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Groups, Geometry, and Dynamics

Actions of groups of homeomorphisms on one-manifolds

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Abstract. In this article, we describe all the group morphisms from the group of compactly-supported homeomorphisms isotopic to the identity of a manifold to the group of homeomorphisms of the real line or of the circle.

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1. Introduction

Fix a connected manifold M (without boundary). For an integer $r \geq 0$, we denote by $\mathrm{Diff}^r(M)$ the group of C^r-diffeomorphisms of M. When $r = 0$, this group will also be denoted by $\mathrm{Homeo}(M)$. For a homeomorphism f of M, the *support* of f is the closure of the set:

$$\{x \in M,\ f(x) \neq x\}.$$

We say that a homeomorphism f in $\mathrm{Diff}^r(M)$ with compact support is *compactly isotopic to the identity* if there exists a C^r map

$$F \colon M \times [0,1] \longrightarrow M$$

such that

(1) for any $t \in [0,1]$, $F(\cdot, t)$ belongs to $\mathrm{Diff}^r(M)$;

(2) there exists a compact subset $K \subset M$ such that, for any $t \in [0,1]$, the support of the diffeomorphism $F(\cdot, t)$ is contained in K;

(3) $F(\cdot, 0) = \mathrm{Id}_M$ and $F(\cdot, 1) = f$.

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We denote by $\mathrm{Diff}^r_0(M)$ ($\mathrm{Homeo}_0(M)$ if $r = 0$) the group of compactly supported C^r-diffeomorphisms of M which are compactly isotopic to the identity. The main reason why we are considering these groups is the following difficult theorem by Fisher, Mather and Thurston (see [1], [2], [5], [10], and [11]).

Theorem (Fisher, Mather, and Thurston). *Let M be a connected manifold. The group $\mathrm{Diff}^r_0(M)$ is simple if $r \neq \dim(M) + 1$.*

This theorem will be used throughout the article. It implies for instance that any group morphism from a group of the form $\mathrm{Diff}^r_0(M)$ to another group is either one-to-one or trivial: the kernel of such a morphism is a normal subgroup of $\mathrm{Diff}^r_0(M)$ and hence is either trivial or the whole group. As an application of this theorem, let us prove that any group morphism $\mathrm{Homeo}_0(\mathbb{T}^2) \to \mathrm{Homeo}(\mathbb{R})$ is trivial. Notice that the group $\mathrm{Homeo}_0(\mathbb{T}^2)$ contains finite order elements (the rational translations) whereas the group of homeomorphisms of the real line does not. Hence, a finite order element has to be sent to the identity under such a morphism which is not one-to-one. Therefore, it is trivial.

In [7], Étienne Ghys asked whether the following statement was true: if M and N are two closed manifolds and if there exists a non-trivial morphism $\mathrm{Diff}^\infty_0(M) \to \mathrm{Diff}^\infty_0(N)$, then $\dim(M) \leq \dim(N)$. In [9], Kathryn Mann proved the following theorem. Take a connected manifold M of dimension greater than 1 and a one-dimensional connected manifold N. Then any morphism $\mathrm{Diff}^\infty_0(M) \to \mathrm{Diff}^\infty_0(N)$ is trivial: she answers Ghys's question in the case where the manifold N is one-dimensional. Mann also describes all the group morphisms $\mathrm{Diff}^r_0(M) \to \mathrm{Diff}^r_0(N)$ for $r \geq 3$ when M as well as N are one-dimensional. The techniques involved in the proofs of these theorems are Kopell's lemma (see [16] Theorem 4.1.1) and Szekeres's theorem (see [16] Theorem 4.1.11). These theorems are valid only for a regularity at least C^2. In this article, we prove similar results in the case of a C^0 regularity. The techniques used are different.

Theorem 1.1. *Let M be a connected manifold of dimension greater than 2 and let N be a connected one-manifold. Then any group morphism*

$$\mathrm{Homeo}_0(M) \longrightarrow \mathrm{Homeo}(N)$$

is trivial.

The case where the manifold M is one-dimensional is also well-understood.

Using bounded cohomology techniques, Matsumoto proved the following theorem (see [13] Theorem 5.3) which is also a key point in the proof of our theorems.

Theorem (Matsumoto). *Every group morphism*

$$\mathrm{Homeo}_0(\mathbb{S}^1) \longrightarrow \mathrm{Homeo}_0(\mathbb{S}^1)$$

is a conjugation by a homeomorphism of the circle.

Notice that any group morphism $\mathrm{Homeo}_0(\mathbb{S}^1) \to \mathrm{Homeo}(\mathbb{R})$ is trivial. Recall that, as the group $\mathrm{Homeo}_0(\mathbb{S}^1)$ is simple, such a group morphism is either one-to-one or trivial. However, the group $\mathrm{Homeo}_0(\mathbb{S}^1)$ contains torsion elements whereas the group $\mathrm{Homeo}(\mathbb{R})$ does not: such a morphism cannot be one-to-one.

It remains to study the case of a morphism defined on $\mathrm{Homeo}_0(\mathbb{R})$.

Theorem 1.2. *Let N be a connected one-manifold. For any group morphism*

$$\varphi\colon \mathrm{Homeo}_0(\mathbb{R}) \longrightarrow \mathrm{Homeo}(N),$$

there exists a closed set $K \subset N$ such that

(1) *the set K is pointwise fixed under any homeomorphism in $\varphi(\mathrm{Homeo}_0(\mathbb{R}))$;*

(2) *for any connected component I of $N - K$, there exists a homeomorphism*

$$h_I\colon \mathbb{R} \longrightarrow I$$

such that

$$\varphi(f)_{|I} = h_I f h_I^{-1}, \quad \textit{for all } f \in \mathrm{Homeo}_0(\mathbb{R}).$$

Notice that the set K has to be the set of points which are fixed under every element in $\varphi(\mathrm{Homeo}_0(\mathbb{R}))$.

The following remark will be used repeatedly in the proof of Theorems 1.1 and 1.2. Consider a nontrivial morphism φ from a group G to the group $\mathrm{Homeo}_+(\mathbb{R})$. Denote by F the closed set of points of the real line which are fixed under every element in $\varphi(G)$. Take a connected component I of $\mathbb{R} - F$. Any homeomorphism in $\varphi(G)$ preserves this connected component I. Consider then the morphism

$$\begin{aligned} G &\longrightarrow \mathrm{Homeo}_+(I), \\ g &\longmapsto \varphi(g)_{|I}. \end{aligned}$$

Notice that the image of this morphism has no global fixed point and that the interval I is homeomorphic to the real line. We have just seen that any morphism $G \to \mathrm{Homeo}_+(\mathbb{R})$ induces by restriction a morphism without global fixed point. Hence, to prove that any morphism $G \to \mathrm{Homeo}_+(\mathbb{R})$ is trivial, it suffices to prove that any such morphism has a fixed point.

2. Proofs of Theorems 1.1 and 1.2

Fix integers $d \geq k \geq 0$. We will call *embedded k-dimensional ball of* $\mathbb{R}^d$ the image of the closed unit ball of $\mathbb{R}^k = \mathbb{R}^k \times \{0\}^{d-k} \subset \mathbb{R}^d$ under a homeomorphism of $\mathbb{R}^d$. Take an embedded k-dimensional ball $D \subset \mathbb{R}^d$ (which is a single point if $k = 0$). We denote by G_D^d the group of homeomorphisms of $\mathbb{R}^d - D$ with compact support which are compactly isotopic to the identity. As any homeomorphism in the group G_D^d is equal to the identity near the embedded ball D, it can be continuously extended by the identity on the ball D. Hence the group G_D^d can be seen as a subgroup of $\mathrm{Homeo}_0(\mathbb{R}^d)$.

Finally, if G denotes a subgroup of $\mathrm{Homeo}(\mathbb{R}^d)$, a point $p \in \mathbb{R}^d$ is said to be fixed under the group G if it is fixed under all the elements of this group. We denote by $\mathrm{Fix}(G)$ the (closed) set of fixed points of G.

The theorems will be deduced from the following propositions. The two first propositions will be proved respectively in Sections 3 and 4.

Proposition 2.1. *Let $d > 0$ and let $\varphi\colon \mathrm{Homeo}_0(\mathbb{R}^d) \to \mathrm{Homeo}(\mathbb{R})$ be a group morphism. Suppose that no point of the real line is fixed under the group $\varphi(\mathrm{Homeo}_0(\mathbb{R}^d))$. Then, for any embedded $(d-1)$-dimensional ball $D \subset \mathbb{R}^d$, the group $\varphi(G_D^d)$ admits at most one fixed point.*

Proposition 2.2. *Let $d > 0$ and*

$$\varphi\colon \mathrm{Homeo}_0(\mathbb{R}^d) \longrightarrow \mathrm{Homeo}(\mathbb{R})$$

be a group morphism. Then, for any point p in $\mathbb{R}^d$, the group $\varphi(G_p^d)$ admits at least one fixed point.

Proposition 2.3. *Let $d > 0$. For any group morphism*

$$\psi\colon \mathrm{Homeo}_0(\mathbb{R}^d) \longrightarrow \mathrm{Homeo}(\mathbb{S}^1),$$

the group $\psi(\mathrm{Homeo}_0(\mathbb{R}^d))$ has a fixed point.

Proof of Proposition 2.3. Recall that the group $\mathrm{Homeo}_0(\mathbb{R}^d)$ is infinite and simple and that the group $\mathrm{Homeo}(\mathbb{S}^1)/\mathrm{Homeo}_0(\mathbb{S}^1)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Hence any morphism $\mathrm{Homeo}_0(\mathbb{R}^d) \to \mathrm{Homeo}(\mathbb{S}^1)/\mathrm{Homeo}_0(\mathbb{S}^1)$ is trivial. Therefore, the image of a morphism $\mathrm{Homeo}_0(\mathbb{R}^d) \to \mathrm{Homeo}(\mathbb{S}^1)$ is contained in $\mathrm{Homeo}_0(\mathbb{S}^1)$.

The rest of the proof of this proposition uses a result by Ghys. Ghys associates to any morphism from a group G to the group $\mathrm{Homeo}_0(\mathbb{S}^1)$ an element of the second bounded cohomology group $H_b^2(G,\mathbb{Z})$ of the discrete group G, which we call the bounded Euler class of this action of G. This class vanishes if and only if the action has a global fixed point on the circle. For some more background about the bounded cohomology of groups and the bounded Euler class of a group acting on a circle, see Section 6 in [6].

By a theorem by Matsumoto and Morita (see Theorem 3.1 in [14]):

$$H_b^2(\mathrm{Homeo}_0(\mathbb{R}^d),\mathbb{Z}) = \{0\}\,.$$

Therefore, the bounded Euler class of a morphism $\mathrm{Homeo}_0(\mathbb{R}^d) \to \mathrm{Homeo}_0(\mathbb{S}^1)$ vanishes: this action has a fixed point. □

Proof of Theorem 1.1. Let $d = \dim(M)$. The theorem will be deduced from the following lemma.

Lemma 2.4. *For any $d > 1$, any group morphism* $\mathrm{Homeo}_0(\mathbb{R}^d) \to \mathrm{Homeo}(\mathbb{R})$ *is trivial.*

Using Proposition 2.3, we obtain the following immediate corollary.

Corollary 2.5. *For any $d > 1$, any group morphism* $\mathrm{Homeo}_0(\mathbb{R}^d) \to \mathrm{Homeo}(\mathbb{S}^1)$ *is trivial.*

Let us see why this lemma and this corollary imply the theorem. Consider a morphism $\mathrm{Homeo}_0(M) \to \mathrm{Homeo}_0(N)$. Take an open set $U \subset M$ homeomorphic to $\mathbb{R}^d$ and let us denote by $\mathrm{Homeo}_0(U)$ the subgroup of $\mathrm{Homeo}_0(M)$ consisting of homeomorphisms supported in U. By Lemma 2.4 and Corollary 2.5, the restriction of this morphism to the subgroup $\mathrm{Homeo}_0(U)$ is trivial. Moreover, as the group $\mathrm{Homeo}_0(M)$ is simple, such a group morphism is either one-to-one or trivial: it is necessarily trivial in this case. □

Proof of Lemma 2.4. Take a group morphism $\varphi\colon \mathrm{Homeo}_0(\mathbb{R}^d) \to \mathrm{Homeo}(\mathbb{R})$. Suppose by contradiction that this morphism is nontrivial. Replacing if necessary $\mathbb{R}$ with a connected component of the complement of the closed set $\mathrm{Fix}(\varphi(\mathrm{Homeo}_0(\mathbb{R}^d)))$, we can suppose that the group $\varphi(\mathrm{Homeo}_0(\mathbb{R}^d))$ has no fixed points.

Claim 2.6. *For any points $p_1 \neq p_2$ in $\mathbb{R}^d$,*

$$\mathrm{Fix}(\varphi(G^d_{p_1})) \cap \mathrm{Fix}(\varphi(G^d_{p_2})) = \emptyset.$$

Proof. The proof of this claim requires the following lemma which will be proved afterwards.

Lemma 2.7. *Let $d \geq 1$ and $d \geq k \geq 0$ be integers. Let D_1 and D_2 be two disjoint embedded k-dimensional balls of $\mathbb{R}^d$. Then, for any homeomorphism f in $\mathrm{Homeo}_0(\mathbb{R}^d)$, there exist homeomorphisms f_1, f_3 in $G^d_{D_1}$ and f_2 in $G^d_{D_2}$ such that*

$$f = f_1 f_2 f_3.$$

Take two points p_1 and p_2 in $\mathbb{R}^d$. Suppose by contradiction that

$$\mathrm{Fix}(\varphi(G^d_{p_1})) \cap \mathrm{Fix}(\varphi(G^d_{p_2})) \neq \emptyset.$$

By Lemma 2.7 applied to the 0-dimensional balls $\{p_1\}$ and $\{p_2\}$, a point in this set is a fixed point of the group $\varphi(\mathrm{Homeo}_0(\mathbb{R}^d))$, a contradiction. $\square$

By Proposition 2.2, the sets $\mathrm{Fix}(\varphi(G^d_p))$, for $p \in \mathbb{R}^d$ are nonempty. We just saw that they are pairwise disjoint. Recall that, for any embedded $(d-1)$-dimensional ball D, the set $\mathrm{Fix}(\varphi(G^d_D))$ contains the union of the sets $\mathrm{Fix}(\varphi(G^d_p))$ over the points p in the closed set D. Hence, this set has infinitely many points as $d \geq 2$, a contradiction with Proposition 2.1. $\square$

Proof of Lemma 2.7. To, prove this lemma, we use the following theorem by Brown and Gluck (see Theorem 7.1 in [3]), which is also a consequence of the annulus theorem by Kirby and Quinn (see [8] and [17]).

Theorem (Brown–Gluck). *Let $d \geq 1$ and let B_1 and B_2 be two d-dimensional balls of $\mathbb{R}^d$ such that the ball B_1 is contained in the interior of B_2. Let h be any homeomorphism in $\mathrm{Homeo}_0(\mathbb{R}^d)$ such that the ball $h(B_1)$ is also contained in the interior of B_2. Then there exists a homeomorphism $\tilde{h}$ in $\mathrm{Homeo}_0(\mathbb{R}^d)$ with the following properties*:

(1) *the homeomorphism $\tilde{h}$ is supported in B_2*;

(2) $\tilde{h}_{|B_1} = h_{|B_1}$.

Take a homeomorphism f in $\mathrm{Homeo}_0(\mathbb{R}^d)$.

Claim 2.8. *There exists a homeomorphism f_1 $G^d_{D_1}$ such that f_1^{-1} sends the k-dimensional embedded ball $f(D_1)$ to a k-dimensional embedded ball which lies in the same connected component of $\mathbb{R}^d - D_2$ as the embedded ball D_1.*

Notice that, if $d \neq 1$, the set $\mathbb{R}^d - D_2$ is connected. In the case $d \neq 1$, this lemma amounts to finding a homeomorphism which sends the ball $f(D_1)$ to a ball which is disjoint from D_2.

Proof. First suppose that $d = 1$. If $\sup(D_1) < \inf(D_2)$, take as f_1^{-1} any homeomorphism in $\mathrm{Homeo}_0(\mathbb{R}^d)$ supported in $(\sup(D_1), +\infty)$ which sends the point $\sup(h(D_1))$ to a point $x < \inf(D_2)$. If $\sup(D_2) < \inf(D_1)$, take as f_1^{-1} any homeomorphism in $\mathrm{Homeo}_0(\mathbb{R}^d)$ supported in $(-\infty, \inf(D_1))$ which sends the point $\inf(h(D_1))$ to a point $x > \inf(D_2)$.

Now suppose that $d \geq 2$. It is not difficult to find a d-dimensional embedded ball B which contains the k-dimensional ball D_2 and a point p outside $f(D_1)$ in its interior: using the definition of an embedded ball, find first a d-dimensional B_0 which contains D_2 in its interior. If this ball is not contained in $f(D_1)$ take $B = B_0$. Otherwise take any point p which does not belong to $f(D_1)$ and consider a tubular neighbourhood T of a path in $\mathbb{R}^d - D_1$ which joins the ball B_0 and the point p to construct the ball B out of T and B.

Changing coordinates if necessary, we can suppose that $p = 0 \in \mathbb{R}^d$ and that the ball B is the unit ball. Consider any vector field X of $\mathbb{R}^d$ which is supported in B and which is equal to $x \mapsto x$ on a ball centered at 0 containing D_2. Let V be a neighbourhood of the point 0 which is disjoint from the embedded ball $f(D_1)$. Denote by φ_X^t the time t of the flow of the vector field X. Observe that there exists $T > 0$ such that $\varphi_X^T(B - V) \cap D_2 = \emptyset$. Hence $\varphi_X^T(f(D_1)) \cap D_2 = \emptyset$. It suffices to take $f_1^{-1} = \varphi_X^T$. $\square$

Take a d-dimensional ball B_2 with the following properties:

(1) it contains D_1 and $f_1^{-1}f(D_1)$;

(2) it is disjoint from the embedded ball D_2.

Consider a d-dimensional ball B_1 contained in the interior of the embedded ball B_2 such that $f_1^{-1}f(B_1)$ is contained in the interior of B_2. Apply the theorem by Brown and Gluck above with the balls B_1, B_2 and the homeomorphism $h = f_1^{-1}f$: there exists a homeomorphism $\hat{f}_2$ in $G_{D_2}^d$ which is equal to $f_1^{-1}f$ in a neighbourhood of the k-dimensional embedded ball D_1.

Notice that the homeomorphism $\hat{f}_2^{-1}f_1^{-1}f$ pointwise fixes a neighbourhood of the embedded ball D_1. However, its restriction to $\mathbb{R}^d - D_1$ might not be compactly isotopic to the identity. Nevertheless, this homeomorphism of $\mathbb{R}^d - D_1$ is compactly isotopic to a homeomorphism η whose support is contained in a small neighbourhood of the embedded ball D_1 and is disjoint from the embedded ball D_2: in order to see it, conjugate the homeomorphism $\hat{f}_2^{-1}f_1^{-1}f$ with the flow at a sufficiently large time of a vector field for which a small neighbourhood of the embedded ball D_1 is an attractor.

Let us check that the homeomorphism $\eta_{|\mathbb{R}^d - D_2}$ is compactly isotopic to the identity. To prove it, it suffices to conjugate this homeomorphism by a continuous family of homeomorphisms $(h_t)_{t\in[0,+\infty)}$ supported in $\mathbb{R}^d - D_2$ such that

(1) $h_0 = \mathrm{Id}$ and

(2) the family of compact sets $(h_t(\mathrm{supp}(\eta)))_{t\geq 0}$ converges to a point for the Hausdorff topology as $t \to +\infty$.

Hence the continuous family of homeomorphisms $h_t \eta h_t^{-1}$ converges to the identity as $t \to +\infty$ (this the well-known Alexander trick).

To finish the proof of the lemma, it suffices to take

$$f_2 = \hat{f}_2 \eta$$

and

$$f_3 = f_2^{-1} f_1^{-1} f. \qquad \square$$

Proof of Theorem 1.2. Let $\varphi\colon \mathrm{Homeo}_0(\mathbb{R}) \to \mathrm{Homeo}(N)$ be a nontrivial group morphism. By Proposition 2.3, we can suppose that the manifold N is the real line $\mathbb{R}$. Replacing $\mathbb{R}$ with a connected component of the complement of the closed set $\mathrm{Fix}(\varphi(\mathrm{Homeo}_0(\mathbb{R})))$ if necessary, we can suppose that the group $\varphi(\mathrm{Homeo}_0(\mathbb{R}))$ has no fixed point (see the remark at the end of the introduction). Recall that the group $\mathrm{Homeo}_0(\mathbb{R})$ is simple. Hence any morphism $\mathrm{Homeo}_0(\mathbb{R}) \to \mathbb{Z}/2\mathbb{Z}$ is trivial. Thus, any element of the group $\varphi(\mathrm{Homeo}_0(\mathbb{R}))$ preserves the orientation of $\mathbb{R}$.

By Propositions 2.1 and 2.2, for any real number x, the group $\varphi(G_x^1)$ has a unique fixed point $h(x)$. Take a homeomorphism f in $\mathrm{Homeo}_0(\mathbb{R})$ which sends a point x in $\mathbb{R}$ to a point y in $\mathbb{R}$. Then $fG_x^1 f^{-1} = G_y^1$ and, taking the image under φ, $\varphi(f)\varphi(G_x^1)\varphi(f)^{-1} = \varphi(G_y^1)$. Hence $\varphi(f)(\mathrm{Fix}(\varphi(G_x^1))) = \mathrm{Fix}(\varphi(G_y^1))$. Therefore, for any homeomorphism f in $\mathrm{Homeo}_0(\mathbb{R})$, $\varphi(f)h = hf$.

Let us prove that the map h is one-to-one. Suppose by contradiction that there exist real numbers $x \neq y$ such that $h(x) = h(y)$. The point $h(x)$ is fixed under the groups $\varphi(G_x^1)$ and $\varphi(G_y^1)$. However, the groups G_x^1 and G_y^1 generate the group $\mathrm{Homeo}_0(\mathbb{R})$ by Lemma 2.7. Therefore, the point $h(x)$ is fixed under the group $\varphi(\mathrm{Homeo}_0(\mathbb{R}))$, a contradiction.

Now we prove that the map h is either strictly increasing or strictly decreasing. Fix two points $x_0 < y_0$ of the real line. For any two points $x < y$ of the real line, let us consider a homeomorphism $f_{x,y}$ in $\mathrm{Homeo}_0(\mathbb{R})$ such that $f_{x,y}(x_0) = x$ and

$f_{x,y}(y_0) = y$. As $\varphi(f_{x,y})h = hf_{x,y}$, the homeomorphism $\varphi(f_{x,y})$ sends the ordered pair $(h(x_0), h(y_0))$ to the ordered pair $(h(x), h(y))$. As the homeomorphism $\varphi(f_{x,y})$ is strictly increasing:

$$h(x) < h(y) \iff h(x_0) < h(y_0)$$

and

$$h(x) > h(y) \iff h(x_0) > h(y_0).$$

Hence the map h is either strictly increasing or strictly decreasing.

Now, it remains to prove that the map h is onto to complete the proof. Suppose by contradiction that the map h is not onto. Notice that the set $h(\mathbb{R})$ is preserved under the group $\varphi(\mathrm{Homeo}_0(\mathbb{R}))$. If this set had a lower bound or an upper bound, then the supremum of this set or the infimum of this set would provide a fixed point for the group $\varphi(\mathrm{Homeo}_0(\mathbb{R}))$, a contradiction. This set has neither upper bound nor lower bound. Let C be a connected component of the complement of the set $h(\mathbb{R})$. Replacing if necessary h by $-h$ and the morphism φ by its conjugate under $-\mathrm{Id}$, we can suppose that the map h is increasing. Let us denote by x_0 the supremum of the set of points x such that the real number $h(x)$ is lower than any point in the interval C. Then the point $h(x_0)$ is necessarily in the closure of C: otherwise, there would exist an interval in the complement of $h(\mathbb{R})$ which strictly contains the interval C. Hence the point $h(x_0)$ is either the infimum or the supremum of the interval C. As the proof is analogous in these two cases, we suppose from now on that the point $h(x_0)$ is the supremum of the interval C.

Choose, for each couple (z_1, z_2) of real numbers, a homeomorphism g_{z_1,z_2} in $\mathrm{Homeo}_0(\mathbb{R})$ which sends the point z_1 to the point z_2. Then the sets $g_{x_0,x}(C)$, for x in $\mathbb{R}$, are pairwise disjoint: they are pairwise distinct as their suprema are pairwise distinct (the supremum of the set $g_{x_0,x}(C)$ is the point $h(x)$). Moreover, those sets do not contain any point of $h(\mathbb{R})$ and the infima of those sets are accumulated by points in $h(\mathbb{R})$. Hence, these sets are pairwise disjoint. Then the set C has necessarily an empty interior as the topological space $\mathbb{R}$ is second-countable. Therefore $C = \{h(x_0)\}$, which is not possible. $\square$

3. Proof of Proposition 2.1

Fix $d > 0$ and a group morphism $\varphi\colon \mathrm{Homeo}_0(\mathbb{R}^d) \to \mathrm{Homeo}(\mathbb{R})$. We want to prove that, for any $(d-1)$-dimensional embedded ball D, the group $\varphi(G^d_D)$ has at most one global fixed point.

The proof of the proposition is similar to the proofs of Lemmas 3.6 and 3.7 in [15]. For an embedded $(d-1)$-dimensional ball D, let $F_D = \mathrm{Fix}(\varphi(G^d_D))$. Let us prove that these sets are pairwise homeomorphic. Take two embedded $(d-1)$-dimensional balls D an D' and take a homeomorphism h in $\mathrm{Homeo}_0(\mathbb{R}^d)$ which sends the set D onto D'. Observe that

$$hG^d_D h^{-1} = H^d_{D'}$$

and that

$$\varphi(h)\varphi(G^d_D)\varphi(h)^{-1} = \varphi(H^d_{D'}).$$

Therefore,

$$\varphi(h)(F_D) = F_{D'}.$$

In the case where these sets are all empty, there is nothing to prove. We suppose in what follows that they are not empty.

Given two disjoint embedded $(d-1)$-dimensional balls D and D', Lemma 2.7 implies, as in the proof of Lemma 2.4,

$$F_D \cap F_{D'} = \emptyset.$$

Lemma 3.1. *Fix an embedded $(d-1)$-dimensional ball D_0 of $\mathbb{R}^d$. For any connected component C of the complement of F_{D_0}, there exists an embedded $(d-1)$-dimensional ball D disjoint from D_0 such that the set F_D meets C.*

Proof. Let (a_1, a_2) be a connected component of the complement of F_{D_0}. It is possible that either $a_1 = -\infty$ or $a_2 = +\infty$. Consider a homeomorphism $e\colon \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R}^d$ such that $e(B^{d-1} \times \{0\}) = D_0$, where B^{d-1} denotes the unit closed ball in $\mathbb{R}^{d-1}$. For any real number x, let $D_x = e(B^{d-1} \times \{x\})$. Given two real $x \neq y$, take a homeomorphism $\eta_{x,y}$ in $\mathrm{Homeo}_0(\mathbb{R})$ which sends the point x to the point y. Consider a homeomorphism $h_{x,y}$ such that the following property is satisfied. The restriction of $eh_{x,y}e^{-1}$ to $B^{d-1} \times \mathbb{R}$ is equal to the map

$$\begin{aligned} B^{d-1} \times \mathbb{R} &\longrightarrow \mathbb{R}^{d-1} \times \mathbb{R}, \\ (p, z) &\longmapsto (p, \eta_{x,y}(z)). \end{aligned}$$

Notice that, for any real numbers x and y, $h_{x,y}(D_x) = D_y$

Let us prove by contradiction that there exists a real number $x \neq 0$ such that $F_{D_x} \cap (a_1, a_2) \neq \emptyset$. Suppose that, for any such embedded ball D_x, $F_{D_x} \cap (a_1, a_2) = \emptyset$. We claim that the open sets $\varphi(h_{0,x})((a_1, a_2))$ are pairwise disjoint. It is not possible as there would be uncountably many pairwise disjoint open intervals in $\mathbb{R}$.

Indeed, suppose by contradiction that there exists real numbers $x \neq y$ such that $\varphi(h_{0,x})((a_1, a_2)) \cap \varphi(h_{0,y})((a_1, a_2)) \neq \emptyset$. Notice that the homeomorphism $h_{0,x}^{-1} h_{0,y}$ and $h_{0,y}^{-1} h_{0,x}$ send respectively the set D_0 to sets of the form D_z and $D_{z'}$, where $z, z' \in \mathbb{R}$. Hence, for $i = 1, 2$, the homeomorphisms $\varphi(h_{0,x}^{-1} h_{0,y})$ (respectively $\varphi(h_{0,y}^{-1} h_{0,x})$) sends the point $a_i \in F_{D_0}$ to a point in F_{D_z} (respectively in $F_{D_{z'}}$). By hypothesis, these points do not belong to (a_1, a_2). Therefore

$$\varphi(h_{0,y}^{-1} h_{0,x})(a_1, a_2) = (a_1, a_2)$$

or

$$\varphi(h_{0,x})(a_1, a_2) = \varphi(h_{0,y})(a_1, a_2).$$

But this last equality cannot hold as the real endpoints of the interval on the left-hand side belong to F_{D_x} and the real endpoints point of the interval on the right-hand side belongs to F_{D_y}. Moreover, we saw that these two closed sets were disjoint, a contradiction. $\square$

Lemma 3.2. *Each set F_D contains only one point.*

Proof. Suppose that there exists an embedded $(d-1)$-dimensional ball D such that the set F_D contains two points $p_1 < p_2$. By Lemma 3.1, there exists an embedded $(d-1)$-dimensional ball D' disjoint from D such that the set $F_{D'}$ has a common point with the open interval (p_1, p_2). Take a real number $r < p_1$. Then, for any homeomorphisms g_1 in G_D, g_2 in $G_{D'}$ and g_3 in G_D,

$$\varphi(g_1) \circ \varphi(g_2) \circ \varphi(g_3)(r) < p_2.$$

By Lemma 2.7, this implies that the following inclusion holds:

$$\{\varphi(g)(r), g \in \mathrm{Homeo}_0(\mathbb{R}^d)\} \subset (-\infty, p_2].$$

The supremum of the left-hand set provides a fixed point for the action φ, a contradiction. $\square$

4. Proof of Proposition 2.2

This proof uses the following lemmas. For a subgroup G of $\mathrm{Homeo}_0(\mathbb{R}^d)$, we define the support $\mathrm{Supp}(G)$ of G as the closure of the set

$$\{x \in \mathbb{R}^d, \text{ there exists } g \in G,\ gx \neq x\}.$$

Let $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) = \{f \in \mathrm{Homeo}(\mathbb{R}),\ \text{for all } x \in \mathbb{R}, f(x+1) = f(x) + 1\}$.

To prove Proposition 2.2, we need the following lemmas.

Lemma 4.1. *Let G and G' be subgroups of the group* $\mathrm{Homeo}_+(\mathbb{R})$ *of orientation-preserving homeomorphisms of the real line. Suppose that the following conditions are satisfied.*

(1) *The groups G and G' are isomorphic to the group* $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.

(2) *The subgroups G and G' of* $\mathrm{Homeo}_+(\mathbb{R})$ *commute: $gg' = g'g$ for all $g \in G$, $g' \in G'e$.*

Then $\mathrm{Supp}(G) \subset \mathrm{Fix}(G')$ *and* $\mathrm{Supp}(G') \subset \mathrm{Fix}(G)$.

Lemma 4.2. *Let $d > 0$. Take any nonempty open subset U of $\mathbb{R}^d$. Then there exists a subgroup of* $\mathrm{Homeo}_0(\mathbb{R}^d)$ *isomorphic to* $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ *which is supported in U.*

Lemma 4.1 will be proved in the next section. We now provide a proof of Lemma 4.2.

Proof of Lemma 4.2. Take a closed ball B contained in U. Changing coordinates if necessary, we can suppose that B is the unit closed ball in $\mathbb{R}^d$. Take an orientation-preserving homeomorphism $h\colon \mathbb{R} \to (-1,1)$. For any orientation-preserving homeomorphism $f : \mathbb{R} \to \mathbb{R}$, we define the homeomorphism

$$\lambda_h(f)\colon \mathbb{R}^d \longrightarrow \mathbb{R}^d$$

in the following way.

(1) The homeomorphism $\lambda_h(f)$ is equal to the identity outside the interior of the ball B.

(2) For any $(x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} \cap \mathrm{int}(B)$:

$$\lambda_h(f)(x_1,x') = \left(\sqrt{1-\|x'\|^2}h \circ f \circ h^{-1}\left(\frac{x_1}{\sqrt{1-\|x'\|^2}}\right), x'\right).$$

The map λ_h defines an embedding of the group $\mathrm{Homeo}_+(\mathbb{R})$ into the group $\mathrm{Homeo}_0(\mathbb{R}^d)$. The image under λ_h of the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is a subgroup of $\mathrm{Homeo}_0(\mathbb{R}^d)$ which is supported in U. □

Let us complete now the proof of Proposition 2.2.

Proof of Proposition 2.2. Fix a point p in $\mathbb{R}^d$. Take a closed ball $B \subset \mathbb{R}^d$ which is centered at p. Let us prove that $\mathrm{Fix}(\varphi(G_B^d)) \neq \emptyset$.

Take a subgroup G_1 of $\mathrm{Homeo}_0(\mathbb{R}^d)$ which is isomorphic to $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ and supported in B. Such a subgroup exists by Lemma 4.2. This subgroup commutes with any subgroup G_2 of $\mathrm{Homeo}_0(\mathbb{R}^d)$ which is isomorphic to $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ and supported outside B.

If the group $\varphi(\mathrm{Homeo}_0(\mathbb{R}^d))$ admits a fixed point, there is nothing to prove. Suppose that this group has no fixed point. As the group $\mathrm{Homeo}_0(\mathbb{R}^d)$ is simple, the morphism φ is one-to-one. Moreover, any morphism $\mathrm{Homeo}_0(\mathbb{R}^d) \to \mathbb{Z}/2\mathbb{Z}$ is trivial: the morphism φ takes values in $\mathrm{Homeo}_+(\mathbb{R})$. Hence the subgroups $\varphi(G_1)$ and $\varphi(G_2)$ of $\mathrm{Homeo}(\mathbb{R})$ satisfy the hypothesis of Lemma 4.1. By this lemma,

$$\emptyset \neq \mathrm{Supp}(\varphi(G_1)) \subset \mathrm{Fix}(\varphi(G_2)).$$

Claim 4.3. *The group G_B^d is generated by the union of its subgroups isomorphic to* $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.

This claim implies that

$$\emptyset \neq \mathrm{Supp}(\varphi(G_1)) \subset \mathrm{Fix}(\varphi(G_B^d)).$$

Proof. For $d \geq 2$, observe that the open set $\mathbb{R}^d - B$ is connected. Hence, as we recalled in the introduction, the group G_B^d is simple by a theorem by Fisher (see [5]). The claim is a direct consequence of the simplicity of this group. In the case where $d = 1$, the open set $\mathbb{R} - B$ has two connected components. Denote by $[a, b]$ the compact interval B. The inclusions of the groups $\mathrm{Homeo}_0((-\infty, a))$ and $\mathrm{Homeo}_0((b, +\infty))$ induce an isomorphism

$$\mathrm{Homeo}_0((-\infty, a)) \times \mathrm{Homeo}_0((b, +\infty)) \longrightarrow G_B^d.$$

The simplicity of each factor of this decomposition implies the claim. □

Claim 4.4. *The set* $\mathrm{Fix}(\varphi(G_B^d))$ *is compact.*

Proof. Suppose by contradiction that there exists a sequence $(a_k)_{k\in\mathbb{N}}$ of real numbers in $\mathrm{Fix}(\varphi(G_B^d))$ which tends to $+\infty$ (if we suppose that it tends to $-\infty$, we obtain of course an analogous contradiction). Let us choose a closed ball $B' \subset \mathbb{R}^d$ which is disjoint from B. Observe that the subgroups G_B^d and $G_{B'}^d$ are conjugate in $\mathrm{Homeo}_0(\mathbb{R}^d)$ by a homeomorphism which sends the ball B to the ball B'. Then the subgroups $\varphi(G_B^d)$ and $\varphi(G_{B'}^d)$ are conjugate in the group $\mathrm{Homeo}_+(\mathbb{R})$. Hence the sets $\mathrm{Fix}(\varphi(G_B^d))$ and $\mathrm{Fix}(\varphi(G_{B'}^d))$ are homeomorphic: there exists a sequence $(b_k)_{k\in\mathbb{N}}$ of elements in $\mathrm{Fix}(\varphi(G_{B'}^d))$ which tends to $+\infty$. Take positive integers n_1, n_2 and n_3 such that $a_{n_1} < b_{n_2} < a_{n_3}$. Fix $x_0 < a_{n_1}$. We notice then that for any homeomorphisms $g_1 \in G_B^d$, $g_2 \in G_{B'}^d$ and $g_3 \in G_B^d$, the following inequality is satisfied:

$$\varphi(g_1)\varphi(g_2)\varphi(g_3)(x_0) < a_{n_3}.$$

However, by Lemma 2.7, any element g in $\mathrm{Homeo}_0(\mathbb{R}^d)$ can be written as a product

$$g = g_1 g_2 g_3,$$

where g_1 and g_3 belong to G_B^d and g_2 belongs to $G_{B'}^d$. The proof of this fact is similar to that of Lemma 2.7. Therefore,

$$\overline{\{\varphi(g)(x_0),\ g \in \mathrm{Homeo}_0(\mathbb{R}^d)\}} \subset (-\infty, a_{n_3}].$$

The greatest element of the left-hand set is a fixed point of the image of φ: this is not possible as this image was supposed to have no fixed point. □

Observe that the group $\varphi(G_p^d)$ is the union of its subgroup of the form $\varphi(G_{B'}^d)$, with B' varying over the set $\mathcal{B}_p$ of closed balls centered at the point p. By compactness, the set

$$\mathrm{Fix}(\varphi(G_p^d)) = \bigcap_{B' \in \mathcal{B}_p} \mathrm{Fix}(G_{B'}^d)$$

is nonempty. Proposition 2.2 is proved. □

5. Proof of Lemma 4.1

We start this section by recalling some facts about the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ of homeomorphisms of the real line which commute with integral translations. Observe that the center of the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is the subgroup generated by the translation $x \mapsto x + 1$. The quotient of this group by its center is the group $\mathrm{Homeo}_0(\mathbb{S}^1)$. The following lemma is classical.

Lemma 5.1. *Any group morphism* $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{Z}$ *or* $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{Z}/2\mathbb{Z}$ *is trivial.*

Proof of Lemma 5.1. Actually, any element in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ can be written as a product of commutators, *i.e.* elements of the form $aba^{-1}b^{-1}$, "where a and b belong to the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$. For an explicit construction of such a decomposition, see Section 2 in [4]. □

Lemma 5.2. *Let* $\psi\colon \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}_+(\mathbb{R})$ *be a group morphism. Denote by* F *the closed set of fixed points of the group* $\psi(\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}))$*. Then, for any connected component* K *of the complement of* F*, there exists a homeomorphism*

$$h_K\colon \mathbb{R} \longrightarrow K$$

such that

$$\psi(f)(x) = h_K f h_K^{-1}, \quad \textit{for all } f \in \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}), x \in K.$$

This lemma is similar to Matsumoto's theorem about morphisms $\mathrm{Homeo}_0(\mathbb{S}^1) \to \mathrm{Homeo}_0(\mathbb{S}^1)$ (see introduction) and the proof of this lemma relies heavily on Matsumoto's theorem. Before proving this lemma, let us see how it implies Lemma 4.1.

Proof of Lemma 4.1. Recall that we are given two subgroups G and G' of $\mathrm{Homeo}_+(\mathbb{R})$ isomorphic to the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.

Let α (respectively α') be a generator of the center of G (respectively of G'). Let $A_\alpha = \mathbb{R} - \mathrm{Fix}(\alpha)$ and $A_{\alpha'} = \mathbb{R} - \mathrm{Fix}(\alpha')$.

As the homeomorphisms α and α' commute:

$$\begin{cases} \alpha'(A_\alpha) = A_\alpha, \\ \alpha(A_{\alpha'}) = A_{\alpha'}. \end{cases}$$

Claim 5.3. *Take any connected component I of A_α and any connected component I' of $A_{\alpha'}$. Then the interval I and I' are disjoint.*

This claim is proved below. Let us complete now the proof of Lemma 4.1. By Lemma 5.2, $A_\alpha = \mathrm{Fix}(G)$ and $A_{\alpha'} = \mathrm{Fix}(G')$. Hence, we have proved that any connected component of the complement of $\mathrm{Fix}(G)$ is disjoint from the complement of $\mathrm{Fix}(G')$. Therefore $\mathrm{Supp}(G) \subset \mathrm{Fix}(G')$. We have also proved that $\mathrm{Supp}(G') \subset \mathrm{Fix}(G)$. ☐

Claim 5.3 is a direct consequence of the three following claims.

Claim 5.4. *Either I is contained in I', or I' is contained in I, or I and I' are disjoint.*

Claim 5.5. *The interval I is not strictly contained in the interval I'.*

Of course, the case where the interval I' is strictly contained in I is symmetric and cannot occur.

Claim 5.6. *The interval I and I' are distinct.*

Proof of Claim 5.4. Suppose for a contradiction that the conclusion of this claim does not hold. Changing the roles of α and α' if necessary, we can suppose that the supremum b of I is contained in the open interval I' and the infimum a' of I' is contained in the open interval I. Then either the sequence $(\alpha'^k(b))_{k>0}$ converges to the point a' as $k \to +\infty$ or the sequence $(\alpha'^{-k}(b))_{k>0}$ converges to the point a' as $k \to +\infty$. In any case, a sequence of points in A_α converge to the point a'. As the set A_α is closed, this means that the point a' belongs to A_α. This is not possible as a' belongs to I which is a connected component of the complement of A_α . $\square$

Proof of Claim 5.5. Suppose for a contradiction that the interval I is strictly contained in the interval I'. Let $\sim$ be the equivalence relation defined on I' by

$$x \sim y \iff (\text{there exists } k \in \mathbb{Z},\ x = \alpha'^k(y)).$$

The topological space $I'/\sim$ is homeomorphic to a circle. By Lemma 5.2, the group G' preserves the interval I'. Notice that the group $G'/\langle\alpha'\rangle \approx \mathrm{Homeo}_0(\mathbb{S}^1)$ acts on the circle $I'/\sim$. As the group G' commutes with the homeomorphism α, this action preserves the nonempty set $(A_\alpha \cap I')/\sim$. As $\alpha'(A_\alpha) = A_\alpha$, the points of the interval I are sent to points in the complement of A_α under the iterates of the homeomorphism α'. Hence the set $(A_\alpha \cap I')/\sim$ is not equal to the whole circle $I'/\sim$. However, by Theorem 5.3 in [13] (see the remark below Theorem 1.2), a non-trivial action of the group $\mathrm{Homeo}_0(\mathbb{S}^1)$ on a circle has no non-empty proper invariant subset. Hence, the group $G'/\langle\alpha'\rangle$ acts trivially on the circle $I'/\sim$: for any element β' of G', and any point $x \in I'$, there exists an integer $k(x,\beta') \in \mathbb{Z}$ such that $\beta'(x) = \alpha'^{k(x,\beta')}(x)$. Fixing such a point x, we see that the map

$$\begin{aligned} G' &\longrightarrow \mathbb{Z}, \\ \beta' &\longmapsto k(x,\beta'), \end{aligned}$$

is a group morphism. Such a group morphism is trivial by Lemma 5.1. Therefore, the group G' acts trivially on the interval I', a contradiction. $\square$

Proof of Claim 5.6. Suppose that $I = I'$. Take any element β' in G'. As the homeomorphism β' commutes with α, by Lemma 5.2, the homeomorphism β' is equal to some element of G on I. As the homeomorphism β' commute with any element of G, there exists a unique integer $k(\beta')$ such that $\beta'_{|I} = \alpha_{|I}^{k(\beta')}$. The map $k\colon G \to \mathbb{Z}$ is a nontrivial group morphism. But such a map cannot exist by Lemma 5.1. $\square$

It remains to prove Lemma 5.2.

Proof of Lemma 5.2. Denote by t a generator of the center of $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.

Claim 5.7. *The connected components of the complement of* $\mathrm{Fix}(\psi(t))$ *are each preserved by the group* $\psi(\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}))$. *Moreover*

$$\mathrm{Fix}(\psi(\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}))) = \mathrm{Fix}(\psi(t)).$$

Claim 5.8. *Any action of the group* $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ *on* $\mathbb{R}$ *without fixed points is conjugate to the standard action.*

It is clear that these two claims imply Lemma 5.2. □

Proof of Claim 5.7. The set $\mathrm{Fix}(\psi(t))$ is preserved under any element in $\psi(\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}))$, because any element of this group commutes with the homeomorphism $\psi(t)$. Moreover, any element in $\psi(\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}))$ preserves the orientation by Lemma 5.1. Hence the action ψ induces an action of the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})/\langle t\rangle$, which is isomorphic to $\mathrm{Homeo}_0(\mathbb{S}^1)$, on the set $F = \mathrm{Fix}(\psi(t))$ by increasing homeomorphisms. As the group $\mathrm{Homeo}_0(\mathbb{S}^1)$ is simple, the induced morphism from the group $\mathrm{Homeo}_0(\mathbb{S}^1)$ to the group $\mathrm{Homeo}_<(F)$ of increasing homeomorphisms of F is either one-to-one or trivial. However, the group $\mathrm{Homeo}_0(\mathbb{S}^1)$ contains some non-trivial finite order elements whereas the group $\mathrm{Homeo}_<(F)$ does not: such a morphism is trivial. Hence any element of the group $\psi(\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}))$ fixes any point in $\mathrm{Fix}(\psi(t))$: any element of this group preserves each connected component of the complement of $\mathrm{Fix}(\psi(t))$. □

Proof of Claim 5.8. We denote by $\varphi\colon \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}(\mathbb{R})$ a morphism such that the group $\varphi(\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}))$ of homeomorphisms of $\mathbb{R}$ has no fixed point.

By Claim 5.7, the homeomorphism $\varphi(t)$ has no fixed point. Changing coordinates if necessary, we can suppose that the homeomorphism $\varphi(t)$ is the translation $x \mapsto x+1$. The morphism φ induces an action $\hat{\varphi}$ of the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})/\langle t\rangle \approx \mathrm{Homeo}_0(\mathbb{S}^1)$ on the circle $\mathbb{R}/\mathbb{Z}$. This action is nontrivial: otherwise, there would exist a nontrivial group morphism $\mathrm{Homeo}_0(\mathbb{S}^1) \to \mathbb{Z}$. By the theorem by Matsumoto that we recalled earlier (see the introduction of this article), there exists a homeomorphism h of the circle $\mathbb{R}/\mathbb{Z}$ such that, for any homeomorphism f in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})/\langle t\rangle$ (which can be canonically identified with $\mathrm{Homeo}_0(\mathbb{R}/\mathbb{Z})$):

$$\hat{\varphi}(f) = hfh^{-1}.$$

Take a lift $\tilde{h}\colon \mathbb{R} \to \mathbb{R}$ of h. For any integer n, denote by $T_n : \mathbb{R} \to \mathbb{R}$ the translation $x \mapsto x+n$. For any homeomorphism f in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, there exists an integer $n(f)$ such that

$$\varphi(f) = T_{n(f)}\tilde{h} f \tilde{h}^{-1}.$$

However, the map $n\colon \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{Z}$ is a group morphism: it is trivial by Lemma 5.1. This completes the proof of Claim 5.8. □

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