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An infinitely generated virtual cohomology group for noncocompact arithmetic groups over function fields

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Abstract. Let $\mathbf{G}(\mathcal{O}_S)$ be a noncocompact irreducible arithmetic group over a global function field K of characteristic p, and let Γ be a finite-index, residually p-finite subgroup of $\mathbf{G}(\mathcal{O}_S)$. We show that the cohomology of Γ in the dimension of its associated Euclidean building with coefficients in the field of p elements is infinite.

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1. Introduction

Let K be a global function field that contains the field with p elements, $\mathbb{F}_p$. We let S be a finite nonempty set of inequivalent valuations of K. The ring $\mathcal{O}_S \subseteq K$ will denote the corresponding ring of S-integers. For any $v \in S$, we let K_v be the completion of K with respect to v so that K_v is a locally compact field.

We denote by $\mathbf{G}$ a connected noncommutative absolutely almost simple K-group, and we let

$$k(\mathbf{G}, S) = \sum_{v \in S} \operatorname{rank}_{K_v} \mathbf{G}$$

so that $k(\mathbf{G}, S)$ is the dimension of the Euclidean building on which the arithmetic group $\mathbf{G}(\mathcal{O}_S)$ acts as a lattice. Thus for example, $k(\mathbf{SL_n}, S) = |S|(n-1)$.

If $\mathbf{G}$ is K-anisotropic, then $\mathbf{G}(\mathcal{O}_S)$ contains a torsion-free finite-index subgroup that acts freely and cocompactly on a Euclidean building of dimension $k(\mathbf{G}, S)$. Determining the finiteness properties of arithmetic groups $\mathbf{G}(\mathcal{O}_S)$ in the case that

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$\mathbf{G}$ is K-isotropic has been more difficult. The model for the K-isotropic case was provided by the following theorem of Stuhler [16].

Theorem 1. *The arithmetic group* $\mathbf{SL_2}(\mathcal{O}_S)$ *is of type* $F_{k(\mathbf{SL_2},S)-1}$*, and if* Γ *is any finite-index subgroup of* $\mathbf{SL_2}(\mathcal{O}_S)$ *whose only torsion elements are* p*-elements, then* $\mathrm{H}^{k(\mathbf{SL_2},S)}(\,\Gamma\,;\,\mathbb{F}_p\,)$ *is infinite.*

Recall that a group π is *of type* F_n if there exists a $K(\pi, 1)$ with finite n-skeleton.

It is well-known, by Selberg's Lemma, that $\mathbf{SL_2}(\mathcal{O}_S)$, or that any arithmetic group over function fields $\mathbf{G}(\mathcal{O}_S)$ as above, contains a finite-index subgroup whose only torsion elements are p-elements.

Bux–Köhl–Witzel [6] completely generalized "half" of Theorem 1 with the following theorem.

Theorem 2. *If* $\mathbf{G}$ *is* K*-isotropic, then* $\mathbf{G}(\mathcal{O}_S)$ *is of type* $F_{k(\mathbf{G},S)-1}$.

Important evidence for the theorem of Bux–Köhl–Witzel was contributed by Behr [4], Abels [1], Abramenko [2], and Bux-Wortman [8].

There are now three proofs that $\mathbf{G}(\mathcal{O}_S)$ as in Theorem 2 is not of type $F_{k(\mathbf{G},S)}$ due to Bux-Wortman [7], Bux–Köhl–Witzel [6], and Kropholler [13] as observed by Gandini [10]. However, outside of the case that $k(\mathbf{G}, S) = 1$, the "second half" of Stuhler's Theorem 1 had not been generalized to include any other arithmetic groups. This paper uses the results of Bux–Köhl–Witzel and Schulz [15] to further generalize the results of Stuhler by proving

Theorem 3. *Suppose* $\mathbf{G}$ *is* K*-isotropic. If* Γ *is a finite-index subgroup of* $\mathbf{G}(\mathcal{O}_S)$ *that is residually* p*-finite, then* $\mathrm{H}^{k(\mathbf{G},S)}(\,\Gamma\,;\,\mathbb{F}_p\,)$ *is infinite.*

A group Γ is *residually* p*-finite* if for any nontrivial $\gamma \in \Gamma$, there is a homomorphism of Γ onto a finite p-group that evaluates γ nontrivially. Such finite-index subgroups of $\mathbf{G}(\mathcal{O}_S)$ are well-known to exist, by Platonov's Theorem, and we provide a proof of their existence in Section 8 for completeness.

To compare Theorems 1 and 3, notice that any torsion element of a residually p-finite group has order a power of p. The author does not know of an example of a finite-index subgroup $\Gamma \leq \mathbf{G}(\mathcal{O}_S)$ whose only torsion elements are p-elements, but such that Γ is not residually p-finite.

As an example of Theorem 3, there is a finite-index subgroup of $\mathbf{SL_n}(\mathcal{O}_S)$ whose cohomology in dimension $|S|(n-1)$ with coefficients in $\mathbb{F}_p$ is infinite. In particular, there is a finite-index subgroup Γ of $\mathbf{SL_n}(\mathbb{F}_p[t])$ such that $H^{n-1}(\Gamma; \mathbb{F}_p)$ is infinite.

1.1. Outline of the proof. To prove Theorem 1, Stuhler analyzed the cell stabilizers of the $\mathbf{SL_2}(\mathcal{O}_S)$-action on the associated Euclidean building which is a product of regular $(p+1)$-valent trees. The cell stabilizers of Γ as in Theorem 1 are products of the group $\mathbb{F}_p$, but the cell stabilizers of a random arithmetic group acting on its associated Euclidean building are more difficult to describe and to work with, so our proof of Theorem 3 proceeds in a different direction.

The main tool in our proof of Theorem 3 is the work of Bux–Köhl–Witzel, and we spend a good portion of the beginning of our proof recalling their work. Let $k = k(\mathbf{G}, S)$ and let X be the Euclidean building that $\mathbf{G}(\mathcal{O}_S)$ acts on as a lattice. Bux–Köhl–Witzel finds a $\mathbf{G}(\mathcal{O}_S)$-invariant, cocompact, $(k-2)$-connected complex $X_{k-2} \subseteq X$. We attach k-cells and $(k+1)$-cells to X_{k-2} to produce a k-connected complex X_k endowed with a Γ-action and a Γ-equivariant map $\psi : X_k \to X$.

We find an unbounded sequence of points $\Gamma y_n \in \Gamma\backslash X$, and a sequence of normal subgroups Γ_n of Γ with index a power of p such that each $y_n \in X$ is contained in a neighborhood of X that injects into $\Gamma_n\backslash X$, and such that the p-group Γ/Γ_n acts on the homology of the image of the neighborhood in the quotient, with coefficients in $\mathbb{F}_p$. The action of the p-group on the homology group produces a functional that nontrivially, and Γ-invariantly, evaluates the image under ψ of the attached k-cells in X_k. Therefore, for each n, we have an assignment of k-cells in $\Gamma\backslash X_k$ to elements of $\mathbb{F}_p$. This produces an infinite sequence in $\mathrm{H}^k(\Gamma\backslash X_k ; \mathbb{F}_p)$. The group Γ may not act freely on X_k, but the lack of freeness is confined to a cocompact subspace of X_k, namely X_{k-2}, and that implies that $\mathrm{H}^k(\Gamma ; \mathbb{F}_p)$ is infinite.

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2. Preliminaries on $\mathbf{G}(\mathcal{O}_S)$ and its action on a Euclidean building

This section establishes some conventions for notation.

2.1. Basic group structure. Let K, $\mathcal{O}_S$, and $\mathbf{G}$ be as in Theorem 3. Because $\mathbf{G}$ is K-isotropic, it contains a proper minimal K-parabolic subgroup $\mathbf{J}$. Let $\mathbf{A}$ be a maximal K-split torus in $\mathbf{J}$, and let $\mathbf{P}$ be a maximal proper K-parabolic subgroup of $\mathbf{G}$ that contains $\mathbf{J}$.

Recall the Langlands decomposition that

$$\mathbf{P} = \mathbf{UHT}$$

where $\mathbf{U}$ is the unipotent radical of $\mathbf{P}$, $\mathbf{H}$ is a reductive K-group with K-anisotropic center, $\mathbf{T}$ is a 1-dimensional connected subtorus of $\mathbf{A}$, and $\mathbf{T}$ commutes with $\mathbf{H}$.

In the remainder of this paper we denote the product over S of local points of a K-group by "unbolding," so that, for example,

$$G = \prod_{v \in S} \mathbf{G}(K_v)$$

2.2. Euclidean building. Let X be the Euclidean building for the semisimple group G. We let $k = k(\mathbf{G}, S)$ so that $k = \dim(X)$.

For each $v \in S$ we choose a maximal K_v-split torus in $\mathbf{J}$ that contains $\mathbf{A}$, and name it $\mathbf{A}_v$. We let $\Sigma \subseteq X$ be the apartment corresponding to the group $\prod_{v \in S} \mathbf{A}_v(K_v)$.

3. Review of Bux–Köhl–Witzel and an unbounded sequence of points $y_n \in X$

Our proof makes use of two results from Bux–Köhl–Witzel [6]: the existence of a $\mathbf{G}(\mathcal{O}_S)$-invariant, $(k-2)$-connected subcomplex $X_{k-2} \subseteq X$ that is cocompact modulo $\mathbf{G}(\mathcal{O}_S)$, and a lemma that will allow us to extend certain "local" k-disks about neighborhoods of points in X to "global" k-disks in X – Lemma 9 and Corollary 10 below. Most of this section is devoted to recalling the work of Bux–Köhl–Witzel. For details omitted from the account in this paper, see [6].

We will use the notation of [6] in our Section 3 except for the following: we will refer to cells in the spherical building for G by the parabolic groups they represent. For example, if $g \in G$ and we write that $g \in P$, then we are treating P as a parabolic group, but if x is a point in the visual boundary of X and we write that $x \in P$, then we are treating P as the simplex in the visual boundary of X that corresponds to P. The correct interpretation should always be clear from context.

3.1. Busemann function for P. For each $v \in S$, let X_v be the Euclidean building for $\mathbf{G}(K_v)$, so that $X = \prod_{v \in S} X_v$. If $\mathcal{O}_v \subseteq K_v$ is the ring of integers, then we let x_v be the vertex in X_v stabilized by $\mathbf{G}(\mathcal{O}_v)$.

Let $\mathbb{A}_K$ be the ring of adeles for K, and let $\mathbb{A}_S$ be the subring of S-adeles. The group $\mathbf{G}(\mathbb{A}_S)$ has a natural left action on X. Given a point $y \in X$ we let $\mathbf{G}(\mathbb{A}_S)_y$ be the stabilizer of y in $\mathbf{G}(\mathbb{A}_S)$.

Following Harder ([11]) and [6], for any $y \in \prod_{v \in S} \mathbf{G}(K_v)x_v$ we let

$$\tilde{\beta}_P(y) = \log_q[\mathrm{vol}[\mathbf{U}(\mathbb{A}_K) \cap \mathbf{G}(\mathbb{A}_S)_y]]$$

where q is the cardinality of the field of constants in K.

We let $\chi_{\mathbf{P}}$ be the canonical character of $\mathbf{P}$. (See Section 1.3 [11] for the definition of $\chi_{\mathbf{P}}$.) The essential feature of $\chi_{\mathbf{P}}$ that will be used below is that the determinant of conjugation by $g \in P$ on U is $\chi_{\mathbf{P}}(g)$.

If $g \in P$, then we have the following transformation rule from Harder [11] Satz 1.3.2:

$$\tilde{\beta}_P(gy) = \tilde{\beta}_P(y) + \log_q(||\chi_{\mathbf{P}}(g)||)$$

where $||\cdot||$ denotes the idele norm. (There is a difference in sign in the line above with [11] and [6] that comes from our convention of using left actions in this paper rather than right actions as in [11] and [6].)

Recall that a Busemann function on the Euclidean building X is given by first choosing a unit speed geodesic $\rho \subseteq X$ and then assigning to any point $x \in X$ the limit as $t \to \infty$ of the difference between the distance between $\rho(t)$ and $\rho(0)$ and the distance between $\rho(t)$ and x.

Proposition 4. *There is some $s > 0$ and a Busemann function $\beta_P : X \to \mathbb{R}$ such that*

$$\beta_P(y) = \tilde{\beta}_P(y)$$

for all $y \in \prod_{v \in S} \mathbf{G}(K_v)x_v$, and such that β_P is nonconstant on factors of X.

Proof. This is Proposition 12.2 of [6]. □

Lemma 5. *The Busemann function β_P is invariant under the actions of U, H, and $\mathbf{T}(\mathcal{O}_S)$ on X, and thus is invariant under the action of $\mathbf{P}(\mathcal{O}_S) \leq UH\mathbf{T}(\mathcal{O}_S)$.*

Proof. Any K-defined character on $\mathbf{P}$, including the canonical character $\chi_{\mathbf{P}}$, evaluates $\mathbf{U}$ trivially since it is unipotent and $\mathbf{H}$ trivially since it is reductive with K-anisotropic center. Thus the result for U and H follows from the transformation rule above.

Similarly, we need to observe that $||\chi_{\mathbf{P}}(t)|| = 1$ for any $t \in \mathbf{T}(\mathcal{O}_S)$. This follows from the product formula (since $\chi_{\mathbf{P}}(t) \in K$) and from the fact that $\mathbf{T}(K_w)$ is bounded if $w \notin S$. □

3.2. Descending chambers at a vertex. Given a vertex $x \in X$, we let $\mathrm{St}(x) \subseteq X$ denote the star of x, the union of all chambers in X that contain x. Thus, the boundary of the star – denoted as $\partial\mathrm{St}(x)$ – is the link of x.

We let $\mathrm{St}^{\downarrow}(x)$ denote the union of chambers $\mathfrak{C} \subseteq X$ containing x with the property that $\beta_P(z) < \beta_P(x)$ for all $z \in \mathfrak{C}$ with $z \neq x$. We let

$$B\mathrm{St}^{\downarrow}(x) = \mathrm{St}^{\downarrow}(x) \cap \partial\mathrm{St}(x).$$

Recall that a special vertex $x \in \Sigma$ is a vertex that is contained in a representative from each parallel family of walls in the Coxeter complex Σ. Thus, the Coxeter complex of an apartment in the spherical building $\partial\mathrm{St}(x)$ is isomorphic to the Coxeter complex of an apartment in the boundary of X when x is special.

The following result is due to Schulz [15].

Lemma 6. *If $x \in X$ is a special vertex, then $B\mathrm{St}^{\downarrow}(x)$ is homotopy equivalent to a noncontractible wedge of $(k-1)$-spheres.*

Proof. Recall that the Busemann function β_P is nonconstant on the factors of X. Since x is a special vertex, the join factors of $\partial\mathrm{St}(x)$ correspond to the factors of X. Therefore, β_P is nonconstant on the join factors of $\partial\mathrm{St}(x)$. That is to say, in the terminology used in [6], the "vertical part" of $\partial\mathrm{St}(x)$ is $\partial\mathrm{St}(x)$ in its entirety.

Notice that $B\mathrm{St}^{\downarrow}(x)$ is exactly the maximal subcomplex of $\partial\mathrm{St}(x)$ that is supported on the complement of the closed ball of radius $\frac{\pi}{2}$ around the gradient direction of β_P in $\partial\mathrm{St}(x)$. Thus, by Theorem B of [15] – restated in Theorem 4.6 of [6] – $B\mathrm{St}^{\downarrow}(x)$ is $(k-1)$-dimensional, $(k-2)$-connected, and noncontractible. □

See also Theorem A.2 of Dymara–Osajda [9].

3.3. Reduction datum. If $\mathbf{M_a}$ is a maximal proper K-parabolic subgroup of $\mathbf{G}$, then we can define a Busemann function β_{M_a} with respect to $\mathbf{M_a}$ similarly to how we defined β_P with respect to $\mathbf{P}$.

In [6], and following [11], there are real constants $r < R$ such that the collection of Busemann functions β_{M_a} forms what is called a *uniform* $\mathbf{G}(\mathcal{O}_S)$*-invariant and cocompact reduction datum*. (See Theorem 1.9 of [6].) The remainder of Section 3.3 is a recollection of what this sort of datum entails. In Section 3.3 we will use $\mathbf{M_a}$ to denote a maximal proper K-parabolic subgroup of $\mathbf{G}$. We will use $\mathbf{M_i}$ to denote a minimal K-parabolic subgroup of $\mathbf{G}$.

For $x \in X$ and a K-parabolic subgroup $\mathbf{Q} \leq \mathbf{G}$, we let $\beta_Q(x)$ be the maximum of all $\beta_{M_a}(x)$ with $\mathbf{Q} \leq \mathbf{M_a}$.

Given an apartment $\Sigma' \subseteq X$ that contains Q as a cell in its boundary, and given $t \in \mathbb{R}$, we let

$$Y_{\Sigma',Q}(t) = \{ x \in \Sigma' \mid \beta_Q(x) \leq t \}$$

This set is convex in Σ' as it is the intersection of the convex sets $\Sigma' \cap \beta_{M_a}^{-1}(\mathbb{R}_{\leq t})$ for $\mathbf{M_a}$ containing $\mathbf{Q}$. Thus, there is a closest point projection

$$\mathrm{pr}^t_{\Sigma',Q} : \Sigma' \to Y_{\Sigma',Q}(t)$$

The group $\sigma_t(x, Q)$ is defined to be the group of $\prod_{v \in S} K_v$-points of the intersection of all $\mathbf{M_a}$ that contain $\mathbf{Q}$ and such that $\beta_{M_a}(\mathrm{pr}^t_{\Sigma',Q}(x)) = t$. We have that $\sigma_t(x, Q) \geq Q$ (as groups, not as cells in the boundary) and we say that Q *t-reduces* $x \in X$ if $\sigma_t(x, Q) = Q$.

To say that the collection of β_{M_a} is an (r, R) *reduction datum* for $r < R$ means that if $\mathbf{M_i}$ is a minimal K-parabolic subgroup of $\mathbf{G}$ that r-reduces $x \in X$, then $\mathbf{M_i} \leq \sigma_R(x, M_i)$.

To say that the reduction datum is *uniform* means that there exists a constant d such that any point in a subset of X whose diameter is less than d can be r-reduced by a common minimal K-parabolic. We can assume, as in [6], by perhaps choosing a lesser r, that d is greater than the diameter of closed stars of cells in X.

The reduction datum is $\mathbf{G}(\mathcal{O}_S)$-*invariant* since

$$\beta_{^\gamma M_a}(\gamma x) = \beta_{M_a}(x)$$

for all $x \in X$, $\gamma \in \mathbf{G}(\mathcal{O}_S)$, and maximal proper K-parabolic $\mathbf{M_a}$. (Here $^\gamma M_a = \gamma M_a \gamma^{-1}$.)

That the reduction datum is *cocompact* means that for any real number $t \geq R$, the set of $x \in X$ for which $\beta_{M_i}(x) \leq t$ for all minimal K-parabolics $\mathbf{M_i}$ that r-reduce x is cocompact with respect to the action of $\mathbf{G}(\mathcal{O}_S)$.

3.4. Definition of height. In [6], the reduction datum is used to define a height function $h : X \to \mathbb{R}_{\geq 0}$. In Section 3.4, we recall this definition.

Choose a special vertex $z \in \Sigma$, and let W_z be the spherical Coxeter group that fixes z in Σ.

The affine space Σ may be realized as a vector space with origin z. Let V_z be the set of all differences of vertices in Σ whose closed stars intersect, where we regard vertices in this context as vectors in Σ. Notice that V_z is finite.

We let $D = W_z V_z$. Again, realizing points of D as vectors of the vector space Σ with origin z, we let

$$Z(D) = \Big\{ \sum_{d \in D} a_d d \mid 0 \leq a_d \leq 1 \text{ for all } d \in D \Big\}$$

The set $Z(D) \subseteq \Sigma$ depended on the choice of vertex z, but modulo isometric translations of Σ, $Z(D)$ is defined intrinsically in terms of the geometry of Σ. Furthermore, if $\Sigma' \subseteq X$ is any apartment in X, then Σ' is isometric to Σ as Coxeter complexes, and thus $x + Z(D)$ is a well-defined subset of Σ' for any $x \in \Sigma'$.

To define a height function, a suitably large $R^* > R$ is chosen. For any apartment $\Sigma' \subseteq X$, any $x \in \Sigma'$, and any minimal K-parabolic $\mathbf{M_i}$ such that M_i represents a cell in the boundary of Σ' that r-reduces x; the point x^*_{Σ',M_i} is defined to be the closest point to x in $Y_{\Sigma',M_i}(R^*) - Z(D)$. Then $h(x)$ is defined as the distance between x and x^*_{Σ',M_i}, and it is shown in Proposition 5.2 of [6] to be independent of Σ' or $\mathbf{M_i}$.

If $h(x) > 0$, then $e(x)$ is defined as the point in the visual boundary of Σ' that is determined as the limit point of the geodesic ray in Σ' from x^*_{Σ',M_i} through x. The point $e(x)$ is also shown to be independent of Σ' or $\mathbf{M_i}$ in Proposition 5.2 of [6]. If we let $\sigma(x)$ denote the group of $\prod_{v\in S} K_v$-points of the K-parabolic subgroup of $\mathbf{G}$ that is minimal with respect to the property that $\sigma(x)$ contains every $\sigma_R(x, M_i)$ for which M_i r-reduces x, then $e(x) \in \sigma(x)$.

As the reduction datum used in this section is $\mathbf{G}(\mathcal{O}_S)$-invariant, we have that $h(\gamma x) = h(x)$ for any $\gamma \in \mathbf{G}(\mathcal{O}_S)$. And if $h(x) > 0$, then $e(\gamma x) = \gamma e(x)$ and $\sigma(\gamma x) = {}^{\gamma}\sigma(x)$.

The subsets of X whose values under h are bounded from above are shown to have bounded quotient on $\mathbf{G}(\mathcal{O}_S)\backslash X$ (See Proposition 2.4 and Observation 5.5 of [6]).

3.5. Choice of y_n. We still have more to discuss about the results of [6], but we take a short break from our account of [6] to establish a sequence of points in X that will be used throughout our proof in this paper.

Lemma 7. *Let $N^* > 0$ be twice the maximum diameter of stars in X. We can choose $R^* \gg 0$ as above to satisfy the following: There is a constant $C^* \in \mathbb{R}$, and a geodesic ray $\ell_Y \subseteq \Sigma$ that limits to a point $\ell_Y(\infty)$ in the simplex P and is orthogonal to level sets of β_P in Σ, such that every point z in the N^*-neighborhood of $U\ell_Y$ in X is r-reduced by J, has*

$$h(z) = \beta_P(z) + C^* > 0$$

and

$$e(z) = \ell_Y(\infty) \in P.$$

Furthermore, there is a sequence of special vertices $y_n \in \Sigma$ that are contained in chambers of Σ that intersect ℓ_Y, such that $\beta_P(y_n)$ is a strictly increasing sequence of numbers, and such that the set of all $(y_n)^*_{\Sigma,P}$ is a bounded set.

Proof. There are $\mathrm{rank}_K \mathbf{G} \leq \dim(\Sigma)$ maximal proper K-parabolic subgroups that contain $\mathbf{J}$. The space $Y_{\Sigma,J}(R^*) \subseteq \Sigma$ is the intersection of one half-apartment of Σ for every maximal proper K-parabolic subgroup that contains $\mathbf{J}$, and the set $\beta_P^{-1}(R^*) \cap Y_{\Sigma,J}(R^*)$ is an unbounded face of the boundary of $Y_{\Sigma,J}(R^*)$. We call this face F_{P,R^*}. It has dimension equal to $\dim(\Sigma) - 1$.

We let

$$\Omega(r, R^*, J, P) = \{\, x \in \Sigma \mid \sigma_r(x, J) = J \text{ and } \sigma_{R^*}(x, J) = P \,\}$$

For $x \in \Sigma$, we let $B_\Sigma(x; N^*) \subseteq \Sigma$ be the ball in Σ centered at x with radius N^*. Notice that by replacing R^* with a greater constant, we may assume that there is some $x \in F_{P,R^*} \cap \Omega(r, R^*, J, P)$ such that

$$F_{P,R^*} \cap [B_\Sigma(x; N^*) + Z(D)] \subseteq F_{P,R^*} \cap \Omega(r, R^*, J, P)$$

Furthermore, if y is contained in the geodesic ray $\ell_Y \subseteq \Sigma$ that begins at x, is orthogonal to F_{P,R^*}, and is contained in $\Omega(r, R^*, J, P)$, then

$$B_\Sigma(y; N^*) + Z(D) \subseteq \Omega(r, R^*, J, P)$$

as long as the distance between y and x is sufficiently large. We replace ℓ_Y with a subray so that

$$B_\Sigma(y; N^*) + Z(D) \subseteq \Omega(r, R^*, J, P) \quad \text{for any } y \in \ell_Y.$$

If z is contained in the interior of $\Omega(r, R^*, J, P)$, then $e(z)$ is given by the direction of the gradient of β_P restricted to Σ – which is the direction of $\ell_Y(\infty)$. Thus by Lemma 5,

$$UHe(z) = UH\ell_Y(\infty) = \ell_Y(\infty) = e(z).$$

And $T \leq A$ acts trivially on the boundary of Σ, so we have

$$Pe(z) = UHTe(z) = e(z)$$

which implies that $e(z) \in P$.

We let d_0 be the constant difference of the distance between $y \in \ell_Y$ and $\Sigma \cap \beta_P^{-1}(R^*)$ and the distance between $y + Z(D)$ and $\Sigma \cap \beta_P^{-1}(R^*)$. (Note that the latter of the two distances is $h(y)$.) Then for $z \in B_\Sigma(y; N^*)$,

$$h(z) = \beta_P(z) - R^* - d_0.$$

Thus we let

$$C^* = -R^* - d_0.$$

Again let $y \in \ell_Y$ and now let $z \in B_X(y; N^*)$, where $B_X(y; N^*)$ is the ball in X of radius N^* that is centered at y. We will show that z is r-reduced by J, has $h(z) = \beta_P(z) + C^* > 0$, and has $e(z) = \ell_Y(\infty) \in P$.

For every $v \in S$, let $\mathbf{J}_v \leq \mathbf{G}$ be a minimal K_v-parabolic subgroup of $\mathbf{G}$ such that $\mathbf{A}_v \leq \mathbf{J}_v \leq \mathbf{J}$. We let $\mathbf{U}_v$ be the unipotent radical of $\mathbf{J}_v$, so that $\mathbf{U}_v \leq \mathbf{J} \leq \mathbf{P}$ and $\mathbf{U}_v \leq \mathbf{UH}$.

If X_v is the Euclidean building for $\mathbf{G}(K_v)$, and Σ_v is the apartment that $\mathbf{A}_v(K_v)$ acts on, then because any point in X_v is contained in a $\mathbf{J}_v(K_v)$ translate of Σ_v

$$X_v = \mathbf{J}_v(K_v)\Sigma_v = \mathbf{U}_v(K_v)\mathbf{Z}_{\mathbf{G}}(\mathbf{A}_v)(K_v)\Sigma_v = \mathbf{U}_v(K_v)\Sigma_v$$

where $\mathbf{Z}_{\mathbf{G}}(\mathbf{A}_v)$ is the centralizer of $\mathbf{A}_v$ in $\mathbf{G}$, and thus is a Levi subgroup of $\mathbf{J}_v$. Therefore,

$$X = \prod_{v \in S} \mathbf{U}_v(K_v)\Sigma$$

and there is a distance nonincreasing retraction

$$\varrho \colon X \to \Sigma$$

defined on each $u\Sigma$ for $u \in \prod_{v \in S} \mathbf{U}_v(K_v)$ as the map $u^{-1} \colon u\Sigma \to \Sigma$.

So for $z \in B_X(y; N^*)$ we choose $u \in \prod_{v \in S} \mathbf{U}_v(K_v)$ such that $u^{-1}z \in \Sigma$. Because ϱ is distance nonincreasing and $\varrho(y) = y$, we have that $u^{-1}z \in B_\Sigma(y; N^*)$. By Lemma 5

$$\beta_P(z) + C^* = \beta_P(u^{-1}z) + C^* = h(u^{-1}z) > 0$$

If $\mathbf{Q}$ is a proper K-parabolic subgroup of $\mathbf{G}$ containing $\mathbf{J}$, then $\mathbf{Q}$ contains $\mathbf{U}_v$ and thus $u^{-1}\mathbf{Q}u = \mathbf{Q}$, so applying the clear analogue of Lemma 5 to each maximal proper K-parabolic group containing $\mathbf{J}$ yields $uY_{\Sigma,J}(R^*) = Y_{u\Sigma,J}(R^*)$ and that $z \in u\Omega(r, R^*, J, P)$ since $u^{-1}z \in B_\Sigma(y; N^*) \subseteq \Omega(r, R^*, J, P)$. Thus, z is r-reduced by $uJu^{-1} = J$ and $u^{-1}(z^*_{u\Sigma,J}) = (u^{-1}z)^*_{\Sigma,J}$ and

$$h(z) = h(u^{-1}z) = \beta_P(z) + C^*.$$

Furthermore, as the set $\Omega(r, R^*, J, P)$ limits to the cell P and $u \in P$, the set $u\Omega(r, R^*, J, P)$ also limits to P and thus

$$e(z) = e(u^{-1}z) = \ell_Y(\infty) \in P.$$

To review, we have shown that for any z in the N^*-neighborhood of ℓ_Y in X that z is r-reduced by J, has $h(z) = \beta_P(z)+C^* > 0$, and has $e(z) = \ell_Y(\infty) \in P$. We still need to show the same results apply to the weaker condition that z is contained in the N^*-neighborhood of $U\ell_Y$ in X. For that, recall that $\mathbf{U}$ is unipotent, so $\mathbf{U}(\mathcal{O}_S)$ is a cocompact lattice in U. That is, there is a compact set $B \subseteq U$ such that $\mathbf{U}(\mathcal{O}_S)B = U$. Since ℓ_Y limits to P and $\mathbf{U}$ is the unipotent radical of $\mathbf{P}$, any element of U fixes pointwise a subray of ℓ_Y. Therefore, there is a common subray of ℓ_Y that is fixed pointwise by every element of B. Thus, by replacing ℓ_Y with a subray we may assume that B fixes ℓ_Y and thus that

$$U\ell_Y = \mathbf{U}(\mathcal{O}_S)B\ell_Y = \mathbf{U}(\mathcal{O}_S)\ell_Y$$

Hence, if $z \in UB_X(\ell_Y; N^*) = \mathbf{U}(\mathcal{O}_S)B_X(\ell_Y; N^*)$ then $uz \in B_X(\ell_Y; N^*)$ for some $u \in \mathbf{U}(\mathcal{O}_S)$, and since h is $\mathbf{G}(\mathcal{O}_S)$-invariant and β_P is U-invariant,

$$h(z) = h(uz) = \beta_P(uz) + C^* = \beta_P(z) + C^*$$

Since the reduction datum is $\mathbf{G}(\mathcal{O}_S)$-invariant and uz is r-reduced by J, we see that z is r-reduced by $u^{-1}Ju = J$. Last, since $u \in \mathbf{U}(\mathcal{O}_S) \leq P$ and $e(uz) \in P$ we have $e(z) = u^{-1}e(uz) = e(uz) = \ell_Y(\infty)$.

To find the sequence of y_n, just choose an unbounded sequence of chambers in Σ that intersect ℓ_Y. Any chamber in X contains a special vertex, and this produces the sequence of y_n. Because each of the $y_n \in \Sigma$ are a uniformly bounded distance from ℓ_Y, each $(y_n)^*_{\Sigma,P} \in F_{P,R^*}$ is a uniformly bounded distance from the point $x \in F_{P,R^*}$. □

In the remainder of this paper, we shall abbreviate $\mathrm{St}(y_n)$ as S_n. Similarly, we shall abbreviate $\mathrm{St}^{\downarrow}(y_n)$ and $B\mathrm{St}^{\downarrow}(y_n)$ as $S_n^{\downarrow}$ and $BS_n^{\downarrow}$ respectively.

3.6. Morse function. Section 3.6 is the final section in which we recount the work of Bux–Köhl–Witzel. In this section we recall the definition of a combinatorial Morse function from [6] that is defined on the vertices of the barycentric subdivision of X and used to deduce connectivity properties of subsets of X.

For any cell $\tau \in X$ we let $\dim(\tau)$ be its dimension. There is also a number defined in [6] as $\mathrm{dp}(\tau)$ which refers to the "depth" of a cell. We refer the reader to Section 8 of [6] for the definition of the depth of a cell.

We let $\mathring{X}$ be the barycentric subdivision of the Euclidean building X. For any cell $\tau \subseteq X$, we let $\mathring{\tau}$ be its barycenter. Bux–Köhl–Witzel assigned to $\mathring{\tau}$ the triple of real numbers

$$f_{\mathrm{BKW}}(\mathring{\tau}) = \left(\max_{x\in\tau}(h(x)), \mathrm{dp}(\tau), \dim(\tau)\right)$$

The function f_{BKW} is a combinatorial Morse function when triples of real numbers are ordered lexicographically.

For any triple of real numbers s that is greater than or equal to the triple $s_0 = (1,0,0)$, we let $\mathring{X}(s)$ be the subcomplex of $\mathring{X}$ spanned by the $\mathring{\tau}$ for which $f_{\mathrm{BKW}}(\mathring{\tau}) \leq s$. Since f_{BKW} is $\mathbf{G}(\mathcal{O}_S)$-invariant, so too is $\mathring{X}(s)$. Since $\mathring{X}(s)$ is a closed subset of $\mathring{X}$ whose height is bounded, it is cocompact modulo $\mathbf{G}(\mathcal{O}_S)$. The values of f_{BKW} are finite below any given bound, and we let $s+1$ denote the least value of f_{BKW} that is greater than s.

We let $\mathrm{Lk}(\mathring{\tau})$ be the link of $\mathring{\tau}$ in $\mathring{X}$, and we define the *Morse descending link* of $\mathring{\tau}$ with respect to the Morse function f_{BKW} to be the complex of simplices $\sigma \subseteq \mathrm{Lk}(\mathring{\tau})$ such that

$$f_{\mathrm{BKW}}(v) < f_{\mathrm{BKW}}(\mathring{\tau})$$

for every vertex $v \in \sigma$. To obtain $\mathring{X}(s+1)$ we attach to $\mathring{X}(s)$ the descending links of cells $\mathring{\tau} \subseteq \mathring{X}$ with $f_{\mathrm{BKW}}(\mathring{\tau}) = s+1$. The work of Bux–Köhl–Witzel is to have defined f_{BKW} in such a way as to utilize the work of Schulz [15] in showing that the Morse descending links of vertices in $\mathring{X}$ are either contractible or spherical of dimension $(k-1)$. Thus, up to homotopy equivalence, $\mathring{X}(s+1)$ is obtained by attaching k-cells to $\mathring{X}(s)$. This process induces an isomorphism of homotopy groups

$$\pi_i(\mathring{X}(s)) \cong \pi_i(\mathring{X}(s+1)) \quad \text{for } i \leq k-2.$$

Since X is contractible and the union of the $\mathring{X}(s)$, we have that $\mathring{X}(s)$ is $(k-2)$-connected for any $s \geq s_0$. It is the existence of a $\mathbf{G}(\mathcal{O}_S)$-cocompact $(k-2)$-connected space that can be viewed as the main result of [6] as it immediately implies that $\mathbf{G}(\mathcal{O}_S)$ is of type F_{k-1}.

In what remains, we will let $X_{k-2} = X(s_0)$. In particular, X_{k-2} is a $(k-2)$-connected subcomplex of X that is invariant and cocompact under the action of $\mathbf{G}(\mathcal{O}_S)$. We will also pass to a subsequence of the y_n to assume that $S_n \cap X_{k-2} = \emptyset$ for all n.

The following lemma demonstrates the compatibility of β_P and f_{BKW} on S_n.

Lemma 8. *The Morse descending link of y_n with respect to f_{BKW} equals $BS_n^{\downarrow}$.*

Proof. As in Section 6 of [6], the height function h forces a decomposition of the link of $y_n \in X$ into a join of a "horizontal link" of y_n and a "vertical link" of y_n where the horizontal link of y_n is the join of all factors of the link of y_n whose points are evaluated by h as $h(y_n)$.

By Lemma 7, the restriction of β_P to the horizontal link of y_n is constant. But y_n is a special vertex, so Proposition 4 implies that the horizontal link of y_n is trivial, and therefore, that the vertical link of y_n equals the link of y_n.

Now by Proposition 9.6 of [6], the Morse descending link of y_n is the subcomplex of the link of y_n in X that is spanned by all vertices v in the link of y_n such that $h(v) < h(y_n)$. (Keep in mind that any vertex of X is "significant.") Again, by Lemma 7, this complex is equal to $BS_n^{\downarrow}$. □

3.7. Extending local disks near y_n. In addition to the existence of X_{k-2}, we shall utilize the results of [6] to extend "local" disks near y_n to "global" disks in X. More precisely, we have

Lemma 9. *Let $\sigma: S^{k-1} \to X$ be a continuous map of a $(k-1)$-sphere into X. Suppose there is some triple $s > s_0$ such that $\sigma(S^{k-1}) \subseteq \mathring{X}(s)$. Then there is a homotopy*

$$F: S^{k-1} \times [0,1] \longrightarrow X$$

such that, for all $x \in S^{k-1}$,

$$F(x,t) \in \mathring{X}(s),$$

$$F(x,0) = \sigma(x),$$

and

$$F(x,1) \in \mathring{X}(s_0) = X_{k-2}.$$

Proof. Let $c_1^0, \ldots, c_m^0 \subseteq X$ be the image under σ of the 0-cells of S^{k-1}. Let $c_{i,F}^0 \subseteq \mathring{X}(s)$ be paths from c_i^0 to X_{k-2}. The boundary of each $c_{i,F}^0$ is c_i^0 and b_i^0 for some $b_i^0 \in X_{k-2}$.

If $k = 1$, then $m = 2$, and $c_{1,F}^0 \cup c_{2,F}^0$ is the image of the homotopy F.

If $k \geq 2$, then let $c_i^1 \subseteq \sigma(S^{k-1})$ be the image of the 1-cell with boundary c_ℓ^0 and c_j^0. Since $\mathring{X}(s)$ is obtained from X_{k-2} by attaching k-cells, there is a homotopy relative b_ℓ^0 and b_j^0 between $c_i^1 \cup c_{\ell,F}^0 \cup c_{j,F}^0$ and a 1-cell $b_i^1 \subseteq X_{k-2}$. We name the image of this homotopy $c_{i,F}^1$.

If $k = 2$, then the union of the $c_{i,F}^1$ defines the homotopy F.

If $k \geq 3$, then we proceed as above by induction on the skeleta of S^{k-1}. □

We let $I_n = S_n - \partial S_n$ be the interior of S_n. As a consequence of the above lemma, we have

Corollary 10. *For $n \gg 0$, there is a k-disk $D_n^k \subseteq S_n^{\downarrow} \cup (X - \mathbf{G}(\mathcal{O}_S)I_n)$ with $\partial D_n^k \subseteq X_{k-2}$ and such that $D_n^k \cap S_n^{\downarrow}$ is a k-disk that represents a noncontractible k-sphere in the quotient space $S_n^{\downarrow}/BS_n^{\downarrow}$.*

Proof. Let s_n be the triple such that $f_{\mathrm{BKW}}(y_n) = s_n$. By Lemma 7, and the definition of the Morse function f_{BKW}, we have for any cell $\tau \subseteq S_n$ that is not contained in ∂S_n that $f_{\mathrm{BKW}}(\mathbf{G}(\mathcal{O}_S)\tau) = f_{\mathrm{BKW}}(\tau) \geq s_n$ since $y_n \in \tau$. That is, $\mathbf{G}(\mathcal{O}_S)I_n \cap \mathring{X}(s_n - 1) = \emptyset$.

By Lemmas 6 and 8, there is a noncontractible $(k-1)$-sphere $\sigma_n^{k-1} \subseteq BS_n^{\downarrow}$. We let $d_n^k \subseteq S_n^{\downarrow}$ be the cone at $y_n \in S_n^{\downarrow}$ on

$$\sigma_n^{k-1} \subseteq BS_n^{\downarrow} \subseteq \mathring{X}(s_n - 1)$$

By Lemma 9, there is a homotopy F between ∂d_n^k and a $(k-1)$-sphere in X_{k-2} whose image is contained in $\mathring{X}(s_n - 1)$. We let D_n^k be the union of d_n^k and F. Then

$$D_n^k \subseteq S_n^{\downarrow} \cup \mathring{X}(s_n - 1) \subseteq S_n^{\downarrow} \cup (X - \mathbf{G}(\mathcal{O}_S)I_n)$$

That $D_n^k \cap S_n^{\downarrow} = d_n^k$ represents a noncontractible k-sphere in $S_n^{\downarrow}/BS_n^{\downarrow}$ follows from the natural identification of $d_n^k/\partial d_n^k$ and $S_n^{\downarrow}/BS_n^{\downarrow}$ with the suspensions of σ_n^{k-1} and $BS_n^{\downarrow}$ respectively. $\square$

Lemma 11. *Suppose that $\mathfrak{C}_a, \mathfrak{C}_b \subseteq S_n^{\downarrow}$ are chambers in X, and that there is some $\gamma \in \mathbf{G}(\mathcal{O}_S)$ such that $\gamma\mathfrak{C}_a = \mathfrak{C}_b$. Then*

$$\gamma y_n = y_n.$$

Proof. The vertex y_n is the only vertex of any chamber in $S_n^{\downarrow}$ with

$$f_{\mathrm{BKW}}(v) = f_{\mathrm{BKW}}(y_n).$$

Since f_{BKW} is $\mathbf{G}(\mathcal{O}_S)$ invariant, we have for $\gamma y_n \in \mathfrak{C}_b$ that

$$f_{\mathrm{BKW}}(\gamma y_n) = f_{\mathrm{BKW}}(y_n)$$

so that $\gamma y_n = y_n$. $\square$

4. Construction of a k-connected $\mathbf{G}(\mathcal{O}_S)$-complex

Bux–Köhl–Witzel gives us a $(k-2)$-connected complex that $\mathbf{G}(\mathcal{O}_S)$ acts on properly and cocompactly, namely X_{k-2}. In order to determine the cohomology of finite-index subgroups of $\mathbf{G}(\mathcal{O}_S)$ in dimension k, we will create a k-connected space that $\mathbf{G}(\mathcal{O}_S)$ acts on. In this section we will construct such a space by attaching k-cells to X_{k-2} and then attaching $(k+1)$-cells after that.

4.1. Construction of X_k. We let

$$\psi\colon X_{k-2} \longrightarrow X$$

be the inclusion. In the process of our construction of a k-connected space that contains X_{k-2}, we will be extending ψ to a map from that k-connected space into X.

Let

$$\sigma\colon S^{k-1} \longrightarrow X_{k-2}$$

be a continuous map of a $(k-1)$-sphere into the $(k-1)$-skeleton of X_{k-2}. We regard σ as an attaching map for a k-cell that we name $D^k_{1,\sigma}$.

For each nontrivial $\gamma \in \mathbf{G}(\mathcal{O}_S)$, we attach another k-cell $D^k_{\gamma,\sigma}$ to X_{k-2} using the attaching map $\gamma \circ \sigma$. We assign a homeomorphism

$$\gamma : D^k_{1,\sigma} \longrightarrow D^k_{\gamma,\sigma}$$

that restricts to the γ-action on $\partial D^k_{1,\sigma}, \partial D^k_{\gamma,\sigma} \subseteq X_{k-2}$. Then for any $\lambda \in \mathbf{G}(\mathcal{O}_S)$, we let

$$\lambda\colon D^k_{\gamma,\sigma} \longrightarrow D^k_{\lambda\gamma,\sigma}$$

be the homeomorphism defined by $\lambda = (\lambda\gamma)\gamma^{-1}$. In this way, we have defined a $\mathbf{G}(\mathcal{O}_S)$-action on the complex

$$X_{k-2} \cup \bigcup_{\gamma\in\mathbf{G}(\mathcal{O}_S)} D^k_{\gamma,\sigma}$$

We repeat the process above for every continuous $\sigma\colon S^{k-1} \to X_{k-2}$ with image in the $(k-1)$-skeleton of X_{k-2}. The resulting union of X_{k-2} with the union of every $D^k_{\gamma,\sigma}$ for every pair of γ and σ is a k-complex that we will denote by X_{k-1}. Notice that X_{k-1} is a $(k-1)$-connected, $\mathbf{G}(\mathcal{O}_S)$-complex. The group $\mathbf{G}(\mathcal{O}_S)$ will not in general act freely on X_{k-1}, but any nontrivial point stabilizers correspond to points in X_{k-2} since the interiors of each of the $D^k_{\gamma,\sigma}$ are disjoint.

We extend ψ to each $D^k_{\gamma,\sigma}$ – and thus to all of X_{k-1} – by assigning arbitrary continuous maps $\psi : D^k_{1,\sigma} \to X$ that agree with ψ on $\partial D^k_{1,\sigma} \subseteq X_{k-2}$ and then by defining $\psi : D^k_{\gamma,\sigma} \to X$ as $\gamma \circ \psi \circ \gamma^{-1}$. Notice that $\gamma \circ \psi = \psi \circ \gamma$ so that ψ is $\mathbf{G}(\mathcal{O}_S)$-equivariant.

Now repeat the above process, this time attaching $(k+1)$-cells $D^{k+1}_{\gamma,\sigma}$ to X_{k-1} with attaching maps $\sigma : S^k \to X_{k-1}$ to obtain a k-connected complex X_k that $\mathbf{G}(\mathcal{O}_S)$ acts on with a $\mathbf{G}(\mathcal{O}_S)$-equivariant map $\psi : X_k \to X$ that restricts to $X_{k-2} \subseteq X$ as the inclusion map. The action of $\mathbf{G}(\mathcal{O}_S)$ on $X_k - X_{k-2}$ is free.

5. Assigning attaching disks to cycles in a finite complex

In this section we will begin to focus some attention on a given finite-index subgroup Γ of $\mathbf{G}(\mathcal{O}_S)$ from the statement of our main result, Theorem 3. That is, we let Γ be any finite-index subgroup of $\mathbf{G}(\mathcal{O}_S)$ that is residually p-finite.

Our goal in proving our main result is to show that $H^k(\Gamma\backslash X_k; \mathbb{F}_p)$ is infinite. In the penultimate section of this paper we explain why this implies that $H^k(\Gamma; \mathbb{F}_p)$ is infinite.

5.1. Definition of Γ_n. Our proof of our main result relies on forming a sequence of finite quotients of the group Γ. These quotients are described in the following

Lemma 12. *For any $n \geq 0$, there is a normal subgroup $\Gamma_n \trianglelefteq \Gamma$ such that Γ/Γ_n is a finite p-group and Γ_n acts cocompactly and freely on ΓS_n.*

Proof. The group Γ acts cocompactly on ΓS_n.

For any cell $\tau \subseteq S_n$, let Γ_τ be the finite stabilizer of τ in Γ, and let $Z_n \subseteq \Gamma$ be the finite set of the union of Γ_τ over the finite set of cells $\tau \subseteq S_n$.

Since Γ is residually p-finite, there is for each nontrivial $\gamma \in Z_n$ a finite p-group, G_γ, and a homomorphism

$$\phi_\gamma\colon \Gamma \longrightarrow G_\gamma$$

such that

$$\phi_\gamma(\gamma) \neq 1.$$

Now let

$$\phi\colon \Gamma \longrightarrow \prod_\gamma G_\gamma$$

be the product of the ϕ_γ, and let Γ_n be the kernel of ϕ. Then $\Gamma_n \trianglelefteq \Gamma$, Γ/Γ_n is a finite p-group, and $Z_n \cap \Gamma_n = \{1\}$.

Since Γ_n is finite-index in Γ, it acts cocompactly on ΓS_n. Furthermore, if $\gamma \in \Gamma_n$ and $\gamma g\tau = g\tau$ for some $g \in \Gamma$ and some cell $\tau \subseteq S_n$, then $g^{-1}\gamma g \in \Gamma_n$ is contained in $\Gamma_\tau \subseteq Z_n$, and thus $g^{-1}\gamma g$, and hence γ, is trivial. $\square$

5.2. Definition of θ_n. We define

$$\theta_n\colon X \longrightarrow \Gamma_n\backslash X$$

to be the quotient map. Notice that Γ acts on $\Gamma_n\backslash X$ since Γ_n is normal in Γ. Furthermore, θ_n is Γ-equivariant.

Also note that Γ acts on the pair $(X, X - \Gamma I_n)$ and thus on the pair $(\theta_n(X), \theta_n(X - \Gamma I_n))$, and therefore on the homologies of these pairs. (All homologies of complexes in this paper are cellular.)

5.3. Definition of $\Theta_n(D^k_{\gamma,\sigma})$. Given a k-cell $D^k_{\gamma,\sigma}$ attached to X_{k-2} in the construction of X_k, we have that $\psi(\partial D^k_{\gamma,\sigma}) \subseteq X_{k-2}$.

By Lemma 7, the sequence of $h(y_n)$, and hence of $f_{\mathrm{BKW}}(\Gamma y_n)$ is unbounded. Thus we may assume that X_{k-2} intersects each ΓS_n trivially, which implies $\partial\psi(D^k_{\gamma,\sigma}) \subseteq X - \Gamma I_n$ and thus that $\psi(D^k_{\gamma,\sigma})$ represents a class in the homology group $H_k(X, X - \Gamma I_n; \mathbb{F}_p)$, and further, that $\theta_n \circ \psi(D^k_{\gamma,\sigma})$ represents a class in the homology group $H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$. In the remainder we shall let

$$\Theta_n(D^k_{\gamma,\sigma}) = [\theta_n \circ \psi(D^k_{\gamma,\sigma})] \in H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$$

Recall that ψ is Γ-equivariant, and that θ_n is Γ-equivariant. Therefore, the group Γ acts on the set of all $\Theta_n(D^k_{\gamma,\sigma})$ by the rule that if $g \in \Gamma$, then

$$\begin{aligned} g\Theta_n(D^k_{\gamma,\sigma}) &= g[\theta_n \circ \psi(D^k_{\gamma,\sigma})] \\ &= [\theta_n \circ \psi(gD^k_{\gamma,\sigma})] \\ &= [\theta_n \circ \psi(D^k_{g\gamma,\sigma})] \\ &= \Theta_n(D^k_{g\gamma,\sigma}) \end{aligned}$$

5.4. Definition of W_n. We let W_n be the vector subspace of the space $H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$ generated by the classes $\Theta_n(D^k_{\gamma,\sigma})$ for every pair γ and σ.

By the above, the Γ-action on $H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$ restricts to a Γ-action on W_n. Since Γ_n acts trivially on $\theta_n(X)$, the action of Γ on W_n factors through the finite p-group Γ/Γ_n.

Lemma 13. *The vector space W_n is finite-dimensional and nonzero.*

Proof. The space X is the union of ΓS_n and $X - \Gamma I_n$, so ΓS_n surjects via θ_n onto the quotient $\theta_n(X)/\theta_n(X - \Gamma I_n)$. Lemma 12 gives us that $\theta_n(\Gamma S_n)$ is a finite complex, and thus, $\theta_n(X)/\theta_n(X - \Gamma I_n)$ is finite. The finite dimensionality of W_n now follows from the finite dimensionality of $H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$.

Let $D^k_n \subseteq X$ be as in Corollary 10. We claim that $\theta_n(D^k_n)$ represents a nonzero class in $H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$. Indeed, $BS^{\downarrow}_n \subseteq X - \Gamma I_n$ and it suffices to prove that

$$(\theta_n)_* : H_k(S^{\downarrow}_n, BS^{\downarrow}_n; \mathbb{F}_p) \longrightarrow H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$$

is injective. As $\theta_n(X)$ is a k-dimensional complex, this reduces to showing that $\theta_n(\mathfrak{C}_a) \neq \theta_n(\mathfrak{C}_b)$ for distinct chambers $\mathfrak{C}_a, \mathfrak{C}_b \subseteq S_n^{\downarrow}$. In other words, we want to show that $\gamma\mathfrak{C}_a = \mathfrak{C}_b$ for any $\gamma \in \Gamma_n$ and any pair of chambers $\mathfrak{C}_a, \mathfrak{C}_b \subseteq S_n^{\downarrow}$ implies that $\mathfrak{C}_a = \mathfrak{C}_b$. By Lemma 11, any such $\gamma \in \Gamma_n$ fixes $y_n \in \Gamma S_n$, and by Lemma 12, γ is trivial so that $\mathfrak{C}_a = \mathfrak{C}_b$.

Now let $\sigma_n \colon S^{k-1} \to X_{k-2}$ represent ∂D_n^k, and let D^k_{1,σ_n} be the k-disk attached to X_{k-2} by σ_n in the construction of X_k. Since X is contractible and k-dimensional, and since D_n^k and $\psi(D^k_{1,\sigma_n})$ share a common boundary, they are equal in the group of cellular k-chains in X. Therefore, by the above paragraph,

$$\Theta_n(D^k_{1,\sigma_n}) = [\theta_n \circ \psi(D^k_{1,\sigma_n})] = [\theta_n(D^k_n)]$$

is a nonzero class in $W_n \leq H_k(\theta_n(X), \theta_n(X - \Gamma I_n); \mathbb{F}_p)$. □

6. A sequence of cycles and cocycles for $\Gamma \backslash X_k$

The action of Γ on W_n induces an action of Γ on the dual vector space W_n^* by

$$\gamma\phi(x) = \phi(\gamma^{-1}x)$$

for $\gamma \in \Gamma$, $\phi \in W_n^*$, and $x \in W_n$.

Lemma 14. *For each n, there is a Γ-invariant $\varphi_n \in W_n^*$ and some $\lambda_n \in \mathbf{G}(\mathcal{O}_S)$ and $\tau_n \colon S^{k-1} \to X_{k-2}$ such that*

$$\varphi_n(\Theta_n(D^k_{\lambda_n,\tau_n})) \neq 0.$$

Furthermore, after passing to a subsequence, if $m > n$ then

$$\varphi_m(\Theta_m(D^k_{\lambda_n,\tau_n})) = 0.$$

Proof. A linear transformation of a finite-dimensional nonzero vector space of characteristic p is unipotent if and only if it has order p^k for some k (see e.g. 15.1 in [12]). Since the action of Γ on W_n^* factors through the p-group Γ/Γ_n, the elements of Γ act on W_n^* as unipotent transformations. By Kolchin's Theorem (see e.g. 2.5 in [3]), any group of unipotent transformations on a finite-dimensional nonzero vector space fixes a nonzero vector. That is, there is some Γ-invariant $\varphi_n \in W_n^*$ and some k-disk $D^k_{\lambda_n,\tau_n}$ from the construction of X_k such that $\varphi_n(\Theta_n(D^k_{\lambda_n,\tau_n})) \neq 0$.

Given the disk $D^k_{\lambda_n,\tau_n}$ above, we may assume that the f_{BKW}-values of the cells in S_{n+1}, and hence of those in ΓS_{n+1} exceed the f_{BKW}-values of the finitely many cells in $\psi(D^k_{\lambda_n,\tau_n})$. Thus, if $m > n$ we have that $\psi(D^k_{\lambda_n,\tau_n}) \subseteq X - \Gamma I_m$ and thus $\Theta_m(D^k_{\lambda_n,\tau_n}) = 0$ in W_m. □

6.1. Cocycles. Let $D^k_{\gamma,\sigma}$ be a k-cell that was attached to X_{k-2} in the construction of X_k. Recall that $\Theta_n(D^k_{\gamma,\sigma})$ represents a class in W_n and that φ_n is a Γ-invariant functional on W_n.

Lemma 15. *For any* $n \geq 0$, $\gamma \in \mathbf{G}(\mathcal{O}_S)$, $g \in \Gamma$, *and* $D^k_{\gamma,\sigma}$, *we have*

$$\varphi_n(\Theta_n(D^k_{\gamma,\sigma})) = \varphi_n(\Theta_n(gD^k_{\gamma,\sigma})).$$

Proof. This is immediate since ψ is Γ-equivariant, θ_n is Γ-equivariant, and φ_n is Γ-invariant. □

Let $q\colon X_k \to \Gamma\backslash X_k$ be the quotient map. Note that any k-cell in $\Gamma\backslash X_k$ is contained in $\Gamma\backslash X_{k-2}$ or else is of the form $q(D^k_{\gamma,\sigma})$ for some $D^k_{\gamma,\sigma} \subseteq X_k$. We define the k-cochain Φ_n on k-chains in $\Gamma\backslash X_k$ with values in $\mathbb{F}_p$ as 0 on $\Gamma\backslash X_{k-2}$ and

$$\Phi_n(q(D^k_{\gamma,\sigma})) = \varphi_n(\Theta_n(D^k_{\gamma,\sigma}))$$

for any $q(D^k_{\gamma,\sigma})$, and then we extend linearly. The previous lemma tells us that Φ_n is well-defined.

Lemma 16. Φ_n *is a cocycle.*

Proof. The $(k+1)$-cells of $\Gamma\backslash X_k$ are of the form $q(D^{k+1}_{\gamma,\sigma})$, so we must check that Φ_n evaluates the boundary of any $q(D^{k+1}_{\gamma,\sigma})$ trivially.

Let $\mathfrak{C}_1, \ldots, \mathfrak{C}_m$ be a collection of k-cells in X_{k-2} such that the chain $\partial D^{k+1}_{\gamma,\sigma}$ equals $\sum_j \mathfrak{C}_j + \sum_i D^k_{\gamma_i,\sigma_i}$ for some $D^k_{\gamma_i,\sigma_i}$ where we suppress in this notation the orientation of k-cells. Then $\partial q(D^{k+1}_{\gamma,\sigma}) = \sum_j q(\mathfrak{C}_j) + \sum_i q(D^k_{\gamma_i,\sigma_i})$.

Note that $\psi(\partial D^{k+1}_{\gamma,\sigma})$ is a k-sphere in the k-dimensional and contractible X, and hence it represents the 0-chain. That is, the chain $\psi(\sum_j \mathfrak{C}_j + \sum_i D^k_{\gamma_i,\sigma_i}) \cap \Gamma S_n$, and hence $\psi(\sum_i D^k_{\gamma_i,\sigma_i}) \cap \Gamma S_n$, is the 0-chain. Therefore, $\Theta_n(\sum_i D^k_{\gamma_i,\sigma_i})$ is the 0-chain, which implies

$$\begin{aligned}
\Phi_n\big(\partial q(D^{k+1}_{\gamma,\sigma})\big) &= \Phi_n\Big(\sum_j q(\mathfrak{C}_j) + \sum_i q(D^k_{\gamma_i,\sigma_i})\Big) \\
&= \Phi_n\Big(\sum_i q(D^k_{\gamma_i,\sigma_i})\Big) \\
&= \varphi_n \circ \Theta_n\Big(\sum_i D^k_{\gamma_i,f_i}\Big) \\
&= \varphi_n(0) \\
&= 0
\end{aligned}$$

□

6.2. Cycles. Given $D^k_{\lambda_n,\tau_n}$ as in Lemma 14, the k-chain $D^k_{\lambda_n,\tau_n} - D^k_{\lambda_0,\tau_0}$ is the difference of two k-disks in X_k. We let

$$C_n = q(D^k_{\lambda_n,\tau_n}) - q(D^k_{\lambda_0,\tau_0})$$

which is a k-chain in $\Gamma\backslash X_k$.

Lemma 17. *After passing to a subsequence in n, each C_n is a k-cycle over $\mathbb{F}_p$ in $\Gamma\backslash X_k$.*

Proof. Notice that $q(\partial D^k_{\gamma_n,\sigma_n})$ is a $(k-1)$-cycle in $\Gamma\backslash X_{k-2}$. Since $\Gamma\backslash X_{k-2}$ is compact, there are only finitely many cellular $(k-1)$-chains in $\Gamma\backslash X_{k-2}$ with coefficients in $\mathbb{F}_p$. Therefore, we may pass to a subsequence and assume that $q(\partial D^k_{\lambda_n,\tau_n})$ is a constant $\mathbb{F}_p$-cycle for $n \geq 0$. □

We can now prove

Proposition 18. *$H^k(\Gamma\backslash X_k; \mathbb{F}_p)$ and $H_k(\Gamma\backslash X_k; \mathbb{F}_p)$ are infinite.*

Proof. Let $m \geq n > 0$. By the definitions of Φ_n and C_n, and by Lemma 14,

$$\begin{aligned}
\Phi_m(C_n) &= \Phi_m(q(D^k_{\lambda_n,\tau_n})) - \Phi_m(q(D^k_{\lambda_0,\tau_0})) \\
&= \varphi_m(\Theta_m(D^k_{\lambda_n,\tau_n})) - \varphi_m(\Theta_m(D^k_{\lambda_0,\tau_0})) \\
&= \varphi_m(\Theta_m(D^k_{\lambda_n,\tau_n}))
\end{aligned}$$

does not equal 0 if $m = n$, but does equal 0 if $m > n$. Thus, each of the terms in the sequences $[\Phi_n] \in H^k(\Gamma\backslash X_k; \mathbb{F}_p)$ and $[C_n] \in H_k(\Gamma\backslash X_k; \mathbb{F}_p)$ are distinct. □

7. Proof of Theorem 3

If Γ acts freely on X_k, then Theorem 3 is immediate from Proposition 18. And one can always choose a finite-index, residually p-finite subgroup of $\mathbf{G}(\mathcal{O}_S)$ that acts freely on X_k (see the following section). However, to show Theorem 3 holds for any, and not just some, finite-index, residually p-finite subgroup of $\mathbf{G}(\mathcal{O}_S)$, we need to apply one more technique. That is the goal of this section.

By our construction of X_k, the group Γ acts freely on $X_k - X_{k-2}$, and while it may not be true that Γ acts freely on X_{k-2}, it does act cocompactly on X_{k-2}. That is, there are only finitely many k-cells in the quotient $\Gamma\backslash X_{k-2}$. This will imply Theorem 3 after the application of a spectral sequence.

The material from this section is taken from Chapter VII of Brown's text on Cohomology of Groups [5].

We begin by subdividing X_k such that individual cells in X_k inject into $\Gamma\backslash X_k$.

We let $H_k^\Gamma(X_k;\mathbb{F}_p)$ be the k-th equivariant homology group of Γ and X_k with coefficients in $\mathbb{F}_p$. That is, if $C_*(X_k;\mathbb{F}_p)$ is the chain complex for the homology of X_k with coefficients in $\mathbb{F}_p$, and if F_* is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$, then

$$H_k^\Gamma(X_k;\mathbb{F}_p) = H_k(F_* \otimes_\Gamma C_*(X_k;\mathbb{F}_p))$$

Lemma 19. $H_k^\Gamma(X_k;\mathbb{F}_p) = H_k(\Gamma;\mathbb{F}_p)$

Proof. The complex $F_* \otimes_\Gamma C_*(X_k;\mathbb{F}_p)$ is a double complex with an associated spectral sequence

$$E^1_{\ell,q} = H_q(F_\ell \otimes_\Gamma C_*(X_k;\mathbb{F}_p)) = F_\ell \otimes_\Gamma H_q(X_k;\mathbb{F}_p)$$

and

$$E^2_{\ell,q} = H_\ell(\Gamma; H_q(X_k;\mathbb{F}_p))$$

Notice that if $0 < q \le k$ then $E^2_{\ell,q} = H_\ell(\Gamma;0) = 0$ since X_k is k-connected. It follows that $E^r_{\ell,q} = 0$ when $r \ge 2$ and $0 < q \le k$. Hence,

$$H_k(\Gamma;\mathbb{F}_p) = E^2_{k,0} = E^\infty_{k,0} = \bigoplus_{\ell+q=k} E^\infty_{\ell,q}$$

The lemma follows since the spectral sequence converges to $H_*^\Gamma(X_k;\mathbb{F}_p)$. □

The complex $F_* \otimes_\Gamma C_*(X_k;\mathbb{F}_p)$ is also a double complex with an associated spectral sequence where $E^1_{\ell,q} = H_q(F_* \otimes_\Gamma C_\ell(X_k;\mathbb{F}_p))$. The spectral sequence converges to $H_*^\Gamma(X_k;\mathbb{F}_p)$, and in particular,

$$H_k(\Gamma;\mathbb{F}_p) = H_k^\Gamma(X_k;\mathbb{F}_p) = \bigoplus_{\ell+q=k} E^\infty_{\ell,q}$$

As in VII.7.7 of [5],

$$E^1_{\ell,q} = \bigoplus_{c\in Y_\ell} H_q(\Gamma_c;\mathbb{F}_p)$$

where Y_ℓ is a set of representatives of ℓ-cells in X_k modulo Γ, and Γ_c is the stabilizer in Γ of c.

Lemma 20. *If $r, q \geq 1$, then $E^r_{\ell,q}$ is finite.*

Proof. Since Γ acts cocompactly on X_{k-2} and freely on $X_k - X_{k-2}$, there are only finitely many $c \in Y_\ell$ such that $\Gamma_c \neq 1$. Thus, $E^1_{\ell,q}$ is finite as it is a finite sum of homology groups of finite groups with coefficients in a finite field. The lemma follows since the dimension of $E^r_{\ell,q}$ is bounded by that of $E^1_{\ell,q}$. □

Lemma 21. $E^2_{\ell,0} = H_\ell(\Gamma\backslash X_k; \mathbb{F}_p)$. *In particular, by Proposition 18, $E^2_{k,0}$ is infinite.*

Proof. Let ∂' be the boundary operator for $C_*(X_k; \mathbb{F}_p)$, and for any $(\ell-1)$-cell $d \subseteq X_k$, let π_d be the projection of $C_{\ell-1}(X_k; \mathbb{F}_p)$ onto the coordinate represented by d.

We let ∂ be the boundary operator for the chain complex of $\Gamma\backslash X_k$, denoted as $C_*(\Gamma\backslash X_k; \mathbb{F}_p)$.

Notice that $E^2_{*,0}$ is the homology of the complex $(E^1_{k,0}, d^1)$ where

$$d^1 \colon E^1_{\ell,0} \to E^1_{\ell-1,0}.$$

There is a natural identification of

$$E^1_{\ell,0} = \bigoplus_{c \in Y_\ell} H_0(\Gamma_c; \mathbb{F}_p) = \bigoplus_{c \in Y_\ell} \mathbb{F}_p$$

with

$$C_\ell(\Gamma\backslash X_k; \mathbb{F}_p)$$

given by

$$(a_c)_{c \in Y_\ell} \longmapsto \sum_{\Gamma c \subseteq \Gamma\backslash X_k} a_c(\Gamma c)$$

where $a_c \in \mathbb{F}_p$. Below we apply this identification liberally.

Our goal is to show that d^1 can be identified with ∂. For this, if $c \in Y_\ell$ then we let $\mathcal{D}_c$ be the set of $(\ell-1)$-cells in X_k contained in c. Then VII.8.1 of [5] tells us that if $a_c \in \mathbb{F}_p = H_0(\Gamma_c; \mathbb{F}_p)$ then, up to sign,

$$d^1(a_c) = \sum_{d \in \mathcal{D}_c} v_d \circ u_{cd} \circ t_c(a_c)$$

where

$$t_c \colon H_0(\Gamma_c; \mathbb{F}_p) \longrightarrow H_0(\Gamma_c; \mathbb{F}_p)$$

is transfer – and thus is the identity – and where

$$v_d \colon H_0(\Gamma_d; \mathbb{F}_p) \longrightarrow H_0(\Gamma_{d_0}; \mathbb{F}_p)$$

for $d_0 \in Y_{\ell-1}$ is such that $\Gamma d = \Gamma d_0$ and v_d is induced by conjugation in Γ – and thus is the identity – and where

$$u_{cd} : H_0(\Gamma_c; \mathbb{F}_p) \longrightarrow H_0(\Gamma_d; \mathbb{F}_p)$$

is induced by $\Gamma_c \hookrightarrow \Gamma_d$ and $\pi_d \circ \partial'|_c$ – and thus is identified with

$$\pi_d \circ \partial'|_c : \{ a_c c \mid a_c \in \mathbb{F}_p \} \longrightarrow \{ a_d d \mid a_d \in \mathbb{F}_p \}$$

Therefore,

$$\begin{aligned} d^1(a_c) &= \sum_{d \in \mathcal{D}_c} u_{cd}(a_c) \\ &= \sum_{d \in \mathcal{D}_c} \pi_d \circ \partial'(a_c) \\ &= \partial(a_c(\Gamma c)). \end{aligned}$$

□

7.1. Proof of Theorem 3. By the two preceding lemmas, we have for each $r \geq 2$ that the kernel of

$$d^r : E^r_{k,0} \longrightarrow E^r_{k-r,r-1}$$

is infinite, which implies the infiniteness of

$$E^\infty_{k,0} \leq \bigoplus_{\ell+q=k} E^\infty_{\ell,q} = H_k(\Gamma; \mathbb{F}_p) \cong H^k(\Gamma; \mathbb{F}_p).$$

8. Existence of finite-index, residually p-finite subgroups of $\mathbf{G}(\mathcal{O}_S)$

In this section we give a sketch of the well-known existence statement from the title of this section. The existence essentially follows from Platonov's Theorem on finitely-generated matrix groups. We took our account below from Nica [14].

Let w be a valuation of K that is not contained in S, and let $\mathfrak{m} \subseteq \mathcal{O}_S$ be the ideal $\{ x \in \mathcal{O}_S \mid |x|_w < 1 \}$. Note that $\cap_k \mathfrak{m}^k = 0$. Furthermore, $\mathcal{O}_S/\mathfrak{m}$ is identified with the values of elements of $\mathcal{O}_S$ at w, and hence is finite. Similarly, $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is finite for any $k \geq 1$, so that $\mathcal{O}_S/\mathfrak{m}^k$ is a finite ring.

For $k \geq 1$, let Λ_k be the kernel of

$$\alpha_k : \mathbf{GL_n}(\mathcal{O}_S) \longrightarrow \mathbf{GL_n}(\mathcal{O}_S/\mathfrak{m}^k)$$

Since $\mathcal{O}_S/\mathfrak{m}^k$ is a finite ring, Λ_k is a finite-index normal subgroup of $\mathbf{GL_n}(\mathcal{O}_S)$. Also note that if $m > k$ then Λ_m is a normal subgroup of Λ_k since Λ_m is the kernel of α_m restricted to Λ_k.

We claim that Λ_k/Λ_{k+1} is a p-group. Indeed, if $g \in \Lambda_k$ then the matrix entries of $g-1$ are contained in $\mathfrak{m}^k$. Thus, the matrix entries of $(g-1)^p$ are contained in $\mathfrak{m}^{k+1}$. Since $\mathcal{O}_S \subseteq K$ has characteristic p, $g^p - 1 = (g-1)^p$ so that $g^p \in \Lambda_{k+1}$, establishing our claim.

Note that $\cap_k \mathfrak{m}^k = 0$ implies $\cap_k \Lambda_k = 1$. Thus, if $Z \subseteq \Lambda_1$ is finite we can choose $k \gg 0$ such that $Z \cap \Lambda_k \subseteq \{1\}$, and

$$[\Lambda_1 : \Lambda_k] = \prod_{i=1}^{k-1} [\Lambda_i : \Lambda_{i+1}]$$

is a power of p. Therefore, Λ_1 is a finite-index, residually p-finite subgroup of $\mathbf{GL_n}(\mathcal{O}_S)$.

For general $\mathbf{G}(\mathcal{O}_S)$ we have an embedding of K-groups $\mathbf{G} \leq \mathbf{GL_n}$ and we replace Λ_k in the above with $\Lambda_k \cap \mathbf{G}(\mathcal{O}_S)$.

References

[1] H. Abels, Finiteness properties of certain arithmetic groups in the function field case. *Israel J. Math.* **76** (1991), no. 1-2, 113–128. Zbl 0819.20051 MR 1177335

[2] P. Abramenko, Finiteness properties of Chevalley groups over $\mathbb{F}_q[t]$. *Israel J. Math.* **87** (1994), no. 1-3, 203–223. Zbl 0809.20034 MR 1286827

[3] J. Alperin and and R. Bell, Graduate Texts in Mathematics, 162. Springer, New York, N.Y., 1995. Zbl 0839.20001 MR 1369573

[4] H. Behr, Arithmetic groups over function fields. I. A complete characterization of finitely generated and finitely presented arithmetic subgroups of reductive algebraic groups. *J. Reine Angew. Math.* **495** (1998), 79–118. Zbl 0923.20038 MR 1603845

[5] K. Brown, *Cohomology of groups*. Graduate Texts in Mathematics, 87. Springer, Berlin etc., 1982. Zbl 0584.20036 MR 0672956

[6] K.-U. Bux, R. Köhl, and S. Witzel, Higher finiteness properties of reductive arithmetic groups in positive characteristic: the rank theorem. *Ann. of Math.* (2) **177** (2013), no. 1, 311–366. Zbl 1290.20039 MR 2999042

[7] K.-U. Bux and K. Wortman, Finiteness properties of arithmetic groups over function fields. *Invent. Math.* **167** (2007), no. 2, 355–378. Zbl 1126.20030 MR 2270455

[8] K.-U. Bux and K. Wortman, Connectivity properties of horospheres in Euclidean buildings and applications to finiteness properties of discrete groups. *Invent. Math.* **185** (2011), no. 2, 395–419. Zbl 1237.20041 MR 2819164

[9] J. Dymara and D. Osajda, Boundaries of right-angled hyperbolic buildings. *Fund. Math.* **197** (2007), 123–165. Zbl 1177.20042 MR 2365885

[10] G. Gandini, Bounding the homological finiteness length. *Bull. Lond. Math. Soc.* **44** (2012), no. 6, 1209–1214. Zbl 1272.20052 MR 3007653

[11] G. Harder, Minkowskische Reduktionstheorie über Funtionenkörpern. *Invent. Math.* **7** (1969), 33–54. Zbl 0284441 MR 0284441

[12] J. Humphreys, *Linear algebraic groups.* Graduate Texts in Mathematics, 21. Springer, Berlin etc., 1975. Zbl 0325.20039 MR 0396773

[13] P. Kropholler, On groups of type $(\mathrm{FP})_\infty$. *J. Pure Appl. Algebra* **90** (1993), no. 1, 55–67. Zbl 0816.20042 MR 1246274

[14] B. Nica, Linear groups - Malcev's theorem and Selberg's lemma. Preprint 2013. arXiv:1306.2385 [math.GR]

[15] B. Schulz, Spherical subcomplexes of spherical buildings. *Geom. Topol.* **17** (2013), no. 1, 531–562. Zbl 1271.51006 MR 3039769

[16] U. Stuhler, Homological properties of certain arithmetic groups in the function field case. *Invent. Math.* **57** (1980), no. 3, 263–281. Zbl 0432.14026 MR 0568936

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