pi(h) \iff [g] = [h] \in \underline{\ell^p H}^1(\Gamma)$;

(2) $g \in \ell^\infty(G) \implies \pi(g) \in \ell^\infty$;

(3) $\pi(g) \in \mathsf{D}^q(\Gamma)$ *for any $q > p$.*

Using Theorem 1.2, it also easy to find *graphs* of polynomial growth which have $\underline{\ell^p H}^1(\Gamma) \neq \{0\}$.

Example 4.4. Take Γ to be two copies of the usual Cayley graph of $\mathbb{Z}^d$ (where $d \geq 3$) and join them by an edge (say between their respective identity elements). So a vertex $x \in \Gamma$ can be written as $x = (z, i)$ with $i \in 1, 2$ and $z \in \mathbb{Z}^d$. Consider $f = \sum_{i \geq 0} P_e^{(i)}$ on $\mathbb{Z}^d$ (where $e \in \mathbb{Z}^d$ is the identity element). Define

$$g(z, i) = \begin{cases} f(z) & \text{if } i = 1, \\ K + 2f(0) - f(z) & \text{if } i = 2, \end{cases}$$

where $K = \nabla^* \nabla f(0)$. Then g is harmonic and $\nabla g \in \ell^p(E)$ for $p < 2d$ (since $f \in \ell^p(\mathbb{Z}^d)$ for $p < 2d$). Since g is non-constant, $\underline{\ell^p H}^1(\Gamma) \neq \{0\}$ for any $p < d/2$.

Actually, since Γ has two ends, using Proposition A.1 one has that $\underline{\ell^1 H}^1(\Gamma) \neq \{0\}$. Since a corollary of Theorem 1.2 is that $\underline{\ell^q H}^1(\Gamma) \hookrightarrow \underline{\ell^p H}^1(\Gamma)$ for $1 \leq q \leq p < d/2$, one also sees that $\ell^p H^1(\Gamma) \neq \{0\}$ for $p \in [1, d/2[$.

In fact, the previous example could have been done for any Cayley graph (as soon as the growth is faster than quadratic).

Another quotient which is sometimes studied is the $\ell^{p,q}$-cohomology. This is the quotient,

$$\ell^{p,q} H^1(\Gamma) := \frac{\mathsf{D}^p(\Gamma)}{\ell^q(X) + \mathbb{K}}.$$

Recall that in Lemma 3.13, if $g \in \mathsf{D}^p(\Gamma)$ and Γ has IS_d, then $\tilde{g} - g \in \ell^q(X)$ for $q > \frac{dp}{d-2p}$. As a corollary

Corollary 4.5. *If Γ is Liouville and has IS_d, then $\ell^{p,q} H^1(\Gamma)$ is trivial for all $q > \frac{dp}{d-2p}$.*

There is also an analogue of Theorem 1.3:

Proposition 4.6. *Assume Γ' has IS_d and Γ' is quasi-isometric to a spanning subgraph of Γ. Let $p < \infty$ and $q > \frac{dp}{d-2p}$. Then non-trivial $\ell^{p,q}$-classes of Γ are given by functions $g \in \mathsf{D}^p(\Gamma)$ whose class is non-trivial in $\ell^{p,q} H^1(\Gamma')$.*

4.3. Normal subgroups. The aim here is to prove Theorem 1.4. A preliminary result on graphs "stitched" together is necessary.

Lemma 4.7. *Assume $d/2 > p \geq 1$. Let Γ_i be a family of graphs all satisfying IS_d with the same constant. Let Γ^Π be the disjoint union of the Γ_i. Fix some $k \in \mathbb{Z}_{\geq 2}$ and let Γ' be obtained by adding $\leq k$ edges to each Γ_i inside Γ^Π so that the resulting graph is connected. Then $[g] \neq 0 \in \underline{\ell^p H}^1(\Gamma')$ if and only at least one of the following holds:*

- *there exists i such that $[g_{|\Gamma_i}] \neq 0 \in \underline{\ell^p H}^1(\Gamma_i)$;*

- *there exists $i \neq j$ such that $g_{|\Gamma_i}$ and $g_{|\Gamma_j}$ are constant at ∞ (in the sense of Lemma 3.1) but not the same constant.*

If the Γ_i have IS_ω, the statement holds for unreduced cohomology.

Proof. If all Γ_i satisfy IS_d with the same constant then so do Γ^Π and Γ' (a consequence of the concavity of $t \mapsto t^{1-1/d}$).

Without loss of generality, one may assume that g is bounded (by the Holopainen and Soardi truncation lemma, see Lemma 2.1). If g is non-trivial in cohomology, then, by Lemma 3.9, the boundary value of g in Γ' is not constant. In particular, $g \not\to c$ as $|x| \to \infty$.

On the other hand, one could see g and $\tilde{g}$ (its boundary value in Γ') as functions on Γ^Π or restrict them to the Γ_i. In each Γ_i, note that there are at most k vertices which are connected to some other Γ_j in Γ'. Since the Γ_i are transient ($d > 2$), the probability that a random walker starting at x hits one of those k vertices decreases to 0 as x tends to infinity. This means that the boundary value of $g_{|\Gamma_i}$ will (for large enough x) be arbitrarily close to $(\tilde{g})_{|\Gamma_i}$.

Thus, a first possibility is that the boundary of g restricted to one of the Γ_i will be non-trivial. If this is not the case, then $g_{|\Gamma_i} \to c_i$ as $|x| \to \infty$. Since the boundary value of g in Γ' is non-trivial, the constants c_i must be different.

Similarly, if all $[g_{|\Gamma_i}] = 0$, then, by Lemma 3.8, all boundary values of the copies of Γ_i are constant. By Lemma 3.9, g is trivial exactly when these constant are the same.

Lastly, if the Γ_i have IS_ω, boundary values of g are in the same unreduced class as g (see Remark 3.12), so unreduced cohomology may be considered. $\square$

Proof of Theorem 1.4. Let $N \lhd G$ be an infinite normal subgroup of G (a finitely generated group). If N is finitely generated as a group, it is possible to pick a Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ so that S generates N and G. Denote by $\Gamma_N = \mathrm{Cay}(N, S \cap N)$. Then Γ admits $[G : N]$ (in the present case, infinitely many) copies of Γ_N as a subgraph.

Assume N has polynomial growth at least d (where $d \geq 3$). Let $p \in [1, d/2[$. Let Γ and Γ_N be as above. Pick the Γ_i to be the various copies of Γ_N in Γ: Γ_i is the graph restricted to the vertices $N\gamma_i$ where γ_i are representatives for the cosets of N. Since they are all isomorphic (edges are defined on the left), they have IS_d with the same constant. Consider two graphs: the graph of the disjoint copies of Γ_N, say Γ^Π as above, and this same graph with edges added between the different copies so that the result is connected, call it Γ' as above. Note that only finitely many edges need to be added to each copy of Γ_N to construct Γ' (in a way that Γ' is connected) as G is finitely generated and N is normal.

Assume there is a non-trivial element g in $\mathsf{D}^p(\Gamma)$. By Corollary 4.1, it is also non-trivial in $\underline{\ell^p H}^1(\Gamma')$. By Lemma 4.7, there are two possibilities.

First possibility: one of its restriction to some $N\gamma_0$ possesses a non-constant boundary value. But this would imply that $\underline{\ell^p H}^1(\Gamma_N)$ is non-trivial; this contradicts the hypothesis.

Second possibility: the restrictions all have constant boundary values, but not the same constant. This would imply that, outside large enough finite sets, these restrictions are arbitrarily close to their constant value. But the gradient of g on the initial graph Γ would then not be in $\ell^p(E)$ (in fact, not even in $c_0(E)$) but in $\ell^\infty(E)$. Indeed, since N is normal, the distance from $x \in \Gamma_i$ to Γ_j is always the same. This contradicts $g \in \mathsf{D}^p(\Gamma)$.

As before, in the case when N is non-amenable, boundary values belong to the same unreduced class (see Remark 3.12), so the result will extend to unreduced cohomology in this case. □

A. Ends and degree one reduced ℓ^1-cohomology

This section is devoted to the reduced degree one ℓ^1-cohomology. This result is known (even well-known, if one adheres to the rule that at least 3 persons were aware of it): P. Pansu was aware of this, the main argument of non-vanishing is present in Martin and Valette [31, Example 3 in §4] who mention hearing it from M. Bourdon. It is included here for the sake of completion.

The ends of a graph are the infinite components of a group which cannot be separated by a finite (*i.e.* compact) set. More precisely, an end ξ is a function from finite sets to infinite connected components of their complement so that $\xi(F) \cap \xi(F') \neq \varnothing$ (for any F and F'). It may also be seen as an equivalence class of (infinite) rays who eventually leave any finite set. Two rays r and r' are equivalent if, for any finite set F, the infinite part of r and r' lie in the same (infinite) connected component.

Thanks to Stallings' theorem, groups with infinitely many ends contain an (non-trivial) amalgamated product or a (non-trivial) HNN extension. Being without ends is equivalent to being finite, and amenable groups may not have infinitely many ends. This may be seen using Stallings' theorem, see also Moon and Valette [32] for a direct proof. An intuitive idea is that a Cayley graph with infinitely many ends contains has a quasi-isometry to a tree T with strong isoperimetric constant, and hence cannot be amenable. Groups with two ends admit $\mathbb{Z}$ as a finite index subgroup. These groups are peculiar, as they have non-trivial reduced ℓ^1-cohomology in degree 1, even if their reduced ℓ^p-cohomology (in all degrees) vanishes for $1 < p < \infty$.

So outside virtually-$\mathbb{Z}$ groups, all infinite amenable groups have one end.

Proposition A.1. *Let Γ be a connected graph, then $\underline{\ell^1 H}^1(\Gamma) = 0$ if and only if the number of ends of Γ is ≤ 1. More precisely, let $\mathcal{N} = \mathbb{K}^{\mathrm{ends}(\Gamma)}/\mathbb{K}$ be the vector space of functions on ends modulo constants. There is a boundary value map $\beta : \mathsf{D}^1(\Gamma) \to \mathcal{N}$ such that $\beta(g) = \beta(h) \iff [g] = [h] \in \underline{\ell^1 H}^1(\Gamma)$.*

Note that the isomorphism is in the category of vector spaces, not of normed vector spaces. In a few cases, the norm on $\mathcal{N}$ resembles the norm of the quotient $\ell^\infty(|\mathrm{ends}|)/\mathbb{K}$. The proof is slightly different than the argument from M. Bourdon found in Martin and Valette [31, Example 3 in §4].

Proof. Note that $\mathsf{D}^1(\Gamma) \subset \ell^\infty(X)$: if $g \in \mathsf{D}^1(\Gamma)$, then, for P a path from x to y,

$$|g(y)| = |g(x) + \sum_{e \in P} g(e)| \leq |g(x)| + \|\nabla g\|_{\ell^1(E)}.$$

In fact, $\|g\|_{\ell^\infty(X)} \leq \|g\|_{\mathsf{D}^1(\Gamma)} + \inf_{x \in X} |g(x)|$. Since functions in ℓ^1 decrease at ∞, if one removes a large enough finite set, the function g on the resulting graph is almost constant. In particular, it is possible to define a value of g on each end: let B_n be the ball of radius n at some fixed vertex (root) o, then

$$\beta g(\xi) := \lim_{n \to \infty} g(x_n) \text{ where } x_n \in \xi(B_n)$$

Alternatively, if $r : \mathbb{Z}_{\geq 0} \to X$ is a ray representing the end ξ, then the value at ξ can also be defined as $\lim_{n \to \infty} g(r(n))$. It is fairly straightforward to check these limit do not depend on the choice (of x_n and o or of the ray r).

Fix an end ξ_0. Then, define $\beta : \mathsf{D}^1(\Gamma) \to \mathcal{N}$ by changing with a constant the value of g to be 0 at ξ_0 and then looking at the values at the ends. This map is continuous and trivial on $\ell^1(X) + \mathbb{K}$ (since functions in $\ell^1(X)$ have trivial value at the ends). By continuity, $\overline{\ell^1(X) + \mathbb{K}}^{\mathsf{D}^1(\Gamma)} \subset \ker \beta$.

Assume, $\beta(f) = 0$, this means that, for all $\varepsilon > 0$, there exists $X_\varepsilon \subset X$ a finite set such that $|f(X_\varepsilon^c)| < \varepsilon$. The proof then follows *verbatim*, as in Lemma 3.1. $\quad\square$

Amenable groups with two ends step strangely out of the crowd: although their ℓ^p-cohomology is always trivial if $p > 1$, it is non-trivial for $p = 1$ (actually isomorphic to the base field). An amusing corollary is

Corollary A.2. *Let G be a finitely generated group. G has infinitely many ends if and only if for some (and hence all) Cayley graph Γ, for all $p \in [1, \infty[$, $\underline{\ell^p H}^1(\Gamma) \neq 0$. G has two ends if and only if for some (and hence all) Cayley graph Γ, for all $p \in]1, \infty[$, $\underline{\ell^p H}^1(\Gamma) = 0$ but $\underline{\ell^1 H}^1(\Gamma) = \mathbb{K}$.*

Proof. Use Proposition A.1 for reduced ℓ^1-cohomology, use any vanishing theorem on groups of polynomial growth (Kappos [24], [16, Theorem 3.2], or Tessera [44]) to get the remaining values of p for groups with two ends, and finally use Theorem 1.3 on groups with infinitely many ends (which are in particular non-amenable). □

It is worth noting that Bekka and Valette showed in [4, Lemma 2, p. 316] that (for G discrete) the cohomology $H^1(G, \mathbb{C}G)$ is also isomorphic (as a vector space) to $\mathcal{N}$. Furthermore, by [4, Proposition 1], there is an embedding $H^1(G, \mathbb{C}G) \hookrightarrow \ell^1 H^1(G)$. A careful reading would probably reveal this remains injective in reduced cohomology (the only case to check is when G has two ends).

For completion, let us also mention an other extremal case:

Proposition A.3. *Let Γ be an infinite graph, then $\underline{\ell^\infty H}^1(\Gamma) \neq \{0\}$.*

Proof. $\ell^\infty H^1(\Gamma)$ is the quotient of Lipschitz functions by bounded Lipschitz functions so is manifestly never trivial. Further, if one takes g to be the distance to a fixed vertex r, *i.e.* $g(\gamma) = d(\gamma, r)$, then $[g]$ is not trivial in the reduced cohomology. Indeed, a function with $\|g - h\| < 1/2$ has positive gradient on all edges between the spheres around r. As a consequence h may not be bounded, and no element of $\ell^\infty(X)$ may be close to g. □

It seems quite plausible that $\underline{c_0 H}^1(\Gamma) = \{0\}$ for any graph. Indeed, let Ω_ε a big ball such that $\|f\|_{\mathrm{D}\infty(\Gamma \setminus \Omega_\varepsilon)} \leq \varepsilon$. Start constructing g_ε by making it equal to f on Ω_ε. If these functions can be extended so that its gradient is always $< \varepsilon$ outside Ω_ε and that it is finitely supported, then $\|f - g_\varepsilon\|_{\mathrm{D}\infty} < 2\varepsilon$. Hence the class of f would be trivial. It might be useful to use Lemma 2.1 (*i.e.* f may be assumed bounded) to conclude.

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