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Hyperbolic relatively hyperbolic graphs and disk graphs

Ursula Hamenstädt[1]

Abstract. We show that a relatively hyperbolic graph with uniformly hyperbolic peripheral subgraphs is hyperbolic. As an application, we show that the disk graph and the electrified disk graph of a handlebody H of genus $g \geq 2$ are hyperbolic, and we determine their Gromov boundaries.

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Contents

1. Introduction

Consider a connected metric graph $\mathcal{G}$ in which a family $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ of complete connected subgraphs has been specified. Here $\mathcal{C}$ is a countable, finite or empty index set. The graph $\mathcal{G}$ is *hyperbolic relative to the family* $\mathcal{H}$ if the following properties are satisfied.

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Define the $\mathcal{H}$-*electrification* $\mathcal{E}\mathcal{G}$ of $\mathcal{G}$ to be the graph which is obtained from $\mathcal{G}$ by adding for every $c \in \mathcal{C}$ a new vertex v_c which is connected to each vertex $x \in H_c$ by an edge and which is not connected to any other vertex. We require that the graph $\mathcal{E}\mathcal{G}$ is hyperbolic in the sense of Gromov and that moreover a property called *bounded penetration* holds true (see [8] for perhaps the first formulation of this property). We refer to [25] for a consolidation of the various notions of relative hyperbolicity found in the literature.

If $\mathcal{G}$ is a hyperbolic metric graph and if $\mathcal{H}$ is a family of connected uniformly quasi-convex subgraphs of $\mathcal{G}$ whose fixed size neighborhoods intersect in set of uniformly bounded diameter then $\mathcal{G}$ is hyperbolic relative to $\mathcal{H}$. This fact is probably folklore; implicitly it was worked out in a slightly modified form in [17].

Vice versa, Farb showed in [8] that if $\mathcal{G}$ is the Cayley graph of a finitely generated group and if the graphs H_c are δ-hyperbolic for a number $\delta > 0$ not depending on $c \in \mathcal{C}$ then $\mathcal{G}$ is hyperbolic. In [6] it is noted that using a result of Bowditch [3], the argument in [8] can be extended to arbitrary (possibly locally infinite) relatively hyperbolic metric graphs.

Our first goal is to give a different and self-contained proof of this result which gives effective estimates for the hyperbolicity constant as well as explicit control on uniform quasi-geodesics. We show

Theorem 1. *Let $\mathcal{G}$ be a metric graph which is hyperbolic relative to a family $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ of complete connected subgraphs. If there is a number $\delta > 0$ such that each of the graphs H_c is δ-hyperbolic then $\mathcal{G}$ is hyperbolic. Moreover, the subgraphs H_c $(c \in \mathcal{C})$ are uniformly quasi-convex.*

The control we obtain allows to use the result inductively. Moreover, the Gromov boundary of $\mathcal{G}$ can easily be determined from the Gromov boundaries of the $\mathcal{H}$-electrification $\mathcal{E}\mathcal{G}$ and the Gromov boundaries of the quasi-convex subgraphs H_c.

We next discuss applications of Theorem 1.

Let S be a closed surface of genus $g \geq 2$. For a number $k < g$ define the *graph of non-separating k-multicurves* to be the following metric graph $\mathcal{NC}(k)$. Vertices are k-tuples of of essential pairwise non-homotopic simple closed curves on S which cut S into a single connected component. Two such non-separating multicurves c_1, c_2 are connected by an edge if $c_1 \cup c_2$ is a non-separating multicurve with $k+1$ components. In [13] we used Theorem 1 to show

Theorem 2. *For $k < g/2 + 1$ the graph $\mathcal{NC}(k)$ is hyperbolic.*

We also observed that the bound $k < g/2 + 1$ is sharp. The same argument applies to the graph of non-separating multicurves on a surface with punctures.

In this article we use Theorem 1 to investigate the geometry of graphs of disks in a handlebody. A handlebody of genus $g \geq 1$ is a compact three-dimensional manifold H which can be realized as a closed regular neighborhood in $\mathbb{R}^3$ of an embedded bouquet of g circles. Its boundary ∂H is an oriented surface of genus g.

An *essential disk* in H is a properly embedded disk $(D, \partial D) \subset (H, \partial H)$ whose boundary ∂D is an essential simple closed curve in ∂H.

A subsurface X of the compact surface ∂H is called *essential* if it is a complementary component of an embedded multicurve in ∂H. Note that the complement of a non-separating simple closed curve in ∂H is essential in this sense, i.e. the inclusion $X \to \partial H$ need not induce an injection on fundamental groups.

Define a connected essential subsurface X of the boundary ∂H of H to be *thick* if the following properties hold true.

(1) Every disk intersects X.

(2) X is filled by boundaries of disks.

The boundary surface ∂H of H is thick. An example of a proper thick subsurface of ∂H is the complement in ∂H of a suitably chosen simple closed curve which is not diskbounding.

Definition. Let $X \subset \partial H$ be a thick subsurface. The *electrified disk graph* of X is the graph $\mathcal{EDG}(X)$ whose vertices are isotopy classes of essential disks in H with boundary in X. Two vertices D_1, D_2 are connected by an edge of length one if there is an essential simple closed curve in X which can be realized disjointly from both $\partial D_1, \partial D_2$.

If $X = \partial H$ then we call $\mathcal{EDG}(X)$ the *electrified disk graph* of H. Using Theorem 1 we show

Theorem 3. *The electrified disk graph $\mathcal{EDG}(X)$ of a thick subsurface $X \subset \partial H$ of the boundary ∂H of a handlebody H of genus $g \geq 2$ is hyperbolic.*

The electrified disk graph of the thick subsurface X is moreover of infinite diameter [11].

For the investigation of the *handlebody group*, i.e. the group of isotopy classes of homeomorphisms of H, a more natural graph to consider is the so-called disk graph which is defined as follows.

Definition. The *disk graph* $\mathcal{DG}$ of H is the graph whose vertices are isotopy classes of essential disks in H. Two such disks are connected by an edge of length one if and only if they can be realized disjointly.

Since for any two disjoint essential simple closed curves c, d on ∂H there is a simple closed curve on ∂H which can be realized disjointly from c, d (e.g. one of the curves c, d), the electrified disk graph is obtained from the disk graph by adding some edges. This observation allows to apply Theorem 1 inductively to the graphs $\mathcal{EDG}(X)$ where X passes through the thick subsurfaces of ∂H and deduce in a bottom-up inductive procedure hyperbolicity of the disk graph from hyperbolicity of the electrified disk graph. In this way we obtain a new, completely combinatorial and significantly simpler proof of the following result which was first established by Masur and Schleimer [22].

Theorem 4. *The disk graph $\mathcal{DG}$ of a handlebody H of genus $g \geq 2$ is hyperbolic.*

We also determine the Gromov boundary of the disk graph. Namely, recall from [18, 9] that the Gromov boundary of the curve graph of an essential subsurface X of ∂H can be identified with the space of minimal geodesic laminations λ in X which *fill* X, i.e. are such that every essential simple closed curve in X has non-trivial intersection with λ. The Gromov topology on this space of geodesic laminations is the *coarse Hausdorff topology* which can be defined as follows. A sequence λ_i converges to λ if and only if every limit in the usual Hausdorff topology of a subsequence of λ_i contains λ as a sublamination. Notice that the coarse Hausdorff topology is defined on the entire space $\mathcal{L}(\partial H)$ of geodesic laminations on ∂H, however it is not Hausdorff.

We observe that for every thick subsurface X of ∂H the Gromov boundary $\partial\mathcal{EDG}(X)$ of the electrified disk graph $\mathcal{EDG}(X)$ can be identified with a subspace of the space of geodesic laminations on X, equipped with the coarse Hausdorff topology. Moreover we show

Theorem 5. *The Gromov boundary $\partial\mathcal{DG}$ of the disk graph equals the subspace*

$$\partial\mathcal{DG} = \bigcup_X \partial\mathcal{EDG}(X) \subset \mathcal{L}(\partial H)$$

equipped with the coarse Hausdorff topology. The union is over all thick subsurfaces X of ∂H.

There is no analog of this result for handlebodies with *spots*, i.e. with marked points on the boundary. Indeed, we showed in [12] that the disk graph of a handlebody with one or two spots on the boundary is not hyperbolic. The electrified disk graph is not hyperbolic for handlebodies with one spot on the boundary, and the same holds true for sphere graphs.

The organization of this paper is as follows. In Section 2 we show Theorem 1. Section 3 discusses some relative version of results from [11]. In Section 4, we show the Theorem 3. In Section 5 we construct a graph whose vertices are disks and which is obtained from the electrified disk graph by removing some edges and from the disk graph by adding edges. We show that this graph is hyperbolic. The argument can be used inductively and yields the proof of Theorem 4 as well as of Theorem 5 in Section 6.

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2. Hyperbolic thinnings of hyperbolic graphs

In this section we show Theorem 1 from the introduction. Consider a (not necessarily locally finite) metric graph $\mathcal{G}$ (i.e. edges have length one) and a family $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ of complete connected subgraphs, where $\mathcal{C}$ is any countable, finite or empty index set.

Define the $\mathcal{H}$-*electrification* of $\mathcal{G}$ to be the metric graph $(\mathcal{E}\mathcal{G}, d_{\mathcal{E}})$ which is obtained from $\mathcal{G}$ by adding vertices and edges as follows. For each $c \in \mathcal{C}$ there is a unique vertex $v_c \in \mathcal{E}\mathcal{G} - \mathcal{G}$. This vertex is connected with each of the vertices of H_c by a single edge of length one, and it is not connected with any other vertex.

In the sequel all parametrized paths γ in $\mathcal{G}$ or $\mathcal{E}\mathcal{G}$ are supposed to be *simplicial*. This means that the image of every integer is a vertex, and the image of an integral interval $[k, k+1]$ is an edge or a single vertex.

Call a simplicial path γ in $\mathcal{E}\mathcal{G}$ *efficient* if for every $c \in \mathcal{C}$ we have $\gamma(k) = v_c$ for at most one k. Note that if γ is an efficient simplicial path in $\mathcal{E}\mathcal{G}$ which passes through $\gamma(k) = v_c$ for some $c \in \mathcal{C}$ then $\gamma(k-1) \in H_c, \gamma(k+1) \in H_c$.

The following definition is an adaptation of a definition from [8].

Definition 2.1. The family $\mathcal{H}$ has the *bounded penetration property* if for every $L > 1$ there is a number $p(L) > 0$ with the following property. Let γ be an efficient L-quasi-geodesic in $\mathcal{E}\mathcal{G}$, let $c \in \mathcal{C}$ and let $k \in \mathbb{Z}$ be such that $\gamma(k) = v_c$.

If the distance in H_c between $\gamma(k-1)$ and $\gamma(k+1)$ is at least $p(L)$ then every efficient L-quasi-geodesic γ' in $\mathcal{EG}$ with the same endpoints as γ passes through v_c. Moreover, if $k' \in \mathbb{Z}$ is such that $\gamma'(k') = v_c$ then the distance in H_c between $\gamma(k-1), \gamma'(k'-1)$ and between $\gamma(k+1), \gamma'(k'+1)$ is at most $p(L)$.

The definition below of relative hyperbolicity for a graph is taken from [25] where it is shown to be equivalent to other definitions of relative hyperbolicity found in the literature.

Definition 2.2. Let $\mathcal{H}$ be a family of complete connected subgraphs of a metric graph $\mathcal{G}$. The graph $\mathcal{G}$ is *hyperbolic relative to* $\mathcal{H}$ if the $\mathcal{H}$-electrification of $\mathcal{G}$ is hyperbolic and if moreover $\mathcal{H}$ has the bounded penetration property.

From now on we always consider a metric graph $\mathcal{G}$ which is hyperbolic relative to a family $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ of complete connected subgraphs.

We say that the family $\mathcal{H}$ is *r-bounded* for a number $r > 0$ if

$$\operatorname{diam}(H_c \cap H_d) \leq r \quad \text{for } c \neq d \in \mathcal{C},$$

where the diameter is the minimum of the diameters for the intrinsic path metrics on H_c and H_d. A family which is r-bounded for some $r > 0$ is simply called bounded.

The following is a consequence of the main theorem of [25] (the equivalence of definition RH0 and RH2).

Proposition 2.3. *If $\mathcal{G}$ is hyperbolic relative to the family $\mathcal{H}$ then $\mathcal{H}$ is bounded.*

Let $\mathcal{H}$ be as in Definition 2.1. Define an *enlargement* $\hat{\gamma}$ of an efficient simplicial L-quasi-geodesic $\gamma\colon [0,n] \to \mathcal{EG}$ with endpoints $\gamma(0), \gamma(n) \in \mathcal{G}$ as follows. Let $0 < k_1 < \cdots < k_s < n$ be those points such that $\gamma(k_i) = v_{c_i}$ for some $c_i \in \mathcal{C}$. Then $\gamma(k_i - 1), \gamma(k_i + 1) \in H_{c_i}$. For each $i \leq s$ replace $\gamma[k_i - 1, k_i + 1]$ by a simplicial geodesic in H_{c_i} with the same endpoints.

For a number $k > 0$ define a subset Z of the metric graph $\mathcal{G}$ to be *k-quasi-convex* if any geodesic with both endpoints in Z is contained in the k-neighborhood of Z. In particular, up to perhaps increasing the number k, any two points in Z can be connected in Z by a (not necessarily continuous) path which is a k-quasi-geodesic in $\mathcal{G}$. The goal of this section is to show

Theorem 2.4. *Let $\mathcal{G}$ be a metric graph which is hyperbolic relative to a family $\mathcal{H} = \{H_c \mid c\}$ of complete connected subgraphs. If there is a number $\delta > 0$ such that each of the graphs H_c is δ-hyperbolic then $\mathcal{G}$ is hyperbolic. Enlargements of geodesics in $\mathcal{E}\mathcal{G}$ are uniform quasi-geodesics in $\mathcal{G}$. The subgraphs H_c are uniformly quasi-convex.*

For the remainder of this section we assume that $\mathcal{G}$ is a graph which satisfies the assumptions in Theorem 2.4.

By Proposition 2.3 there is some $r > 0$ so that the family $\mathcal{H}$ is r-bounded. In the sequel we always assume that for $L > 1$ the constant $p(L)$ as in Definition 2.1 is bigger than $2r$.

For a number $R > 2r$ call $c \in \mathcal{C}$ *R-wide* for an efficient L-quasi-geodesic γ in $\mathcal{E}\mathcal{G}$ if the following holds true. There is some $k \in \mathbb{Z}$ such that $\gamma(k) = v_c$, and the distance between $\gamma(k-1), \gamma(k+1)$ in H_c is at least R. Note that since $\mathcal{H}$ is r-bounded, c is uniquely determined by $\gamma(k-1), \gamma(k+1)$. If $R = p(L)$ is as in Definition 2.1 then we simply say that c is *wide*.

Lemma 2.5. *Let $L \geq 1$ and let γ_1, γ_2 be two efficient L-quasi-geodesics in $\mathcal{E}\mathcal{C}$ with the same endpoints. If $c \in \mathcal{C}$ is $3p(L)$-wide for γ_1 then c is wide for γ_2.*

Proof. By definition, if c is $3p(L)$-wide for γ_1 then there is some k so that $\gamma_1(k) = v_c$ and that the distance in H_c between $\gamma_1(k-1)$ and $\gamma_1(k+1)$ is at least $3p(L)$. Since γ_2 is an efficient L-quasi-geodesic with the same endpoints as γ_1, by the bounded penetration property there is some k' so that $\gamma_2(k') = v_c$, moreover the distance in H_c between $\gamma_1(k-1)$ and $\gamma_2(k'-1)$ and between $\gamma_1(k+1)$ and $\gamma_2(k'+1)$ is at most $p(L)$. Thus by the triangle inequality, the distance in H_c between $\gamma_2(k'-1)$ and $\gamma_2(k'+1)$ is at least $p(L)$ which is what we wanted to show. ☐

Define the *Hausdorff distance* between two closed subsets A, B of a metric space to be the infimum of the numbers $b > 0$ such that A is contained in the b-neighborhood of B and B is contained in the b-neighborhood of A.

The following lemma was known before in the context of relatively hyperbolic groups. We refer to [16] (Corollary 8.14 and Corollary 8.15) and [23] for such versions and more.

Lemma 2.6. *For every $L > 0$ there is a number $\kappa(L) > 0$ with the following property. Let γ_1, γ_2 be two efficient simplicial L-quasi-geodesics in $\mathcal{E}\mathcal{G}$ connecting the same points in $\mathcal{G}$, with enlargements $\hat{\gamma}_1, \hat{\gamma}_2$. Then the Hausdorff distance in $\mathcal{G}$ between the images of $\hat{\gamma}_1, \hat{\gamma}_2$ is at most $\kappa(L)$.*

Proof. Let $\gamma\colon [0,n] \to \mathcal{EG}$ be an efficient simplicial L-quasi-geodesic with endpoints $\gamma(0), \gamma(n) \in \mathcal{G}$. Let $R > p(L)$ and assume that $c \in \mathcal{C}$ is not R-wide for γ. If there is some $u \in \{1,\dots,n-1\}$ such that $\gamma(u) = v_c$ then $\gamma(u-1), \gamma(u+1) \in H_c$. Since c is not R-wide for γ, $\gamma(u-1)$ can be connected to $\gamma(u+1)$ by an arc in H_c of length at most R. In particular, if no $c \in \mathcal{C}$ is R-wide for γ then an enlargement $\hat\gamma$ of γ is an $\widehat{L}$-quasi-geodesic in $\mathcal{EG}$ for a universal constant $\widehat{L} = \widehat{L}(L,R) > 0$. Then $\hat\gamma$ is a $\widehat{L}$-quasi-geodesic in $\mathcal{G}$ as well (note that the inclusion $\mathcal{G} \to \mathcal{EG}$ is 1-Lipschitz).

Let $\gamma_i\colon [0,n_i] \to \mathcal{EG}$ be efficient L-quasi-geodesics ($i = 1,2$) with the same endpoints in $\mathcal{G}$. Assume that no $c \in \mathcal{C}$ is wide for γ_1. By Lemma 2.5, no $c \in \mathcal{C}$ is $R = 3p(L)$-wide for γ_2. Let $\hat\gamma_i$ be an enlargement of γ_i. By the above discussion, the arcs $\hat\gamma_i$ are $\widehat{L} = \widehat{L}(L, 3p(L))$-quasi-geodesics in $\mathcal{EG}$. In particular, by hyperbolicity of $\mathcal{EG}$, the Hausdorff distance in $\mathcal{EG}$ between the images of $\hat\gamma_i$ is bounded from above by a constant $b-1>0$ only depending on L.

We have to show that the Hausdorff distance in $\mathcal{G}$ between these images is also uniformly bounded. For this let $x = \hat\gamma_1(u)$ be any vertex on $\hat\gamma_1$ and let $y = \hat\gamma_2(w)$ be a vertex on $\hat\gamma_2$ of minimal distance in $\mathcal{EG}$ to x. Then $d_{\mathcal{E}}(x,y) \le b$ (here as before, $d_{\mathcal{E}}$ is the distance in $\mathcal{EG}$, and we let d be the distance in $\mathcal{G}$). Let ζ be a geodesic in $\mathcal{EG}$ connecting x to y. Since y is a vertex on $\hat\gamma_2$ of minimal distance to x, ζ intersects $\hat\gamma_2$ only at its endpoints.

We claim that there is a universal constant $\chi > 0$ such that no $c \in \mathcal{C}$ is χ-wide for ζ. Namely, since $\hat\gamma_1$ does not pass through any of the special vertices in $\mathcal{EG}$, the concatenation $\xi = \zeta \circ \hat\gamma_1[0,u]$ is efficient (here $\hat\gamma_1[0,u]$ is the restriction of the arc $\hat\gamma_1$ to its initial subsegment connecting $\hat\gamma_1(0) = \gamma_1(0)$ to $\hat\gamma_1(u)$). Thus ξ is an efficient L'-quasi-geodesic in $\mathcal{EG}$ with the same endpoints as $\hat\gamma_2[0,w]$ where $L' > \widehat{L} > L$ only depends on L. Hence by the bounded penetration property, if $c \in \mathcal{C}$ is $p(L')$-wide for ζ then the $\widehat{L}$-quasi-geodesic $\hat\gamma_2[0,w]$ passes through the vertex v_c which is a contradiction.

As a consequence of the above discussion, the length of an enlargement of ζ is bounded from above by a fixed multiple of $d_{\mathcal{E}}(\hat\gamma_1(u), \hat\gamma_2(w))$, i.e. it is uniformly bounded. This shows that $d(\hat\gamma_1(u), \hat\gamma_2(w))$ is uniformly bounded. Thus the image of $\hat\gamma_1$ is contained in a neighborhood of uniformly bounded diameter in $\mathcal{G}$ of the image of $\hat\gamma_2$.

Now γ_2 is such that no $c \in \mathcal{C}$ is $3p(L)$-wide for γ_2. Thus up to adjusting constants, we can exchange γ_1 and γ_2 in the above argument. This shows that indeed the Hausdorff distance in $\mathcal{G}$ between the images of the enlargements $\hat\gamma_1, \hat\gamma_2$ is bounded by a number only depending on L.

Let $\gamma_j : [0, n_j] \to \mathcal{E}\mathcal{G}$ be arbitrary efficient L-quasi-geodesics ($j = 1, 2$) connecting the same points in $\mathcal{G}$. Then there are numbers $0 < u_1 < \cdots < u_k < n_1$ such that for every $i \leq k$, $\gamma_1(u_i) = v_{c_i}$ where $c_i \in \mathcal{C}$ is wide for γ_1, and there are no other wide points for γ_1. Put $u_0 = -1$ and $u_{k+1} = n_1 + 1$.

By the bounded penetration property, there are numbers $w_i \in \{1, \ldots, n_2 - 1\}$ such that $\gamma_2(w_i) = \gamma_1(u_i) = v_{c_i}$ for all i. Moreover, the distance in H_{c_i} between $\gamma_1(u_i - 1)$ and $\gamma_2(w_i - 1)$ and between $\gamma_1(u_i + 1)$ and $\gamma_2(w_i + 1)$ is at most $p(L)$. Since γ_1, γ_2 are L-quasi-geodesics by assumption, we may assume that the special vertices v_{c_i} appear along γ_2 in the same order as along γ_1, i.e. that $0 < w_1 < \cdots < w_k < n_2$. Namely, for each i the concatenation $\gamma_2[w_i, n_2] \circ \gamma_1[0, u_i]$ is an L'-quasi-geodesic with the same endpoints as γ_1 for a number $L' > 0$ only depending on L. If there is some $j > i$ so that $w_j < w_i$ then this quasi-geodesic does not pass through v_{c_j} which violates the bounded penetration property, once again up to adjusting constants. Put $w_0 = -1$ and $w_{k+1} = n_2 + 1$.

For each $i \leq k$, define a simplicial edge path $\zeta_i : [a_i, a_{i+1}] \to \mathcal{E}\mathcal{G}$ connecting $\zeta_i(a_i) = \gamma_1(u_i + 1) \in H_{c_i}$ to $\zeta_i(a_{i+1}) = \gamma_1(u_{i+1} - 1) \in H_{c_{i+1}}$ as the concatentation of the following three arcs. A geodesic in H_{c_i} connecting $\gamma_1(u_i + 1)$ to $\gamma_2(w_i + 1)$ (whose length is at most $p(L)$), the arc $\gamma_2[w_i + 1, w_{i+1} - 1]$ and a geodesic in $H_{c_{i+1}}$ connecting $\gamma_2(w_{i+1} - 1)$ to $\gamma_2(u_{i+1} - 1)$. Let moreover $\eta_i = \gamma_1|[u_i + 1, u_{i+1} - 1]$ ($i \geq 0$). Then η_i, ζ_i are efficient uniform quasi-geodesics in $\mathcal{E}\mathcal{G}$ with the same endpoints, and η_i does not have wide points.

For each i let $\hat{\nu}_i$ be an enlargement of the arc $\nu_i = \gamma_2[w_i + 1, w_{i+1} - 1]$. By construction, there is an enlargement $\hat{\zeta}_i$ of the efficient quasi-geodesic ζ_i which contains $\hat{\nu}_i$ as a subarc and whose Hausdorff distance in $\mathcal{G}$ to $\hat{\nu}_i$ is uniformly bounded. Let $\hat{\eta}_i$ be an enlargement of η_i. Then $\hat{\zeta}_i, \hat{\eta}_i$ are enlargements of the efficient uniform quasi-geodesics ζ_i, η_i in $\mathcal{E}\mathcal{G}$ with the same endpoints, and η_i does not have wide points. Therefore by the first part of this proof, the Hausdorff distance in $\mathcal{G}$ between $\hat{\eta}_i$ and $\hat{\zeta}_i$ is uniformly bounded. Hence the Hausdorff distance between $\hat{\eta}_i$ and $\hat{\nu}_i$ is uniformly bounded as well.

There is an enlargement $\hat{\gamma}_1$ of γ_1 which can be represented as

$$\hat{\gamma}_1 = \hat{\eta}_k \circ \sigma_k \circ \cdots \circ \sigma_1 \circ \hat{\eta}_0$$

where for each i, σ_i is a geodesic in H_{c_i} connecting $\gamma_1(u_i - 1)$ to $\gamma_1(u_i + 1)$. Similarly, there is an enlargement $\hat{\gamma}_2$ of γ_2 which can be represented as

$$\hat{\gamma}_2 = \hat{\nu}_k \circ \tau_k \circ \cdots \circ \tau_1 \circ \hat{\nu}_0$$

where for each i, τ_i is a geodesic in H_{c_i} connecting $\gamma_2(w_i - 1)$ to $\gamma_2(w_i + 1)$.

For each i the distance in H_{c_i} between $\gamma_1(u_i-1)$ and $\gamma_2(w_i-1)$ is at most $p(L)$, and the same holds true for the distance between $\gamma_1(u_i+1)$ and $\gamma_2(w_i+1)$. Since H_{c_i} is δ-hyperbolic for a constant $\delta>0$ not depending on c_i, the Hausdorff distance in H_{c_i} between any two geodesics connecting $\gamma_1(u_i-1)$ to $\gamma_1(u_i+1)$ and connecting $\gamma_2(w_i-1)$ to $\gamma_2(w_i+1)$ is uniformly bounded. Together with the above discussion, this shows the lemma. □

Let for the moment X be an arbitrary geodesic metric space. Assume that for every pair of points $x,y\in X$ there is a fixed choice of a path $\rho_{x,y}$ connecting x to y. The *thin triangle property* for this family of paths states that there is a universal number $C>0$ so that for any triple x,y,z of points in X, the image of $\rho_{x,y}$ is contained in the C-neighborhood of the union of the images of $\rho_{y,z},\rho_{z,x}$.

For two vertices $x,y\in\mathcal{G}$ let $\rho_{x,y}$ be an enlargement of a geodesic in $\mathcal{EG}$ connecting x to y. We have

Proposition 2.7. *The thin triangle property holds true for the paths $\rho_{x,y}$.*

Proof. Let x_1,x_2,x_3 be three vertices in $\mathcal{G}$ and for $i=1,2,3$ let $\gamma_i\colon[0,n_i]\to\mathcal{EG}$ be a geodesic connecting x_i to x_{i+1}.

By hyperbolicity of $\mathcal{EG}$ there is a number $L>0$ not depending on the points x_i, and there is a vertex $y\in\mathcal{EG}$ with the following property. For $i=1,2,3$ let $\beta_i\colon[0,p_i]\to\mathcal{EG}$ be a geodesic in $\mathcal{EG}$ connecting x_i to y. Then for all i, $\alpha_i=\beta_{i+1}^{-1}\circ\beta_i$ is an L-quasi-geodesic connecting x_i to x_{i+1}.

We claim that without loss of generality we may assume that the quasi geodesics α_i are efficient. Namely, since the arcs β_i are geodesics, they do not backtrack. Thus if α_1 is *not* efficient then there is a common point y on β_1 and β_2. Let $s_1<p_1$ be the smallest number so that $\beta_1(s_1)=\beta_2(s_2)$ for some $s_2\in[0,p_2]$. Then the distance between y and $\beta_i(s_i)$ $(i=1,2)$ is uniformly bounded, and $\tilde\alpha_1=(\beta_2[0,s_2])^{-1}\circ\beta_1[0,s_1]$ is an efficient L-quasi-geodesic connecting x_1 to x_2. Replace y by $\beta_1(s_1)$, replace β_i by $\tilde\beta_i=\beta_i[0,s_i]$ $(i=1,2)$ and replace β_3 by a geodesic $\tilde\beta_3\colon[0,\tilde p_3]\to\mathcal{EG}$ connecting x_3 to $\beta_1(s_1)$. Thus up to increasing the number L by a uniformly bounded amount we may assume that the quasi-geodesic α_1 is efficient.

Assume from now on that β_1,β_2,β_3 are such that the quasi-geodesic $\alpha_1=\beta_2^{-1}\circ\beta_1$ is efficient. Using the notation from the second paragraph of this proof, if there is some $s<p_3$ such that $\beta_3(s)$ is contained in α_1 then let s_3 be the smallest number with this property. Replace the point $y=\beta_i(p_i)$ by $\beta_3(s_3)$, replace β_3 by $\beta_3[0,s_3]$ and for $i=1,2$ replace β_i by the subarc of α_1 connecting x_i to $\beta_3(s_3)$. With this construction, up to increasing the number L by a uniformly bounded amount and perhaps replacing β_1,β_2 by uniform quasi-geodesics

rather than geodesics we may assume that all three quasi-geodesics $\tilde{\alpha}_i = \tilde{\beta}_{i+1}^{-1} \circ \tilde{\beta}_i$ $(i = 1, 2, 3)$ are efficient.

Resuming notation, assume from now on that the quasi-geodesics α_i are efficient. By Lemma 2.6, the Hausdorff distance between an enlargement of the geodesic γ_i and any choice of an enlargement of the efficient uniform quasi-geodesic α_i with the same endpoints is uniformly bounded. Thus it suffices to show the thin triangle property for enlargements of the quasi-geodesics α_i.

If $y \in \mathcal{G}$ then an enlargement of the quasi-geodesic α_i is the concatenation of an enlargement of the quasi-geodesic β_i and an enlargement of the quasi-geodesic β_{i+1}^{-1} which have endpoints in $\mathcal{G}$. Hence in this case the thin triangle property follows once more from Lemma 2.6.

If $y = v_c$ for some $c \in \mathcal{C}$ then we distinguish two cases.

Case 1. $c \in \mathcal{C}$ is wide for each of the quasi-geodesics α_i.

Recall that $y = \beta_i(p_i)$. By hyperbolicity of H_c, there is a number $R > 0$ not depending on c such that for all $i \in \{1, 2, 3\}$ the image of any geodesic in H_c connecting $\beta_i(p_i - 1)$ to $\beta_{i+1}(p_{i+1} - 1)$ is contained in the R-neighborhood of the union of the images of any two geodesics connecting $\beta_j(p_j - 1)$ to $\beta_{j+1}(p_{j+1} - 1)$ for $j \neq i$ and where indices are taken modulo three. In other words, the thin triangle property holds true for such geodesics.

Now let $\hat{\alpha}_i$ be an enlargement of α_i and let ζ_i be the subarc of $\hat{\alpha}_i$ which connects $\beta_i(p_i - 1)$ to $\beta_{i+1}(p_{i+1} - 1)$. By the definition of an enlargement, ζ_i is a geodesic in H_c. Thus by the discussion in the previous paragraph and by the fact that we may use the same enlargement of the arc $\beta_{i+1}[0, p_{i+1} - 1]$ for the construction of an enlargement of α_i and α_{i+1}, the thin triangle property holds true for some suitable choice of an enlargement of the quasi-geodesics α_i. It then holds true for every chocie which is what we wanted to show.

Case 2. For at least one i, $c \in \mathcal{C}$ is not wide for the quasi-geodesic α_i.

Assume that this holds true for the quasi-geodesic α_1. Then the distance in H_c between $\beta_1(p_1 - 1)$ and $\beta_2(p_2 - 1)$ is uniformly bounded (depending on the quasi-geodesic constant for α_1). Replace the point y by $\beta_1(p_1 - 1)$, replace the quasi-geodesic β_1 by $\tilde{\beta}_1 = \beta_1[0, p_1 - 1]$, replace the quasi-geodesic β_2 by the concatentation $\tilde{\beta}_2$ of $\beta_2[0, p_2 - 1]$ with a geodesic in H_c connecting $\beta_2(p_2 - 1)$ to $\beta_1(p_1 - 1)$, and replace the geodesic β_3 by the concatentation $\tilde{\beta}_3$ of β_3 with the edge connecting v_c to $\beta_1(p_1 - 1)$. The resulting arcs $\tilde{\beta}_i$ are efficient uniform quasi-geodesics in $\mathcal{EG}$, and they connect the points x_i to $y \in \mathcal{G}$. Moreover, the quasi-geodesics $\tilde{\beta}_{i+1}^{-1} \circ \tilde{\beta}_i$ are efficient as well and hence we are done by the above proof for the case $y \in \mathcal{G}$. ☐

Now we are ready to show

Corollary 2.8. *$\mathcal{G}$ is hyperbolic. Enlargements of geodesics in $\mathcal{E}\mathcal{G}$ are uniform quasi-geodesics in $\mathcal{G}$.*

Proof. For any pair (x, y) of vertices in $\mathcal{G}$ let $\eta_{x,y}$ be a reparametrization on $[0, 1]$ of the path $\rho_{x,y}$. By Proposition 3.5 of [10] and Theorem 3.15 of [22] (which is essentially due to Bowditch), it suffices to show that there is some $n > 0$ such that the paths $\eta_{x,y}$ have the following properties (where d is the distance in $\mathcal{G}$).

(1) If $d(x, y) \leq 1$ then the diameter of $\eta_{x,y}[0, 1]$ is at most n.

(2) For all vertices x, y, z the set $\eta_{x,y}[0, 1]$ is contained in the n-neighborhood of $\eta_{x,y}[0, 1] \cup \eta_{y,z}[0, 1]$.

Property 1) above is immediate from Lemma 2.6. The thin triangle property 2) follows from Proposition 2.7. □

The following corollary is an immediate consequence of Corollary 2.8.

Corollary 2.9. *There is a number $k > 0$ such that each of the subgraphs H_c ($c \in \mathcal{C}$) is k-quasi-convex.*

We complete this section with a description of the Gromov boundary of $\mathcal{G}$.

Let as before $\mathcal{E}\mathcal{G}$ be the $\mathcal{H}$-electrification of $\mathcal{G}$. Denote by $\partial\mathcal{E}\mathcal{G}$ the Gromov boundary of $\mathcal{E}\mathcal{G}$. For each $c \in \mathcal{C}$ let moreover ∂H_c be the Gromov boundary of H_c. We equip

$$\partial\mathcal{G} = \partial\mathcal{E}\mathcal{G} \cup \bigcup_c \partial H_c$$

with a topology which is defined by describing for each point $\xi \in \partial\mathcal{G}$ a neighborhood basis as follows.

Let first $\xi \in \partial\mathcal{E}\mathcal{G}$. Let $L > 1$ be such that every point $x \in \mathcal{G}$ can be connected in $\mathcal{E}\mathcal{G}$ to every point $\zeta \in \partial\mathcal{E}\mathcal{G}$ by an L-quasi-geodesic in $\mathcal{E}\mathcal{G}$. Let $R > 0$ be sufficiently large.

Let $\delta_{\mathcal{E}}$ be a Gromov metric on $\partial\mathcal{E}\mathcal{G}$ based at a fixed point $x \in \mathcal{G}$. Let $\gamma\colon [0, \infty) \to \mathcal{E}\mathcal{G}$ be an L-quasi- geodesic ray connecting $x = \gamma(0)$ to ξ. For $\epsilon > 0$ let $\mathcal{C}(\xi, \epsilon)$ be the collection of all $c \in \mathcal{C}$ such that there exists a geodesic in $\mathcal{E}\mathcal{G}$ connecting x to v_c which passes through the $2R$-neighborhood of $\gamma[-\log\epsilon, \infty)$. Define $B_\epsilon(\xi) \subset \partial\mathcal{G}$ by

$$B_\epsilon(\xi) = \{\zeta \subset \partial\mathcal{E}\mathcal{G}, \delta_{\mathcal{E}}(\zeta, \xi) < \epsilon\} \cup \bigcup_{c \in \mathcal{C}(\xi,\epsilon)} \partial H_c.$$

Clearly we have $\bigcap_{\epsilon>0} B_\epsilon(\xi) = \{\xi\}$. Declare the family of sets $B_\epsilon(\xi)$ to be a neighborhood basis of ξ in $\partial\mathcal{G}$. Note that changing the basepoint x yields an equivalent neighborhood basis.

If $c \in \mathcal{C}$ and $\xi \in \partial H_c$ then choose a basepoint $x \in H_c$. By enlarging L we may assume that x can be connected to every point in ∂H_c by a quasi-geodesic in H_c which is an L-quasi-geodesic in $\mathcal{G}$. Choose such a quasi-geodesic $\gamma\colon [0,\infty) \to H_c$ connecting $\gamma(0) = x$ to ξ.

For $\epsilon > 0$ let $\mathcal{C}(\xi,\epsilon)$ be the collection of all $d \in \mathcal{C} - \{c\}$ such that there exists a geodesic in $\mathcal{E}\mathcal{G}$ connecting x to v_d which passes through the $2R$-neighborhood of $\gamma[-\log\epsilon,\infty)$ in H_c.

Let $\widehat{B}_\epsilon(\xi)$ be the set of all $\zeta \in \partial\mathcal{E}\mathcal{G}$ such that an L-quasi-geodesic in $\mathcal{E}\mathcal{G}$ connecting x to ζ passes through the R-neighborhood of $\gamma[-\log\epsilon,\infty)$ in H_c. Let $D_c(\xi,\epsilon)$ be the open ball of radius ϵ about ξ in the Gromov boundary of H_c with respect to a Gromov metric based at x. Define

$$B_\epsilon(\xi) = D_c(\xi,\epsilon) \cup \widehat{B}_\epsilon(\xi) \cup \bigcup_{d\in C(\xi,\epsilon)} \partial H_d .$$

As before, we have

$$\bigcap_{\epsilon>0} B_\epsilon(\xi) = \{\xi\}.$$

Declare the family of sets $B_\epsilon(\xi)$ to be a neighborhood basis of $\xi \in \partial\mathcal{G}$. We have

Proposition 2.10. *$\partial\mathcal{G}$ is the Gromov boundary of $\mathcal{G}$.*

Proof. For a number $L > 1$ define an *unparametrized L-quasi-geodesic* in the graph $\mathcal{E}\mathcal{G}$ to be a path $\eta\colon [0,\infty) \to \mathcal{E}\mathcal{G}$ with the following property. There is some $n \in (0,\infty]$, and there is an increasing homeomorphism $\rho\colon [0,n) \to [0,\infty)$ such that $\eta \circ \rho$ is an L-quasi-geodesic in $\mathcal{E}\mathcal{G}$.

Let $x \in \mathcal{G}$ be a vertex and let $q > 1$ be sufficiently large that x can be connected to every point in the Gromov boundary of $\mathcal{G}$ by a q-quasi-geodesic ray in $\mathcal{G}$. Let $\gamma\colon [0,\infty) \to \mathcal{G}$ be such a simplicial q-quasi-geodesic ray. We claim that there is a number $q' > 1$ such that γ viewed as a path in $\mathcal{E}\mathcal{G}$ is an unparametrized q'-quasi-geodesic in $\mathcal{E}\mathcal{G}$.

Namely, for each $i > 0$ let ζ_i be an enlargement of a geodesic in $\mathcal{E}\mathcal{G}$ with endpoints $\gamma(0),\gamma(i)$. Then there is a number $b > 1$ such that ζ_i is a b-quasi-geodesic in $\mathcal{G}$. By hyperbolicity, the Hausdorff distance in $\mathcal{G}$ between $\gamma[0,i]$ and the image of ζ_i is uniformly bounded. Hence the same holds true if this Hausdorff distance is measured with respect to the distance in $\mathcal{E}\mathcal{G} \supset \mathcal{G}$. Thus the Hausdorff distance in $\mathcal{E}\mathcal{G}$ between $\gamma[0,i]$ and a geodesic in $\mathcal{E}\mathcal{G}$ with the same endpoints is uniformly bounded. Since $i > 0$ was arbitrary, this implies that indeed γ is an unparametrized q'-quasi-geodesic in $\mathcal{E}\mathcal{G}$ for a number $q' > 0$ only depending on q.

As a consequence, if the diameter of $\gamma[0,\infty)$ in $\mathcal{E}\mathcal{G}$ is infinite then up to parametrization, $\gamma[0,\infty)$ is a q'-quasi-geodesic ray in $\mathcal{E}\mathcal{G}$ and hence it converges as $i \to \infty$ to a point $\xi \in \partial\mathcal{E}\mathcal{G} \subset \partial\mathcal{G}$.

Now assume that the diameter of $\gamma[0,\infty)$ in $\mathcal{E}\mathcal{G}$ is finite. By Corollary 2.9, there is a number $M > 0$ not depending on γ or on $c \in \mathcal{C}$ with the following properties.

(1) If $x, y \in \mathcal{G}$ are any two vertices and if $c \in \mathcal{C}$ is such that the distance in H_c of some shortest distance projections of x, y into H_c is at least M then a geodesic in $\mathcal{E}\mathcal{G}$ connecting x to y passes through the special vertex v_c defined by c.

(2) If there is some $k > 0$ and some $c \in \mathcal{C}$ such that the distance in H_c of some shortest distance projections of $\gamma(0), \gamma(k)$ into H_c is at least $2M$ then for each $\ell > k$ the distance in H_c of any shortest distance projections of $\gamma(0), \gamma(\ell)$ into H_c is at least M.

For $k > 0$ let $\mathcal{C}_1(k)$ (or $\mathcal{C}_2(k)$) be the set of all $c \in \mathcal{C}$ so that the distance in H_c between some shortest distance projections of $\gamma(0), \gamma(k)$ into H_c is at least M (or $2M$). By property (2) above, for $\ell \geq k$ we have $\mathcal{C}_2(k) \subset \mathcal{C}_1(\ell)$.

The diameter of the image of any simplicial geodesic in $\mathcal{E}\mathcal{G}$ equals the length of the geodesic and hence it is bounded from below by the number of special vertices it passes through. Since the diameter of $\gamma[0,\infty)$ in $\mathcal{E}\mathcal{G}$ is finite by assumption, by property (1) and (2) the set $P = \bigcup_{k>0} \mathcal{C}_2(k)$ is finite. Moreover, there is some $k_0 > 0$ such that

$$\bigcup_{k>0} \mathcal{C}_2(k) \subset \bigcup_{k \leq k_0} \mathcal{C}_1(k).$$

As a consequence, if $c \in \mathcal{C}$ and $\ell > k_0$ are such that the distance in H_c of some shortest distance projection of $\gamma(0), \gamma(\ell)$ into H_c is at least $2M$ then the distance of the shortest distance projection of $\gamma(0), \gamma(k_0)$ is at least M.

Now the diameter of $\gamma[k_0,\infty)$ in $\mathcal{E}\mathcal{G}$ is finite and therefore there is some $c \in \mathcal{C}$ so that $\gamma[k_0,\infty)$ is contained in a uniformly bounded neighborhood of H_c.

On the other hand, γ is a q-quasi-geodesic in $\mathcal{G}$ and the inclusion $H_c \to \mathcal{G}$ is a quasi-isometric embedding. By hyperbolicity there is a quasi-geodesic ray ζ in H_c whose Hausdorff distance to $\gamma[k_0,\infty)$ is bounded. As a consequence, γ determines a point $\mu \in \partial H_c \subset \partial\mathcal{G}$.

To summarize, there is a map Λ from the Gromov boundary of $\mathcal{G}$ into $\partial\mathcal{G}$. It is easily seen from the above discussion that the map Λ is injective. Corollary 2.9 then shows that Λ is in fact a bijection.

We claim that Λ is moreover continuous and open. To this end let again $\gamma\colon [0,\infty) \to \mathcal{G}$ be a q-quasi-geodesic. By the above discussion, we may assume that either γ has infinite diameter in $\mathcal{E}\mathcal{G}$ or there is some $k_0 \geq 0$ and some $c \in H_c$ so that $\gamma[k_0,\infty) \subset H_c$.

A neighborhood basis for the endpoint of γ in the Gromov boundary of $\mathcal{G}$ consists of the family $D(m)$ $(m \geq 1)$ of sets where $D(m)$ contains all endpoints of uniform quasi-geodesics β in $\mathcal{G}$ which pass through a fixed size neighborhood of $\gamma(m)$. Up to replacing β by a quasi-geodesic of uniformly controlled Hausdorff distance to β, we may assume that one of the following two possibilities is satisfied.

(1) β is an enlargement of a quasi-geodesic in $\mathcal{E}\mathcal{G}$ of infinite diameter and hence it defines a point in the set $\{\delta_\xi(\zeta,\xi) < \epsilon\}$ where $\epsilon > 0$ is determined by the distance in $\mathcal{E}\mathcal{G}$ between $\gamma(m)$ and $\gamma(0)$.

(2) There is some $d \in \mathcal{C}$ with the properties described in the definition of the sets $B_\epsilon(\xi)$ so that the tail of β is contained in H_d and hence defines a boundary point of H_d as specified in the description of the neighboorhood basis of $\gamma(\infty)$ in the definition of $\partial\mathcal{G}$.

From this description is it immediate that the image under Λ of a neighborhood basis of $\gamma(\infty)$ in the Gromov boundary of $\mathcal{G}$ equals a neighborhood basis of $\Lambda(\gamma(\infty))$ in $\partial\mathcal{G}$. □

3. Thick subsurfaces

In this section we consider a handlebody H of genus $g \geq 2$. By a *disk* in H we mean an essential disk in H.

Two disks $D_1, D_2 \subset H$ are in *normal position* if their boundary circles intersect in the minimal number of points and if every component of $D_1 \cap D_2$ is an embedded arc in $D_1 \cap D_2$ with endpoints in $\partial D_1 \cap \partial D_2$. In the sequel we always assume that disks are in normal position; this can be achieved by modifying one of the two disks with an isotopy.

As in the introduction, call a connected essential subsurface X of ∂H *thick* if the following conditions are satisfied.

(1) Every disk intersects X.

(2) X is filled by boundaries of disks.

The first property says that no essential disk can be isotoped off X. The second property implies that $\partial H - X$ is not thick. An example of a thick subsurface is the complement in ∂H of a suitably chosen simple closed curve which is not diskbounding. The entire boundary surface ∂H is thick as well.

For a thick subsurface X of ∂H define $\mathcal{EDG}(X)$ to be the graph whose vertices are disks with boundary contained in X. By property (1) in the definition of a thick subsurface, the boundary of each such vertex is an essential simple closed curve in X. Two such disks D, E are connected by an edge of length one if and only if there is an essential simple closed curve γ in X which can be realized disjointly from both D, E (e.g. the boundary of D if the disks D, E are disjoint).

Denote by $d_{\mathcal{E},X}$ the distance in $\mathcal{EDG}(X)$. The disk graph $\mathcal{DG}(X)$ of X is defined in the obvious way, and we denote by $d_{\mathcal{D},X}$ its distance function.

In the sequel we always assume that all curves and multicurves on $X \subset \partial H$ are essential. For two simple closed multicurves c, d on ∂H let $\iota(c, d)$ be the geometric intersection number between c, d.

The following lemma [21] implies that for every thick subsurface X of ∂H the graph $\mathcal{DG}(X)$ is connected. For its proof and later use, let D, E be disks in minimal position. Define an *outer component* of E with respect to D to be a component $\hat{E}$ of $E - D$ which is a disk whose boundary consists of a single subarc of ∂E and a single subarc α of D. The arc α intersects the boundary of D precisely at its endpoints. Surgery of D at this outer component $\hat{E}$ replaces D by the union of $\hat{E}$ with one of the two components of $D - \alpha$ (compare e.g. [21, 11]).

Lemma 3.1. *Let $X \subset \partial H$ be a thick subsurface. Let $D, E \subset H$ be disks with boundaries in X. Then D can be connected to a disk E' which is disjoint from E by at most $\iota(\partial D, \partial E)/2$ simple surgeries. In particular,*

$$d_{\mathcal{D},X}(D, E) \le \iota(\partial D, \partial E)/2 + 1.$$

Proof. Let D, E be two disks in normal position with boundary in X. Assume that D, E are not disjoint. Then there is an outer component of $E - D$. A disk D' obtained by surgery of D at this component is essential in H and its boundary is contained in X, i.e. $D' \in \mathcal{EDG}(X)$. Moreover, D' is disjoint from D, i.e. we have $d_{\mathcal{D},X}(D', D) = 1$, and

$$\iota(\partial E, \partial D') \le \iota(\partial D, \partial E) - 2. \tag{1}$$

The lemma now follows by induction on $\iota(\partial D, \partial E)$. ☐

Remark 3.2. Lemma 3.1 implies that a thick subsurface X of ∂H can not be a four-holed sphere or a one-holed torus. Namely, if X is a four-holed sphere or a one-holed torus and if X contains the boundaries of two distinct disks D, E then these disks intersect. Surgery of D at an outer component of $E - D$ results in an essential disk $D' \neq D$ which up to homotopy is disjoint from the disk D and whose boundary is contained in X. Since any two essential simple closed curves in X intersect, the boundary of D' is peripheral in X which violates the assumption that no boundary component of X is diskbounding.

A simple closed multicurve γ in a thick subsurface X of ∂H is called *diskbusting* if γ intersects every disk with boundary in X.

Consider an oriented I-bundle $\mathcal{J}(F)$ over a compact (not necessarily orientable) surface F with (not necessarily connected) boundary ∂F. The boundary $\partial\mathcal{J}(F)$ of $\mathcal{J}(F)$ decomposes into the *horizontal boundary* and the *vertical boundary*. The vertical boundary is the interior of the restriction of the I-bundle to ∂F and consists of a collection of pairwise disjoint open incompressible annuli. The horizontal boundary is the complement of the vertical boundary in $\partial\mathcal{J}(F)$.

For a given boundary component α of F, the union of the horizontal boundary of $\mathcal{J}(F)$ with the I-bundle over α is a compact connected orientable surface $F_\alpha \subset \partial\mathcal{J}(F)$. The boundary of F_α is empty if and only if the boundary of F is connected. If the boundary of F is not connected then F_α is properly contained in the boundary $\partial\mathcal{J}(F)$ of $\mathcal{J}(F)$. The complement $\partial\mathcal{J}(F) - F_\alpha$ is a union of incompressible annuli.

Definition 3.3. An *I-bundle generator* in a thick subsurface $X \subset \partial H$ is an essential simple closed curve $\gamma \subset X$ with the following property. There is a compact surface F with non-empty boundary ∂F, there is a boundary component α of ∂F, and there is an orientation preserving embedding Ψ of the oriented I-bundle $\mathcal{J}(F)$ over F into H which maps α to γ and which maps F_α onto the complement in X of a tubular neighborhood of the boundary ∂X of X.

We call the surface F the *base* of the I-bundle generated by γ.

Example 3.4. 1) An orientable I-bundle over an orientable base is a trivial bundle. Thus if ∂H admits an I-bundle generator γ with orientable base surface F then the genus g of ∂H is even and equals twice the genus of F. The boundary of F is connected. The I-bundle over every essential arc in F with endpoints in ∂F is an embedded disk in H. There is an orientation reversing involution $\Phi\colon H \to H$ whose fixed point set intersects ∂H precisely in γ. This involution acts as a reflection in the fiber. The union of any essential arc β in F with endpoints in ∂F with its image under Φ is the boundary of a disk in H (there is a small abuse of notation here since the fixed point set of Φ intersects ∂H in a subset of the fibre over ∂F). This disk is just the I-bundle over the arc β. We refer to [14] for more information on I-bundles.

2) Let F be an oriented surface of genus $k \geq 1$ with two boundary components α, β. The oriented I-bundle $\mathcal{J}(F) = F \times [0,1]$ over F is homeomorphic to a handlebody H of genus $2k + 1$. The boundary component β of F is neither diskbounding nor diskbusting in H. Namely, as in 1) above, the I-bundle over every essential simple arc in F with both endpoints on α is an essential disk in H. The subsurface $X = \partial H - \beta \subset \partial H$ is thick. The boundary component α of F intersects every disk with boundary in X and is an I-bundle generator for X whose base is the surface F. The thick surface X is naturally homeomorphic to F_α, the complement of the I-bundle over β in the boundary of $\mathcal{J}(F)$. The image of $F \times [0,1] = \mathcal{J}(F)$ under the embedding $\mathcal{J}(F) \to H$ is the complement of a neighborhood of β in H which is homeomorphic to a solid torus.

3) Let F be the connected sum of g copies of the real projective plane with a disk. The orientable I-bundle over F is a handlebody H of genus g. The vertical boundary of the I-bundle is an annulus whose core curve γ is non-separating. The complement of the annulus is the two-sheeted orientation cover of F. The I-bundle over any simple arc in F with both endpoints on the boundary of F is an embedded disk in H.

4) Let γ be a non-separating I-bundle generator for a proper thick subsurface X of ∂H, with base F. Then F is non-orientable. Up to isotopy, the thick subsurface X of ∂H is the intersection of the boundary $\partial\mathcal{J}(F)$ of the bundle $\mathcal{J}(F) \subset H$ with ∂H. It can be obtained from the orientation cover $\widehat{F}$ of F by glueing an annulus to the two preimages of the preferred boundary component α of F. The I-bundle over every essential embedded arc β in F with endpoints on α is a disk in H. Its boundary is the preimage of β in $F_\alpha \subset \partial\mathcal{J}(F)$, viewed as the orientation cover of F (here we use the same small abuse of terminology as before).

For a thick subsurface X of ∂H let $\mathcal{SDG}(X)$ be the graph whose vertices are disks with boundaries contained in X and where two such disks D, E are connected by an edge of length one if one of the following two possibilities is satisfied.

(1) There is an essential simple closed curve $\alpha \subset X$ (i.e. which is essential as a curve in the subsurface X of ∂H) which is disjoint from $D \cup E$ (for example, ∂D if D, E are disjoint).

(2) There is an I-bundle generator $\gamma \subset X$ which intersects both D, E in precisely two points.

We denote by $d_{\mathcal{S},X}$ the distance in $\mathcal{SDG}(X)$. If $X = \partial H$ then we simply write $d_{\mathcal{S}}$ instead of $d_{\mathcal{S},\partial H}$.

The following was proved in [11] in the case $X = \partial H$. The proof of the result carries over to an arbitrary thick subsurface without modification.

Proposition 3.5. *Let $X \subset \partial H$ be a thick subsurface. The vertex inclusion defines a quasi-isometric embedding of $\mathcal{SDG}(X)$ into the curve graph of X. In particular, $\mathcal{SDG}(X)$ is δ-hyperbolic for a number $\delta > 0$ only depending on the genus of H.*

4. Hyperbolicity of the electrified disk graph

As in Section 3, we consider a handlebody H of genus $g \geq 2$, with boundary ∂H. The goal of this section is to use Theorem 1 to show hyperbolicity of the electrified disk graph $\mathcal{EDG}(X)$ of a thick subsurface X of ∂H. We also determine the Gromov boundary of $\mathcal{EDG}(X)$.

In the sequel for a number $L > 1$ we call a map $\varphi : X \to Y$ between metric spaces X, Y an L-quasi-isometry if for all $x, y \in X$ we have

$$d(x, y)/L - L \leq d(\varphi(x), \varphi(y)) \leq L d(x, y) + L$$

and if moreover for every $y \in Y$ there is some $x \in X$ with $d(\varphi(x), y) \leq L$.

Let $X \subset \partial H$ be a thick subsurface. Recall that X is connected, and by the remark after Lemma 3.1, X is distinct from a sphere with at most four holes and from a torus with a single hole. Denote by $d_{\mathcal{CG},X}$ the distance in the curve graph $\mathcal{CG}(X)$ of X, by $d_{\mathcal{S},X}$ the distance in the graph $\mathcal{SDG}(X)$ and by $d_{\mathcal{E},X}$ the distance in the electrified disk graph $\mathcal{EDG}(X)$ of X.

If X does not contain any I-bundle generator then $\mathcal{EDG}(X) = \mathcal{SDG}(X)$ and there is nothing to show. Thus assume that there is an I-bundle generator $\gamma \subset X$. Let

$$\mathcal{E}(\gamma) \subset \mathcal{EDG}(X)$$

be the complete subgraph of $\mathcal{EDG}(X)$ whose vertices are disks intersecting γ in precisely two points. Define

$$\mathcal{E} = \{\mathcal{E}(\gamma) \mid \gamma\}$$

where γ runs through all I-bundle generators in X. By definition, $\mathcal{SDG}(X)$ is 2-quasi-isometric to the $\mathcal{E}$-electrification of $\mathcal{EDG}(X)$. Thus by Theorem 2.4, to show hyperbolicity of $\mathcal{EDG}(X)$ it suffices to show that each of the graphs $\mathcal{E}(\gamma)$ is δ-hyperbolic for a number $\delta > 0$ not depending on γ and that the bounded penetration property holds true.

We begin with establishing hyperbolicity of the graphs $\mathcal{E}(\gamma)$. To this end, for a compact (not necessarily orientable) surface F with boundary ∂F and for a fixed boundary component α of F, define the *electrified arc graph* $C'(F,\alpha)$ as follows. Vertices of $C'(F,\alpha)$ are essential embedded arcs in F with both endpoints in α. Two such arcs are connected by an edge of length one if either they are disjoint or if they are disjoint from a common essential simple closed curve. If F is non-orientable, then we require that an essential simple closed curve does not bound a Möbius band in F.

The following statement is well known but hard to find in the literature. We give a proof for completeness.

Lemma 4.1. *Let F be a compact surface with boundary ∂F. Assume that F is not a sphere with at most three holes or a projective plane with at most three holes. Let α be a boundary circle of F. Then $C'(F,\alpha)$ is* 4*-quasi-isometric to the curve graph of F.*

Proof. Define the *arc and curve graph* $\mathcal{A}(F,\alpha)$ of F to be the graph whose vertices are arcs with endpoints on α or essential simple closed curves in F. Two such arcs or curves are connected by an edge of length one if they can be realized disjointly.

Consider first the case that F either is a one-holed torus, a one-holed Klein bottle, a four holed sphere or a four-holed projective plane. In this case two simple closed curves in F are connected by an edge in the curve graph of F if they intersect in the minimal number of points (one or two). Let β be an essential simple closed curve in F. Cutting F open along β yields a three-holed sphere (if F is a one-holed torus or a one-holed Klein bottle), the disjoint union of two three holed spheres (if F is a four-holed sphere) or the disjoint union of a three holed sphere and a three holed projective plane (if F is a four-holed projective plane).

Thus there is a unique essential arc $\Lambda(\beta) \subset F$ with endpoints on α which is disjoint from β. The distance between two essential simple closed curves β,γ in the curve graph of F equals one if and only if the arcs $\Lambda(\beta)$, $\Lambda(\gamma)$ are disjoint. This means that the map Λ which associates to a simple closed curve β in F the unique arc $\Lambda(\beta)$ with endpoints on α which is disjoint from β defines an isometry of the curve graph of F onto the *arc graph* of (F,α). This arc graph is the complete subgraph of $\mathcal{A}(F,\alpha)$ whose vertex set consists of arcs with endpoints on α. Moreover, in the special case at hand, this arc graph is just the graph $C'(F,\alpha)$. This yields the statement of the lemma for one-holed tori, one-holed Klein bottles, four-holed spheres and four-holed projective planes.

Now assume that the surface F is such that two vertices in the curve graph of F are connected by an edge if they can be realized disjointly. Then for any two disjoint essential simple closed curves β, γ in F there is an essential arc with endpoints on α which is disjoint from both β, γ. In particular, for every simplicial path c in the arc and curve graph $\mathcal{A}(F,\alpha)$ connecting two vertices in $\mathcal{A}(F,\alpha)$ which are arcs with endpoints on α, there is a path of at most double length in $C'(F,\alpha)$ connecting the same endpoints. This path can be obtained from c as follows. If $c(i), c(i+1)$ are both simple closed curves then replace $c[i,i+1]$ by a simplicial path in $\mathcal{A}(F,\alpha)$ of length 2 with the same endpoints whose midpoint is an arc disjoint from $c(i), c(i+1)$. In the resulting path, a simple closed curve $\beta \subset F$ is adjacent to two arcs disjoint from β and hence we can view this path as a path in $C'(F,\alpha)$. Thus the vertex inclusion $C'(F,\alpha) \to \mathcal{A}(F,\alpha)$ is a 2-quasi-isometry.

We are left with showing that the inclusion of the curve graph of F into $\mathcal{A}(F,\alpha)$ is a 2-quasi-isometry. However, this is well known, and in the case of an oriented surface, it can be found in [20]. A sketch of a proof is as follows. Construct from a simplicial path in $\mathcal{A}(F,\alpha)$ connecting two simple closed curves a new path by replacing any edge connecting two arcs by a path of length two with the same endpoints whose middle vertex is a disjoint simple closed curve. Then replace any arc β by an essential simple closed curve which is composed of β and one of the two components of $\alpha - \beta$ (at least one of the two choices of such curves will be essential). □

A thick subsurface X of ∂H is not a four-holed sphere. Thus if γ is a separating I-bundle generator for X then the base of the I-bundle (which is an oriented surface with boundary) either has positive genus or is a sphere with at least four holes. Similarly, if γ is a non-separating I-bundle generator for X then we may assume that the base F of the I-bundle is not a projective plane with three holes. Namely, if F is a projective plane with three holes and if α is a distinguished boundary component of F then there is up to homotopy a unique essential arc β in F with boundary on α. The I-bundle over β is the unique disk in the oriented I-bundle over F which intersects the curve α in precisely two points.

For the formulation of the following lemma, for an I-bundle generator γ in a thick subsurface X of ∂H, with base surface F, denote again by γ the distinguished boundary component of F. A disk $D \subset H$ with boundary $\partial D \subset X$ which intersects γ in precisely two points is an I-bundle over a simple arc $\beta \subset F$ with boundary on γ. Namely, if F is oriented then the inclusion $F \to \mathfrak{J}(F)$ induces an isomorphism of fundamental groups. As the boundary of a disk defines the trivial element in the fundamental group of $\mathcal{J}(F)$, the two components of $\partial D - \gamma$ define inverse elements in the fundamental group of F relative to γ. A similar argument can be used in the case that F is non-orientable. We call β the *projection* of ∂D to F. With these notations we show.

Lemma 4.2. *Let $X \subset \partial H$ be a thick subsurface and let γ be an I-bundle generator in X, with base surface F. Then the map which associates to a disk $D \in \mathcal{E}(\gamma)$ the projection of ∂D to F extends to a 2-quasi-isometry of $\mathcal{E}(\gamma)$ onto the electrified arc graph $\mathcal{C}'(F,\gamma)$ of F.*

Proof. Let γ be an I-bundle generator in X and let F be the base surface of the I-bundle generated by γ. Let $\mathcal{J}(F)$ be the oriented I-bundle over F as in the definition of an I-bundle generator and let $\Psi\colon \mathcal{J}(F) \to H$ be a corresponding embedding. Up to isotopy, we have $\Psi(\partial\mathcal{J}(F)) \cap \partial H = X$. There is an orientation reversing bundle involution Φ of $\mathcal{J}(F)$ which exchanges the endpoints of the fibres. The involution preserves $\partial\mathcal{J}(F)$ and the curve γ. The quotient of $\partial\mathcal{J}(F)$ under this involution equals the base surface F of the I-bundle. The projection of γ is the distinguished boundary component of F, again denoted by γ.

Up to isotopy, if the boundary ∂D of a disk D in H is contained in X and intersects the curve γ in precisely two points then $\Psi^{-1}(\partial D)$ is invariant under the involution Φ (see the comment preceding this lemma). Thus the map

$$\Theta\colon \mathcal{V}(C'(F,\gamma)) \longrightarrow \mathcal{V}(\mathcal{E}(\gamma))$$

which associates to an arc β in F with endpoints on γ the I-bundle over β is a bijection. Here $\mathcal{V}(C'(F,\gamma))$ (or $\mathcal{V}(\mathcal{E}(\gamma))$) is the set of vertices of $C'(F,\gamma)$ (or $\mathcal{E}(\gamma)$).

If $\alpha,\beta \in \mathcal{V}(C'(F,\gamma))$ are connected by an edge then either α,β are disjoint and so are $\Theta(\alpha),\Theta(\beta)$, or α,β are disjoint from an essential simple closed curve ρ in F and therefore the disks $\Theta(\alpha),\Theta(\beta)$ are disjoint from $\Psi(\rho) \subset X$. Thus Θ extends to a 1-Lipschitz map $C'(F,\gamma) \to \mathcal{E}(\gamma)$.

We are left with showing that $\Theta^{-1}\colon \mathcal{V}(\mathcal{E}(\gamma)) \to \mathcal{V}(C'(F,\gamma))$ is 2-Lipschitz where $\mathcal{V}(\mathcal{E}(\gamma))$ and $\mathcal{V}(C'(F,\gamma))$ are equipped with the restriction of the metric on $\mathcal{E}(\gamma), C'(F,\gamma)$. To this end let $\alpha,\beta \in \mathcal{V}(C'(F,\gamma))$ be such that $\Theta(\alpha),\Theta(\beta)$ are connected by an edge in $\mathcal{E}(\gamma)$. If $\Theta(\alpha),\Theta(\beta)$ are disjoint then the same holds true for α,β and hence α,β are connected by an edge in $C'(F,\gamma)$. Otherwise $\Theta(\alpha),\Theta(\beta)$ are disjoint from an essential simple closed curve ρ in X.

The boundaries $\partial\Theta(\alpha),\partial\Theta(\beta)$ of the disks $\Theta(\alpha),\Theta(\beta)$ are invariant under the involution $\Psi \circ \Phi \circ \Psi^{-1}$ and therefore $\partial\Theta(\alpha) \cup \partial\Theta(\beta)$ is disjoint from $\rho \cup \Psi \circ \Phi \circ \Psi^{-1}(\rho)$. As a consequence, the projection of $\Psi^{-1}(\rho) \cup \Theta\Psi^{-1}(\rho)$

to the base surface F is a union of essential arcs with endpoints on γ and closed curves (not necessarily simple) which are disjoint from $\alpha \cup \beta$. If there is a component which is a simple arc with endpoints on γ or if there is a component which is a simple closed curve then the distance in $C'(F,\gamma)$ between α and β is at most two as claimed (note that this always holds true if F is orientable). Otherwise the projection of $\Psi^{-1}(\rho) \cup \Theta\Psi^{-1}(\rho)$ consists of arcs and closed curves which are not simple. However, any simple closed loop which is embedded in the graph defined by these arcs and curves is essential and disjoint from $\alpha \cup \beta$. Thus the claim follows as before. The lemma is proven. □

From Lemma 4.2, Lemma 4.1 and hyperbolicity of the curve graph of X ([19], and [2] for the curve graph of a non-orientable surface) we immediately obtain

Corollary 4.3. *There is a number $\delta > 0$ such that each of the graphs $\mathcal{E}(\gamma)$ is δ-hyperbolic.*

Note that the number $\delta > 0$ in the statement of the corollary only depends on H but not on X. In fact, the main result of [1, 4, 7, 15] together with Lemma 4.2 shows that it can even be chosen independent of H.

We are left with the verification of the bounded penetration property. To this end recall from [20] the definition of a *subsurface projection*. Namely, let again $X \subset \partial H$ be a thick subsurface and let $Y \subset X$ be an essential, open connected subsurface which is distinct from X, a three-holed sphere and an annulus. We call such a subsurface Y a *proper* subsurface of X. The arc and curve graph $\mathcal{AC}(Y)$ of Y (here we do not specify a boundary component) is the graph whose vertices are isotopy classes of arcs with endpoints on ∂Y or essential simple closed curves in Y, and two such vertices are connected by an edge of length one if they can be realized disjointly. The vertex inclusion of the curve graph of Y into the arc and curve graph is a 2-quasi-isometry [20].

There is a projection π_Y of the curve graph $\mathcal{CG}(X)$ of X into the space of subsets of $\mathcal{AC}(Y)$ which associates to a simple closed curve in X the homotopy classes of its intersection components with Y. For every simple closed multicurve c in X, the diameter of $\pi_Y(c)$ in $\mathcal{AC}(Y)$ is at most one. If c can be realized disjointly from Y then $\pi_Y(c) = \emptyset$.

As before, call a path ρ in a metric graph G simplicial if ρ maps each interval $[k, k+1]$ (where $k \in \mathbb{Z}$) isometrically onto an edge of G or a single vertex. The following lemma is a version of Theorem 3.1 of [20].

Lemma 4.4. *There is a number $B > 0$ with the following property. Let Y be a proper subsurface of X and let ρ be a simplicial path in $\mathcal{CG}(X)$ which is an L-quasi-geodesic for some $L \geq 1$. If $\pi_Y(v) \neq \emptyset$ for every vertex v on ρ then*

$$\operatorname{diam} \pi_Y(\rho) < BL^3.$$

Proof. By hyperbolicity, for every $L > 1$ there is a number $n(L) > 0$ so that for every L-quasi-geodesic ρ in $\mathcal{CG}(X)$ of finite length, the Hausdorff distance between the image of ρ and the image of a geodesic ρ' with the same endpoints does not exceed $n(L)$. Indeed, there is a number $k > 0$ only depending on the hyperbolicity constant for $\mathcal{CG}(X)$ such that we can choose $n(L) = kL^2$ (Proposition III.H.1.7 in [5]).

Now let $Y \subset X$ be a proper subsurface. By Theorem 3.1 of [20], there is a number $M > 0$ with the following property. If ζ is any simplicial geodesic in $\mathcal{CG}(X)$ and if $\pi_Y(\zeta(s)) \neq \emptyset$ for all $s \in \mathbb{Z}$ in the domain of ζ then

$$\operatorname{diam}(\pi_Y(\zeta)) \leq M.$$

Let $L > 1$, let $\rho\colon [0,k] \to \mathcal{CG}(X)$ be a simplicial path which is an L-quasi-geodesic and assume that

$$\operatorname{diam}(\pi_Y(\rho(0) \cup \rho(k))) \geq 2M + L(2n(L) + 6).$$

Our goal is to show that ρ passes through the set $A \subset \mathcal{CG}(X)$ of all essential simple closed curves in $X - Y$. The diameter of A in $\mathcal{CG}(X)$ is at most two.

To this end let ρ' be a simplicial geodesic in $\mathcal{CG}(X)$ with the same endpoints as ρ. Theorem 3.1 of [20] shows that there is some $u \in \mathbb{Z}$ such that $\rho'(u) \in A$. Then ρ passes through the $n(L)$-neighborhood of A.

Let $s + 1 \leq t - 1$ be the smallest and the biggest number, respectively, so that $\rho(s+1), \rho(t-1)$ are contained in the $n(L)$-neighborhood of A. Then $\rho[0,s]$ (or $\rho[t,k]$) is contained in the complement of the $n(L)$-neighborhood of A. Since ρ is an L-quasi-geodesic, a geodesic connecting $\rho(0)$ to $\rho(s)$ (or connecting $\rho(t)$ to $\rho(k)$) is contained in the $n(L)$-neighborhood of $\rho[0,s]$ (or of $\rho[t,k]$) and hence it does not pass through A. In particular,

$$\operatorname{diam}(\pi_Y(\rho(0) \cup \rho(s))) \leq M \text{ and } \operatorname{diam}(\pi_Y(\rho(t) \cup \rho(k))) \leq M.$$

As a consequence, we have

$$\operatorname{diam}(\pi_Y(\rho(s) \cup \rho(t))) \geq L(2n(L) + 6). \tag{2}$$

Since $d_{\mathcal{CG},X}(\rho(s+1),A)\le n(L)$ and $d_{\mathcal{CG},X}(\rho(t-1),A)\le n(L)$ and since the diameter of A is at most 2, we obtain $d_{\mathcal{CG},X}(\rho(s),\rho(t))\le 2n(L)+4$. Now ρ is a simplicial L-quasi-geodesic in $\mathcal{CG}(X)$ and hence the length $t-s$ of $\rho[s,t]$ is at most $L(2n(L)+4)+L=L(2n(L)+5)$. For all $\ell\in\mathbb{Z}$ the curves $\rho(\ell),\rho(\ell+1)$ are disjoint and therefore if $\rho(\ell),\rho(\ell+1)$ both intersect Y then the diameter of $\pi_Y(\rho(\ell)\cup\rho(\ell+1))$ is at most one. Thus if $\rho(\ell)$ intersects Y for all ℓ then

$$\operatorname{diam}(\pi_Y(\rho(s)\cup\rho(t)))\le L(2n(L)+5).$$

This contradicts inequality (2) and completes the proof of the lemma. □

For simplicity of notation, in the remainder of this section we identify disks in H with their boundaries. In other words, for a thick subsurface X of ∂H we view the vertex sets of the graphs $\mathcal{SDG}(X)$, $\mathcal{EDG}(X)$ as subsets of the vertex set of the curve graph $\mathcal{CG}(X)$ of X.

Let $\mathcal{SDG}_0(X)$ be the $\mathcal{E}$-electrification of $\mathcal{EDG}(X)$. For each I-bundle generator γ in X, the graph $\mathcal{SDG}_0(X)$ contains a special vertex v_γ. The vertex set of $\mathcal{SDG}_0(X)$ is the union of the set of all diskbounding simple closed curves in X with the set $\{v_\gamma \mid \gamma\}$. In particular, there is a natural vertex inclusion $\mathcal{V}(\mathcal{SDG}_0(X))\to\mathcal{CG}(X)$ which maps the special vertex v_γ to the simple closed curve γ. Since $\mathcal{SDG}(X)$ is quasi-isometric to the $\mathcal{E}$-electrification of $\mathcal{EDG}(X)$, Proposition 3.5 shows that this vertex inclusion extends to a quasi-isometric embedding $\mathcal{SDG}_0(X)\to\mathcal{CG}(X)$.

We aim at replacing the special vertices on a geodesic in $\mathcal{SDG}_0(X)$ by geodesic segments in the peripheral graphs $\mathcal{E}(\gamma)$ while keeping track of subsurface projections. To this end we associate to an efficient L-quasi-geodesic $\rho\colon[0,n]\to\mathcal{SDG}_0(X)$ a simplicial path $\tilde\rho$ in the curve graph $\mathcal{CG}(X)$ as follows.

A vertex $\rho(j)$ in $\mathcal{SDG}_0(X)$ which is not one of the special vertices v_γ also defines a vertex in $\mathcal{CG}(X)$. If $\rho(j),\rho(j+1)$ are two vertices of this kind which are connected in $\mathcal{SDG}_0(X)$ by an edge then they are connected in $\mathcal{EDG}(X)\subset\mathcal{SDG}_0(X)$ by an edge. By the definition of the electrified disk graph, this means that there is a simple closed curve α in X which is disjoint from $\rho(j)\cup\rho(j+1)$. Thus $\rho(j)$ and $\rho(j+1)$ can be connected in $\mathcal{CG}(X)$ by an edge path of length at most two. We replace the edge $\rho[j,j+1]$ by such a path.

Similarly, if $\rho(j)=v_\gamma$ for an I-bundle generator γ in X, then $\rho(j-1)$, $\rho(j+1)$ are vertices in $\mathcal{EDG}(X)$, i.e. diskbounding simple closed curves, Moreover, $\rho(j-1),\rho(j+1)$ intersect γ in precisely two points. Replace $\rho[j-1,j+1]$ by an edge path in $\mathcal{CG}(X)$ with the same endpoints whose length is at most four and which passes through γ. The arc $\tilde\rho$ constructed in this way from ρ is a uniform quasi-geodesic in $\mathcal{CG}(X)$ which passes through any I-bundle generator γ at most once, and it passes through γ if and only if it passes through a simple closed curve which is disjoint from γ. We call $\tilde{\rho}$ a *canonical modification* of ρ. By Proposition 3.5, the canonical modification of an efficient L-quasi-geodesic in $\mathcal{SDG}_0(X)$ is an L'-quasi-geodesic in $\mathcal{CG}(X)$ for a number $L' > 0$ only depending on L.

Now we are ready to show

Lemma 4.5. *For every thick subsurface X of ∂H the family $\mathcal{E}$ has the bounded penetration property.*

Proof. Let γ be a separating I-bundle generator in X. Then $X-\gamma$ has two homeomorphic components X_1, X_2 with a distinguished boundary component γ. Denote by $d_{\mathcal{AC}(X_i)}$ the distance in the arc and curve graph $\mathcal{AC}(X_i)$ of X_i $(i = 1,2)$ for this boundary component of X_i. Every simple closed curve α in X which has an essential intersection with γ projects to a collection of arcs α_1, α_2 in X_1, X_2 which define subsets of $\mathcal{AC}(X_i)$ $(i = 1,2)$. If β is another simple closed curve intersecting γ then define

$$d_{\mathcal{AC}(X-\gamma)}(\alpha,\beta) = \max\{d_{\mathcal{AC}(X_1)}(\alpha_1,\beta_1), d_{\mathcal{AC}(X_2)}(\alpha_2,\beta_2)\}.$$

If $\pi^\gamma\colon \mathcal{CG}(X) \to \mathcal{AC}(X-\gamma) = \mathcal{AC}(X_1) \cup \mathcal{AC}(X_2)$ denotes the subsurface projection then by Proposition 3.5 and Lemma 4.4, there is a number $M(L) > 0$ with the following property.

Let $\rho\colon [0,n] \to \mathcal{SDG}_0(X)$ be an efficient simplicial L-quasi-geodesic, with canonical modification $\tilde{\rho}$. If

$$d_{\mathcal{AC}(X-\gamma)}(\pi^\gamma(\rho(0)), \pi^\gamma(\rho(n))) \geq M(L)$$

then there is some $k_0 \in \mathbb{Z}$ such that $\tilde{\rho}(k_0) = \gamma$. Equivalently, there is some $k < n$ such that $\rho(k) = v_\gamma$. Moreover,

$$d_{\mathcal{AC}(X_i)}(\pi^\gamma(\rho(0)), \pi^\gamma(\rho(k-1))) \leq M(L)\ (i = 1,2),$$

and similarly

$$d_{\mathcal{AC}(X_i)}(\pi^\gamma(\rho(k+1)), \pi^\gamma(\rho(n))) \leq M(L)\ (i = 1,2).$$

As a consequence, if $\rho'\colon [0,n'] \to \mathcal{SDG}_0(X)$ is another efficient simplicial L-quasi-geodesic with the same endpoints, then there is some $k' < n'$ such that $\rho'(k') = v_\gamma$, and

$$d_{\mathcal{AC}(X-\gamma)}(\pi^\gamma(\rho(k-1)), \pi^\gamma(\rho'(k'-1))) \leq 2M(L),$$

$$d_{\mathcal{AC}(X-\gamma)}(\pi^\gamma(\rho(k+1)), \pi^\gamma(\rho'(k'+1))) \leq 2M(L).$$

Lemma 4.2 and Lemma 4.1 now show that the distance in $\mathcal{E}(\gamma)$ between $\rho(k-1), \rho'(k'-1)$ and between $\rho(k+1), \rho'(k'+1)$ is uniformly bounded. In particular, the bounded penetration property holds true for the subgraph $\mathcal{E}(\gamma)$ and for quasi-geodesics connecting two disks whose boundaries have projections of large diameter into $X-\gamma$.

On the other hand, if $\rho\colon [0,n] \to \mathcal{SDG}_0(X)$ is any efficient simplicial L-quasi-geodesic and if $\rho(k) = v_\gamma$ for some I-bundle generator γ then using once more Lemma 4.4, we conclude that

$$d_{\mathcal{AC}(X-\gamma)}(\pi^\gamma(\rho(0)), \pi^\gamma(\rho(k-1))) \leq M(L).$$

Therefore the reasoning in the previous paragraph shows that whenever the distance in $\mathcal{E}(\gamma)$ between $\rho(k-1), \rho(k+1)$ is sufficiently large then

$$d_{\mathcal{AC}(X-\gamma)}(\pi^\gamma(\rho(0)), \pi^\gamma(\rho(n))) \geq M(L).$$

In other words, the conclusion in the previous paragraph holds true, and the bounded penetration property for separating I-bundle generators follows.

Now assume that γ is non-separating. Let $\pi^\gamma\colon \mathcal{CG}(X) \to \mathcal{AC}(X-\gamma)$ be the subsurface projection. Using the notations from the beginning of this proof, if the distance in $\mathcal{AC}(X-\gamma)$ between $\pi^\gamma(\rho(0))$ and $\pi^\gamma(\rho(n))$ is at least $M(L)$ then there is some k so that $\rho(k) = v_\gamma$. Moreover, we have $\rho(k-1) \in \mathcal{E}(\gamma), \rho(k+1) \in \mathcal{E}(\gamma)$. As a consequence, the curves $\rho(k-1), \rho(k+1)$ are invariant under the orientation reversing involution φ of X which preserves γ and extends to an involution of the I-bundle defined by γ.

Let F be the base of the I-bundle defined by γ and let $\alpha, \beta \in C'(F,\gamma)$ be the projections of $\rho(k-1), \rho(k+1)$. By Lemma 4.2, the distance in $\mathcal{E}(\gamma)$ between $\rho(k-1), \rho(k+1)$ is uniformly equivalent to the distance in $C'(F,\gamma)$ between α, β. Since $\rho(k-1), \rho(k+1)$ are invariant under the involution Φ, the main result of [24] shows that this distance is also uniformly equivalent to the distance between $\pi^\gamma(\rho(k-1))$ and $\pi^\gamma(\rho(k+1))$ in $\mathcal{AC}(X-\gamma)$.

In particular, if ρ' is any other efficient L-quasi-geodesic in $\mathcal{SDG}_0(X)$ with the same endpoints, then there is some k' with $\rho(k') = v_\gamma$, and the distance in $\mathcal{E}(\gamma)$ between $\rho(k-1), \rho'(k'-1)$ and between $\rho(k+1)$ and $\rho'(k'+1)$ is uniformly bounded. The bounded penetration property follows in this case.

Finally, as in the case of a separating I-bundle generator, this argument can be inverted. Together this completes the proof of the lemma. □

We can now apply Theorem 2.4 to conclude

Corollary 4.6. *For every thick subsurface X of ∂H, the graph $\mathcal{EDG}(X)$ is δ-hyperbolic for a number $\delta > 0$ not depending on X. Enlargements of geodesics in $\mathcal{SDG}_0(X)$ are uniform quasi-geodesics in $\mathcal{EDG}(X)$. There is a number $k > 0$ such that for every I-bundle generator γ in X, the subgraph $\mathcal{E}(\gamma)$ of $\mathcal{EDG}(X)$ is k-quasi-convex.*

Proof. By Proposition 3.5, the $\mathcal{E}$-electrification of $\mathcal{EDG}(X)$ is hyperbolic. The bounded penetration property holds true by Lemma 4.5 and hence $\mathcal{EDG}(X)$ is hyperbolic relative to $\mathcal{E}$. By Corollary 4.3, there is a number $\delta > 0$ such that each of the subgraphs $\mathcal{E}(\gamma)$ is δ-hyperbolic. Thus the conditions in Theorem 2.4 are satisfied. □

In the remainder of this section, we specialize to the case $X = \partial H$. We begin with establishing a distance estimate for the electrified disk graph $\mathcal{EDG} = \mathcal{EDG}(\partial H)$.

If γ is an I-bundle generator in ∂H then let π^γ be the subsurface projection of a simple closed curve in ∂H into the arc and curve-graph of $\partial H - \gamma$.

For a subset A of a metric space Y and a number $C > 0$ define $\mathrm{diam}(A)_C$ to be the diameter of A if this diameter is at least C and let $\mathrm{diam}(A)_C = 0$ otherwise. The notation $\asymp$ means equality up to a universal multiplicative constant.

Corollary 4.7. *Let H is a handlebody of genus $g \geq 2$. Then there is a number $C > 0$ such that for any two disks D, E in H we have*

$$d_{\mathcal{E}}(D, E) \asymp d_{\mathcal{CG}}(\partial D, \partial E) + \sum_{\gamma} \mathrm{diam}(\pi^\gamma(\partial D \cup \partial E))_C$$

where γ passes through all I-bundle generators on ∂H.

Proof. Let $\mathcal{SDG}_0$ be the $\mathcal{E}$-electrification of $\mathcal{EDG}$. For an I-bundle generator γ in ∂H denote by v_γ the special vertex in $\mathcal{SDG}_0$ defined by γ.

Let $\rho\colon [0,k] \to \mathcal{SDG}_0$ be a geodesic. By Corollary 2.8 and Corollary 4.6, an enlargement $\hat\rho$ of ρ is a uniform quasi-geodesic in $\mathcal{EDG}$. Thus it suffices to show that the length of $\hat\rho$ is uniformly comparable to the right hand side of the formula in the corollary.

By Proposition 3.5, there is a number $L > 1$ such that a simplicial arc $\tilde\rho$ in the curve graph $\mathcal{CG}$ of ∂H constructed from ρ as in the proof of Lemma 4.5 is an

L-quasi-geodesic in $\mathcal{CG}$. Lemma 4.5 shows that if $\hat{\rho}$ is an enlargement of ρ then the diameter of the intersection of $\hat{\rho}$ with $\mathcal{E}(\gamma)$ equals the diameter of $\pi^{\gamma}(\rho(0) \cup \rho(k))$ up to a universal multiplicative and additive constant. This is what we wanted to show. □

We complete this section with determining the Gromov boundary of the electrified disk graph of H. To this end let H be a handlebody of genus $g \geq 2$. Let $\mathcal{L}$ be the space of all geodesic laminations on ∂H equipped with the *coarse Hausdorff topology* [9]. In this topology, a sequence of laminations λ_i converges to λ if every accumulation point of λ_i in the usual Hausdorff topology for compact subsets of ∂H contains λ as a sublamination. This topology is not Hausdorff.

Let

$$\mathcal{H} \subset \mathcal{L}$$

be the subspace of all minimal laminations which fill up ∂H, i.e. such that complementary components are simply connected, and which are limits in the coarse Hausdorff topology of diskbounding simple closed curves. The restriction to $\mathcal{H}$ of the coarse Hausdorff topology is Hausdorff (see [9] for a discussion of this fact).

For an I-bundle generator γ let $\partial\mathcal{E}(\gamma) \subset \mathcal{L}$ be the set of all geodesic laminations which consist of two minimal components filling up $\partial H - \gamma$ and which are limits in the coarse Hausdorff topology of boundaries of disks contained in $\mathcal{E}(\gamma)$. Each lamination $\mu \in \partial\mathcal{E}(\gamma)$ is invariant under the orientation reversing involution Φ_γ of ∂H which fixes γ pointwise and exchanges the endpoints of the fibres of the defining I-bundle.

Define

$$\partial\mathcal{EDG} = \mathcal{H} \cup \bigcup_{\gamma} \partial\mathcal{E}(\gamma) \subset \mathcal{L}$$

where the union is over all I-bundle generators $\gamma \subset \partial H$. The handlebody group $\mathrm{Map}(H)$ acts on $\partial\mathcal{EDG}$ equipped with the coarse Hausdorff topology as a group of homeomorphisms.

Proposition 2.10 can now be applied to show

Theorem 4.8. *The Gromov boundary of $\mathcal{EDG}$ is* $\mathrm{Map}(H)$*-equivariantly homeomorphic to $\partial\mathcal{EDG}$.*

Proof. We show first that the subspace $\partial\mathcal{EDG}$ of $\mathcal{L}$ is Hausdorff.

A point $\lambda \in \partial\mathcal{EDG}$ either is a minimal geodesic lamination which fills up ∂H, or it is a geodesic lamination with two minimal components which fill up $\partial H - \gamma$ for some I-bundle generator γ. Let $\nu \neq \lambda$ be another such lamination. We claim that ν and λ intersect. This means that for some fixed hyperbolic metric on ∂H, the geodesic representatives of ν, λ intersect transversely.

If either ν or λ fills up ∂H (i.e. if the complementary components of ν or λ are simply connected) then this is obvious. Otherwise ν fills up the complement of an I-bundle generator γ, and λ fills up the complement of an I-bundle generator γ'. Now the simple closed curve γ is the only minimal geodesic lamination which does not intersect ν and which is distinct from a component of ν. The lamination λ consists of two minimal components which are not simple closed curves and therefore the geodesic laminations ν, λ indeed intersect.

Since $\nu, \lambda \in \partial\mathcal{EDG}$ intersect, by the definition of the coarse Hausdorff topology there are neighborhoods U of λ, V of ν in $\mathcal{L}$ so that any two laminations $\lambda' \in U, \nu' \in V$ intersect. In particular, the neighborhoods U, V are disjoint. This shows that $\partial\mathcal{EDG}$ is Hausdorff.

Proposition 2.10 shows that there is a natural bijection Λ between $\partial\mathcal{EDG}$ and the Gromov boundary of $\mathcal{EDG}$. That this bijection is in fact a homeomorphism follows from the description the Gromov boundary of the curve graph of ∂H as discussed in [18, 9] and Proposition 2.10.

To be more precise, let γ be a separating I-bundle generator for ∂H. The orientation reversing involution Φ_γ of the I-bundle determined by γ restricts to a homeomorphism of $\partial H - \gamma$ which exchanges the two components of $\partial H - \gamma$. By Lemma 4.1 and Lemma 4.2, the graph $\mathcal{E}(\gamma)$ can be identified with the complete subgraph of $\mathcal{CG}$ whose vertex set is the set of all simple closed curves α in ∂H which intersect γ in precisely in two points and are invariant under Φ_γ. Using again Lemma 4.1, Lemma 4.2 and the description of the Gromov boundary of the curve graph of a component of $X - \gamma$ in [18, 9], the Gromov boundary of $\mathcal{E}(\gamma)$ has a natural identification with the space $\partial\mathcal{E}(\gamma)$ of all Φ_γ-invariant geodesic laminations which consist of two minimal components, each of which fills a component of $\partial H - \gamma$. The topology on this space is the coarse Hausdorff topology. A similar description is valid for the Gromov boundary of $\mathcal{E}(\zeta)$ where ζ is an orientation reversing I-bundle generator.

Proposition 2.10 shows that the Gromov boundaries of the subspaces $\mathcal{E}(\gamma)$ are embedded subspaces of the Gromov boundary of $\mathcal{EDG}$. The Gromov boundary $\mathcal{H}$ of $\mathcal{SDG}$ is embedded in the Gromov boundary of $\mathcal{EDG}$ as well. For every $\xi \in \mathcal{H}$, a neighborhood basis of ξ in the Gromov boundary of $\mathcal{EDG}$ consists of sets which are unions of a neighborhood of ξ in $\mathcal{H}$ with sets $\partial\mathcal{E}(\gamma)$ where the curves γ are contained in some neighborhood of ξ in $\mathcal{CG} \cup \partial\mathcal{CG}$. By the description of neighborhood bases of ξ in $\mathcal{CG} \cup \partial\mathcal{CG}$ as neighborhoods of ξ in the space of geodesic laminations, equipped with the coarse Hausdorff topology [18, 9], a neighborhood basis of ξ as a point in the Gromov boundary of $\mathcal{EDG}$ maps to a neighborhood basis of $\Lambda(\xi)$ in lamination space equipped with the coarse Hausdorff topology. The same holds true for $\xi \in \partial\mathcal{E}(\gamma)$ where γ is any I-bundle generator. □

5. Intermediate hyperbolic graphs

In this section we construct a graph whose vertices are disks and which can be obtained from the disk graph by adding edges and from the electrified disk graph by deleting edges. We use Corollary 2.8 to show that this graph is hyperbolic. The construction in this section can be iterated inductively and yields hyperbolicity of the disk graph as explained in Section 6.

First we slightly weaken the definition of thick subsurface of ∂H as follows. Namely, define a connected properly embedded subsurface Y of ∂H to be *visible* if every disk intersects Y and if moreover Y contains the boundary of at least one disk. Thus a thick subsurface is visible, but a visible subsurface may not be filled by boundaries of disks and hence may not be thick. Note that if Y is visible then the electrified disk graph $\mathcal{EDG}(Y)$ of Y is defined. However, if Y is not thick then its diameter equals one.

Let as before X be a thick subsurface of ∂H. Recall from Remark 3.2 that X is not a four holed sphere or a one holed torus. Define $\mathcal{EDG}(2, X)$ to be the graph whose vertices are isotopy classes of essential disks with boundary in X. Two such disks D, E in $\mathcal{EDG}(2, X)$ are connected by an edge of length one if either D, E are disjoint or if $\partial D, \partial E$ are disjoint from an essential multicurve $\beta \subset \partial X$ consisting of two components which are not freely homotopic.

Call a simple closed curve γ in X *admissible* if γ has the following properties.

(1) γ is neither diskbounding nor diskbusting.

(2) Either γ is non-separating or γ decomposes X into a three-holed sphere X_1 and a visible second component X_2.

For an admissible simple closed curve γ in X write $\mathcal{EDG}(X - \gamma)$ to denote the electrified disk graph of the component of $X - \gamma$ which is not a three-holed sphere. Define $\mathcal{F}(\gamma)$ to be the complete subgraph of $\mathcal{EDG}(2, X)$ whose vertex set consists of all disks which are disjoint from γ. As γ is not diskbounding by assumption, the boundary of such a disk is not freely homotopic to γ. A disk $D \in \mathcal{F}(\gamma)$ defines a vertex in $\mathcal{EDG}(X - \gamma)$.

Lemma 5.1. *The vertex inclusion defines an isometry of* $\mathcal{F}(\gamma)$ *and* $\mathcal{EDG}(X-\gamma)$.

Proof. By Remark 3.2, if X is a five-holed sphere or a two-holed torus then $\mathcal{F}(\gamma)$ and $\mathcal{EDG}(X-\gamma)$ contain at most one vertex, so there is nothing to show. Thus assume that X is different from a five holed sphere or a two holed torus.

Two disks $D, E \in \mathcal{F}(\gamma)$ are connected by an edge in $\mathcal{EDG}(2, X)$ if and only if either they are disjoint or if there is a pair β_1, β_2 of disjoint not homotopic essential simple closed curves in X which are disjoint from both D, E.

If D, E are disjoint then they are connected in $\mathcal{EDG}(X-\gamma)$ by an edge, so assume that D, E are disjoint from two disjoint not homotopic curves β_1, β_2. If one of the curves β_1, β_2, say the curve β_1, is disjoint from γ, then one of the two curves β_1 (if β_1 is not homotopic to γ) or β_2 (if β_1 is homotopic to γ) is an essential simple closed curve $X-\gamma$. This curve must be contained in the component $\hat{X}$ of $X-\gamma$ which is not a three holed sphere (note that if γ is non-separating then $\hat{X} = X-\gamma$). Thus by definition, D, E viewed as vertices in $\mathcal{EDG}(X-\gamma)$ are connected by an edge.

Now assume that both β_1, β_2 intersect γ. We claim that there is an essential simple closed curve contained in the intersection of a tubular neighborhood of $\gamma \cup \beta_1$ with $\hat{X}$ which is disjoint from γ and not homotopic to γ. This curve is then essential in $\hat{X}$ and disjoint from γ, D, E and once again, D, E are connected by an edge in $\mathcal{EDG}(X-\gamma)$.

To show the claim let η be a component of $\beta_1-\gamma$. In the case that γ is separating we require that η is contained in $\hat{X}$. As $\hat{X}$ is not a three-holed sphere or a one-holed torus, one of the boundary components of a tubular neighborhood of $\gamma \cup \eta$ is an essential simple closed curve in $\hat{X}$ disjoint from γ and not freely homotopic to γ.

As a consequence, the vertex inclusion $\mathcal{F}(\gamma) \to \mathcal{EDG}(X-\gamma)$ extends to a 1-Lipschitz embedding. By definition, this embedding is surjective on vertices. Moreover, any two vertices which are connected in $\mathcal{EDG}(X-\gamma)$ by an edge are also connected in $\mathcal{F}(\gamma)$ by an edge. In other words, the 1-Lipschitz embedding $\mathcal{F}(\gamma) \to \mathcal{EDG}(X-\gamma)$ is in fact an isometry. ☐

Lemma 5.1 and Corollary 4.6 imply

Corollary 5.2. *There is a number* $\delta > 0$ *so that each of the graphs* $\mathcal{F}(\gamma)$ *is* δ*-hyperbolic.*

Let $\mathcal{F} = \{\mathcal{F}(\gamma) \mid \gamma\}$ be the family of all these subgraphs of $\mathcal{EDG}(2, X)$ where γ passes through all admissible curves in X. Our goal is to apply Theorem 2.4 to $\mathcal{F}$.

Lemma 5.3. *$\mathcal{EDG}(X)$ is 2-quasi-isometric to the $\mathcal{F}$-electrification of $\mathcal{EDG}(2, X)$.*

Proof. Let $\mathcal{G}$ be the $\mathcal{F}$-electrification of $\mathcal{EDG}(2, X)$. We show first that the vertex inclusion $\mathcal{EDG}(X) \to \mathcal{G}$ is 2-Lipschitz.

To this end let D, E be any two vertices in $\mathcal{EDG}(X)$ which are connected by an edge. Then either D, E are disjoint, or they are disjoint from a common essential simple closed curve γ in X.

If D, E are disjoint then D, E viewed as vertices in $\mathcal{EDG}(2, X)$ are connected by an edge in $\mathcal{EDG}(2, X)$.

Now assume that D, E are disjoint from a common essential simple closed curve γ in X. If γ is diskbounding then the distance between D, E in the disk graph of X is at most two and hence the same holds true for the distance in $\mathcal{G}$. If γ is admissible then $D, E \in \mathcal{F}(\gamma)$ and hence by the definition of the $\mathcal{F}$-electrification of $\mathcal{EDG}(2, X)$, their distance in $\mathcal{G}$ is at most two.

On the other hand, if γ is neither admissible nor diskbounding, then γ is a separating simple closed curve in X. The surface $X - \gamma$ is a disjoint union of essential surfaces X_1, X_2 which are distinct from three-holed spheres. The boundaries of D, E are contained in $X_1 \cup X_2$.

If $\partial D, \partial E$ are contained in distinct components of $X - \gamma$ then D, E are disjoint and hence D, E are connected by an edge in $\mathcal{EDG}(2, X)$. If $\partial D, \partial E$ are contained in the same component of $X - \gamma$, say in X_1, then the second component X_2 contains an essential simple closed curve η, and $\partial D, \partial E$ are disjoint from the multi-curve $\gamma \cup \eta$ with two components. Once more, this implies that D, E are connected in $\mathcal{EDG}(2, X)$ by an edge. As a consequence, the vertex inclusion $\mathcal{EDG}(X) \to \mathcal{G}$ is indeed 2-Lipschitz.

That this inclusion is in fact a 2-quasi-isometry is immediate from the definitions. Namely, if $\gamma \subset X$ is admissible then by the definition of $\mathcal{EDG}(X)$, any two vertices in $\mathcal{F}(\gamma)$ are connected in $\mathcal{EDG}(X)$ by an edge. ☐

Our goal is to use Lemma 5.3 and Theorem 2.4 to show hyperbolicity of the graph $\mathcal{EDG}(2, X)$. To verify the bounded penetration property using the strategy from Section 4 we have to carefully keep track of subsurface projections. To make this control quantitative, for $\kappa > 0$ define a simplicial path $\zeta\colon [0, n] \to \mathcal{EDG}(X)$ to be *κ-good* if for every thick subsurface Y of X there is a number $u = u(Y) \in [0, n)$ with the following properties.

(1) For every $j \le u$, $\mathrm{diam}(\pi_Y(\zeta(0) \cup \zeta(j))) \le \kappa$.

(2) For every $j > u$, $\mathrm{diam}(\pi_Y(\zeta(j) \cup \zeta(n))) \le \kappa$.

As before, let $\mathcal{SDG}_0(X)$ be the $\mathcal{E}$-electrification of $\mathcal{EDG}(X)$. By Corollary 2.8, enlargements of geodesics in $\mathcal{SDG}_0(X)$ are uniform quasi-geodesics in $\mathcal{EDG}(X)$.

Lemma 5.4. *There is a number $\kappa > 0$ not depending on X such that an enlargement of a geodesic in $\mathcal{SDG}_0(X)$ is κ-good.*

Proof. By Proposition 3.5 and Lemma 4.4, a geodesic ζ in $\mathcal{SDG}_0(X)$ is κ_0-good for a number κ_0 not depending on X.

Let $\tilde{\zeta}\colon [0, m] \to \mathcal{CG}(X)$ be a canonical modification of ζ. Suppose that $Y \subset X$ is a thick subsurface such that $\mathrm{diam}(\pi_Y(\tilde{\zeta}(0) \cup \tilde{\zeta}(m))) \ge \kappa_0$. Since ζ is κ_0-good and Y is thick, there is a unique number $k > 0$ so that $\tilde{\zeta}(k)$ is disjoint from Y, and $\tilde{\zeta}(k)$ is not a diskbounding curve.

Recall from Section 3 the definition of the family $\mathcal{E} = \{\mathcal{E}(\gamma) \mid \gamma\}$ of complete subgraphs of $\mathcal{EDG}(X)$. Let ρ be an enlargement of ζ, let γ be an I-bundle generator in X and let $\rho[s, t]$ be a maximal subarc of ρ contained in $\mathcal{E}(\gamma)$. By maximality and the definition of a canonical modification, there is some $\ell \ge 0$ such that $\rho(s) = \tilde{\zeta}(\ell)$ and $\rho(t) = \tilde{\zeta}(\ell + 4)$. It now suffices to show that $k \notin [\ell, \ell + 4]$ and that there is a number $\kappa_1 > 0$ such that the diameter of the projection $\pi_Y(\rho[s, t])$ is at most κ_1.

Since Y is thick and hence contains the boundary of some disk and since γ is an I-bundle generator we have $Y \not\subset X - \gamma$. We use this fact to show that $k \notin [\ell, \ell + 4]$.

For this we argue by contradiction and we assume otherwise. Then up to exchanging the orientation of ζ and ρ we have $k = \ell + 1$ and $\tilde{\zeta}(k)$ is a simple closed curve which is disjoint from the thick subsurface Y, from γ and from $\rho(s) = \tilde{\zeta}(\ell)$. Moreover, Y intersects both $\rho(s)$ (since $\rho(s)$ is diskbounding) and γ (by assumption).

Now let us assume that there is some simple closed curve α which is disjoint from $\rho(s)$ and γ and which intersects Y. Then we can replace $\tilde{\zeta}(k)$ by α and obtain another uniform quasi-geodesic in $\mathcal{CG}(X)$ with the same endpoints. Each of the vertices of this new quasi-geodesic intersects Y. By Lemma 4.4, this is impossible if the diameter of the subsurface projection of the endpoints of $\tilde{\zeta}$ is sufficiently large.

As a consequence, any simple closed curve disjoint from both γ and $\rho(s)$ is disjoint from Y. Since γ and $\rho(s)$ intersect in precisely two points, a tubular neighborhood of $\gamma \cup \rho(s)$ is a four-holed sphere (if γ is separating) or a two-holed torus (if γ is non-separating). This tubular neighborhood then must be the subsurface Y. However, by Remark 3.2, a four-holed sphere can not be thick. If Y is a two-holed torus then Y is the boundary of an I-bundle over a two-holed projective plane and once again, Y can not be thick. This is a contradiction and shows that $k \notin [\ell, \ell + 4]$.

Thus assume without loss of generality that $\ell + 4 < k$ (the case $k < \ell$ is treated in the same way). If $u \in (s,t)$ is arbitrary then the path obtained from $\tilde{\zeta}$ by replacing $\tilde{\zeta}[\ell, \ell+4]$ by an edge path of length at most eight with the same endpoints which contains $\rho(u)$ as a vertex is a uniform quasi-geodesic in $\mathcal{CG}(X)$. The lemma now follows from Lemma 4.4, applied to this modification of $\tilde{\zeta}[0,k]$. □

By Corollary 4.6, enlargements of geodesics in $\mathcal{SDG}_0(X)$ are uniform quasi-geodesics in $\mathcal{EDG}(X)$. Define an *level-2 hierarchy path* to be a simplicial path in $\mathcal{EDG}(2,X)$ which is an enlargement of an enlargement of a geodesic in $\mathcal{SDG}_0(X)$.

Theorem 5.5. *$\mathcal{EDG}(2,X)$ is hyperbolic. Level-2 hierarchy paths of geodesics in $\mathcal{SDG}_0(X)$ are uniform quasi-geodesics in $\mathcal{EDG}(2,X)$.*

Proof. By Lemma 5.3, Lemma 5.1, Corollary 4.6 and Theorem 2.4, it suffices to show that the family $\mathcal{F} = \{\mathcal{F}(\gamma) \mid \gamma \text{ admissible}\}$ satisfies the bounded penetration property.

To this end let γ be an admissible simple closed curve in X and let ρ be an enlargement of a geodesic ζ in $\mathcal{SDG}_0(X)$. By Lemma 5.4 and the definition of an enlargement, ρ passes through two points of large distance in $\mathcal{F}(\gamma)$ if and only if the diameter of the subsurface projections into $X - \gamma$ of the endpoints of ρ is large, and this can explicitly be made quantitative. In other words, enlargements of geodesics in $\mathcal{SDG}_0(X)$ satisfy the bounded penetration property.

We have to show that this property holds true for any L-quasi-geodesic with the same endpoints, with quantitative control only depending on L. Thus let $L > 1$ be arbitrary and let $\beta\colon [0,m] \to \mathcal{EDG}(X)$ be any L-quasi-geodesic in $\mathcal{EDG}(X)$ with the same endpoints as ρ. As ρ is a uniform quasi-geodesic in $\mathcal{EDG}(X)$, by hyperbolicity there is a number $n(L) > 0$ only depending on L so that the Hausdorff distance between the image of ρ and the image of β is at most $n(L)$.

Now if γ is an admissible simple closed curve and if the diameter of the subsurface projection of the endpoints of ρ into $X - \gamma$ (i.e. into the component of $X - \gamma$ which is different from a three-holed sphere if γ is separating) is large then by Lemma 5.4, ρ passes through $\mathcal{F}(\gamma)$ and hence the quasi-geodesic β passes through the $n(L)$-neighborhood of $\mathcal{F}(\gamma)$.

Let $s_0 + 1 \le t_0 - 1$ be the smallest and biggest number, respectively, so that $\beta(s_0+1), \beta(t_0-1)$ are contained in the $n(L)$-neighborhood of $\mathcal{F}(\gamma)$. The distance between $\beta(s_0)$ and $\beta(t_0)$ is at most $2n(L) + 2$. An enlargement of a geodesic in $\mathcal{SDG}_0(X)$ with endpoints $\beta(0), \beta(s_0)$ and $\beta(t_0), \beta(m)$, respectively, does not pass through $\mathcal{F}(\gamma)$. Thus the diameter of the subsurface projection of $\beta(s_0) \cup \beta(t_0)$ into $X - \gamma$ is large.

A canonical modification of $\beta[s_0, t_0]$ is an edge path in the curve graph of X of uniformly bounded length. As the diameter of the subsurface projection of its endpoints into $X - \gamma$ is large, by Lemma 4.4 this path passes through the complement of $\mathcal{F}(\gamma)$. Then $\beta[s_0, t_0]$ passes through $\mathcal{F}(\gamma)$. Moreover, by the same argument, $\beta[s_0, t_0]$ contains two points in $\mathcal{F}(\gamma)$ whose distance in the curve graph of $X - \gamma$ to the two points on ρ is uniformly bounded.

As a consequence, given any admissible curve γ, a large subsurface projection into $X - \gamma$ of the endpoints of a uniform quasi-geodesic β in $\mathcal{EDG}(X)$ implies that β passes through $\mathcal{F}(\gamma)$, moreover entry and exit points are contained in a fixed size neighborhood (only depending on L) of points determined by the subsurface projections of the endpoints. We refer to the proof of Proposition 3.1 in [13] for a version of this argument with all estimates explicit.

On the other hand, the same argument with the roles of ρ and β exchanged also shows that for an admissible curve γ, a uniform quasi-geodesic in $\mathcal{EDG}(X)$ passes through two points in $\mathcal{F}(\gamma)$ of large distance if and only if this is true for the enlargement of a geodesic in $\mathcal{SDG}_0(X)$. However, the latter holds true if and only if the diameter of the subsurface projection of the endpoints is large. This shows the bounded penetration property and completes the proof of the theorem. □

As an illustration of the method used to establish hyperbolicity we observe

Corollary 5.6. *The disk graph of a handlebody of genus 2 is hyperbolic.*

Proof. Let H be a handlebody of genus two. By Theorem 5.5, it suffices to show that the vertex inclusion $\mathcal{DG} \to \mathcal{EDG}(2, \partial H)$ is a quasi-isometry.

To this end observe that a connected component of the complement in ∂H of a simple multicurve c with two components which are not freely homotopic either is a four-holed sphere or a one-holed torus or a three-holed disk. By Remark 3.2, if both components of c are not diskbounding then $X - c$ contains at most one boundary of a disk. Thus two disks which are connected by an edge in $\mathcal{EDG}(2, \partial H)$ are disjoint and hence connected by a disk in $\mathcal{DG}$. The corollary now follows from Theorem 5.5. □

6. Hyperbolicity of the disk graph

The goal of this section is to complete the proof of Theorem 4 using an argument which is new and simpler than the argument of Masur and Schleimer [22].

The idea is to define a finite chain of intermediate graphs lying geometrically between the electrified disk graph and the disk graph. This chain begins with the electrified disk graph and ends with the disk graph. With an inductive application of the construction in Section 5 we show that each of these graphs is hyperbolic.

We next introduce the chain of graphs. Namely, for a thick subsurface X of ∂H and for $k \geq 1$ define $\mathcal{EDG}(k, X)$ to be the graph whose vertex set is the set of all disks with boundary in X and where two such disks are connected by an edge of length one if and only if either they are disjoint or they are disjoint from an essential multicurve in X with a least k components. Note that if k equals the cardinality of a pants decomposition for X then $\mathcal{EDG}(k, X)$ is just the disk graph of X. The graph $\mathcal{EDG}(1, X)$ is the electrified disc graph $\mathcal{EDG}(X)$ of X which is hyperbolic by Corollary 4.6. The graph $\mathcal{EDG}(2, X)$ is hyperbolic by Theorem 5.5.

The strategy is to deduce by induction on k hyperbolicity of $\mathcal{EDG}(k+1, X)$ from hyperbolicity of $\mathcal{EDG}(k, Y)$ where Y runs through all (not necessarily proper) thick subsurfaces of X.

To this end define inductively a hierarchy of connected subsurfaces of X as follows. A level-one subsurface is the complementary component of an admissible curve which is not a three holed sphere. By induction, a level-k subsurface is a level-one subsurface of a level-$(k-1)$ subsurface.

Let $\mathcal{F}(k, X) = \{\mathcal{EDG}(Y) \mid Y\}$ where Y runs through all level-$(k-1)$ subsurfaces in X. Corollary 4.6 implies

Lemma 6.1. *$\mathcal{F}(k, X)$ is a family of δ-hyperbolic graphs for a number $\delta = \delta(k) > 0$ only depending on k.*

We now use the family $\mathcal{F}(k, X)$ and Lemma 6.1 to show

Theorem 6.2. *The disk graph $\mathcal{DG}(X)$ of a thick subsurface X of ∂H is hyperbolic.*

Proof. Define inductively a level-k hierarchy path in $\mathcal{EDG}(k, X)$ to be an enlargement of a level-$(k-1)$ hierarchy path in $\mathcal{EDG}(k-1, X)$. Here a level-2 hierarchy path was defined in Section 5.

For a number $\kappa > 0$ call a quasi-geodesic $\zeta\colon [0, n] \to \mathcal{EDG}(X, k)$ (κ, k)-*good* if the following holds true. Let Y be a thick subsurface of X which is properly contained in a level $(k-1)$-subsurface. Then there is a number $u = u(Y) \in [0, m)$ with the following properties.

(a) For every $j \leq u$, $\mathrm{diam}(\pi_Y(\zeta(0) \cup \zeta(j))) \leq \kappa$.

(b) For every $j > u$, $\mathrm{diam}(\pi_Y(\zeta(j) \cup \zeta(n))) \leq \kappa$.

As in Section 5, we show by induction on k the following.

(1) The $\mathcal{F}(k-1, X)$-electrification of $\mathcal{EDG}(k, X)$ is 2-quasi-isometric to $\mathcal{EDG}(k-1, X)$.

(2) The graph $\mathcal{EDG}(k, X)$ is hyperbolic relative to the family $\mathcal{F}(k, X)$ of complete subgraphs. Level-k hierarchy paths are uniform quasi-geodesics.

(3) There is a number $\kappa_k > 0$ such that level-k hierarchy paths are (κ_k, k)-good.

Properties (1) and (2) for $k = 1$ is just Corollary 4.6. Property (3) for $k = 1$ is the Lemma 5.3. Properties (1) and (2) for the case $k = 2$ was shown in Corollary 5.5.

By induction, assume that Properties (1),(2),(3) hold true for $k-1 \in [1, 3g-3)$.

Property (1) for k follows as in the proof of Lemma 5.3. Namely, let D, E be disks with boundary in X which are connected in $\mathcal{EDG}(k-1, X)$ by an edge. We may assume that D, E are not disjoint. Then D, E are disjoint from a multicurve α with $k-1$ components. Let Y be the smallest subsurface of X filled by $\partial D \cup \partial E$. If $X - Y$ contains a diskbounding curve then the distance between D, E in $\mathcal{EDG}(k, X)$ is at most two. If Y is properly contained in a level-$(k-1)$-subsurface of X then there is a multicurve in X with k components which is disjoint from $D \cup E$ and hence D, E are connected by an edge in $\mathcal{EDG}(k, X)$. Otherwise Y is a level-$(k-1)$-subsurface and hence D, E have distance at most two in the $\mathcal{F}(k-1, X)$-electrification of $\mathcal{EDG}(k, X)$. Thus the vertex inclusion of $\mathcal{EDG}(k-1, X)$ into the $\mathcal{F}(k-1, X)$-electrification of $\mathcal{EDG}(k, X)$ is two-Lipschitz and in fact a two-quasi-isometry.

To show the bounded penetration property required in (2) above, let D, E be any two disks with boundary in X and let ρ be a level-$(k-1)$-hierarchy path in $\mathcal{EDG}(k-1, X)$ connecting D to E. By induction hypothesis, this path is a uniform quasi-geodesic, moreover it is $(\kappa_{k-1}, k-1)$-good.

By the reasoning in the proof of Theorem 5.5, such a path has the bounded penetration property for the subgraphs from the family $\mathcal{F}(k-1, X)$ with constants only depending on $k-1$. By induction hypothesis, level-$(k-1)$-hierarchy paths are uniform quasi-geodesics in $\mathcal{EDG}(k-1, X)$ and therefore the bounded penetration property for the family $\mathcal{F}(k-1, X)$ follows from the argument in the proof of Theorem 5.5 without modification.

The reasoning in the proof of Lemma 5.4 implies moreover Property (3) above (see also [13] where this argument is used in a more complicated situation).

As remarked earlier, if k is the number of simple closed curves in a pants decomposition of X then $\mathcal{EDG}(k, X) = \mathcal{DG}(X)$. This completes the proof of Theorem 6.2. $\square$

For a thick subsurface Y of ∂H denote as before by π_Y the subsurface projection of simple closed curves into the arc and curve graph of Y. If γ is an I-bundle generator in a thick subsurface Y then let π^γ be the subsurface projection into $Y - \gamma$.

The following corollary is now immediate from our construction. It was earlier obtained by Masur and Schleimer (Theorem 19.9 of [22]).

Corollary 6.3. *There is a number $C > 0$ such that*

$$d_{\mathcal{D}}(D, E) \asymp \sum_Y \operatorname{diam}(\pi_Y(E \cup D))_C + \sum_\gamma \operatorname{diam}(\pi^\gamma(E \cup D))_C$$

where Y passes through all thick subsurfaces of ∂H, where γ passes through all I-bundle generators in thick subsurfaces of ∂H, and the diameter is taken in the arc and curve graph.

For a thick subsurface Y of ∂H let $\partial\mathcal{EDG}(Y)$ be the Gromov boundary of $\mathcal{EDG}(Y)$. Define

$$\partial\mathcal{DG} = \bigcup_Y \partial\mathcal{EDG}(Y) \subset \mathcal{L}$$

where the union is over all thick subsurfaces of ∂H and where this union is viewed as a subspace of $\mathcal{L}$, i.e. it is equipped with the coarse Hausdorff topology. The proof of the following statement is completely analogous to the proof of Proposition 4.8 and will be omitted.

Corollary 6.4. *$\partial\mathcal{DG}$ is the Gromov boundary of $\mathcal{DG}$.*

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Ursula Hamenstädt, Mathematisches Institut der Universität Bonn,
Endenicher Allee 60, 53115 Bonn, Deutschland

e-mail: ursula@math.uni-bonn.de