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The congruence subgroup problem for the free metabelian group on two generators

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Abstract. In this paper we describe the profinite completion of the free solvable group on m generators of solvability length $r \geq 1$. Then, we show that for $m = r = 2$, the free metabelian group on two generators does not have the Congruence Subgroup Property.

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0. Introduction

Let $\Gamma := \mathrm{GL}_m(\mathbb{Z})$ be the matrix group of the $m \times m$ invertible matrices over $\mathbb{Z}$. For $n \in \mathbb{N}$, set $\Gamma[n] := \ker(\Gamma \to \mathrm{GL}_m(\mathbb{Z}_n))$ where $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$. The classical congruence subgroup problem asks whether every subgroup of Γ of finite index contains a congruence subgroup $\Gamma[n]$, for some $n \in \mathbb{N}$?

By referring to $\mathrm{GL}_m(\mathbb{Z})$ as the automorphism group of $\mathbb{Z}^m$, one can generalize the congruence subgroup problem as follows. Let G be a finitely generated group, and let $\Gamma := \mathrm{Aut}(G)$ be the automorphism group of G. For a characteristic subgroup K of finite index in G, set $\Gamma[K] := \ker(\Gamma \to \mathrm{Aut}(G/K))$. Now, the generalized congruence subgroup problem asks: does every finite index subgroup of Γ contain a congruence subgroup of the form $\Gamma[K]$ for some characteristic subgroup K of finite index in G? In the case where the answer to this question is positive we say that G has the CSP (Congruence Subgroup Property).

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We prefer to say that G has the CSP rather than Γ, as there could potentially exist two groups G_1, G_2 with $\mathrm{Aut}(G_1)$, $\mathrm{Aut}(G_2)$ isomorphic such that just one of these automorphism groups has the CSP. But when there is no danger of ambiguities we will also say that $\Gamma = \mathrm{Aut}(G)$ has the CSP.

It is well known that $\mathbb{Z}^2$, the free abelian group on two generators, does not have the CSP. This result was known already in the 19^{th} century ([15] and [9]). On the other hand, in 2001 Asada [1] showed, by using methods of algebraic geometry, that F_2, the free group on two generators does have the CSP. Later (2011), Bux, Ershov, and Rapinchuk [3] provided a group theoretic version of this result and proof.

Our paper deals with an intermediate case: the free metabelian group on two generators (or more generally, the free solvable group of length $r \geq 2$ on two generators). As we will see, a large part of Bux–Ershov–Rapinchuk's proof works for the automorphism group of the free metabelian group, which initially caused us to believe that it has the CSP. However, as we will show below, it does not.

Theorem 0.1. *The free metabelian group on two generators Φ does not have the CSP.*

Here is the main line of the proof: the generalized congruence subgroup problem asks whether the profinite topology of $\mathrm{Aut}(G)$ is the same as the congruence topology. By some standard arguments from [3], one can pass to $\mathrm{Out}(G)$ (if G is residually finite and $\widehat{G}$ is centerless, which is the case for our groups, see Theorem 2.11 and Proposition 2.10). In §2 we will give an explicit description of the profinite topology and completion of the free solvable group $G = \Phi_{m,r}$ on m generators and solvability length r. This enables one to describe the congruence topology of $\mathrm{Aut}(G)$. What makes now the case $m = 2$ so special is that $\mathrm{Out}(\Phi_{2,r}) \cong \mathrm{GL}_2(\mathbb{Z})$, and the congruence subgroup problem for $\Phi_{2,r}$ boils down to the question as to whether the congruence topology of $\mathrm{Out}(\Phi_{2,r})$ induces the full profinite topology of $\mathrm{GL}_2(\mathbb{Z})$. Note that for F_2, the free group on two generators, this is indeed the case: the congruence topology of $\mathrm{Out}(F_2)$ is the full profinite topology of $\mathrm{Out}(F_2) \cong \mathrm{GL}_2(\mathbb{Z})$.

On the other hand, we will show that for $\Phi := \Phi_{2,2}$, the free metabelian group on two generators, the congruence topology of $\mathrm{Out}(\Phi) \cong \mathrm{GL}_2(\mathbb{Z})$ is equal to the classical congruence topology of $\mathrm{GL}_2(\mathbb{Z})$. As it was pointed out before, it is well known that this topology is much weaker than the profinite topology of $\mathrm{GL}_2(\mathbb{Z})$. Hence Φ fails to have the CSP.

Our method and result suggest conjecturing that all the free solvable groups $\Phi_{2,r}$ (on two generators) do not have the CSP.

The paper is organized as follows: in §1 we present the Romanovskii embedding, a generalization of the well-known Magnus embedding, and use it in §2 to describe the profinite completion of a finitely generated free solvable group. In §3 we quote a result of Bux, Ershov and Rapinchuk which they used to prove that F_2 does have the CSP. In §4 we formulate a conjecture on $\Phi_{2,r}$ that would imply that $\Phi_{2,r}$ does not have the CSP. Finally, in §5 we prove this conjecture for $\Phi := \Phi_{2,2}$, and conclude that Φ does not have the CSP.

1. Romanovskii embedding

Let F be the free group on m generators, and let R be a normal subgroup of F. Denote by R' the commutator subgroup of R. Following the well-known Magnus embedding (see [2], [12], and [8]) and its generalization by Šhmel′kin (see [13] and [14]), Romanovskii found a faithful embedding of $F/(R'R^n)$, $n \in \mathbb{N}\cup\{0\}$ (with the notation $R^0 = 1$), into a matrix group, which enables to describe $F/(R'R^n)$ through F/R. In this section we will present the Romanovskii embedding, and as mentioned, we will use it in the next section to describe the profinite completion of a finitely generated free solvable group.

Define

$$G := F/R$$

and

$$\tilde{G}_n := F/R'R^n, \quad n \in \mathbb{N} \cup \{0\}.$$

Denote by $\{f_1, \ldots, f_m\}$, $\{g_1, \ldots, g_m\}$ and $\{\tilde{g}_1, \ldots, \tilde{g}_m\}$ the appropriate generators of F, G and $\tilde{G}_n$ respectively. By the mapping $f_i \mapsto \tilde{g}_i \mapsto g_i$, $i = 1, \ldots, m$, we get a natural epimorphism $F \to \tilde{G}_n \to G$. For $n \in \mathbb{N} \cup \{0\}$ denote $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. Now, for the group G, denote by $\mathbb{Z}_n[G]$ the group ring of G over $\mathbb{Z}_n$, and by

$$T_n := \mathbb{Z}_n[G]t_1 + \cdots + \mathbb{Z}_n[G]t_m$$

the left free $\mathbb{Z}_n[G]$ module with basis $\{t_1, t_2, \ldots, t_m\}$. Now, define

$$R_n(G) := \left\{ \begin{pmatrix} g & t \\ 0 & 1 \end{pmatrix} \;\middle|\; g \in G, t \in T_n \right\}.$$

One can notice that $R_n(G)$ has a group structure under formal matrix multiplication given by the formula

$$\begin{pmatrix} g & t \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} h & s \\ 0 & 1 \end{pmatrix} := \begin{pmatrix} gh & gs + t \\ 0 & 1 \end{pmatrix}.$$

Theorem 1.1 ([11], Theorem 2). (1) *By the mapping* α*, defined by*

$$\tilde{g}_i \longmapsto \begin{pmatrix} g_i & t_i \\ 0 & 1 \end{pmatrix}, \quad i = 1, \ldots, m,$$

we get a faithful representation of $\tilde{G}_n$ *as a subgroup of* $R_n(G)$.

(2) *An element of the form*

$$\begin{pmatrix} g & \sum_{i=1}^{m} a_i t_i \\ 0 & 1 \end{pmatrix} \in R_n(G)$$

belongs to $\mathrm{Im}(\alpha)$ *if and only if* $1 - g = \sum_{i=1}^{m} a_i(1 - g_i)$. *In particular,*

$$\begin{pmatrix} 1 & \sum_{i=1}^{m} a_i t_i \\ 0 & 1 \end{pmatrix} \in \mathrm{Im}(\alpha) \iff \sum_{i=1}^{m} a_i(1 - g_i) = 0.$$

Considering the formula for multiplication in $R_n(G)$, it is easy to see that under α, the subgroup $R/R'R^n \leqslant \tilde{G}_n$ is isomorphic to $\left\{\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Im}(\alpha)\right\}$.

2. The profinite completion of a finitely generated free solvable group

As before, let F be the free group on m generators. Denote by

$$F^{(1)} := [F, F]$$

the commutator subgroup of F, and by induction, define

$$F^{(r+1)} := [F^{(r)}, F^{(r)}].$$

Denote by $\Phi_r := F/F^{(r)}$ the free solvable group on m generators of solvability length r, and define

$$\Phi_0 := F/F = \{e\}.$$

To simplify notations, we write F, Φ_r, $\Phi_{r,n}$ instead of F_m, $\Phi_{m,r}$, $\Phi_{m,r,n}$ respectively, as the results of this section hold for every finite number of generators bigger than 1. Sometime we will use also the notation $\Phi_{r,0} = \Phi_r$. In this section we will present, using the Romanovskii embedding, an "explicit" description of $\widehat{\Phi}_r$, in the sense that we will find finite quotients $\{\Phi_{r,n}\}_{n=1}^{\infty}$, of Φ_r, such that

$$\widehat{\Phi}_r \cong \varprojlim_{n \in \mathbb{N}} \Phi_{r,n}.$$

This explicit description will help us later to describe some properties of $\widehat{\Phi}_r$, and to prove the main theorem.

Definition 2.1. For $n \in \mathbb{N}$, define by induction on r the following groups: for $r = 0$,

$$K_{0,n} := F$$

and for $r + 1$,

$$K_{r+1,n} := K'_{r,n}(K_{r,n})^n.$$

Denote

$$\Phi_{r,n} := F/K_{r,n}.$$

Proposition 2.2. *For $n \in \mathbb{N}$ we have for $r = 0$, $|\Phi_{0,n}| = 1$, and for $r > 0$,*

$$|\Phi_{r+1,n}| = |\Phi_{r,n}| \cdot n^{|\Phi_{r,n}|\cdot(m-1)+1}.$$

Proof. For $r = 0$ the result is obvious. For $r + 1$, by a well-known theorem of Schreier, we know that $K_{r,n}$ is free, and its rank is $|\Phi_{r,n}| \cdot (m-1) + 1$. Therefore, $K_{r,n}/K'_{r,n}(K_{r,n})^n$ isomorphic to $(\mathbb{Z}_n)^{|\Phi_{r,n}|\cdot(m-1)+1}$ and thus, $|\Phi_{r+1,n}| = |\Phi_{r,n}| \cdot n^{|\Phi_{r,n}|\cdot(m-1)+1}$. □

Proposition 2.3. *For $n \in \mathbb{N}$, denote $\Phi^{\mathrm{ab}}_{r,n} := \Phi_{r,n}/[\Phi_{r,n}, \Phi_{r,n}]$. Then*

$$\Phi^{\mathrm{ab}}_{r,n} = (\mathbb{Z}/n^r\mathbb{Z})^m.$$

Proof. We need to show that

$$[F, F] \cdot K_{r,n} = [F, F] \cdot F^{(n^r)}.$$

By induction on r. For $r = 1$ the result is obvious. For $r + 1$,

$$\begin{aligned} [F, F] \cdot K_{r+1,n} &= [F, F] \cdot [K_{r,n}, K_{r,n}] \cdot (K_{r,n})^n \\ &= [F, F] \cdot (F^{(n^r)})^n \\ &= [F, F] \cdot F^{(n^{r+1})}. \end{aligned}$$

□

The next proposition follows easily from the definitions.

Proposition 2.4. *For $n \in \mathbb{N}$, denote $L_{r,n} := \ker(\Phi_r \to \Phi_{r,n})$. Then, $L_{r,n}$ is a characteristic subgroup of Φ_r.*

We need now few notations.

Definition 2.5. For $r, n \in \mathbb{N} \cup \{0\}$ denote

- $x_{r,n,1}, \ldots, x_{r,n,m}$ the m standard generators of $\Phi_{r,n}$;
- $\mathbb{Z}_n[\Phi_{r,n}]$ the group ring of $\Phi_{r,n}$ over $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$;
- $T_{r,n} := \mathbb{Z}_n[\Phi_{r,n}]t_{r,n,1} + \ldots + \mathbb{Z}_n[\Phi_{r,n}]t_{r,n,m}$, the $\mathbb{Z}_n[\Phi_{r,n}]$ left free module with the base $t_{r,n,1}, t_{r,n,2}, \ldots, t_{r,n,m}$;
- $R(\Phi_{r,n}) := \left\{ \left(\begin{smallmatrix} g & t \\ 0 & 1 \end{smallmatrix}\right) \mid g \in \Phi_{r,n}, t \in T_{r,n} \right\}$.

The next proposition is a special case of Theorem 1.1.

Proposition 2.6. *For $r, n \in \mathbb{N}$, the correspondence*

$$x_{r+1,n,i} \longleftrightarrow \begin{pmatrix} x_{r,n,i} & t_{r,n,i} \\ 0 & 1 \end{pmatrix}, \quad i = 1, \ldots, m,$$

gives a faithful representation of $\Phi_{r+1,n}$ as a subgroup of $R(\Phi_{r,n})$.

Proposition 2.7. *Denote by $\pi_{r,n}$ the natural homomorphism from Φ_r to $\Phi_{r,n}$. Then every finite quotient $\pi\colon \Phi_r \to G$ whose order divides n factorizes through $\Phi_{r,n}$, i.e. there is $\tilde{\pi}$,*

$$\Phi_r \xrightarrow{\pi_{r,n}} \Phi_{r,n} \xrightarrow{\tilde{\pi}} G,$$

such that

$$\tilde{\pi} \circ \pi_{r,n} = \pi.$$

In particular,

$$\widehat{\Phi}_r \cong \varprojlim_{n \in \mathrm{N}} \Phi_{r,n}.$$

Proof. Let G be a quotient of Φ_r whose order divides n. Set $G = F/R$ where $R \geq F^{(r)}$ and $R \geq F^n$ - i.e. $R \geq F^{(r)} \cdot F^n$. We recall that $\Phi_{r,n} = F/K_{r,n}$. According to these notations we actually need to prove that $R \geq K_{r,n}$. We will show by induction on l that for every $l \in \mathbb{N}$ we have $R \cdot F^{(l)} \geq K_{l,n}$. Therefore, as $R \geq F^{(r)}$ we will deduce that in particular, $R \geq K_{r,n}$ as required.

For $l = 1$ we have $K_{1,n} = F'F^n$ and in this case we have

$$K_{1,n} = F^n \cdot F' \leq R \cdot F^{(1)}.$$

For $l + 1$,

$$\begin{aligned} K_{l+1,n} &= [K_{l,n}, K_{l,n}] \cdot (K_{l,n})^n \\ &\leq [R \cdot F^{(l)}, R \cdot F^{(l)}] \cdot (R \cdot F^{(l)})^n \\ &\leq (R \cdot F^{(l+1)}) \cdot R \\ &= R \cdot F^{(l+1)}. \end{aligned}$$

□

The last proposition gives us an "explicit" description of the structure of $\hat{\Phi}_r$. For example:

Corollary 2.8. *Let $z_{n,1}, \ldots, z_{n,m}$ be the standard generators of $(\mathbb{Z}_n)^m$, then*

- $\hat{\Phi}_1 \cong \varprojlim_{n\in\mathbb{N}} \langle z_{n,1}, \ldots, z_{n,m}\rangle$.
- $\hat{\Phi}_2 \cong \varprojlim_{n\in\mathbb{N}} \left\langle \begin{pmatrix} z_{n,1} & t_{1,n,1} \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} z_{n,m} & t_{1,n,m} \\ 0 & 1 \end{pmatrix} \right\rangle$.
- $\hat{\Phi}_3 \cong \varprojlim_{n\in\mathbb{N}} \left\langle \begin{pmatrix} \begin{pmatrix} z_{n,1} & t_{1,n,1} \\ 0 & 1 \end{pmatrix} & t_{2,n,1} \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} \begin{pmatrix} z_{n,m} & t_{1,n,m} \\ 0 & 1 \end{pmatrix} & t_{2,n,m} \\ 0 & 1 \end{pmatrix} \right\rangle$

By using this "explicit" description of $\hat{\Phi}_r$, we will show a few properties of $\hat{\Phi}_r$, which will help us later solve the congruence subgroup problem for $\Phi_{2,2}$, the free metabelian group on two generators.

Proposition 2.9. *Let $n \in \mathbb{N}$ and let $x_{r,n,i}$ be one of the generators of $\Phi_{r,n}$. Then, $O(x_{r,n,i})$, the order of $x_{r,n,i}$, is equals n^r.*

Proof. By induction on r. For $r = 0$, the result is obvious. For $r + 1$ by Proposition 2.6, one can write $x_{r+1,n,i} = \begin{pmatrix} x_{r,n,i} & t_{r,n,i} \\ 0 & 1 \end{pmatrix}$. By the induction hypothesis $O(x_{r,n,i}) = n^r$. For $0 < k \in \mathbb{N}$ we can uniquely write: $k = k_1 n^r + k_2$, when $k_1 \in \mathbb{N} \cup \{0\}$ and $1 \leq k_2 \leq n^r$. Thus, simple calculation shows that for $0 < k \in \mathbb{N}$

$$\begin{aligned}
&(x_{r+1,n,i})^k \\
&= \begin{pmatrix} x_{r,n,i} & t_{r,n,i} \\ 0 & 1 \end{pmatrix}^k \\
&= \begin{pmatrix} (x_{r,n,i})^{k_1 n^r + k_2} & \left(\sum_{j=0}^{k_1 n^r + k_2 - 1} (x_{r,n,i})^j \right) t_{r,n,i} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} (x_{r,n,i})^{k_2} & (1 + \ldots + (x_{r,n,i})^{k_2 - 1} + k_1(1 + \ldots + (x_{r,n,i})^{n^r - 1}))t_{r,n,i} \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Looking at the left upper coordinate, one can see that by the induction hypothesis $k_2 = n^r$ is a necessary condition for $x_{r+1,n,i}$ to satisfy $(x_{r+1,n,i})^k = e$. Therefore, if we want $x_{r+1,n,i}$ to satisfy $(x_{r+1,n,i})^k = e$, we need that the right upper coordinate will equal 0 under the condition $k_2 = n^r$, i.e. $(k_1 + 1)(1 + \ldots + (x_{r,n,i})^{n^r - 1}) = 0$. The right upper coordinate belongs to $T_{r,n}$ which is a free module over $\mathbb{Z}_n[\Phi_{r,n}]$, and therefore, it happens for the first time when $k_1 = n - 1$, i.e. $k = (n-1)n^r + n^r = n^{r+1}$. □

Proposition 2.10. *Let $\widehat{\Phi}_r$ be the free profinite solvable group on $m \geq 2$ generators of solvability length $r \geq 2$. Then, $Z(\widehat{\Phi}_r)$, the center of $\widehat{\Phi}_r$, is trivial.*

Proof. Denote by $\hat{x}_{r,1}, \ldots, \hat{x}_{r,m}$ the standard generators of $\widehat{\Phi}_r$ as a profinite group, i.e. $\hat{x}_{r,i} = \left(\left(\begin{smallmatrix} x_{r-1,n,i} & t_{r-1,n,i} \\ 0 & 1 \end{smallmatrix}\right)\right)_{n=1}^{\infty}$ for $1 \leq i \leq m$. We will show that

$$\bigcap_{i=1}^{m} Z_{\widehat{\Phi}_r}(\hat{x}_{r,i}) = \{e\}$$

(when $Z_{\widehat{\Phi}_r}(\hat{x}_{r,i})$ is the centralizer of $\hat{x}_{r,i}$ in $\widehat{\Phi}_r$), and this actually means that $Z(\widehat{\Phi}_r) = \{e\}$.

Let

$$\left(\begin{pmatrix} g_n & \sum_{i=1}^{m} a_{n,i} t_{r-1,n,i} \\ 0 & 1 \end{pmatrix}\right)_{n=1}^{\infty} \qquad (*)$$

be an element of $\bigcap_{i=1}^{m} Z_{\widehat{\Phi}_r}(\hat{x}_{r,i})$ such that for every $n' \in \mathbb{N}$,

$$\begin{pmatrix} g_{n'} & \sum_{i=1}^{m} a_{n',i} t_{r-1,n',i} \\ 0 & 1 \end{pmatrix} \in \Phi_{r,n'}.$$

We will show that $a_{n',i'} = 0$ and $g_{n'} = e$ for every $n' \in \mathbb{N}$ and $1 \leq i' \leq m$.

Assume negatively that for the element of the form $(*)$ there are $n' \in \mathbb{N}$ and $1 \leq i' \leq m$ such that $a_{n',i'} \neq 0$. Consider Φ_{r,n'^2}. Now, take any $j \neq i'$, and from the condition that

$$\left(\begin{pmatrix} g_n & \sum_{i=1}^{m} a_{n,i} t_{r-1,n,i} \\ 0 & 1 \end{pmatrix}\right)_{n=1}^{\infty} \in Z_{\widehat{\Phi}_r}(\hat{x}_{r,j}),$$

we have

$$\begin{pmatrix} g_{n'^2} & \sum_{i=1}^{m} a_{n'^2,i} t_{r-1,n'^2,i} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{r-1,n'^2,j} & t_{r-1,n'^2,j} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} x_{r-1,n'^2,j} & t_{r-1,n'^2,j} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_{n'^2} & \sum_{i=1}^{m} a_{n'^2,i} t_{r-1,n'^2,i} \\ 0 & 1 \end{pmatrix}.$$

After opening the last multiplication and comparing the coefficients of $t_{r-1,n'^2,i'}$ one can conclude that $a_{n'^2,i'} = x_{r-1,n'^2,j}a_{n'^2,i'}$. As remembered, $a_{n'^2,i'} \in \mathbb{Z}_{n'^2}[\Phi_{r-1,n'^2}]$, and thus we can write $a_{n'^2,i'} = \sum_{g\in\Phi_{r-1,n'^2}} \alpha_g g$ when $\alpha_g \in \mathbb{Z}_{n'^2}$ for every $g \in \Phi_{r-1,n'^2}$. Therefore, the equality $a_{n'^2,i'} = x_{r-1,n'^2,j}a_{n'^2,i'}$ implies that for every $g, h \in \Phi_{r-1,n'^2}$ one has $h \in \langle x_{r-1,n'^2,j}\rangle g \implies \alpha_g = \alpha_h$. Now, according to Proposition 2.9, $O(x_{r-1,n'^2,j}) = (n'^2)^{r-1}$. Therefore, by denoting the representatives of the right cosets of $\langle x_{r-1,n'^2,j}\rangle$ in the group Φ_{r-1,n'^2} in $\{g_u\}_{u\in U}$, one can write

$$a_{n'^2,i'} = \sum_{g\in\Phi_{r-1,n'^2}} \alpha_g g = \sum_{u\in U} \alpha_{g_u}\Big(\sum_{l=0}^{(n'^2)^{r-1}-1} (x_{r-1,n'^2,j})^l\Big) g_u.$$

Now the natural map $\Phi_{r-1,n'^2} \to \Phi_{r-1,n'}$, induces a natural map

$$\mathbb{Z}_{n'^2}[\Phi_{r-1,n'^2}] \longrightarrow \mathbb{Z}_{n'}[\Phi_{r-1,n'}],$$

which according to the inverse limit property, maps $a_{n'^2,i'} \mapsto a_{n',i'}$. Considering that $O(x_{r-1,n',j}) = n'^{r-1}$, and that $a_{n',i'} \in \mathbb{Z}_{n'}[\Phi_{r-1,n'}]$, we deduce that, under the above map,

$$\sum_{l=0}^{n'^{2(r-1)}-1} (x_{r-1,n'^2,j})^l \longmapsto \sum_{l=0}^{n'^{2(r-1)}-1} (x_{r-1,n',j})^l = n'^{r-1} \sum_{l=0}^{n'^{r-1}-1} (x_{r-1,n',j})^l = 0.$$

Thus, $a_{n'^2,i'} \mapsto 0$, and therefore $a_{n',i'} = 0$.

Now, after we proved that $a_{n',i'} = 0$ for every $n' \in \mathbb{N}$ and for every $1 \leq i' \leq m$, it is obvious that according to Theorem 1.1, also $g_{n'} = e$ for every $n' \in \mathbb{N}$. □

The following theorem is well known in more general cases ([6], Theorem 7.1, and [10], Theorem 9.44). But one can notice that using the explicit description we gave for $\hat{\Phi}_r$, it is easy to give another proof for our specific case.

Theorem 2.11. *For every $r \geq 0$, Φ_r is residually finite p for all primes p.*

3. Methods from the solution of the CSP for F_2

As mentioned in §0, Bux, Ershov, and Rapinchuk [3] showed that F_2 has the CSP. In this section, we quote one of their results. We will use it in the opposite way: to show that Φ, the free metabelian group on two generators, does *not* have the CSP.

Proposition 3.1 ([3], Lemma 3.1). *Let G be a finitely generated group. Then*

(1) *G has the CSP if and only if the natural map* $\hat{\imath}\colon \widehat{\mathrm{Aut}(G)} \to \mathrm{Aut}(\widehat{G})$ *is injective.*

(2) *The map* $\hat{\imath}$ *induces* $\hat{\jmath}\colon \widehat{\mathrm{Out}(G)} \to \mathrm{Out}(\widehat{G})$, *and* $\hat{\jmath}$ *is injective if and only if every subgroup of* $\mathrm{Out}(G)$ *of finite index contains a subgroup of* $\mathrm{Out}(G)$ *of the form* $\Delta[K] := \ker(\mathrm{Out}\, G \to \mathrm{Out}\,(G/K))$, *such that K is a characteristic subgroup of G of finite index.*

(3) *Assume G is also residually finite, and that* $\widehat{G}$ *is centerless. Then the map* $\hat{\imath}\colon \widehat{\mathrm{Aut}(G)} \to \mathrm{Aut}(\widehat{G})$ *is injective if and only if* $\hat{\jmath}\colon \widehat{\mathrm{Out}(G)} \to \mathrm{Out}(\widehat{G})$ *is injective.*

This proposition, together with Propositions 2.4, 2.7, 2.10, and Theorem 2.11, give:

Lemma 3.2. *Let* Φ_r *be the free solvable group on* $m > 1$ *generators of solvability length* $r > 0$. *Then,* Φ_r *has the CSP if and only if every subgroup of* $\mathrm{Out}(\Phi_r)$ *of finite index contains a subgroup of the form* $\ker(\mathrm{Out}(\Phi_r) \to \mathrm{Out}(\Phi_{r,n}))$ *for some* $n \in \mathbb{N}$.

4. The CSP for free solvable groups on two generators

As mentioned, Lemma 3.2 is valid for every $r, m > 1$, but from now on we will concentrate on the case $m = 2$. So, in this section, Φ_r denotes the free solvable group on **two** generators of solvability length r, and the generators of Φ_r will be denoted by x_r and y_r. We will denote the generators of $\mathbb{Z}^2$ by x and y.

Consider the natural map $\Phi_r \to \mathbb{Z}^2$, defined by $x_r \mapsto x$, $y_r \mapsto y$. This map induces the natural maps $\mathrm{Aut}(\Phi_r) \to \mathrm{Aut}(\mathbb{Z}^2)$ and $\mathrm{Out}(\Phi_r) \to \mathrm{Out}(\mathbb{Z}^2)$. It is easy to check that the map $\mathrm{Aut}(\Phi_r) \to \mathrm{Aut}(\mathbb{Z}^2)$ is surjective, as $\mathrm{Aut}(\mathbb{Z}^2)$ is generated by the automorphisms

$$\alpha\colon \begin{cases} x \longmapsto xy, \\ y \longmapsto y, \end{cases} \qquad \beta\colon \begin{cases} x \longmapsto x^{-1}, \\ y \longmapsto y, \end{cases} \qquad \gamma\colon \begin{cases} x \longmapsto y, \\ y \longmapsto x, \end{cases}$$

which are the images of the automorphisms

$$\alpha_r\colon \begin{cases} x_r \longmapsto x_r y_r, \\ y_r \longmapsto y_r, \end{cases} \qquad \beta_r\colon \begin{cases} x_r \longmapsto x_r^{-1}, \\ y_r \longmapsto y_r, \end{cases} \qquad \gamma_r\colon \begin{cases} x_r \longmapsto y_r, \\ y_r \longmapsto x_r. \end{cases}$$

Therefore, the map $\mathrm{Out}(\Phi_r) \to \mathrm{Out}(\mathbb{Z}^2) = \mathrm{Aut}(\mathbb{Z}^2)$ is also surjective. We want to show that this map is also injective, and to conclude that $\mathrm{Out}(\Phi_r) \cong \mathrm{Out}(\mathbb{Z}^2) = \mathrm{Aut}(\mathbb{Z}^2) \cong \mathrm{GL}_2(\mathbb{Z})$.

Definition 4.1. Let G be a group and $\mathrm{IA}(G) := \ker(\mathrm{Aut}(G) \to \mathrm{Aut}(G/G'))$. An element of $\mathrm{IA}(G)$ is called an IA-*automorphism.*

The sequence $1 \to \mathrm{IA}(\Phi_r) \to \mathrm{Aut}(\Phi_r) \to \mathrm{Aut}(\mathbb{Z}^2) \to 1$ is an exact sequence, and so is the sequence $1 \to \mathrm{Inn}(\Phi_r) \to \mathrm{Aut}(\Phi_r) \to \mathrm{Out}(\Phi_r) \to 1$. Clearly, $\mathrm{Inn}(G) \subseteq \mathrm{IA}(G)$, and in particular, $\mathrm{Inn}(\Phi_r) \subseteq \mathrm{IA}(\Phi_r)$. Bachmuth, Formanek, and Mochizuki, gave few conditions for equality

Theorem 4.2 ([4]). *Let F be the free group on two generators, and let R be a normal subgroup of F satisfying*

- $R \leqslant F'$;
- $\mathbb{Z}(F/R)$*, the integral group ring of F/R, is a domain.*

Then, $\mathrm{IA}(F/R') = \mathrm{Inn}(F/R')$.

Let us now mention:

Theorem 4.3 (Kropholler, Linnell, and Moody [7]). *Let k be a division ring and let G be a solvable by finite group. If G is torsion-free, then $k[G]$ is a domain.*

Corollary 4.4. *For every $r \in \mathbb{N}$,*

(1) $\mathrm{IA}(\Phi_r) = \mathrm{Inn}(\Phi_r)$;

(2) *the map* $\mathrm{Out}(\Phi_r) \to \mathrm{GL}_2(\mathbb{Z})$ *is an isomorphism.*

Conjecture 4.5. [1] *With the above isomorphism, for $r, n \in \mathbb{N}$,*

$$\ker(\mathrm{Out}(\Phi_r) \to \mathrm{Out}(\Phi_{r,n})) = \ker(\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}_{n^r})).$$

As $\mathbb{Z}^2$ does not have the CSP, there is a subgroup of $\mathrm{Aut}(\mathbb{Z}^2) \cong \mathrm{GL}_2(\mathbb{Z})$ of finite index containing no subgroup of the form $\ker(\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}_n))$. Therefore, if the conjecture is true, the same subgroup of $\mathrm{GL}_2(\mathbb{Z})$ does not contain any subgroup of the form $\ker(\mathrm{Out}(\Phi_r) \to \mathrm{Out}(\Phi_{r,n}))$, and thus, by Lemma 3.2, Φ_r does not have the CSP.

We consider now the following notations.

Definition 4.6. Define

- $\mathrm{GL}_2'(\mathbb{Z}_n) = \mathrm{Im}(\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}_n))$;
- $\mathrm{Aut}'(\Phi_{r,n}) = \mathrm{Im}(\mathrm{Aut}(\Phi_r) \to \mathrm{Aut}(\Phi_{r,n}))$;
- $\mathrm{Out}'(\Phi_{r,n}) = \mathrm{Im}(\mathrm{Out}(\Phi_r) \to \mathrm{Out}(\Phi_{r,n}))$;
- $\mathrm{IA}'(\Phi_{r,n}) = \mathrm{IA}(\Phi_{r,n}) \cap \mathrm{Aut}'(\Phi_{r,n})$.

[1] *Note added in galley proofs.* This conjecture is not true for $r \geq 3$. See D. E.-C. Ben-Ezra and A. Lubotzky, The congruence subgroup problem for low rank free and free metabelian groups, in preparation.

With the above notations we have:

Lemma 4.7. *Conjecture 4.5 is true if and only if* $\mathrm{IA}'(\Phi_{r,n}) = \mathrm{Inn}(\Phi_{r,n})$.

Proof. As was shown in Proposition 2.3, $\Phi_{r,n}^{\mathrm{ab}} = \Phi_{r,n}/[\Phi_{r,n}, \Phi_{r,n}] \backsimeq \mathbb{Z}_{n^r}$. Thus, the natural surjective map $\mathrm{Aut}(\Phi_r) \to \mathrm{GL}_2(\mathbb{Z})$ induces a natural surjective map $\mathrm{Aut}'(\Phi_{r,n}) \to \mathrm{GL}_2'(\mathbb{Z}_{n^r})$ whose kernel is exactly $\mathrm{IA}'(\Phi_{r,n})$. This map induces another natural surjective map $\mathrm{Out}'(\Phi_{r,n}) \to \mathrm{GL}_2'(\mathbb{Z}_{n^r})$, and this map is an isomorphism if and only if $\mathrm{IA}'(\Phi_{r,n}) = \mathrm{Inn}(\Phi_{r,n})$. On the other hand, the latter is an isomorphism if and only if Conjecture 4.5 is true. So the lemma follows. □

Theorem 4.8. *For* $r = 2$ *we have* $\mathrm{IA}'(\Phi_{2,n}) = \mathrm{Inn}(\Phi_{2,n})$ *for every* $n \in \mathbb{N}$.

Thus, once we prove Theorem 4.8, Theorem 0.1 is also proven.

5. Proof of Theorem 4.8

In this section $\Phi = \Phi_0 = \Phi_{2,2}$ denotes the free metabelian group on two generators. We also write Φ_n instead of $\Phi_{2,2,n}$. Let $n \in \mathbb{N} \cup \{0\}$ and let x_n, y_n be the generators of $(\mathbb{Z}_n)^2$. By Theorem 1.1 we can consider Φ_n as the group generated by $x_n^* = \begin{pmatrix} x_n & t_{x_n} \\ 0 & 1 \end{pmatrix}$ and $y_n^* = \begin{pmatrix} y_n & t_{y_n} \\ 0 & 1 \end{pmatrix}$, and the elements of Φ_n are the matrices of the form $\begin{pmatrix} x_n^i y_n^j & a_1(x_n,y_n)t_{x_n} + a_2(x_n,y_n)t_{y_n} \\ 0 & 1 \end{pmatrix}$, such that

(1) $i, j \in \mathbb{Z}$

(2) $a_1(x_n, y_n), a_2(x_n, y_n) \in \mathbb{Z}_n[\mathbb{Z}_n^2] = \mathbb{Z}_n[x_n^{\pm 1}, y_n^{\pm 1}]$

(3) $1 - x_n^i y_n^j = a_1(x_n, y_n) \cdot (1 - x_n) + a_2(x_n, y_n) \cdot (1 - y_n)$

Every automorphism of Φ_n is determined by its action on the two generators $\begin{pmatrix} x_n & t_{x_n} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} y_n & t_{y_n} \\ 0 & 1 \end{pmatrix}$. Therefore, when we want to describe an automorphism, σ, of Φ_n we can write

$$\sigma = \begin{cases} \begin{pmatrix} x_n & t_{x_n} \\ 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} x_n^i y_n^j & a_1 t_{x_n} + a_2 t_{y_n} \\ 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} y_n & t_{y_n} \\ 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} x_n^k y_n^l & b_1 t_{x_n} + b_2 t_{y_n} \\ 0 & 1 \end{pmatrix}. \end{cases}$$

Definition 5.1. Let $n \in \mathbb{N} \cup \{0\}$. For $\sigma \in \mathrm{Aut}(\Phi_n)$ define $d(\sigma) := \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$. Notice that $d(\sigma)$ is well defined, as $a_1, a_2, b_1, b_2 \in \mathbb{Z}_n[\mathbb{Z}_n^2]$ and $\mathbb{Z}_n[\mathbb{Z}_n^2]$ is a commutative ring.

Lemma 5.2. *For* $\sigma \in \mathrm{Aut}(\Phi_n)$*,* $d(\sigma)$ *is invertible in* $\mathbb{Z}_n[\mathbb{Z}_n^2]$.

Proof. As σ is an automorphism, we can write $\begin{pmatrix} x_n & t_{x_n} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} y_n & t_{y_n} \\ 0 & 1 \end{pmatrix}$ as words with the matrices $\begin{pmatrix} x_n^i y_n^j & a_1 t_{x_n} + a_2 t_{y_n} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} x_n^k y_n^l & b_1 t_{x_n} + b_2 t_{y_n} \\ 0 & 1 \end{pmatrix}$. Therefore, it is easy to check, by induction on the length of these words, that there are polynomials $c_1, c_2, d_1, d_2 \in \mathbb{Z}_n[\mathbb{Z}_n^2]$ such that

$$\begin{pmatrix} x_n & t_{x_n} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_n & (c_1 a_1 + d_1 b_1) t_{x_n} + (c_1 a_2 + d_1 b_2) t_{y_n} \\ 0 & 1 \end{pmatrix}$$
$$\Longrightarrow \; c_1 a_1 + d_1 b_1 = 1, c_1 a_2 + d_1 b_2 = 0,$$

$$\begin{pmatrix} y_n & t_{y_n} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y_n & (c_2 a_1 + d_2 b_1) t_{x_n} + (c_2 a_2 + d_2 b_2) t_{y_n} \\ 0 & 1 \end{pmatrix}$$
$$\Longrightarrow \; c_2 a_1 + d_2 b_1 = 0, c_2 a_2 + d_2 b_2 = 1.$$

i.e., we got that $\begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = I$, and therefore, $\det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ is invertible in $\mathbb{Z}_n[\mathbb{Z}_n^2]$. □

From the last lemma it follows that if $\sigma \in \mathrm{Aut}(\Phi)$ then $d(\sigma) = \pm x^r y^s$ for some $r, s \in \mathbb{Z}$, as these are the invertible elements of $\mathbb{Z}[\mathbb{Z}^2]$, see [5], Chapter 8.

Lemma 5.3. *Let* $n \in \mathbb{N} \cup \{0\}$*. For the commutator subgroup of* Φ_n *we have*

$$[\Phi_n, \Phi_n] = \left\{ \begin{pmatrix} 1 & (1 - y_n) \cdot p \cdot t_{x_n} - (1 - x_n) \cdot p \cdot t_{y_n} \\ 0 & 1 \end{pmatrix} \,\middle|\, p \in \mathbb{Z}_n[\mathbb{Z}_n^2] \right\}. \quad (1)$$

Proof. Let us write $g^* = \begin{pmatrix} g & a_1 t_{x_n} + a_2 t_{y_n} \\ 0 & 1 \end{pmatrix}$ and $h^* = \begin{pmatrix} h & b_1 t_{x_n} + b_2 t_{y_n} \\ 0 & 1 \end{pmatrix}$ for two arbitrary elements in Φ_n. Then, by direct computation and by using the identities

$$1 - g = a_1(1 - x_n) + a_2(1 - y_n),$$
$$1 - h = b_1(1 - x_n) + b_2(1 - y_n),$$

we deduce

$$[g^*, h^*] = g^* h^* g^{*-1} h^{*-1}$$
$$= \begin{pmatrix} 1 & (1 - y_n)(a_1 b_2 - b_1 a_2) t_{x_n} - (1 - x_n)(a_1 b_2 - b_1 a_2) t_{y_n} \\ 0 & 1 \end{pmatrix}.$$

This shows that multiplications of commutators of Φ_n are of the form (1).

For the other direction, one should notice that by the notations $x_n^* = \begin{pmatrix} x_n & t_{x_n} \\ 0 & 1 \end{pmatrix}$ and $y_n^* = \begin{pmatrix} y_n & t_{y_n} \\ 0 & 1 \end{pmatrix}$, we have

$$[x_n^*, y_n^*] = \begin{pmatrix} 1 & (1 - y_n) t_{x_n} - (1 - x_n) t_{y_n} \\ 0 & 1 \end{pmatrix}.$$

Moreover, conjugation by x_n^* or y_n^* multiply the right upper coordinate by x_n or y_n respectively. That shows that we can reach every $p \in \mathbb{Z}_n[\mathbb{Z}_n^2]$. □

We can now prove Theorem 4.8:

Proof. It is obvious that $\mathrm{IA}'(\Phi_n) \supseteq \mathrm{Inn}(\Phi_n)$. On the other hand, consider an element $\sigma \in \mathrm{IA}'(\Phi_n)$. As $\sigma \in \mathrm{IA}(\Phi_n)$, by Lemma 5.3, we can describe σ as follows:

$$\sigma = \begin{cases} \begin{pmatrix} x_n & t_{x_n} \\ 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} x_n & t_{x_n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (1-y_n)(x_n^{-1}p)t_{x_n} - (1-x_n)(x_n^{-1}p)t_{y_n} \\ 0 & 1 \end{pmatrix} \\ \qquad = \begin{pmatrix} x_n & [1+(1-y_n)p]t_{x_n} + -(1-x_n)pt_{y_n} \\ 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} y_n & t_{y_n} \\ 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} y_n & t_{y_n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (1-y_n)(y_n^{-1}q)t_{x_n} - (1-x_n)(y_n^{-1}q)t_{y_n} \\ 0 & 1 \end{pmatrix} \\ \qquad = \begin{pmatrix} y_n & (1-y_n)qt_{x_n} + [1-(1-x_n)q]t_{y_n} \\ 0 & 1 \end{pmatrix}. \end{cases}$$

for some $p, q \in \mathbb{Z}_n[\mathbb{Z}_n^2]$.

Now, one can compute the determinant of σ and deduce that

$$\begin{aligned} d(\sigma) &= [1+(1-y_n)p] \cdot [1-(1-x_n)q] + (1-x_n)p \cdot (1-y_n)q \\ &= 1 + (1-y_n)p - (1-x_n)q. \end{aligned}$$

But, by considering that $\sigma \in \mathrm{Aut}'\,\Phi_n$ and by applying the conclusion of Lemma 5.2, we conclude that there are $i, j \in \mathbb{Z}_n$ such that

$$1 + (1-y_n)p - (1-x_n)q = \pm x_n^i y_n^j.$$

Moreover, one should notice that if $x_n \mapsto 1,\ y_n \mapsto 1$ then

$$1 + (1-y_n)p - (1-x_n)q \mapsto 1$$

and therefore

$$1 + (1-y_n)p - (1-x_n)q = x_n^i y_n^j.$$

Thus,

$$1 - x_n^i y_n^j = (1-x_n)q - (1-y_n)p. \tag{2}$$

Now, according to Theorem 1.1, we conclude that the element

$$\begin{pmatrix} x_n^i y_n^j & qt_{x_n} - pt_{y_n} \\ 0 & 1 \end{pmatrix}$$

is in the image of Φ_n in $R_n(\mathbb{Z}_n)$. Direct computation and use of equation (2) shows that

$$\sigma = \mathrm{Inn}\begin{pmatrix} x_n^i y_n^j & qt_{x_n} - pt_{y_n} \\ 0 & 1 \end{pmatrix}$$

so $\sigma \in \mathrm{Inn}(\Phi_n)$ as required. □

That completes the main result of the paper. Now, one can ask whether $\mathrm{IA}(\Phi_n) = \mathrm{Inn}(\Phi_n)$. The next explicit example shows that this is not true in general.

Let us consider the map

$$\sigma = \begin{cases} x_n^* \longrightarrow y_n^* x_n^* y_n^{*-1} = x_n^*[x_n^{*-1}, y_n^*] = \begin{pmatrix} x_n & y_n t_{x_n} + (1-x_n)t_{y_n} \\ 0 & 1 \end{pmatrix}, \\ y_n^* \longrightarrow x_n^* y_n^* x_n^{*-1} = y_n^*[y_n^{*-1}, x_n^*] = \begin{pmatrix} y_n & (1-y_n)t_{x_n} + x_n t_{y_n} \\ 0 & 1 \end{pmatrix}. \end{cases}$$

One should notice that σ defines a homomorphism of Φ_n, as the relations that define Φ_n are satisfied by all the elements of Φ_n, and in particular, by $\sigma(x_n^*)$ and $\sigma(y_n^*)$. We are now going to show that when $n = p$ is prime, σ defines an IA-automorphism of Φ_n which is not inner.

Lemma 5.4. *When p is prime, the polynomial $x_p + y_p - 1$ is invertible in $\mathbb{Z}_p[\mathbb{Z}_p^2]$.*

Proof. When p is prime, we have

$$\begin{aligned} &(x_p + y_p - 1) \cdot \sum_{i=0}^{p-1} (x_p + y_p)^i \\ &\quad = (x_p + y_p)^p - 1 \\ &\quad = x_p^p + \binom{p}{1} x_p^{p-1} y_p + \cdots + \binom{p}{p-1} x_p y_p^{p-1} + y_p^p - 1 \\ &\quad = 1 + 0 + \cdots + 0 + 1 - 1 \\ &= 1. \end{aligned}$$

□

Proposition 5.5. *When p is prime, σ is onto. In particular, σ is an isomorphism.*

Proof. A direct computation shows that

$$[\sigma(x_p^*), \sigma(y_p^*)] = \begin{pmatrix} 1 & (1-y_p)(x_p + y_p - 1)t_{x_p} - (1-x_p)(x_p + y_p - 1)t_{y_p} \\ 0 & 1 \end{pmatrix}.$$

Moreover, direct computation shows that conjugation by $\sigma(x_p^*)$ and $\sigma(y_p^*)$ multiply the right upper coordinate by x_p and y_p respectively. By the previous lemma $x_p + y_p - 1$ is invertible in $\mathbb{Z}_p[\mathbb{Z}_p^2]$. Now, putting all together and by considering Lemma 5.3 we conclude that σ reaches all $[\Phi_p, \Phi_p]$.

On the other hand, the projection of σ to $\Phi_p^{\mathrm{ab}} = \Phi_p/[\Phi_p, \Phi_p]$ is the identity map, so σ reaches also all Φ_p^{ab}. Therefore, σ reaches all Φ_p, as required.

As Φ_p is a finite group we deduce that σ is an isomorphism. □

Proposition 5.6. *When $n = p$ is prime, the automorphism σ is an* IA*-automorphism which is not inner.*

Proof. In the proof of the previous proposition we saw that σ is an IA-automorphism. On the other hand, assume that σ is inner. By direct computation we deduce that the determinant of σ equals $d(\sigma) = x_p + y_p - 1$. On the other hand, as $\mathrm{Inn}(\Phi_p) \subseteq \mathrm{Aut}'(\Phi_p)$, we conclude that $d(\sigma) = \pm x_p^r y_p^s$, and this is a contradiction. □

References

[1] M. Asada, The faithfulness of the monodromy representations associated with certain families of algebraic curves. *J. Pure Appl. Algebra* **159** (2001), no. 2-3, 123–147. Zbl 1045.14013 MR 1828935

[2] J. S. Birman, *Braids, links, and mapping class groups.* Annals of Mathematics Studies, 82. Based on lecture notes by J. Cannon. Princeton University Press, Princeton, N.J., and University of Tokyo Press, Tokyo, 1975. Zbl 0305.57013 MR 0375281

[3] K.-U. Bux, M. V. Ershov, and A. S. Rapinchuk, The congruence subgroup property for Aut F_2: a group-theoretic proof of Asada's theorem. *Groups Geom. Dyn.* **5** (2011), no. 2, 327–353. Zbl 1251.20035 MR 2782176

[4] S. Bachmuth, E. Formanek, and H. Y. Mochizuki, IA--automorphism of certain two-generator torsion-free groups. *J. Algebra* **40** (1976), no. 1, 19–30. Zbl 0328.20034 MR 0409668

[5] R. Crowell and R. H. Fox, *Introduction to knot theory.* Ginn and Co., Boston, MA, etc., 1963. Zbl 0126.39105 MR 0146828

[6] K. W. Gruenberg, Residual properties of infinite soluble groups. *Proc. London Math. Soc.* (3) **7** (1957), 29–62. Zbl 0077.02901 MR 0087652

[7] P. H. Kropholler, P. A. Linnell, and J. A. Moody, Applications of a new K-theoretic theorem to soluble group rings. *Proc. Amer. Math. Soc.* **104** (1988), no. 3, 675–684. Zbl 0691.16013 MR 0964842

[8] W. Magnus, On a theorem of Marshall Hall. *Ann. of Math.* (2) **40**, (1939). 764–768. Zbl 0022.31403 JFM 65.0068.01 MR 0000262

[9] M. S. Raghunathan, The congruence subgroup problem. *Proc. Indian Acad. Sci. Math. Sci.* **114** (2004), no. 4, 299–308. Zbl 1086.20024 MR 2067695

[10] D. Robinson, *Finiteness conditions and generalized soluble groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 62 (Part I) and 63 (Part II). Springer-Verlag, Berlin etc., 1972. Zbl 0243.20032 (Part I) Zbl 0243.20033 (Part II) MR 0332989 (Part I) MR 0332990 (Part II)

[11] N. S. Romanovskiĭ, Smelʹkin embeddings for abstract and profinite groups. *Algebra Log.* **38** (1999), no. 5, 598–612, 639–640, in Russian. English translation, *Algebra and Logic* **38** (1999), no. 5, 326–334. MR 1766704 Zbl 0943.20020

[12] V. N. Remeslennikov and V. G. Sokolov, Some properties of a Magnus embedding. *Algebra i Logika* **9** (1970), 566–578, in Russian. English translation, *Algebra and Logic* **9** (1970), 342–349. Zbl 0247.20026 MR 0292920

[13] A. L. Šhmel′kin, Wreath products and varieties of groups. *Izv. Akad. Nauk SSSR Ser. Mat.* **29** (1965), 149–170, in Russian. Zbl 0135.04701 MR 0193131

[14] A. L. Šhmel′kin, A remark on my paper "Wreath products and varieties of groups." *Izv. Akad. Nauk SSSR Ser. Mat.* **31** (1967), 443–444, in Russian. Zbl 0165.03701 MR 0210769

[15] B. Sury, *The congruence subgroup problem.* Texts and Readings in Mathematics, 24. Hindustan Book Agency, New Delhi, 2003. Zbl 1046.20035 MR 1978430

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