Groups Geom. Dyn. 10 (2016), 601–618
DOI 10.4171/GGD/358

Groups, Geometry, and Dynamics

On the growth of a Coxeter group

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Abstract. For a Coxeter system (W, S) let $a_n^{(W,S)}$ be the cardinality of the sphere of radius n in the Cayley graph of W with respect to the standard generating set S. It is shown that, if $(W, S) \preceq (W', S')$ then $a_n^{(W,S)} \leq a_n^{(W',S')}$ for all $n \in \mathbb{N}_0$, where $\preceq$ is a suitable partial order on Coxeter systems (cf. Theorem A).

It is proven that there exists a constant $\tau = 1.13\ldots$ such that for any non-affine, non-spherical Coxeter system (W, S) the growth rate $\omega(W, S) = \limsup \sqrt[n]{a_n}$ satisfies $\omega(W, S) \geq \tau$ (cf. Theorem B). The constant τ is a Perron number of degree 127 over $\mathbb{Q}$.

For a Coxeter group W the Coxeter generating set is not unique (up to W-conjugacy), but there is a standard procedure, the *diagram twisting* (cf. [3]), which allows one to pass from one Coxeter generating set S to another Coxeter generating set $\mu(S)$. A generalisation of the diagram twisting is introduced, the *mutation*, and it is proven that Poincaré series are invariant under mutations (cf. Theorem C).

Mathematics Subject Classification (2010). Primary: 20F55; Secondary: 20F32, 05C25.

Keywords. Coxeter groups, growth of groups.

Introduction

The growth of finitely generated groups has been the subject of intensive investigations (cf. [11, 12], [13], [9]) and led to ground-breaking results, e.g., M. Gromov showed that a finitely generated group has polynomial growth if, and only if, it is virtually nilpotent (cf. [14]).

For a group G being generated by a finite symmetric set $X \subseteq G$ not containing the identity $1 \in G$, the growth rate[1] is defined by $\omega(G, X) = \limsup_n \sqrt[n]{a_n}$, where a_n is the number of elements in G which can be written as a product of n elements in X but which cannot be written as a product of less than n elements in X. If G is of subexponential growth, i.e., polynomial or intermediate growth, then $\omega(G, X) \leq 1$.

[1] The growth rate is often called *exponential growth rate*.

The set of isomorphism classes of Coxeter systems admits a partial order $\preceq$, and the corresponding monotonicity result for growth sequences is proven.

Theorem (A). *Let* (W,S) *and* (W',S') *be Coxeter systems. If* $(W,S)\preceq(W',S')$ *then* $a_n^{(W,S)} \leq a_n^{(W',S')}$ *for all* $n \in \mathbb{N}_0$.

Spherical and affine Coxeter systems have, respectively, growth rate zero and one. One of the main results of this paper can be stated as follows.

Theorem (B). *Let* (W,S) *be a non-affine, non-spherical Coxeter system. Then its growth rate satisfies* $\omega(W,S) \geq \tau$, *where* $\tau = 1.13\ldots$ *is an algebraic integer of degree* 127 *over* $\mathbb{Q}$, *which is also a Perron number with minimal polynomial* $m_\tau(t)$ *given in* §4. *Moreover,* $\tau = \omega(W,S)$, *where* (W,S) *is the hyperbolic Coxeter system* E_{10}.

A remarkable coincidence occurs (cf. Rem. 4.1). Besides having the smallest minimal growth rate among Coxeter systems, E_{10} is also known to minimise a certain function λ_ρ which reflects, in the hyperbolic case, the metric properties of the orbifold defined by Tits' representation ρ (cf. [22]).

For a group G with a finite symmetric generating set $X \subseteq G \setminus \{1\}$ one defines the growth series by $p_{(G,X)}(t) = \sum_{n\in\mathbb{N}_0} a_n t^n \in \mathbb{Z}[\![t]\!]$, thus $\omega(G,X)$ coincides with the inverse of the radius of convergence of $p_{(G,X)}(t)$, considered as a power series over $\mathbb{C}$. For a Coxeter system (W,S) the growth series is also called the *Poincaré series* of (W,S).

In §5 we define the new notion of a *mutation* $\mu(M,X,Y,\sigma)$ of a Coxeter matrix M, which induces an equivalence relation $\sim$ on Coxeter systems. Mutations generalise diagram twisting (cf. [3]) but in general they do not preserve the isomorphism class of the group. Nevertheless, the Poincaré series is invariant under mutations of the Coxeter matrix.

Theorem (C). *Let* (W,S) *and* (W',S') *be Coxeter systems satisfying* $(W,S) \sim (W',S')$. *Then* $p_{(W,S)}(t) = p_{(W',S')}(t)$.

Thus, mutations provide a tool to produce finitely many non-isomorphic Coxeter groups with the same growth series. It is an open problem whether there exist infinitely many groups with the same growth series (cf. [21, Chapter 1, Problems 1 and 2]).

Acknowledgements. I wish to thank J. Parkinson, P. Spiga and Th. Weigel for helpful discussions, and M. Bucher, T. Smirnova-Nagnibeda and A. Talambutsa for the simulating conversations during the conference "Geometric and analytic group theory" in Ventotene, Italy.
I would also like to thank R. Howlett for providing a copy of [17], and the anonymous referees for helpful comments.

1. Growth of finitely generated groups

Let G be a finitely generated group, and let $X = X^{-1} \subseteq G \setminus \{1\}$ be a finite, symmetric set of generators. The *length* of $g \in G$ with respect to X is the minimal n such that $g = x_1 x_2 \dots x_n$ with $x_i \in X$; the *length function* will be denoted by $\ell_{(G,X)} \colon G \to \mathbb{N}_0$. It has a natural interpretation in terms of the metric on the Cayley graph $\operatorname{Cay}(G, X)$.

For $n \in \mathbb{N}_0$, the ball in $\operatorname{Cay}(G, X)$ centred around 1_G with radius n will be denoted by

$$B_n^{(G,X)} = \{g \in G \mid \ell_{(G,X)}(g) \leq n\},$$

and the corresponding sphere by

$$A_n^{(G,X)} = \{g \in G \mid \ell_{(G,X)}(g) = n\}.$$

Their sizes are $a_n^{(G,X)} = |A_n^{(G,X)}|$ and $b_n^{(G,X)} = |B_n^{(G,X)}|$.

The central objects under investigation are the growth series

$$p_{(G,X)}(t) = \sum_{n \in \mathbb{N}_0} a_n^{(G,X)} t^n \in \mathbb{Z}[\![t]\!],$$

and the growth rate

$$\omega(G, X) = \limsup_{n \to \infty} \sqrt[n]{a_n^{(G,X)}}.$$

Note that G has *exponential growth* if $\omega(G, X) > 1$ for some (and hence any) generating set X. The present paper only deals with finitely generated linear groups G. Therefore, G has *polynomial growth* with respect to some (and hence any) generating system X if $\omega(G, X) \leq 1$ (cf. [28, Corollary 5]).

The *minimal growth rate* $\omega(G)$ is the infimum of $\omega(G, X)$, as X runs over all finite, symmetric generating sets of G.

2. Coxeter groups

Standard references for Coxeter groups include [4, 18].

2.1. Coxeter systems. Let S be a finite set, and let M be an $(S \times S)$-matrix such that $m_{s,s} = 1$, and $m_{s,r} = m_{r,s} \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$ for all $s, r \in S$, $s \neq r$. Then M is a *Coxeter matrix* over S.

The *Coxeter system* associated with a Coxeter matrix M over S is the pair (W, S) where W is the group

$$W = W(M) = \langle S \mid (sr)^{m_{s,r}} \text{ if } m_{s,r} < \infty \rangle. \tag{2.1}$$

The Coxeter matrix M (or, equivalently, the presentation (2.1)) is often encoded in the Coxeter graph $\Gamma(M)$ (cf. [4, Chapter IV, n° 1.9]). Either datum is called the *type* of (W, S).

If $I \subseteq S$ let $W_I = \langle I \rangle \leq W$. The *parabolic subsystem* (W_I, I) is a Coxeter system in its own right, with Coxeter matrix $M_I = (m_{s,r})_{s,r \in I}$. Its Coxeter graph is the graph induced from Γ by the vertices in I, and

$$\ell_{(W_I,I)} = \ell_{(W,S)}|_{W_I}. \tag{2.2}$$

The finite set $\mathcal{F} = \mathcal{F}(W, S) = \{I \subseteq S \mid |W_I| < \infty\}$ is called the set of *spherical residues*.

A *Coxeter-isomorphism* $\varphi\colon (W, S) \to (W', S')$ of Coxeter systems of types M and M' respectively, is a bijection $\varphi\colon S \to S'$ such that $m'_{\varphi(s),\varphi(r)} = m_{s,r}$ for all $s, r \in S$.

Any Coxeter group (W, S) is linear via the Tits' reflection representation $\rho\colon W \to \mathrm{GL}(\mathbb{R}^S)$ (cf. [4, Chapter V, §4]). The representation ρ is determined by the symmetric matrix[2] $B = B_M = \big(-\cos\frac{\pi}{m_{s,r}}\big)_{s,r \in S}$, and the signature of B induces the following tetrachotomy on irreducible Coxeter systems.

(i) If B is positive definite, then (W, S) is *spherical*,

(ii) if B is positive semidefinite with 0 a simple eigenvalue, then (W, S) is *affine*,

(iii) if B has $|S|-1$ positive and 1 negative eigenvalue, then (W, S) is *hyperbolic*,[3] or

(iv) none of the above conditions applies.

The irreducible Coxeter system (W, S) is spherical if, and only if, W is a finite group. The classification of spherical and affine systems is classical (cf. [4, Chapter VI]). For a characterisation of hyperbolic Coxeter systems see §3.3.

2.2. The word problem. If S is a finite set, let S^* be the free monoid[4] over S, equipped with the natural $\mathbb{N}_0$-grading $\deg\colon S^* \mapsto \mathbb{N}_0$, $\deg(s) = 1$ for all $s \in S$, and the ShortLex total order with respect to some total order on S (cf. [10, §2.5]). For $s, t \in S$ and $m \in \mathbb{N}_0$ let $[s, t, m] \in S^*$ be the word

$$[s, t, m] = \begin{cases} (st)^{m/2} & \text{if } 2 \mid m, \\ (st)^{\frac{m-1}{2}} s & \text{if } 2 \nmid m. \end{cases}$$

[2] For short, we put $\frac{\pi}{\infty} = 0$.

[3] There are several non-compatible notions of hyperbolicity, cf. [8, Note 6.9]. In the present work "hyperbolic" coincides with Bourbaki's notion (cf. [4, Chapter V, §4, Ex.13]).

[4] Words in S^* are denoted in boldface: $\boldsymbol{w} = s_1 s_2 \ldots s_n \in S^*$. A subword $\boldsymbol{w}'$ of $\boldsymbol{w}$ is either the empty word $\mathbf{1}$ or a word of the form $\boldsymbol{w}' = s_i s_{i+1} \ldots s_k$ for $1 \leq i \leq k \leq n$.

Let M be a Coxeter matrix over S. The *M-operations* (or *M-moves*) on S^* are modifications of words of the following types:

$$M^{(1)}: \boldsymbol{v}(ss)\boldsymbol{u} \longmapsto \boldsymbol{v}\boldsymbol{u}, \tag{2.3a}$$

$$M^{(2)}: \boldsymbol{v}[s, r, m_{s,r}]\boldsymbol{u} \longmapsto \boldsymbol{v}[r, s, m_{s,r}]\boldsymbol{u}, \quad \text{if } m_{s,r} < \infty. \tag{2.3b}$$

Let (W, S) be the Coxeter system of type M, and let $\pi_M: S^* \to W(M)$ be the canonical projection (of monoids). Then, for all $\boldsymbol{w} \in S^*$,

$$\deg(\boldsymbol{w}) \geq \ell_M(\pi_M(\boldsymbol{w})). \tag{2.4}$$

A word $\boldsymbol{w} \in S^*$ is called *reduced* for (W, S) if equality holds in (2.4). If $w \in W(M)$, there is a unique ShortLex-minimal element $\sigma_M(w) \in S^*$ such that $\pi_M \sigma_M(w) = w$. Thus, $\sigma_M: W(M) \to S^*$ is a section of π_M, with the additional property[5] that

$$\deg(\sigma_M(w)) = \ell_M(w). \tag{2.5}$$

A word $\boldsymbol{w} \in S^*$ is called *M-reduced* if its degree cannot be decreased by applying any finite sequence of M-operations. If two words $\boldsymbol{w}, \boldsymbol{w}'$ are connected by a sequence of M-moves, then they represent the same element in $W(M)$:

$$\pi_M(\boldsymbol{w}) = \pi_M(\boldsymbol{w}'), \tag{2.6}$$

and hence reduced words are M-reduced. Moreover, Tits solved the word problem as follows.

Theorem 2.3 ([27], [4, Chapter IV, §1, Example 13]). *Let (W, S) be the Coxeter system with Coxeter matrix M.*

(i) *A word in S^* is reduced for (W, S) if, and only if, it is M-reduced.*

(ii) *If $\boldsymbol{w}, \boldsymbol{w}' \in S^*$ are reduced words which represent the same element $\pi_M(\boldsymbol{w}) = \pi_M(\boldsymbol{w}') \in W$, then there is a sequence of M-operations taking $\boldsymbol{w}$ to $\boldsymbol{w}'$, and this sequence entirely consists of $M^{(2)}$-operations.*

Following [2, §3.3 and §3.4], let

$$\mathcal{R}_M(w) = \{\boldsymbol{w} \in S^* \mid \pi_M(\boldsymbol{w}) = w \text{ and } \deg(\boldsymbol{w}) = \ell_M(w)\}$$

be the set of the reduced words in S^* representing $w \in W(M)$.

Corollary 2.4. *For $\boldsymbol{w}, \boldsymbol{w}' \in S^*$, and $w \in W(M)$ the following hold.*

(i) $\sigma_M(w) \in \mathcal{R}_M(w)$.

(ii) *If $\boldsymbol{w} \in \mathcal{R}_M(w)$, and there exists a sequence $\boldsymbol{w} \overset{M}{\longmapsto} \boldsymbol{w}'$ of M-moves taking $\boldsymbol{w}$ to $\boldsymbol{w}'$, then $\boldsymbol{w}' \in \mathcal{R}_M(w)$.*

(iii) *If $\pi_M(\boldsymbol{w}) = \pi_M(\boldsymbol{w}')$ and $\boldsymbol{w}'$ is reduced, then there exists a sequence of M-moves*

$$\boldsymbol{w} \longmapsto \boldsymbol{w}'.$$

[5] Actually, any section of π_M with property (2.5) would suffice for the purposes of this paper.

2.5. Poincaré series. Coxeter systems are pairs consisting of a finitely generated group W and a finite, symmetric generating set S, and therefore the machinery described in §1 applies. In the context of Coxeter systems the growth series is also known as the *Poincaré series* $p_{(W,S)}(t)$ of (W,S). If (W,S) is spherical then $p_{(W,S)}(t)$ is a polynomial, which can be explicitly computed in terms of the degrees of the polynomial invariants of (W,S), simply known as the *degrees* of (W,S) (cf. [24], [18, Chapter 3]). For arbitrary Coxeter systems, the Poincaré series can be computed using the following property.

Proposition 2.6 ([25]). *Let (W,S) be a Coxeter system with Poincaré series $p_{(W,S)}(t)$. Then*

$$\frac{1}{p_{(W,S)}(t^{-1})} = \sum_{I\in\mathcal{F}} \frac{(-1)^{|I|}}{p_{(W_I,I)}(t)}, \tag{2.7}$$

where $\mathcal{F} = \mathcal{F}(W,S)$. In particular, the Poincaré series $p_{(W,S)}(t)$ is a rational function.

It is often possible to focus only on irreducible systems.

Lemma 2.7. *Let (W_1,S_1) and (W_2,S_2) be Coxeter systems, and let*

$$(W,S) = (W_1 \times W_2, S_1 \sqcup S_2)$$

be their product. Then

$$\omega(W,S) = \max\{\omega(W_1,S_1), \omega(W_2,S_2)\}.$$

Proof. The factorisation

$$p_{(W,S)}(t) = p_{(W_1,S_1)}(t) \cdot p_{(W_2,S_2)}(t)$$

holds (cf. [4, Chapter IV, n° 1.8 and no 1.9]). Since Poincaré series are series with non-negative coefficients and with degree-zero coefficient equal to one, then

$$\omega(W,S) \geq \max\{\omega(W_1,S_1), \omega(W_2,S_2)\}.$$

On the other hand, the product $p(t)$ of two rational functions $p_1(t)$ and $p_2(t)$ is holomorphic *at least* in the smallest of the open disks centred in zero of radii ρ_1, ρ_2, where each of the two factors are holomorphic: thus

$$\omega(W,S) = \frac{1}{\rho} \leq \frac{1}{\min\{\rho_1,\rho_2\}} = \max\{\omega(W_1,S_1), \omega(W_2,S_2)\}. \qquad \square$$

3. The partial order $\preceq$ on the class of Coxeter systems

The core of the proof of Theorem B is the reduction to a finite set of elementary verifications. The tools which provide this reduction are the partial order $\preceq$ over the set of (Coxeter-isomorphism classes of) Coxeter systems, the corresponding monotonicity results, and the finiteness of the set of minimal non-affine, non-spherical Coxeter systems.

Let (W,S) and (W',S') be Coxeter systems with Coxeter matrices M, M' respectively. Define $(W,S)\preceq(W',S')$ whenever there exists an injective map $\varphi\colon S\to S'$ such that $m_{s,r}\le m'_{\varphi(s),\varphi(r)}$ for all $s,r\in S$ (cf. [22, §6]).

In particular, if (W,S) and (W',S') are Coxeter-isomorphic (cf. §2.1) then $(W,S)\preceq(W',S')$ and $(W',S')\preceq(W,S)$. Therefore the preorder $\preceq$ descends to a partial order on the set of Coxeter-isomorphism classes of Coxeter systems. With a mild abuse of notation we will avoid the distinction between a Coxeter system and its Coxeter-isomorphism class.

3.1. Monotonicity properties. The partial order $\preceq$ has the following important property.

Theorem A. *Let (W,S) and (W',S') be Coxeter systems with Coxeter matrices M and M', respectively. Let $a_k=a_k^{(W,S)}$ and $a'_k=a_k^{(W',S')}$ be the growth sequences with respect to the Coxeter generating set. If $(W,S)\preceq(W',S')$ then $a_k\le a'_k$ for all $k\in\mathbb{N}_0$.*

Proof. Let $\varphi\colon S\to S'$ be an injective map realising the relation $(W,S)\preceq(W',S')$. Let $S''=\operatorname{im}\varphi\subseteq S'$, let $W''=\langle S''\rangle\le W'$, and let (W'',S'') be the corresponding parabolic subsystem of (W',S'). Let $\psi\colon S\to S''$ be given by $\psi(s)=\varphi(s)$ for all $s\in S$. Therefore $\varphi=\iota\circ\psi$, where ι is the inclusion $S''\subseteq S'$, and hence one has $(W,S)\preceq(W'',S'')\preceq(W',S')$.

Let $a''_k=a_k^{(W'',S'')}$. Since (W'',S'') is a parabolic subgroup of (W',S'), then $\ell_{(W'',S'')}=\ell_{(W',S')}|_{W''}$, by (2.2). Hence $A_k^{(W'',S'')}\subseteq A_k^{(W',S')}$, and then

$$a''_k\le a'_k \qquad \text{for all } k\in\mathbb{N}_0. \tag{3.1}$$

We will now prove that $a_k\le a''_k$ for all k. Let M'' be the Coxeter matrix of (W'',S''), and let $N=(m''_{\psi(s),\psi(r)})_{s,r\in S}$. Since ψ is a bijection, $(W(N),S)$ is Coxeter-isomorphic to (W'',S''), and in particular it has growth sequence $a_k^{(W(N),S)}=a''_k$. Let B_k and B_k^N be the balls of radius k in $\operatorname{Cay}(W,S)$ and $\operatorname{Cay}(W(N),S)$, respectively.

By hypothesis $m_{s,r}\le n_{s,r}$ for all $s,r\in S$, and suppose that $N\ne M$. Without loss of generality, assume there exists a unique 2-subset $\{s_0,r_0\}\subseteq S$ such that $m_{s_0,r_0}<n_{s_0,r_0}$. Let $m=m_{s_0,r_0}$ and $n=n_{s_0,r_0}$.

Claim. *For all k, the map $\eta_k = \pi_N \sigma_M|_{B_k} : B_k \to B_k^N$, where σ_M and π_N are defined as in §2.2, is well defined and injective.*

Proof of the claim. First, notice that $\pi_N \sigma_M(B_k) \subseteq B_k^N$, since $\deg(\sigma_M(w)) = \ell_M(w)$ by (2.5) and $\ell_N(\pi_N(\sigma_M(w))) \le \ell_M(w)$ by (2.4). Hence η_k is well defined.

Suppose $v, v' \in B_k$ satisfy $\eta_k(v) = \eta_k(v')$ and let $w = \eta_k(v) \in B_k^N$. Thus

$$\pi_N \sigma_M(v) = \pi_N \sigma_M(v') = w = \pi_N \sigma_N(w).$$

Then, by Corollary 2.4, (iii), there exist sequences of N-moves

$$\sigma_M(v) \longmapsto \sigma_N(w) \longleftarrow \sigma_M(v'). \tag{3.2}$$

Consider first the sequence $\sigma_M(v) \longmapsto \sigma_N(w)$ on the left, and suppose it can be written as the concatenation of elementary N-moves

$$\sigma_M(v) = \boldsymbol{u}_0 \overset{\nu_0}{\longmapsto} \boldsymbol{u}_1 \overset{\nu_1}{\longmapsto} \boldsymbol{u}_2 \longmapsto \ldots \longmapsto \boldsymbol{u}_r \overset{\nu_r}{\longmapsto} \boldsymbol{u}_{r+1} = \sigma_N(w). \tag{3.3}$$

Assume by contradiction, that there exists some t for which ν_t is the $N^{(2)}$-move

$$\nu_t : \boldsymbol{u}_t = \boldsymbol{u}'[s_0, r_0, n]\boldsymbol{u}'' \longmapsto \boldsymbol{u}'[r_0, s_0, n]\boldsymbol{u}'' = \boldsymbol{u}_{t+1}, \tag{3.4}$$

hence $n < \infty$, by (2.3). Let t_0 be the minimum of such t's. Thus, the sequence of moves $\nu_{t_0-1} \circ \nu_{t_0-2} \circ \cdots \circ \nu_1 \circ \nu_0$ is a sequence of M-moves transforming $\sigma_M(v) \in \mathcal{R}_M(v)$ into $\boldsymbol{u}_{t_0}$. Hence, by Corollary 2.4, (i)–(ii), $\boldsymbol{u}_{t_0} \in \mathcal{R}_M(v)$. Since $n > m$ the word $\boldsymbol{u}_{t_0}$ has a subword of the form $[s_0, r_0, m+1]$. Therefore one may apply the M-moves

$$\begin{aligned} \boldsymbol{u}_{t_0} &= \boldsymbol{u}'[s_0, r_0, m+1]\boldsymbol{u}'' = \boldsymbol{u}' s_0 [r_0, s_0, m]\boldsymbol{u}'' \\ &\overset{M^{(2)}}{\longmapsto} \boldsymbol{u}' s_0 [s_0, r_0, m]\boldsymbol{u}'' = \boldsymbol{u}' s_0 s_0 [r_0, s_0, m-1]\boldsymbol{u}'' \\ &\overset{M^{(1)}}{\longmapsto} \boldsymbol{u}'[r_0, s_0, m-1]\boldsymbol{u}'' = \boldsymbol{u}''', \end{aligned}$$

and hence $\deg(\boldsymbol{u}_{t_0}) > \deg \boldsymbol{u}'''$, against the hypothesis that $\boldsymbol{u}_{t_0}$ is (M-)reduced. This gives the desired contradiction and, thus, no $N^{(2)}$-move of the form (3.4) can occur. Since all the remaining N-moves are also M-moves, the sequence (3.3) only consists of M-moves. An analogous argument applies to the sequence $\sigma_M(v') \longmapsto \sigma_N(w)$. Hence, the sequences in (3.2) entirely consist of M-moves, and by (2.6)

$$v = \pi_M \sigma_M(v) = \pi_M \sigma_N(w) = \pi_M \sigma_M(v') = v',$$

which proves the injectivity of η_k. △

Let now $v \in A_k^{(W,S)} \subseteq B_k$. Then $\deg(\sigma_M(v)) = k$, and the previous argument shows that $\sigma_M(v)$ is also N-reduced, therefore $\ell_N(\eta_k(v)) = \ell_N(\pi_N \sigma_M(v)) = k$.

It follows that the maps

$$\vartheta_k = \eta_k|_{A_k} : A_k^{(W,S)} \to A_k^{(W(N),S)}$$

are well defined injections, and hence

$$a_k \leq a_k^{(W(N),S)} = a_k'' \quad \text{for all } k \in \mathbb{N}_0. \tag{3.5}$$

This, together with (3.1), completes the proof. □

Theorem A has the following immediate consequence.

Corollary 3.2. *If* (W, S) *and* (W', S') *are Coxeter systems such that one has* $(W, S) \preceq (W', S')$*, then*

$$\omega(W, S) \leq \omega(W', S').$$

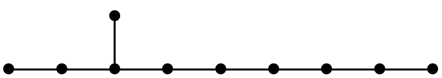
3.3. Minimal non-spherical, non-affine Coxeter systems. Let $\mathcal{X}$ be the set of (Coxeter-isomorphism classes of) non-affine, non-spherical, irreducible Coxeter systems, and let $\mathcal{M} = \min_{\preceq} \mathcal{X}$ be the set of $\preceq$-minimal elements of $\mathcal{X}$.

It is well known that hyperbolic Coxeter systems are characterised as those systems such that every proper irreducible parabolic subsystem is either of spherical or affine type (cf. [4, Chapter V, §4, Example 13]). By minimality, $\mathcal{M}$ consists of hyperbolic Coxeter systems, which are classified in an infinite family of rank-three systems, and 72 exceptions of rank $|S| \geq 4$ (cf. [18, §6.8 and §6.9]). The infinite family consists of the $\langle a, b, c\rangle$-triangle groups with $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} < 1$, and among those only the $\langle 2, 3, 7\rangle$, $\langle 3, 3, 4\rangle$ and $\langle 2, 4, 5\rangle$-triangle groups are $\preceq$-minimal. Among the 72 exceptions, 35 are in $\mathcal{M}$. Therefore,

Proposition 3.4 ([22, Theorem 6.6, Table 5]). $|\mathcal{M}| = 38$.

4. The minimal growth rate of Coxeter groups

Following the notation of [15], let E_{10} be the Coxeter system with Coxeter graph

$\Gamma(E_{10}) =$  .

Theorem B. *If (W, S) is a non-spherical, non-affine Coxeter system, then its growth rate satisfies*

$$\omega(W, S) \geq \tau = 1.138078743\ldots,$$

where τ is the growth rate of the hyperbolic Coxeter system E_{10}. In particular, τ is the inverse of the smallest positive real root of the denominator of the Poincaré series $p_{E_{10}}(t)$ of the Coxeter system E_{10}. Moreover, τ is an algebraic integer of degree 127 over $\mathbb{Q}$, with minimal polynomial

$$\begin{aligned} m_\tau(t) = {} & t^{127} - t^{125} - t^{120} + t^{118} - t^{116} - t^{115} + t^{109} + t^{106} + t^{103} + t^{102} \\ & + 2t^{101} + t^{100} + t^{97} + t^{96} + t^{91} - t^{90} - 2t^{89} - t^{88} - t^{87} - t^{86} \\ & - t^{85} - 2t^{84} - 2t^{83} - t^{82} - 2t^{81} - 3t^{80} - t^{79} - t^{78} - 2t^{77} - t^{76} \\ & - t^{75} - t^{74} - t^{72} - t^{71} + t^{70} + t^{69} + 2t^{67} + 2t^{66} + t^{65} + 2t^{64} \\ & + 2t^{63} + 2t^{62} + 3t^{61} + 2t^{60} + 2t^{59} + 3t^{58} + 3t^{57} + 2t^{56} + 2t^{55} \\ & + 2t^{54} + t^{53} + 2t^{52} + 2t^{51} + t^{46} - t^{45} - 2t^{44} - t^{43} - t^{42} - 2t^{41} \\ & - 2t^{40} - 2t^{39} - 2t^{38} - 2t^{37} - 2t^{36} - 2t^{35} - t^{34} - 2t^{33} - 3t^{32} - t^{31} \\ & - t^{29} - t^{28} - t^{27} + t^{25} + t^{22} + t^{21} + t^{20} + t^{19} + t^{18} + t^{17} \\ & + t^{16} + t^{15} + t^{14} + t^{13} + t^{12} - t - 1. \end{aligned}$$

The integer τ is a Perron number, i.e., an algebraic integer whose module stricly exceeds the module of its algebraic conjugates (*cf.* [19, 20]).

Proof. By monotonicity of the function ω with respect to $\preceq$ (cf. Corollary 3.2) and by Proposition 3.4, it suffices to compute $\omega(W, S)$ for finitely many (W, S).

Moreover, $p_{(W,S)}(t)$ is power series with non-negative coefficients, and also a rational function, by Proposition 2.6. Thus, $\omega(W, S)$ is the inverse of the minimal, positive real root of the denominator of $p_{(W,S)}(t)$. □

Theorem B can be stated in terms of a gap in the set

$$\Omega = \{\omega(W, S) \mid (W, S) \text{ Coxeter system}\} \subseteq \{0, 1\} \cup \mathbb{R}_{\geq \tau}.$$

Remark 4.1. (i) The direct verifications for the 38 relevant Coxeter systems were performed with the help of the computational algebra system Magma (cf. [26]). The code is available at

https://sites.google.com/site/tomterragni/research/computations

(ii) The denominator of $p_{E_{10}}(t)$ is $(t-1)m_{\tau^{-1}}(t)$.

(iii) In many cases $\omega(W, S)$ is an algebraic integer, and also a Perron number. It is known that every Perron number λ is realised as the Perron–Frobenius eigenvalue of an aperiodic, non-negative integral matrix P_λ (cf. [20, Theorem 1]). Lind's proof is constructive, however the algorithm given in the proof may produce a Perron–Frobenius matrix of non-minimal size. It would be interesting to find a minimal-sized Perron–Frobenius matrix for τ.

(iv) The Poincaré series of (all but one) exceptional hyperbolic Coxeter systems are also listed in [7]. In the same paper, some radii of convergence are computed.

(v) It is quite surprising that τ is not realised as growth rate of any of the small rank Coxeter systems, instead it is associated with the Coxeter system E_{10}. However, the growth rate of one of the $\preceq$-minimal rank-three hyperbolic Coxeter groups, namely the one with Coxeter system $\langle 2, 3, 7\rangle$, is Lehmer's number $\lambda_{\text{Lehmer}} = 1.17\ldots$ (cf. [16]), and an interesting coincidence occurs. Let

$$\lambda_\rho(W, S) = \inf(\{\lambda_\rho(w) \mid w \in W\} \cap \mathbb{R}_{>1}),$$

where $\lambda_\rho(w)$ is the spectral radius of the matrix $\rho(w)$, and ρ is Tits' reflection representation.

The number $\lambda_\rho(W, S)$ represents a universal bound for eigenvalues of elements in Coxeter groups. Moreover, if (W, S) is hyperbolic, then $\log \lambda_\rho(W, S)$ is interpreted as a lower bound for the length of non-degenerate, closed hyperbolic geodesics in the orbifold $\mathbb{H}^{|S|-1}/W$.

McMullen proved that

$$\inf_{(W,S)} \lambda_\rho(W, S) = \lambda_{\text{Lehmer}},$$

the infimum being taken as (W, S) runs through the non-affine, non-spherical Coxeter systems (cf. [22]). The infimum is actually a minimum, and it is attained *exactly* for the Coxeter system E_{10}.

It would be interesting to understand this phenomenon.

5. Rigidity and growth

It is well known that there exist non Coxeter-isomorphic Coxeter systems for which the groups are abstractly isomorphic. For a discussion on the isomorphism problem for Coxeter groups, see [6, 23, 1], and references therein.

5.1. Coxeter generating systems. Let G be a group generated by a finite set of involutions $R \subseteq G$. Then $M(R) = (\mathrm{ord}(sr))_{s,r \in R}$ is a Coxeter matrix. Let (W, R) be the Coxeter system with Coxeter matrix $M(R)$. The identity on R induces a surjective homomorphism of groups $j_R \colon W \to G$. Moreover, when j_R is an isomorphism G is a *Coxeter group with Coxeter generating system* R.

If (W, S) is a Coxeter system and σ is either an inner automorphism or the automorphism of W induced by a Coxeter automorphism of (W, S), then $\sigma(S)$ is another Coxeter generating system, and $(W, \sigma(S))$ is Coxeter-isomorphic to (W, S). In general, any inner-by-Coxeter automorphism preserves the Coxeter-isomorphism type. An automorphism which is not inner-by-Coxeter will be called *exotic*.

5.2. Isomorphisms of Coxeter groups. A major problem in the theory of Coxeter groups is to find all possible Coxeter generating systems of a given a Coxeter group W. If, for any two Coxeter generating sets R, S of W, the Coxeter systems (W, S) and (W, R) are Coxeter-isomorphic, then W is called *rigid*. It is well known that there exist non-rigid Coxeter groups, e.g., for n, m odd there are exotic isomorphisms

$$W(I_2(2m)) \simeq W(I_2(m) \times A_1) \quad \text{and} \quad W(B_n) \simeq W(D_n \times A_1). \tag{5.1}$$

There are standard procedures which realise exotic isomorphisms between Coxeter systems, e.g., Brady *et al.* introduced the *diagram twisting* (cf. [3, §4] and §5), and Howlett and Mühlherr introduced a construction, the *elementary reductions*, which deal with exotic isomorphisms $(W, S) \to (W, R)$ for which the set of reflections S^W is different from R^W (cf. [17]). Reductions generalise the exotic isomorphisms (5.1).

Several classes of Coxeter groups are known to be rigid, or rigid up to diagram twisting. For instance, if any of the following conditions is satisfied for a Coxeter generating system S of W, then W is rigid up to diagram twisting (cf. [3, 1, 23]).

(i) (W, S) is right-angled, i.e., $m_{s,r} \in \{2, \infty\}$ for all $s, r \in S$, $s \neq r$;

(ii) (W, S) is infinite and $m_{s,r} < \infty$ for all $s, r \in S$;

(iii) (W, S) can act faithfully, properly and cocompactly on a contractible manifold;

(iv) (W, S) is skew-angled, i.e., $m_{s,r} \neq 2$ for all $s, r \in S$;

(v) $\Gamma_\infty(W, S)$ is a tree, where Γ_∞ is the variant of the Coxeter graph defined in [4, Chapter IV, §1, Example 11].

5.3. Mutations of Coxeter groups

Definition 5.4. Let M be a Coxeter matrix over S, and suppose that there exists a partition $S = X \sqcup Y \sqcup T \sqcup Z$ and a Coxeter-automorphism σ of the subsystem (W_X, X) satisfying

(i) $m_{t,y} = \infty$ for all $t \in T$ and $y \in Y$,

(ii) $m_{z,y} < \infty$ for all $z \in Z$ and $y \in Y$, and

(iii) for all $z \in Z$ and $x \in X$ one has $m_{z,\sigma(x)} = m_{z,x}$.

Then, the 4-tuple (M, X, Y, σ) is called *mutable*. Associated with a mutable tuple (M, X, Y, σ) there is a Coxeter matrix $\mu(M, X, Y, \sigma) = (n_{r,s})_{r,s\in S}$, its *mutation*, given by

$$n_{s,r} = n_{r,s} = \begin{cases} m_{\sigma(r),s} & \text{if } r \in X, s \in Y, \\ m_{\sigma(r),\sigma(s)} & \text{if } r, s \in X, \\ m_{r,s} & \text{otherwise.} \end{cases} \tag{5.2}$$

If (M, X, Y, σ) is mutable, then $(\mu(M, X, Y, \sigma), X, Y, \sigma^{-1})$ is mutable and it is called the *inverse mutable* 4-tuple since $\mu(\mu(M, X, Y, \sigma), X, Y, \sigma^{-1}) = M$. The relation "$N$ is a mutation of M" is symmetric, and therefore its transitive closure is an equivalence relation $\sim$ on Coxeter systems.

Remark 5.5. (i) The partition associated with a mutable tuple (M, X, Y, σ) is determined by X, Y together with conditions (i)–(ii), and therefore T, Z may be omitted from the notation.

(ii) Many Coxeter matrices M only admit trivially mutable tuples, i.e., tuples with $\sigma = \mathrm{id}_X$. Even when a non-trivial tuple exists, it may happen that the associated mutation is Coxeter-isomorphic to M. If this is not the case, (M, X, Y, σ) is called *effective*.

(iii) The operation of mutation is a generalisation of the diagram twisting (cf. [3]). Diagram twists are mutations satisfying the additional conditions (a) W_X is finite, (b) $\sigma(x) = x^{w_0(X)}$ is the conjugation by the longest element of W_X, and (c) $m_{z,x} = 2$ for all $z \in Z$ and $x \in X$. Effective diagram twists determine exotic isomorphisms of Coxeter groups.

Theorem C. *Let (W, S) be a Coxeter system with Coxeter matrix M, and let (M, X, Y, σ) be a mutable tuple for (W, S). Let $N = \mu(M, X, Y, \sigma)$, and let (W', S') be the Coxeter system with Coxeter matrix N.*

Then there is a bijection

$$\_^{\sharp}\colon \mathcal{F} = \mathcal{F}(W, S) \longrightarrow \mathcal{F}(W', S') = \mathcal{F}',$$

such that (W_I, I) is Coxeter-isomorphic to $(W'_{I^\sharp}, I^\sharp)$ for all $I \in \mathcal{F}$. Moreover, if $(W, S) \sim (W', S')$ then

$$p_{(W,S)}(t) = p_{(W',S')}(t). \tag{5.3}$$

Proof. Let $S = X \sqcup Y \sqcup Z \sqcup T$ decompose as in Def. 5.4, and let $I \in \mathcal{F}$. Since every edge of a spherical graph must have a finite label, then either

(a) $I \subseteq X \sqcup T \sqcup Z$, or

(b) $I \subseteq X \sqcup Y \sqcup Z$ and $I \cap Y \neq \emptyset$.

Suppose that (a) holds, then define $I^\sharp = \{r^\sharp = r \mid r \in I\}$. By (5.2), for $r^\sharp, s^\sharp \in I^\sharp$ on has

$$n_{r^\sharp,s^\sharp} = n_{r,s} = \begin{cases} m_{\sigma(r),\sigma(s)} & \text{if } r, s \in X, \\ m_{r,s} & \text{if } r \in X, s \notin X, \\ m_{r,s} & \text{if } r, s \notin X. \end{cases}$$

Since σ is a Coxeter-automorphism of (W_X, X), then $m_{\sigma(r),\sigma(s)} = m_{r,s}$ for $s, r \in X$.

Suppose that (b) holds, then define $I^\sharp = \{r^\sharp \mid r \in I\}$, where now

$$r^\sharp = \begin{cases} \sigma^{-1}(r) & \text{if } r \in X, \\ r & \text{if } r \notin X. \end{cases} \tag{5.4}$$

Then, for $r^\sharp, s^\sharp \in I^\sharp$, by (5.2), (5.4) and Def. 5.4, (iii), one has

$$n_{r^\sharp,s^\sharp} = \begin{cases} m_{\sigma(r^\sharp),\sigma(s^\sharp)} = m_{r,s} & \text{if } r^\sharp, s^\sharp \in X, \\ m_{\sigma(r^\sharp),s^\sharp} = m_{r,s} & \text{if } r^\sharp \in X, s^\sharp \in Y, \\ m_{r^\sharp,s^\sharp} = m_{\sigma^{-1}(r),s} = m_{r,s} & \text{if } r^\sharp \in X, s^\sharp \in Z, \\ m_{r^\sharp,s^\sharp} = m_{r,s} & \text{if } r, s \notin X. \end{cases}$$

Hence, $N_{I^\sharp}$ and M_I determine Coxeter-isomorphic systems. It follows that $I^\sharp \in \mathcal{F}'$ and that (a) holds for $I^\sharp$ if, and only if, (a) holds for I. Thus, the map $I \mapsto I^\sharp$ is a map which preserves the Coxeter-isomorphism type, and it is invertible (its inverse being the $\sharp$-map associated to the inverse mutable tuple). The identity (5.3) then follows from Steinberg's formula (2.7). □

Corollary 5.6. *Suppose that W is rigid up to diagram twisting, and let S, R be Coxeter generating systems for W (cf. §5.1). Then*

$$p_{(W,S)}(t) = p_{(W,R)}(t) \quad \textit{and} \quad \omega(W, S) = \omega(W, R).$$

Let $p_{W,\mathrm{Cox}}(t)$ and $\omega_{\mathrm{Cox}}(W)$ be these common values.

Theorem C implies that effective mutations which are not diagram twists can be regarded as procedures to produce non-isomorphic (and *a fortiori*, non Coxeter-isomorphic) Coxeter systems with the same Poicaré series.

Example 5.7. Consider the rank-seven Coxeter system (W, S) with Coxeter matrix

$$M = \begin{pmatrix} 1 & 3 & 3 & 2 & 3 & 4 & 2 \\ 3 & 1 & 3 & 2 & 3 & 4 & 2 \\ 3 & 3 & 1 & 2 & 2 & 4 & 3 \\ 2 & 2 & 2 & 1 & 3 & 3 & 2 \\ 3 & 3 & 2 & 3 & 1 & 2 & \infty \\ 4 & 4 & 4 & 3 & 2 & 1 & 3 \\ 2 & 2 & 3 & 2 & \infty & 3 & 1 \end{pmatrix}.$$

Let $X = \{s_1, s_2, s_3, s_4\}$, $Y = \{s_5\}$, $Z = \{s_6\}$, $T = \{s_7\}$, and let $\sigma = (1, 2, 3)$. Then (M, X, Y, σ) is mutable, with mutation displayed in Fig. 5.1. Moreover, $N = \mu(M, X, Y, \sigma)$ is a proper mutation, i.e., N is not obtained from M by diagram twisting.

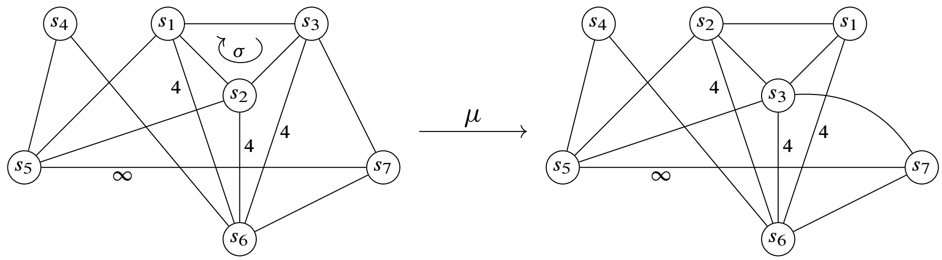


Figure 5.1. A proper mutation.

5.8. A conjecture. Consider the group $\mathrm{PGL}(2, \mathbb{Z}) \simeq (C_2 \times C_2) *_{C_2} S_3$. It is well known that $\mathrm{PGL}(2, \mathbb{Z}) \simeq W$, where (W, S) is the Coxeter system $\langle 2, 3, \infty \rangle$ with Coxeter graph •—•$\overset{\infty}{—}$•. Hence the minimal growth rate satisfies $\omega(\mathrm{PGL}(2, \mathbb{Z})) \leq \omega(W, S) = \alpha$, where α is the *plastic number*, with minimal polynomial $m_\alpha(t) = t^3 - t - 1$. The converse inequality is proven by Bucher and Talambutsa (cf. [5, §6]).

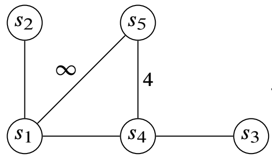
Therefore, the following problem seems to be of some interest.

Conjecture D. *Let W be a Coxeter group rigid up to diagram twisting, and let $\omega_{\mathrm{Cox}}(W)$ be defined as in Corollary 5.6. Then $\omega(W) = \omega_{\mathrm{Cox}}(W)$.*

Remark 5.9. (i) If W is a product of spherical and affine irreducible Coxeter systems, its Poincaré series depends on the chosen generating set. However, the minimal growth rate and the growth rate coincide $\omega(W) = \omega(W, S)$ and their common value is either 0 or 1, depending on the finiteness of the group only.

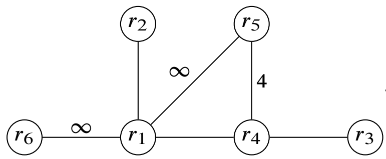
(ii) The rigidity hypothesis in Conj. D cannot be relaxed since, in general, elementary reductions do not preserve the growth rate, as the following example shows. Let

$$M = \begin{pmatrix} 1 & 3 & 2 & 3 & \infty \\ 3 & 1 & 2 & 2 & 2 \\ 2 & 2 & 1 & 3 & 2 \\ 3 & 2 & 3 & 1 & 4 \\ \infty & 2 & 2 & 4 & 1 \end{pmatrix}, \quad \Gamma(M) =$$



Then s_5 is a pseudo-transposition, corresponding to the parabolic subsystem of type B_3 generated by $J = \{s_3, s_4, s_5\}$. Let $r_i = s_i$ for $i \in \{1, \dots, 4\}$, let $r_5 = s_5 s_4 s_5$ and let $r_6 = w_0(J) = s_3 s_4 s_3 s_5 s_4 s_3 s_5 s_4 s_5$ be the longest element of the parabolic subsystem (W_J, J). Then, $R = \{r_i \mid i \in \{1, \dots, 6\}\}$ is a Coxeter generating system for $W(M)$ (cf. [17]). Its Coxeter matrix $M' = M(R)$ is

$$M' = \begin{pmatrix} 1 & 3 & 2 & 3 & \infty & \infty \\ 3 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 1 & 3 & 3 & 2 \\ 3 & 2 & 3 & 1 & 2 & 2 \\ \infty & 2 & 3 & 2 & 1 & 2 \\ \infty & 2 & 2 & 2 & 2 & 1 \end{pmatrix}, \quad \Gamma(M') =$$



By direct computation one sees that $\omega(W, S) = 2.24167\dots$, while $\omega(W, R) = 2.61578\dots$.

References

[1] P. Bahls, *The isomorphism problem in Coxeter groups.* Imperial College Press, London, 2005. Zbl 1113.20034 MR 2162146

[2] A. Björner and F. Brenti, *Combinatorics of Coxeter groups.* Graduate Texts in Mathematics, 231. Springer, New York, 2005. Zbl 1110.05001 MR 2133266

[3] N. Brady, J. P. McCammond, B. Mühlherr, and W. D. Neumann, Rigidity of Coxeter groups and Artin groups. *Geom. Dedicata* **94** (2002), 91–109. Zbl 1031.20035 MR 1950875

[4] N. Bourbaki, *Éléments de mathématique.* Fasc. XXXIV. Chapitres IV–IV. Actualités Scientifiques et Industrielles, 1337, Hermann, Paris, 1968. Reprint, Springer etc., Berlin, 2007. Zbl 0186.33001 MR 0240238

[5] M. Bucher and A. Talambutsa, Exponential growth rates of free and amalgamated products. *Israel J. Math.* **212** (2016), no. 2, 521–546. MR 3505396

[6] R. Charney and M. Davis, When is a Coxeter system determined by its Coxeter group? *J. London Math. Soc.* (2) **61** (2000), no. 2, 441–461. Zbl 0983.20034 MR 1760693

[7] M. Chapovalov, D. Leites, and R. Stekolshchik, The Poincaré series of the hyperbolic Coxeter groups with finite volume of fundamental domains. *J. Nonlinear Math. Phys.* **17** (2010), suppl. 1, 169–215. MR 2827480

[8] M. W. Davis, *The geometry and topology of Coxeter groups.* London Mathematical Society Monographs Series, 32. Princeton University Press, Princeton, N.J., 2008. Zbl 1142.20020 MR 2360474

[9] P. de la Harpe, *Topics in geometric group theory.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000. Zbl 0965.20025 MR 1786869

[10] D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, and W. P. Thurston, *Word processing in groups.* Jones and Bartlett Publishers, Boston, MA, 1992. Zbl 0764.20017 MR 1161694

[11] R. I. Grigorchuk, On Burnside's problem on periodic groups. *Funktsional. Anal. i Prilozhen.* **14** (1980), no. 1, 53–54; in Russian. English translation, *Functional Anal. Appl.* **14** (1980), no. 1, 41–43; in Russian. English translation, *Math. USSR-Izv.* **25** (1985), no. 2, 259–300. Zbl 0595.20029 MR 0565099

[12] R. I. Grigorchuk, Degrees of growth of finitely generated groups and the theory of invariant means. *Izv. Akad. Nauk SSSR Ser. Mat.* **48** (1984), no. 5, 939–985. Zbl 0583.20023 MR 0764305

[13] R. I. Grigorchuk and P. de la Harpe, On problems related to growth, entropy, and spectrum in group theory. *J. Dynam. Control Systems* **3** (1997), no. 1, 51–89. Zbl 0949.20033 MR 1436550

[14] M. Gromov, Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.* **53** (1981), 53–73. Zbl 0474.20018 MR 0623534

[15] B. H. Gross, E. Hironaka, and C. T. McMullen, Cyclotomic factors of Coxeter polynomials. *J. Number Theory* **129** (2009), no. 5, 1034–1043. Zbl 1183.11065 MR 2516970

[16] E. Hironaka, What is … Lehmer's number? *Notices Amer. Math. Soc.* **56** (2009), no. 3, 374–375. Zbl 1163.11069 MR 2494103

[17] R. Howlett and B. Mühlherr, Isomorphisms of Coxeter groups which do not preserve reflections. Preprint 2004.

[18] J. E. Humphreys, *Reflection groups and Coxeter groups.* Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, Cambridge, 1990. Zbl 0725.20028 MR 1066460

[19] D. A. Lind, Entropies and factorizations of topological Markov shifts. *Bull. Amer. Math. Soc.* (*N.S.*) **9** (1983), no. 2, 219–222. Zbl 0524.58034 MR 0707961

[20] D. A. Lind, The entropies of topological Markov shifts and a related class of algebraic integers. *Ergodic Theory Dynam. Systems* **4** (1984), no. 2, 283–300. Zbl 0546.58035 MR 0766106

[21] A. Mann, *How groups grow.* London Mathematical Society Lecture Note Series, 395. Cambridge University Press, Cambridge, 2012. Zbl 1253.20032 MR 2894945

[22] C. T. McMullen, Coxeter groups, Salem numbers and the Hilbert metric. *Publ. Math. Inst. Hautes Études Sci.* **95** (2002), 151–183. Zbl 1148.20305 MR 1953192

[23] B. Mühlherr, The isomorphism problem for Coxeter groups. In Ch. Davis and E. W. Ellers (eds.), *The Coxeter legacy*. Reflections and projections. American Mathematical Society, Providence, R.I., 2006, 1–15. Zbl 1103.20031 MR 2209018

[24] L. Solomon, The orders of the finite Chevalley groups. *J. Algebra* **3** (1966), 376–393. Zbl 0151.02003 MR 0199275

[25] R. Steinberg, *Endomorphisms of linear algebraic groups*. Memoirs of the American Mathematical Society, 80. American Mathematical Society, Providence, R.I., 1968. Zbl 0164.02902 MR 0230728

[26] T. Terragni, Data about hyperbolic Coxeter systems. Preprint 2015. arXiv:1503.08764 [math.GR]

[27] J. Tits, Le problème des mots dans les groupes de Coxeter. Symposia. In *Symposia Mathematica* (INDAM, Rome, 1967/68), Vol. 1, 175–185. MR 0254129

[28] J. Tits, Free subgroups in linear groups. *J. Algebra* **20** (1972), 250–270. Zbl 0236.20032 MR 0286898

Received December 18, 2013

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