

On the space of left-orderings of virtually solvable groups

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Abstract. We show that the space of left-orderings of a countable virtually solvable group is either finite or homeomorphic to a Cantor set. We also provide an explicit description of the space of left-orderings of $\text{SOL} = \mathbb{Z}^2 \rtimes_T \mathbb{Z}$.

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1. Introduction

The space of left-orderings $\mathcal{LO}(G)$ of a left-orderable group G is the set of all possible left-orderings on G endowed with a natural topology that makes it compact, Hausdorff and totally disconnected, see [26] or §2.1. It was proved by Linnell that this space is either finite or uncountable [12]. The problem of relating the topology of $\mathcal{LO}(G)$ with the algebraic structure of G has been of increasing interest since the discovery by Dubrovina and Dubrovin that the space of left-orderings of the braid groups is infinite and yet contains isolated points [6]. Recently, more examples of groups showing these two behaviors have appeared in the literature [4, 9, 10, 17]. All these groups admit a non-abelian free subgroup. However, non-abelian free groups [13], and more generally non-trivial free products of groups have no isolated left-orderings [24].

In any case, the class of groups having isolated orderings is far from being well understood. By contrast, left-orderable groups admitting only finitely many left-orderings have been classified by Tararin [11, Theorem 5.2.1]. For short, we

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shall call these groups “Tarin groups.” Tararin groups form a very restrictive and easy to describe class of finite-rank-solvable groups, see §2.2. As already mentioned, these are the only known examples of left-orderable *amenable* groups with isolated points. Could they be the only ones? This paper brings in a modest contribution to this problem, answering it positively for (virtually) solvable groups.

Theorem 1.1. *The space of left-orderings of a countable virtually solvable group is either finite or a Cantor set.*

In particular we deduce

Corollary 1.2. *If Γ is a (countable) left-orderable virtually solvable group of infinite rank, then $\mathcal{LO}(\Gamma)$ is a Cantor set.*

Recall that a polycyclic group is a solvable group whose successive abelian quotients, in its derived series, are finitely generated, see [21]. Among Tararin groups, those which are virtually polycyclic turn out to be virtually nilpotent. Therefore we deduce

Corollary 1.3. *If Γ is a left-orderable virtually polycyclic group of exponential growth, then $\mathcal{LO}(\Gamma)$ is a Cantor set.*

The study of left-orderable amenable groups is intimately related to that of Conradian orderings (see §2.5 for a definition) and to local indicability. Recall that Morris gave a beautiful proof of the fact that all left-orderable amenable groups admit Conradian orderings [14] (see also [15, 2] for older results in that direction, and [5] for an interesting alternative proof). Together with a fundamental observation of Conrad [3], this provides a very natural characterization of left-orderable groups among amenable groups as those which are locally indicable, i.e. all their finitely generated subgroups have a non-trivial morphism to \mathbb{Z} .

The dichotomy shown in Theorem 1.1 reminds of a similar one, this time for all groups but in restriction to Conradian orderings. Indeed, in [22] the first author proved that the space of Conradian orderings of a countable group is either finite or homeomorphic to the Cantor set. This implies for instance, the general dichotomy for left-orderable groups having only Conradian orderings, such as groups of sub-exponential growth [16].

Remarks about the proof and organization of the paper. It is now well known that countable groups are left-orderable if and only if they act faithfully by order-preserving homeomorphisms on the real line (see for instance [7] or §2.3).

This dynamical approach has shown to be fruitful when trying to understand the topology of the space of left-orderings of a given group or family of groups, see for instance [16, 22, 24]. In this work, this point of view will be crucial.

In a previous version of this paper, Theorem 1.1 was proved for the class of finite-rank solvable groups. An important feature of this subclass is that they are virtually nilpotent-by-abelian [25]. In this case, the proof can be summarized as follows: if the action on the real line associated to an ordering has no discrete orbits, then one can prove that the action is semi-conjugated to an affine action. Then, if this affine action is non-abelian, it is not hard to see that the underlying ordering is non-isolated, see Proposition 2.12 and Corollary 2.13. The case where the action factors through an abelian group of rank greater than one follows from [26]. Finally it is only when the initial action has discrete orbits or when it factors through a rank-one abelian group, that Tararin groups may appear (one needs to work an extra bit to decide whether it is the case).

This approach strongly relies on the work of Plante [20], who shows that any action of a finite-rank solvable group on the real line *quasi preserves* a Radon measure: that is, there is a Radon measure μ such that for every element g in the group there is a positive real number λ_g such that $\mu(g^{-1}A) = \lambda_g\mu(A)$ for every measurable set $A \subseteq \mathbb{R}$.

However, as already noted by Plante, there are actions of solvable groups on the real line (such as $\mathbb{Z} \wr \mathbb{Z}$) in which no non-trivial Radon measure is quasi-preserved. This situation is much more subtle and requires a careful dynamical analysis. In particular, the present proof of Theorem 1.1 is different (and independent!) from the one described above.

In §2 we give some necessary background. Most of the material is well known, but not all. Notably Corollary 2.16 plays a crucial role in our proof. In §3 we prove Theorem 1.1 in a simple example,¹ namely SOL. This example should serve to illustrate the approach for finite-rank solvable groups. We also give an explicit description of its space of left-orderings. In §4 we illustrate the difficulties arising when dealing with solvable groups of infinite rank on a specific example, due to Plante, of an action of $\mathbb{Z} \wr \mathbb{Z}$ on the line. Finally the proof of Theorem 1.1 is carried out in §5. Roughly speaking, the proof consists in showing that the action behaves “at a certain scale” like an affine action (see §4 and Lemma 5.6).

Remark 1.4. We remark that $\mathcal{LO}(G)$ is metrizable when G is a countable group (see [16] or §2.1). Hence, to prove that $\mathcal{LO}(G)$ is homeomorphic to the Cantor

¹ Solvable Baumslag-Solitar groups could have been treated in a similar way, we leave the easy adaptation of the proof to the reader.

set, it is enough to show that it contains no isolated points (see for instance [8]), which is indeed how we proceed. Also, in various places we employ the useful terminology “pseudo-ordering” (Definition 2.4) which means an invariant ordering on a quotient of G by a (not necessarily normal) subgroup, but seen as a partial ordering on G .

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2. Preliminaries

2.1. The topology on $\mathcal{LO}(G)$. A basis of neighborhoods in $\mathcal{LO}(G)$ is the family of the sets $V_{f_1, \dots, f_k} := \{\preceq \mid \text{id} \preceq f_1, \dots, \text{id} \preceq f_k\}$, where $\{f_1, \dots, f_k\}$ runs over all finite subsets of G . If G is countable, then this topology is metrizable [16]. For instance, if G is finitely generated, we may define $\text{dist}(\preceq, \preceq') = 1/2^n$, where n is the first integer such that \preceq and \preceq' do not coincide on n -th ball (with respect to some generator system). For each $g \in G$ and $\preceq \in \mathcal{LO}(G)$, one can define an other element $g(\preceq) \in \mathcal{LO}(G)$ whose *positive cone* is the set of elements $f \in G$ such that $gfg^{-1} \succ \text{id}$. This defines a continuous representation of G in $\text{Homeo}(\mathcal{LO}(G))$ called action by *conjugation* of G on its space of left-orderings.

The following definition is classical [1, 11]. Given a left-ordered group (G, \preceq) and a subgroup H , we say that H is *convex* if for every $g \in G$ such that $h_1 \preceq g \preceq h_2$, for h_1, h_2 in H , we have that $g \in H$. Convex subgroups have the nice property that they induce a total ordering on the left-cosets G/H by

$$g_1H \preceq^* g_2H \iff g_1h_1 \preceq g_2h_2 \quad \text{for all } h_1, h_2 \text{ in } H.$$

This ordering is invariant under the G -action by left translation on G/H . In particular, if H is normal, then G/H is a left-orderable group, in which case we call \preceq^* the *projected* or *quotient* ordering. It follows that \preceq decomposes “lexicographically” as the ordering on the H -cosets and the ordering restricted to H . More precisely we have that

$$\text{id} \prec g \iff \begin{cases} H \prec^* gH \text{ or} \\ H = gH \text{ and } \text{id} \prec g. \end{cases}$$

Elaborating on this, we conclude (see [23] for more details)

Proposition 2.1. *Let \preceq be a left-ordering on G and let H be a convex subgroup. Then there is a continuous injection*

$$\mathcal{LO}(H) \longrightarrow \mathcal{LO}(G),$$

having \preceq in its image. Moreover, if in addition H is normal, then there is a continuous injection

$$\mathcal{LO}(H) \times \mathcal{LO}(G/H) \longrightarrow \mathcal{LO}(G)$$

having \preceq in its image.

Remark 2.2. Let \preceq be a left-ordering on G , and H a normal convex subgroup. Then, it is not hard to check that if the restriction of \preceq to H and the projection of \preceq to G/H are Conradian, then \preceq is also Conradian.

Observe that the set of convex subgroups of a given left-ordering are totally ordered for the inclusion. We call it the convex series of G . The following Corollary is well known, and appears for instance in [22]. For the readers' convenience we sketch a proof. Let G be a group and H be a (not necessarily normal) subgroup. Then to any G -invariant ordering \preceq on G/H , one can define the so-called *opposite* ordering \preceq^{op} defined by

$$g \prec^{\text{op}} f \iff g \succ f.$$

We sometimes say that \preceq^{op} is obtained by *flipping* \preceq . Clearly \preceq^{op} is also G -invariant.

Corollary 2.3. *Suppose that a left-ordered group (G, \preceq) has infinitely many convex subgroups. Then \preceq is non-isolated.*

Sketch of the proof: If the convex series is infinite, then either there exists an infinite increasing sequence of convex subgroups $C_1 < C_2 \dots$ or an infinite decreasing sequence $C'_1 > C'_2 \dots$. Flipping the ordering on C_{n+1}/C_n (resp. on C'_n), one obtains a sequence of orderings \preceq_n , distinct from \preceq , which converges to \preceq when n goes to infinity. \square

Note that we often have to deal with orderings on a quotient G/H , where H is not necessarily normal. It will therefore be convenient to see G -invariant orderings on G/H as a “pseudo-ordering” on G , namely

Definition 2.4 (pseudo-ordering). A pseudo-ordering \preceq on a group G is a left-invariant partial ordering on G induced from a G -invariant ordering \preceq^* on a quotient

$$G/H: \text{id} \prec g \iff H \prec^* gH.$$

Given a pseudo-ordering on G , the set of elements which are not comparable to the neutral element coincides with the subgroup H .

- **CONVEX SUBGROUP.** A convex subgroup C of a pseudo-ordered group is defined similarly as for orderings, with the additional requirement that C must contain H . Note that H itself is convex, and so is the minimal convex subgroup.
- **QUOTIENT BY A CONVEX SUBGROUP.** Given a pseudo-ordering \preceq on G and C a convex subgroup, the pseudo-ordering on G induced from the ordering on G/C will be called the quotient of \preceq by C .

Remark 2.5. If \preceq' is the quotient of \preceq by a convex subgroup C of (G, \preceq) , then convex subgroups of (G, \preceq') are exactly those convex subgroups of (G, \preceq) containing C .

2.2. Tararin groups. We give a slight modification of the original statement of Tararin [11, Theorem 5.2.1], describing groups admitting only finitely many left-orderings. Recall that a series

$$\{1\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_m = G,$$

is said to be *rational* if each quotient G_{i+1}/G_i is torsion-free rank-one abelian.

Theorem 2.6 (Tararin). *Let G be a left-orderable group. If G admits only finitely many left-orderings, then G admits a unique (hence characteristic) rational series*

$$\{1\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_m = G,$$

such that, for every $2 \leq i \leq m$, there is an element of G_i/G_{i-1} whose action by conjugation on G_{i-1}/G_{i-2} is by multiplication by a negative rational number. We shall call such a group a Tararin group.

Remark 2.7. The left-orderings on a Tararin group G are very easy to describe. Indeed, if $\{1\} = G_0 \triangleleft \cdots \triangleleft G_m = G$, is the associated rational series, then on each quotient G_i/G_{i-1} , being rank-one torsion free abelian, there is – up to flipping – a unique left-ordering coming from an embedding into \mathbb{Q} . For every i , let \preceq_i be a choice of an ordering on G_i/G_{i-1} . Then we can produce a left-orderings on G by declaring

$$g \succ \text{id} \iff \begin{cases} g G_{m-1} \succ_m G_{m-1}, \text{ or} \\ g \in G_{m-1}, \text{ and } g \succ_{m-1} \text{id}, \text{ or} \\ \vdots \\ g \in G_1, \text{ and } g \succ_1 \text{id}. \end{cases}$$

It is not hard to check that in this way we can produce all possible (2^m) left-orderings (in fact, it is easy to show that they are all Conradian). Moreover, in any such ordering, the groups G_i are convex, and conversely, every convex subgroup is of this form.

Corollary 2.8. *Let G be a virtually polycyclic group admitting only finitely many left-orderings. Then G admits a unique filtration such that*

$$G_i/G_{i-1} \simeq \mathbb{Z}.$$

The action of (the generator of) G_i/G_{i-1} on G_{i-1}/G_{i-2} is by multiplication by -1 .

Since in a virtually polycyclic group, the group generated by $\{g^2 \mid g \in G\}$ has finite index [21], we deduce

Corollary 2.9. *A virtually polycyclic group having only finitely many left-orderings is virtually nilpotent.*

We finish this section with a rigidity statement for actions of Tararin groups on the line. Let G be a Tararin group and $\{1\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_m = G$ its associated rational series. Since G_{m-1} is convex in every left-ordering of G , and G/G_{m-1} is abelian we have that the sign of $\gamma \in G \setminus G_{m-1}$ is preserved under conjugation. On the other hand, Remark 2.7 says that starting from a left-ordering \preceq , any other left-ordering on G is obtained by flipping the ordering on some of the convex subgroups. It follows from Theorem 2.6, that any flipping on any proper convex subgroup can be realized as conjugations by some element in G . This shows

Proposition 2.10. *The conjugation action of a Tararin group G on $\mathcal{LO}(G)$ has two orbits. Moreover, for any pair of left-orderings \preceq, \preceq' of G there is $g \in G$ such that $g(\preceq)$ and \preceq' coincide over G_{m-1} . Moreover, if we let γ_T be an element in $G \setminus G_{m-1}$ which acts on G_{m-1}/G_{m-2} as multiplication by a negative number, then g can be taken either in G_{m-1} or in $\gamma_T G_{m-1}$.*

2.3. Dynamical realization. As mentioned in the introduction, an important ingredient in our proof of Theorem 1.1 is the fact that countable left-orderable groups naturally act by order-preserving automorphisms of the real line, and vice versa, a group acting faithfully by order-preserving automorphisms of the real line is left-orderable [7].

More precisely, given a left-ordered group (G, \preceq) , there is an embedding of G into $\text{Homeo}_+(\mathbb{R})$, the group of order preserving automorphism of the real line, such that

- G acts without global fixed points,
- for f, g in G , we have that $f \prec g \iff f(0) < g(0)$, and
- the set of fixed points of a non-trivial $f \in G$ has empty interior.

This construction extends to pseudo-orderings in the sense that every pseudo-order on G with minimal convex subgroup H can be induced from an action of G on the real line, where H is the stabilizer of 0.

We call such an action, a *dynamical realization* of (G, \preceq) . Conversely, given an embedding of G into $\text{Homeo}_+(\mathbb{R})$, we can induce a left-ordering as follows: take (x_1, x_2, \dots) a dense sequence in \mathbb{R} , and declare that an element $g \succ_{(x_1, x_2, \dots)}$ id if and only if $g(x_i) > x_i$, where i is such that $g(x_j) = x_j$ for every $j < i$. We call such an ordering, an *induced ordering* from the action. Note that with this procedure we can recover a left-ordering from its dynamical realization by taking $x_1 = 0$.

Remark 2.11. Let $g \in G$, and $\preceq = \preceq_{(x_1, x_2, \dots)}$ be an ordering induced from an action of G on the real line. Then $g(\preceq)$ is the left-ordering induced from the sequence $(g^{-1}(x_1), g^{-1}(x_2), \dots)$.

2.4. Orderings induced by affine actions. A general procedure for trying to approximate a given left-ordering \preceq on a countable group, is to consider its dynamical realization, and to induce an ordering \preceq' from a sequence (x_1, x_2, \dots) where x_1 is close to 0. The fact that \preceq' is close to \preceq when x_1 is close to 0 follows from the continuity of the action, and from the fact that 0 has a free orbit (details are left to the reader). The problem however, is that the two orbits may induce the same ordering. This is the case for instance if \preceq has a least positive element.

Our first step in proving Theorem 1.1 is that left-orderings induced from non-abelian affine actions, are non-isolated.

Proposition 2.12. *Let Γ be a countable group, and suppose a left-ordering \preceq on Γ is induced from a faithful (order-preserving) affine action on the real line. Then, if Γ is non-abelian, \preceq is approximated by its conjugates.*

Proof. Let $\preceq = \preceq_{(x_1, x_2, \dots)}$ be the left-ordering induced from the sequence (x_1, x_2, \dots) . We note that, since the elements in the affine group have at most one fixed point, it is enough to specify two points. So $\preceq = \preceq_{(x_1, x_2)}$.

By assumption, Γ has both non-trivial homotheties and non-trivial translations. It follows that the subgroup made of translations has dense orbits. In particular, the countable set Ω consisting of the points in \mathbb{R} which are fixed by some non-trivial element (homothety) of Γ , is also dense in \mathbb{R} . Therefore, given any two points $x_1, y_1 \in \mathbb{R}$, $x_1 \neq y_1$, there is a non-trivial homothety $h \in \Gamma$, having its unique fixed point between x_1 and y_1 . In particular, the left-orderings $\preceq_{(x_1, x_2)}$ and $\preceq_{(y_1, y_2)}$, induced from (x_1, x_2) and (y_1, y_2) respectively, are different.

We now show that y_1 may be chosen so that $\preceq_{(y_1, y_2)}$ is close to $\preceq_{(x_1, x_2)}$. As noted earlier, this is obvious if x_1 has a free orbit, so let us suppose that it is not the case. Let $\text{Stab}_\Gamma(x_1)$ be the stabilizer of x_1 in Γ , and let $S \subset \Gamma$ be a finite set of \preceq -positive elements. We write $S = S_1 \cup S_2$ where $S_1 = S \cap \text{Stab}_\Gamma(x_1)$, and we call I the open interval between x_1 and x_2 . Since S_2 is finite, there is a small neighborhood U of x_1 such that $\gamma(x) > x$ for every $x \in U$ and every $\gamma \in S_2$. On the other hand, for every $\gamma \in S_1$, we have that $\gamma(x) > x$ for every $x \in I$ (recall that $\gamma \in S_1$ is an homothety). Thus, if we take $y_1 \in I \cap U$ then $\preceq_{(y_1, y_2)}$ and \preceq (y_2 being any point) are in the same open set associated to S . Since $I \cap U$ has non-empty interior, it is easy to see that (y_1, y_2) may be chosen so that $(y_1, y_2) = (\gamma(x_1), \gamma(x_2))$ for some $\gamma \in \Gamma$, which shows that \preceq is approximated by its conjugates. \square

To state the following corollary, recall that given two actions A_1, A_2 of a group Γ on the real line, we say that A_1 is *semi-conjugated* to A_2 if there is an increasing surjective function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$F \circ A_1(\gamma) = A_2(\gamma) \circ F \quad (\text{for all } \gamma \in \Gamma).$$

Corollary 2.13. *Let (Γ, \preceq) be a countable, left-ordered group. Suppose there is an affine (order-preserving) action $A : \Gamma \rightarrow \text{Aff}_+(\mathbb{R})$ whose kernel is convex in \preceq , and whose range is non-abelian. Suppose further that the dynamical realization of (Γ, \preceq) is semi-conjugated to A . Then \preceq is non-isolated.*

Proof: In light of Proposition 2.1, it is enough to show the corollary when A is a faithful action. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be the function that realizes the semi-conjugation, and let $\text{Stab}_{A(\Gamma)}(F(0))$ be the stabilizer of $F(0)$ in $A(\Gamma)$: this is an abelian subgroup of Γ . We claim that it is convex. Indeed, if $\gamma_1 < g < \gamma_2$, then $\gamma_1(0) < g(0) < \gamma_2(0)$. So $F(\gamma_1(0)) \leq F(g(0)) \leq F(\gamma_2(0))$, and hence $A(\gamma_1)(F(0)) \leq A(g)(F(0)) \leq A(\gamma_2)(F(0))$. From where the claim follows.

Now, if $\text{Stab}_{A(\Gamma)}(F(0))$ is trivial then Proposition 2.12 applies directly, since in this case \preceq is realized as the induced ordering from $F(0)$ in the action A . If it

has rank > 1 , then the restriction of \preceq to $\text{Stab}_{A(\Gamma)}(F(0))$ is non-isolated, thus \preceq itself is non-isolated.

Now, if $\text{Stab}_{A(\Gamma)}(F(0))$ has rank exactly 1, then the order restricted to $\text{Stab}_{A(\Gamma)}(F(0))$ is completely determined by the sign of any given non-trivial element, say $A(t) \in \text{Stab}_{A(\Gamma)}(F(0))$. Assume $t > 0$, then, because $A(t)$ acts as a non-trivial homothety, there exists $x \in \mathbb{R}$ such that $A(t)(x) > x$. It follows that \preceq coincides with $\preceq_{(x_1, x_2)}$, the ordering induced from the action A where $x_1 = F(0)$ and $x_2 = x$. So \preceq is non-isolated by Proposition 2.12. \square

2.5. Conradian orderings. There is a special type of left-ordering, introduced in [3], which will be very important in our proof of Theorem 1.1. These are the so called *Conradian* orderings, which are left-orderings satisfying the following additional property:²

$$f \succ \text{id}, g \succ \text{id} \implies fg^2 \succ g.$$

It turns out that Conradian orderings have a very interesting dynamical counterpart. Recall from [16], that $f, g \in \text{Homeo}_+(\mathbb{R})$ are said to be *crossed*, if one of them, say g , has a *domain* I (that is, an open interval, not necessarily bounded, which is fixed by g , and on which g acts without fixed points), such that $f(I)$ is not equal, nor disjoint to I , see Figure 1. A group $G \subset \text{Homeo}_+(\mathbb{R})$ is said to act without crossings, if it does not contains crossed elements.

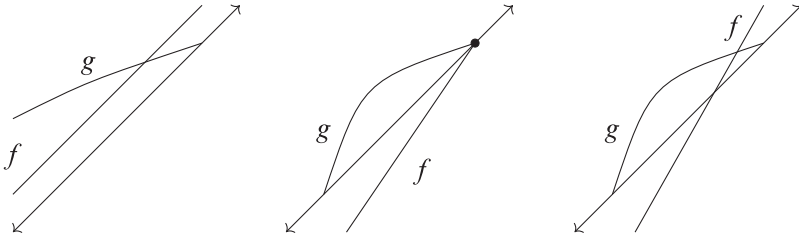


Figure 1. Three different crossings.

Theorem 2.14 (Navas [16]). *The dynamical realization of a Conradian ordering on G is an action without crossings. Conversely, an induced ordering from an action without crossings is Conradian.*

² In fact, in [3] the required property is $f \succ \text{id}, g \succ \text{id} \implies$ there exists $n \geq 1$ such that $fg^n \succ g$. The fact that $n = 2$ is enough is a result from Navas and Jiménez, see [16].

The above theorem implies rather easily that, in a dynamical realization of a Conradian ordering, the set of elements having fixed point is a subgroup (obviously normal). With this, together with a theorem of Hölder stating that every group acting freely on the real line is abelian (see for instance [7]), one can deduce (see for instance [16]. Compare with [3].)

Corollary 2.15. *Let G be a countable group, \preceq be a Conradian ordering of G , and $N \subset G$ be the set of elements having a fixed point in the dynamical realization of (G, \preceq) . Then N is a normal subgroup of G . Moreover, if there is $g \in G$ having no fixed point (for instance if G is finitely generated), then G/N is a non-trivial torsion-free abelian group which acts freely on the (non-empty) set of global fixed points of N .*

Let us state here a last corollary which plays a crucial role in the proof of Theorem 1.1. It describes some constraints on the dynamics of a group acting on the real line, when it has a normal subgroup acting without crossings. More precisely

Corollary 2.16. *Let (G, \preceq) be a left-ordered group, let H be a normal subgroup such that (H, \preceq) is Conradian, and consider a dynamical realization of (G, \preceq) . Let $f \in G$, $g \in H$ and let I be a minimal open interval fixed by g . Then one of the following holds.*

- $f(I) = I$ or
- $f(I)$ is disjoint from I , or
- (up to changing f by its inverse) $\bar{I} \subset f(I)$. In this last case we say that f acts as a **dilation** on I .

Proof. Notice that $f(I)$ is a domain of $g^f = f g f^{-1} \in H$. Hence, it follows from Theorem 2.14 that g and g^f are not crossed. In particular, I and its image by f are either disjoint or one is contained in the other. Indeed, if this is not the case, then $f(I)$ would not be fixed, nor moved disjointly by g . So, up to changing f by f^{-1} , we can assume that $I \subseteq f(I)$.

We still have to rule out the possibility that these two intervals, although different, share a common extremity. This is again easy. Indeed, suppose it is the case that I and $f(I)$ share a common extremity but $I \subsetneq f(I)$. Then g^f can not move I disjoint from itself. But, on the other hand, since g^f have no fixed points inside $f(I)$, we have that I can not be fixed by g^f , contradicting Theorem 2.14. \square

3. The space of orderings of SOL

3.1. The space of orderings of SOL is a Cantor set. In this section we treat the simplest case of a non-virtually nilpotent polycyclic group, namely the group $\text{SOL} = \mathbb{Z}^2 \rtimes_T \mathbb{Z}$, where T is an hyperbolic matrix (that is, a matrix in $\text{SL}_2(\mathbb{Z})$ having trace greater than 2). In particular, T has two irrational eigenvalues. We begin by proving that this group has no isolated orderings, and then we provide a quite explicit description of its set of orderings (describing for instance its bi-invariants orderings, which are left-orderings whose positive cone is preserved under conjugation).

We denote by H the derived subgroup of SOL , which is isomorphic to \mathbb{Z}^2 , and by t the element of \mathbb{Z} acting on H as T . Let \preceq be a left-ordering on SOL , and consider its dynamical realization.

Since H is abelian, the set of elements in H acting with fixed points form a subgroup H' . This subgroup, being finitely generated, actually has a global fixed point. But because T is \mathbb{Q} -irreducible, this subgroup is either trivial or must have finite index. In the latter situation, we have that $H' = H$ as every global fixed point of H' has a finite H -orbit, so must be fixed by H . We therefore have two cases to consider, namely

CASE 1. H has a global fixed point.

Let I be the maximal open interval around 0 without global fixed point of H . Since H is normal in SOL , and SOL acts without global fixed points, we have that the set of global fixed points of H is permuted by SOL and therefore must be infinite. In particular, I is a bounded interval which is either fixed or moved disjointly from itself by the action of SOL . Therefore $H = \text{Stab}(I)$ is convex. Moreover, since H has rank two, the restriction of \preceq to H is non-isolated. Hence, Proposition 2.1 implies that \preceq itself is non-isolated. Observe moreover that \preceq is Conradian.

CASE 2. H has no global fixed point.

It follows that H acts freely on the real line, and so, by Hölder's Theorem [7], it is semi-conjugated to a group of translation. Now recall that Lebesgue's measure is the unique measure, up to scalar multiple, preserved by \mathbb{Z}^2 acting faithfully by translations. In particular H preserves a unique atomless Radon³ measure μ . Moreover, the hyperbolicity of T implies that t does not preserve the measure μ , but it acts on it as a dilation by one of the two eigenvalues of T . We therefore obtain

³ Recall that a *Radon* measure is a measure giving finite mass to compact sets.

a faithful embedding of SOL in $\text{Aff}_+(\mathbb{R})$ which is realized by a semi-conjugation (see equation (1) and (2) respectively). We then conclude from Corollary 2.13. We observe that in this case, the proof of Corollary 2.13 shows that the ordering \preceq is realized as an induced ordering from the associated affine action.

3.2. Description of the space of orderings. It follows from the previous analysis that there are two types of orderings on SOL: those which are Conradian, and those which are induced by affine actions.

• **Conradian orderings.** These always form a closed subset [16]. Here, Conradian orderings are exactly those for which the normal subgroup $H \simeq \mathbb{Z}^2$ is convex. Therefore in $\mathcal{LO}(\text{SOL})$ there are two copies of the Cantor set $\mathcal{LO}(\mathbb{Z}^2)$, each of which corresponding to a choice of sign for t . Let us briefly recall the description of the space of left-orderings of \mathbb{Z}^2 . First observe that each oriented line passing through the origin in \mathbb{R}^2 delimits a unique half-plane (say the one on its right) defining the positive cone of some pseudo-ordering on \mathbb{Z}^2 . The set of elements of \mathbb{Z}^2 which belong to the line form a cyclic convex subgroup, which is trivial precisely when the pseudo-ordering is an ordering (this happens exactly when the slope of the line is irrational). It follows that this space of pseudo-orderings is naturally parametrized by the unit circle. When the slope is rational, one needs to specify an ordering of the convex subgroup, which is determined by a sign. Therefore the space of orderings of \mathbb{Z}^2 can be parametrized by a Cantor set obtained by “doubling” each rational point of the circle (see [26]). For simplicity in Figure 2, we ignore this “blow up” procedure and represent each copy of $\mathcal{LO}(\mathbb{Z}^2)$ as a “vertical” circle.



Figure 2. The space of left-orderings of $\mathbb{Z}^2 \rtimes_T \mathbb{Z}$.

• **Bi-orderings.** Now observe that among these orderings on \mathbb{Z}^2 , those which are invariant under conjugation by t are precisely those corresponding to lines which are eigendirections of the matrix T . The corresponding orderings of SOL are those which are bi-invariant. Taking into account the choices of orientations, this gives precisely eight⁴ bi-orderings.

• **Affine orderings.** The complement of Conradian orderings, namely those induced by affine actions, is an open subset with eight accumulation points, namely SOL's bi-orderings. We represent these affine orderings by four copies of \mathbb{R} , compactified at $\pm\infty$ by pairs of bi-orderings corresponding to two different eigendirections. This requires some explanation.

First, these four intervals are to be thought as Cantor sets. Similarly to our description of orderings of \mathbb{Z}^2 , one can first consider pseudo-orderings $\preceq_x^{(i)}$, $i = 1 \dots 4$, induced by the orbit of one point, x under an affine action of SOL on the line. Such pseudo-ordering is determined by the following data: first, choose an orientation of the line, then one needs to specify the action of t by multiplication by one of the two eigenvalues of T . Hence these pseudo-orderings are naturally parametrized by four copies of the real line. Note that $\preceq_x^{(i)}$ is an ordering precisely when the stabilizer of x is trivial, which happens on the complement of some dense countable subset $D \subset \mathbb{R}$ (this subset corresponds to the possible values of translations of elements of \mathbb{Z}^2). Otherwise, to define an ordering on SOL, one needs to specify an orientation on the convex cyclic subgroup of homotheties fixing x . Therefore the subset of affine orderings is locally a Cantor set, obtained by doubling points belonging to D in each of the four copies of \mathbb{R} .

Consider one of these intervals, corresponding to an action where t acts by dilation (i.e. t^{-1} acts by contraction), and let $\preceq_x^{(1)}$ be the pseudo-ordering associated to the orbit of x . Observe that when x goes to ∞ , the action of t becomes predominant over translations, so that $\preceq_x^{(1)}$ converges to an ordering where translations form a convex subgroup. Since the restriction of $\preceq_x^{(i)}$ to H is conjugation invariant, one easily checks that this limiting ordering is one of our 8 bi-orderings.

Based on this description, it is not difficult to describe the dynamics of the action of SOL on its space of ordering, see for instance Remark 2.11. We leave this as an exercise to the reader.

⁴ A classification of finitely generated groups admitting only finitely many bi-ordering can be found in [1].

4. Plante's action of $\mathbb{Z} \wr \mathbb{Z}$.

The case of SOL which has been described in the previous section is a good representative of what happens for solvable groups with finite rank. The goal of this subsection is to illustrate the difficulties arising when dealing with solvable groups of infinite rank. For simplicity, we shall focus on abelian-by-cyclic groups $H \rtimes \mathbb{Z}$. The prototypical example of such group is

$$\Gamma = \mathbb{Z} \wr \mathbb{Z} = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z} \rtimes \mathbb{Z},$$

where \mathbb{Z} acts by shifting the indices in $H = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}$.

In the finite rank case, we only have to consider two cases: either H has a global fixed point, or has an element which acts freely. Here by contrast, since H is not supposed to have finite rank, a third case can happen for which all elements of H have fixed points, while H does not have any global fixed point.⁵ In [20, Section 5], Plante describes an explicit action of $\mathbb{Z} \wr \mathbb{Z}$ corresponding to this third case (in particular such an action does not quasi-preserve any Radon measure).

We shall now recall the main properties of Plante's action of $\mathbb{Z} \wr \mathbb{Z}$ which are responsible for the fact that the corresponding induced orderings are non-isolated, see Figure 3. Let t be a generator of the cyclic group acting on H . For this action, each non-trivial element h of H preserves a compact interval containing 0 in its interior. Let I_h be the minimal such interval. These intervals are nested, and their intersection is reduced to $\{0\}$. The element t acts as an expanding homeomorphism having 0 as its unique fixed point, and such that

$$t(I_h) = I_{tht^{-1}}.$$

Somehow, this action reminds of an affine action where t would be some kind of homothety, while elements of H would play the role of translations. It turns out that similarly to its affine cousin, this ordering can be approximated by its conjugates. Indeed, take for instance the pseudo-ordering induced by the orbit of 0. The cyclic subgroup generated by t being the minimal convex subgroup, this pseudo-ordering can be completed to an ordering by specifying the sign of t . One easily checks that if $t > 0$ (resp. $t < 0$), then for any sequence of points x_n converging to zero from the right (resp. left), the sequence of orderings \preceq_{x_n} converges to \preceq , while being distinct from it.

⁵ In the proof of Theorem 1.1, the first one of these three cases (concerning Conradian orderings) is treated implicitly as an initial step in our argument.

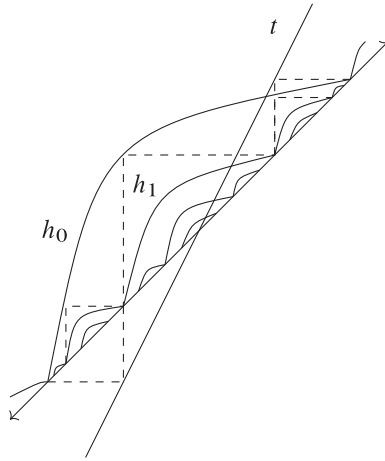


Figure 3. Plante's action: $h_1 = th_0t^{-1}$.

5. Proof of Theorem 1.1

Let \preceq be a left-ordering on virtually solvable group Γ . We assume that the convex series of (Γ, \preceq) is finite, because otherwise, from Corollary 2.3, \preceq is non-isolated. Say the convex series is

$$\{\text{id}\} = C_n \subset C_{n-1} \subset \cdots \subset C_0 = \Gamma.$$

We let $T = C_{j+1} \subseteq \Gamma$ be the *Conradian soul* of (Γ, \preceq) , which is the maximal convex subgroup on which the restriction of \preceq is Conradian. It is known that if T is not a Tararin group, then \preceq is non-isolated (this follows from [16], and appears explicitly in [19, Proposition 2.8]). So we shall assume that T is a Tararin group.

If $T = \Gamma$, then we are done: Γ admits only finitely many left-orderings. So we suppose that $T = C_\ell$ is a proper convex subgroup of Γ . We will show that the restriction of \preceq to $C_{\ell-1}$ is non-isolated. Hence, in light of Proposition 2.1, there is no harm in assuming that there are no convex subgroups between T and Γ .

To show that \preceq is non-isolated, we consider a dynamical realization of (Γ, \preceq) . We let I_T be the minimal interval stable by T and containing 0. The following is a direct consequence of the fact that T is convex.

Lemma 5.1. *Every element of Γ either fixes, or moves I_T disjointly from itself. In particular, the stabilizer of I_T is exactly T .*

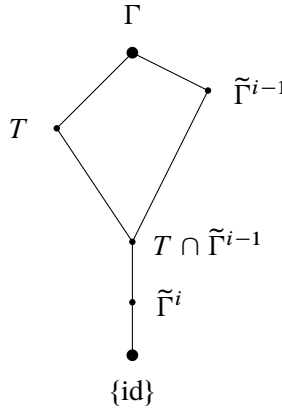
We let $\tilde{\Gamma}$ be a finite index, normal, solvable subgroup of Γ . We let

$$\tilde{\Gamma}^0 = \tilde{\Gamma} \quad \text{and} \quad \tilde{\Gamma}^i = [\tilde{\Gamma}^{i-1}, \tilde{\Gamma}^{i-1}]$$

be the associated derived series

$$\{\text{id}\} = \tilde{\Gamma}^k \triangleleft \tilde{\Gamma}^{k-1} \triangleleft \dots \triangleleft \tilde{\Gamma}^1 \triangleleft \tilde{\Gamma} \triangleleft \Gamma.$$

Note that each $\tilde{\Gamma}^i$ is normal in Γ . We fix once and for all i , being the minimal index such that $\tilde{\Gamma}^i \subseteq T$. Since T is a proper convex subgroup, we have that $k \geq i \geq 1$. In a diagram



Lemma 5.2. *The order restricted to $\tilde{\Gamma}^{i-1}$ is Conradian.*

Proof. Indeed, $\tilde{\Gamma}^{i-1} \cap T$ is convex and normal in $\tilde{\Gamma}^{i-1}$. Moreover, its quotient is abelian, so it admits only Conradian orderings. The lemma then follows from Remark 2.2. □

Lemma 5.3. *The orbit of 0 under $\tilde{\Gamma}^{i-1}$ accumulates on $\pm\infty$.*

Proof. Let I be the smallest open interval containing 0, and stable under $\tilde{\Gamma}^{i-1}$. Since $\tilde{\Gamma}^{i-1}$ is normal in Γ , I is either fixed or moved disjointly by any $\gamma \in \Gamma$. In particular, $\text{Stab}_\Gamma(I)$, the stabilizer of I , is a convex subgroup. Now, if $I \neq \mathbb{R}$, then the maximality of T implies $\text{Stab}_\Gamma(I) \subseteq T$. But this implies $\tilde{\Gamma}^{i-1} \subset T$, which is contrary to our assumptions. □

We have two cases to analyze in order to prove Theorem 1.1.

CASE 1. Suppose there is $g_0 \in \tilde{\Gamma}^{i-1}$ having no fixed points.

Such case occurs for instance if $\tilde{\Gamma}^{i-1}$ has finite rank. Combining the results from [16, 20] we obtain the following proposition (by measure we shall implicitly mean a Radon measure which is finite on compact sets).

Proposition 5.4. *The action of Γ on the real line is semi-conjugated to a non-abelian affine action $\varphi: \Gamma \rightarrow \text{Aff}_+(\mathbb{R})$.*

Proof. Since the action of $\tilde{\Gamma}^{i-1}$ is Conradian and there is $g_0 \in \tilde{\Gamma}^{i-1}$ without fixed points, Corollary 2.15 ensures that $\tilde{\Gamma}^{i-1}$ has a maximal proper convex subgroup $N \supseteq \tilde{\Gamma}^i$, which is normal. In particular, N fixes some open bounded interval I_N around 0. The action of $\tilde{\Gamma}^{i-1}$ either has a discrete orbit or is semi-conjugated to an action factoring through $A = \tilde{\Gamma}^{i-1}/N$. We claim that there is a $\tilde{\Gamma}^{i-1}$ -invariant measure μ . Indeed, in the former case μ may be builded as Dirac masses putted along a discrete orbit of $\tilde{\Gamma}^{i-1}$. In the latter case, since A is abelian and has an element acting without fixed point, [20, Proposition 3.1] implies that the corresponding action has an invariant measure. Lifting back this measure yields a measure μ which is preserved by $\tilde{\Gamma}^{i-1}$ in the original action.

Since g_0 acts without fixed points, the *translation number homomorphism*

$$\tau_\mu: \tilde{\Gamma}^{i-1} \longrightarrow \mathbb{R},$$

given by

$$\tau_\mu(g) = \mu((0, g(0))),$$

is non trivial (here and below, we use the convention $\mu([x, y]) = -\mu([y, x])$ for $y < x$). It now follows from [20, Lemma 4.1 and 4.2] that there is a measure, which we still call μ , which is *quasi-preserved* by Γ , meaning that for every $\gamma \in \Gamma$, there is a positive real number λ_γ such that $\gamma_*(\mu) = \lambda_\gamma \mu$ (where $\gamma_*(\mu)(X) := \mu(\gamma^{-1}(X))$, $X \subseteq \mathbb{R}$). In this way we have a homomorphism

$$\varphi: \Gamma \longrightarrow \text{Aff}_+(\mathbb{R}),$$

which extends τ_μ , given by

$$\varphi(\gamma)(x) = \frac{1}{\lambda_\gamma} x + \mu((0, \gamma(0))). \tag{1}$$

We first check that $\varphi(\Gamma)$ is non-abelian. To this end we let

$$I_\mu := (a, b),$$

where

$$a = \sup\{x < 0 \mid x \in \text{supp}(\mu)\}$$

and

$$b = \inf\{x > 0 \mid x \in \text{supp}(\mu)\}.$$

Then, since μ is quasi-preserved (so in particular its support $\text{supp}(\mu)$ is preserved) we have that $\text{Stab}_\Gamma(I_\mu)$ is either fixed or moved disjointly. In particular, $\text{Stab}_\Gamma(I_\mu)$ is a proper convex subgroup. So, $\text{Ker}(\varphi) \subseteq \text{Stab}_\Gamma(I_\mu) \subseteq T$. Therefore if the affine action of Γ was abelian, then T would be a normal and co-abelian subgroup, so Remark 2.2 would imply that \preceq is Conradian, which is contrary to our assumptions. In particular, there is $\gamma_0 \in \Gamma$ such that $\lambda_{\gamma_0} < 1$.

Finally, we check that this affine action is semi-conjugated to the original dynamical realization action of (Γ, \preceq) . Indeed, if for $x \in \mathbb{R}$ we let

$$F(x) = \mu((0, x]),$$

then a direct computation shows that

$$F(\gamma(x)) = \varphi(\gamma)(F(x)). \tag{2}$$

Thus, it only remains to show that F is continuous, or equivalently that μ has no atoms. But, if there were an atom x_0 , then every non-trivial element of $\tau_\mu(\tilde{\Gamma}^{i-1})$ would have absolute value at least $\mu(\{x_0\})$. Nevertheless,

$$\tau_\mu(\gamma_0^{-n} g_0 \gamma_0^n) = \mu((0, \gamma_0^{-n} g_0 \gamma_0^n(0)]) = \lambda_{\gamma_0}^n \mu(\gamma_0^n(0), g_0 \gamma_0^n(0)) = \lambda_{\gamma_0}^n \tau_\mu(g_0)$$

has absolute value going to 0 (the last equality follows from the $\tilde{\Gamma}^{i-1}$ -invariance of μ , see [20]). This finishes the proof of the proposition. \square

Lemma 5.5. *The kernel of φ is a convex subgroup of (Γ, \preceq) .*

Proof. We keep the notations of the proof of Proposition 5.4.

We first claim that $\text{Stab}_\Gamma(I_\mu) = T$ (equivalently $I_\mu = I_T$). Indeed, let \preceq' be the pseudo-ordering of Γ induced by φ at $F(0) = F(I_\mu)$. Since φ is an affine action, \preceq' has only one convex subgroup, namely

$$\{\gamma \in \Gamma \mid \varphi(\gamma)(F(0)) = F(0)\} = \text{Stab}_\Gamma(I_\mu).$$

However, equation (2) implies that \preceq' is the quotient of \preceq by

$$\text{Stab}_\Gamma(I_\mu): \varphi(g) \succ' \text{id} \implies g \succ \text{id}.$$

Thus, the claim follows from Remark 2.5.

We now show that $\text{Ker}(\varphi)$ is convex in (Γ, \preceq) . First observe that the previous claim implies that $\varphi(T)$ is the abelian subgroup of homotheties centered at $F(0)$. If it was trivial, then we would have that $T = \ker \varphi$ is convex. Let us therefore suppose that $\varphi(T)$ is non-trivial.

We let $\{\text{id}\} = T_0 \triangleleft T_1 \triangleleft \cdots \triangleleft T_m = T$ be the convex series of the Tararin group T . Recall that T_i/T_{i-1} has rank 1 and that the action of T_{i+1} on T_i/T_{i-1} is by multiplication by some negative number. In particular T/T_{m-1} is the unique torsion-free abelian quotient of T . It follows that $T/T_{m-1} = \varphi(T)$ and $\ker \varphi = T_{m-1}$ is convex. \square

Now the proof of Theorem 1.1 in case 1 follows from Corollary 2.13.

CASE 2. Suppose every element $g \in \tilde{\Gamma}^{i-1}$ has a bounded *domain* (i.e. an open interval fixed by g with no fixed point of g in its interior) I_g around 0.

Our proof of Theorem 1.1 in Case 2 consists in showing that \preceq can be approximated by a left-ordering induced from the dynamical realization of (Γ, \preceq) , where the first reference point is chosen outside but very close to I_T . For this purpose, we shall prove that the action is quite similar to the one described in §4 (except that there, I_T was reduced to a point).

It follows from Lemmas 5.2 and 5.3 that the union of the I_g is all of \mathbb{R} . In the sequel, we exploit the facts that $\tilde{\Gamma}^{i-1}$ is normal in Γ and that the order restricted to $\tilde{\Gamma}^{i-1}$ is Conradian to give a more precise picture of the action. First, Corollary 2.16 immediately implies

Lemma 5.6. *Let $f \in \Gamma$ and $g \in \tilde{\Gamma}^{i-1}$. Then one, and only one of the following happen:*

- $f(I_g) = I_g$ or
- $f(I_g)$ is disjoint from I_g or either
- (up to changing f by its inverse) $\overline{I_g} \subset f(I_g)$, i.e. f acts as dilation on I_g .

The following corollaries are easy and left to the reader.

Corollary 5.7. *Let $g \in \tilde{\Gamma}^{i-1} \setminus T$, and I_g its domain. If $t \in T$, then t either fixes I_g or acts on it as a dilation.*

Corollary 5.8. *If I is an interval obtained as a union or an intersection of I_g 's for $g \in \tilde{\Gamma}^{i-1}$, then a weak form of Lemma 5.6 holds for I . Namely every element $f \in \Gamma$ either moves I disjointly from itself, or up to replacing f by its inverse, $I \subset f(I)$ (if the intersection is strict, we say that f weakly dilates I).*

Using the fact that Tararin groups have finite rank, we now obtain a useful description of I_T .

Lemma 5.9. *The intersection of I_g for $g \in \tilde{\Gamma}^{i-1} \setminus T$ coincides with I_T .*

Proof. First observe that Theorem 2.14 implies that the I_g 's, for $g \in \tilde{\Gamma}^{i-1} \setminus T$ are totally ordered for the inclusion, and they all contain I_T . We denote by J the intersection of all I_g for $g \in \tilde{\Gamma}^{i-1} \setminus T$. On the other hand, by Corollary 5.8, if no element weakly dilates J , then the stabilizer of J is convex. Thus it contains T by Corollary 5.7 (which also applies to J), so it must be equal to T . But this implies that $J = I_T$.

It is therefore enough to prove that no element weakly dilates J . Suppose by contradiction that f^{-1} weakly dilates J (i.e. f weakly contracts J). Then there exists $g \in \tilde{\Gamma}^{i-1} \setminus T$ such that $J \cap I_{g^f}$ is strictly contained in J , hence that g^f , and more generally $g_n = g^{f^n}$ belongs to T for all $n \geq 1$. Moreover, $f^n(I_g)$ is a minimal interval (not necessarily containing 0) fixed by the element g_n . Since f acts as a dilation on I_g , we deduce that $f^{n+1}(I_g) \subsetneq f^n(I_g)$. In particular $g_k \in S_n$ for all $k \geq n$, where $S_n = \text{Stab}_T(\overline{f^n(I_g)})$. Let x be a point in the decreasing intersection of compact intervals $\overline{f^n(I_g)}$. We have that $(S_n)_{n \geq 1}$ is a strictly decreasing sequence of convex subgroups of T for the pseudo-ordering \preceq_x , violating the fact that T has only finitely many orderings (and more generally one could check that this is incompatible with the fact that T has finite rank). \square

Since there are no proper convex subgroups above T , we have

Lemma 5.10. *For any $g \in \tilde{\Gamma}^{i-1} \setminus T$, there exists $f \in \Gamma$ acting as a dilation on I_g . Moreover, for every $g, g' \in \tilde{\Gamma}^{i-1} \setminus T$ satisfying $I_g \subseteq I_{g'}$, there exists $f \in \Gamma$ such that $I_{g'} \subseteq f(I_g)$.*

Proof. Looking for a contradiction, suppose that there exists $g \in \tilde{\Gamma}^{i-1} \setminus T$ such that elements of Γ either stabilize I_g or move it disjointly from itself. It follows that the stabilizer S of I_g is convex. By Corollary 5.7, S contains T , and the inclusion is strict as S contains g . On the other hand S cannot be all of Γ since I_g is bounded: a contradiction. This shows the first part of the lemma.

Let us prove the second statement of the lemma. Let

$$\mathcal{F} = \{f \in \Gamma \mid f \text{ dilates } I_g\}.$$

First note that for all $f \in \mathcal{F}$, we have $f(I_g) = I_{g^f}$. Therefore the set of all $f(I_g)$ for $f \in \mathcal{F}$ is nested, let I be its union. We claim that I is either fixed or moved disjointly.

Indeed, let us first suppose that there exists $h \in \tilde{\Gamma}^{i-1}$ such that $h(I)$ strictly contains I . A continuity argument implies that there exists $f \in \mathcal{F}$ such that $h \circ f(I_g)$ contains $f(I_g)$ (hence I_g) and is not contained in I . The first of these statements implies that $h \circ f$ belongs to \mathcal{F} , while the other implies that it does not (by definition of I), so this case cannot occur. Hence, from Corollary 5.8, our claim follows. In particular, the stabilizer of I is a convex subgroup containing T , so I must be all of \mathbb{R} . Hence the lemma. \square

We have given a combinatorial description of the dynamics of Γ around I_T . We now exploit this description to approximate \preceq . First we show that orderings induced by points outside of I_T are distinct from \preceq .

Lemma 5.11. *For all x not in $\overline{I_T}$, there exists $f \in \Gamma$ such that $f \prec_x 0$ and $f \succ 0$. In particular, any left-ordering induced from the dynamical realization of (Γ, \preceq) with x as first reference point is different from \preceq .*

Proof. Let $x \notin \overline{I_T}$, say on its left (the other case is symmetric). It results from Lemma 5.9 that one can find $g, g' \in \tilde{\Gamma}^{i-1} \setminus T$ such that $I_g \subset I_{g'}$ and $x \in I_{g'} \setminus I_g$. On the other hand, Lemma 5.10 provides us with an element f such that $f(I_g)$ contains $I_{g'}$. In particular, if $f \succ \text{id}$, then we are done because $f \prec_x \text{id}$. So we assume that $f \prec \text{id}$. Up to changing g by g^{-1} , we can assume that $g \succ \text{id}$. In particular, $g(I_T)$ is moved to the right of I_T . This implies, that for n large enough, $g^{-n} f g^n \succ \text{id}$. But, in the same time,

$$g^{-n} f g^n(I_g) = g^{-n} f(I_g) = g^{-n}(I_{g^f}) = f(I_g),$$

where the last equality follows because g and g^f are not crossed and $I_g \subset I_{g^f}$. Hence the lemma. \square

The following step consists in showing that when x_n converges to an end point of the interval I_T , then \preceq_{x_n} converges to \preceq outside of T .

Lemma 5.12. *If x_n converges to an end point x of I_T , then \preceq_{x_n} converges to \preceq outside of T . More precisely, for every $g \in \Gamma \setminus T$, we have that $g \succ 0$ if and only if $g \succ_{x_n} 0$ for all n large enough.*

Proof. Since the stabilizer of I_T is precisely T , given any \preceq -positive $g \in \Gamma \setminus T$, we have that g moves I_T disjointly to its right. Therefore $g \succ 0$ if and only if $g \succ_x 0$. The lemma then follows by continuity. \square

To prove Theorem 1.1, we are left to proving that for a well-chosen sequence (x_n) converging to an end point of I_T , the orderings \preceq_{x_n} converges to \preceq in restriction to T . We shall use in a crucial way the fact that Tararin groups have “very few” actions on the real line. More precisely, we have two following lemmas where the action considered is still the dynamical realization of (Γ, \preceq) . Recall that the convex series of T is always given by $\{\text{id}\} = T_0 \triangleleft T_1 \triangleleft \dots \triangleleft T_{m-1} \triangleleft T_m = T$. We let $\gamma_T \in T$ be a non trivial element in T/T_{m-1} which acts on T_{m-1}/T_{m-2} by a multiplication by a negative (rational) number.

Lemma 5.13. *If γ_T fixes some point x , then so does T .*

Proof: Suppose x is fixed by γ_T but not by T . We can then induce a left-ordering on T with reference point (x, x_2, \dots) . In this ordering the stabilizer of x is a proper convex subgroup, which is impossible since it does not coincide with any of the subgroups T_i . \square

Lemma 5.14. *For any $y \in \mathbb{R}$, which is not fixed by γ_T , there is x between $\gamma_T^{-2}(y)$ and $\gamma_T^2(y)$ such that \preceq and \preceq_x coincide over T_{m-1} .*

Proof. Recall that in a dynamical realization, the set of fixed points of a non-trivial element has empty interior. Since T is countable, there is a point z between y and $\gamma_T^{-1}(y)$ whose orbit under T is free. In particular \preceq_z is a total ordering of T . Since T_{m-1} is convex in \preceq_z , there is a minimal open interval I containing z and which is stabilized by T_{m-1} . Being moved disjointly from itself by any non-trivial power of γ_T , I contains at most one point of the orbit of y under $\langle \gamma_T \rangle$. In particular, I is (strictly) contained between $\gamma_T^{-2}(y)$ and $\gamma_T(y)$. Now by Proposition 2.10, there exists an element g either in T_{m-1} or in $\gamma_T T_{m-1}$ such that \preceq and $g(\preceq_z) = \preceq_{gz}$ coincide over T_{m-1} . Clearly $x = gz$ satisfies the conclusion of the lemma. \square

The last step in the proof of Theorem 1.1 is achieved by the following lemma.

Lemma 5.15. *There exists a sequence (x_n) converging to an end point of I_T from outside such that the induced left-ordering $\preceq_{x_n, 0}$ coincides with \preceq over T for all n .*

Indeed, combining Lemmas 5.12 and 5.15, We have that $\preceq_{x_n, 0}$ converges to \preceq . On the other hand, Lemma 5.11 shows that $\preceq_{x_n, 0}$ and \preceq are different. This shows Theorem 1.1 in Case 2.

Proof of Lemma 5.15. The idea is to take $x \notin \overline{I_T}$, x close to ∂I_T , such that the sign of γ_T is preserved and then to use Lemmas 5.13 and 5.14.

By Lemma 5.9, there exists a sequence $g_n \in \tilde{\Gamma}^{i-1} \setminus T$ such that I_{g_n} converges to I_T .

SUBCASE 1. γ_T dilates I_{g_n} .

Up to taking a subsequence, we can assume that $I_{g_n} \subset I_{g_{n-1}}$. For concreteness we suppose $\gamma_T < \text{id}$, and let y_n be the left end-point of I_{g_n} , so that the sequence (y_n) converges to the left end-point z of I_T (in the case $\gamma_T > \text{id}$, we take y_n being right end-points). Since γ_T dilates I_{g_n} , $\gamma_T <_{y_n} \text{id}$. Let x_n be the sequence of points obtained from Lemma 5.14 between $\gamma_T^{-2}(y_n)$ and $\gamma_T^2(y_n)$ such that \leq and \leq_{x_n} coincide in restriction to T_{m-1} . By continuity of γ_T and its inverse, the sequence (x_n) converges to the same limit z . This shows the lemma in Subcase 1.

SUBCASE 2. γ_T (hence T) fixes I_{g_n} .

Again we let x_n be the left end-point of I_{g_n} . Observe that \leq_{x_n} is a pseudo-ordering on Γ , for which the stabilizer S_n of x_n is obviously convex and, from Lemma 5.13, contains T . In particular, the left-ordering $\leq_{x_n,0}$ coincides with \leq over T . This ends the proof of Lemma 5.15. \square

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