Pseudo-Anosov dilatations and the Johnson filtration

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Abstract. Answering a question of Farb, Leininger, and Margalit, we give explicit lower bounds for the dilatations of pseudo-Anosov mapping classes lying in the kth term of the Johnson filtration of the mapping class group.

1. Introduction

Let $\mathrm{Mod}(\Sigma)$ be the mapping class group of a closed orientable surface Σ. Thurston's well-known classification of surface homeomorphisms (see [6]) says that every element of $\mathrm{Mod}(\Sigma)$ is either finite order, reducible, or pseudo-Anosov. By definition, a pseudo-Anosov mapping class $f \in \mathrm{Mod}(\Sigma)$ is one that can be represented by a homeomorphism $F \colon \Sigma \to \Sigma$ such that there exist two transverse singular measured foliations $\mathcal{F}^{\mathrm{u}}$ and $\mathcal{F}^{\mathrm{s}}$ and some $\lambda(f) > 1$ such that $F_*(\mathcal{F}^{\mathrm{u}}) = \lambda(f) \cdot \mathcal{F}^{\mathrm{u}}$ and $F_*(\mathcal{F}^{\mathrm{s}}) = \frac{1}{\lambda(f)} \cdot \mathcal{F}^{\mathrm{s}}$. The number $\lambda(f)$ only depends on f and is known as the *dilatation* of f. The dilatation $\lambda(f)$ of a pseudo-Anosov mapping class f shows up in many places; for instance, the number $\ln(\lambda(f))$ is the translation length with respect to the Teichmüller metric of the action of f on Teichmüller space.

Minimal dilatations. The set of possible dilatations of pseudo-Anosov mapping classes has many interesting properties. Define

$$\mathrm{Spec}(\mathrm{Mod}(\Sigma)) = \{\ln(\lambda(f)) \mid f \in \mathrm{Mod}(\Sigma) \text{ is pseudo-Anosov}\} \subset \mathbb{R}.$$

Arnoux and Yoccoz [1] and Ivanov [8] independently proved that $\mathrm{Spec}(\mathrm{Mod}(\Sigma))$ is closed and discrete, so it has a minimal element $L(\mathrm{Mod}(\Sigma))$. Penner [14] proved that there exists some $C > 1$ such that if Σ has genus g, then

$$\frac{1}{Cg} \leq L(\mathrm{Mod}(\Sigma)) \leq \frac{C}{g}.$$

In particular, $L(\mathrm{Mod}(\Sigma)) \to 0$ as $g \to \infty$.

Johnson filtration. The *Johnson filtration* is an important sequence of subgroups of $\mathrm{Mod}(\Sigma)$. Recall that if G is a group, then the *lower central series* of G is the inductively defined sequence

$$\gamma_1(G) = G \quad \text{and} \quad \gamma_{d+1}(G) = [G, \gamma_d(G)] \text{ for } d \geq 1.$$

Letting $\pi_1 = \pi_1(\Sigma)$, the kth term of the Johnson filtration of $\mathrm{Mod}(\Sigma)$, denoted $\mathcal{N}_k(\Sigma)$, is the kernel of the natural representation $\mathrm{Mod}(\Sigma) \to \mathrm{Out}(\pi_1/\gamma_{k+1}(\pi_1))$. For example, since $\pi_1/\gamma_2(\pi_1) \cong \mathrm{H}_1(\Sigma; \mathbb{Z})$, the group $\mathcal{N}_1(\Sigma)$ is the *Torelli group*, that is, the kernel of the action of $\mathrm{Mod}(\Sigma)$ on $\mathrm{H}_1(\Sigma; \mathbb{Z})$. The groups $\mathcal{N}_k(\Sigma)$ were first defined in [10] and have connections to number theory (see [13]) and 3-manifolds (see [7]); however, many basic questions about them remain open.

Dilatation in the Johnson filtration. If $H < \mathrm{Mod}(\Sigma)$ is a subgroup, then define

$$\mathrm{Spec}(H) = \{\ln(\lambda(f)) \mid f \in H \text{ is pseudo-Anosov}\} \subset \mathbb{R}_{>0}.$$

Possibly $\mathrm{Spec}(H)$ is empty; otherwise, it must have a least element which we will denote by $L(H)$. Farb, Leininger, and Margalit [4] proved that if Σ is a closed surface whose genus is at least 2 and $k \geq 1$, then $\mathrm{Spec}(\mathcal{N}_k(\Sigma))$ is nonempty. They also proved that there exist numbers $n_k > 0$ which depend only on k such that $L(\mathcal{N}_k(\Sigma)) \geq n_k$. Observe that this contrasts sharply with Penner's theorem for the whole mapping class group. Finally, they proved that $n_k \to \infty$ as $k \to \infty$.

Main theorem. The proof of Farb, Leininger, and Margalit that $n_k \to \infty$ as $k \to \infty$ relies on a sort of compactness argument and gives no estimates for n_k. They posed the question of obtaining explicit estimates. Our main theorem answers their question. It says that $L(\mathcal{N}_k(\Sigma))$ is bounded below by a quantity that grows roughly like $\ln(k)$. More precisely, we have the following.

Theorem A. *Let Σ be a closed surface whose genus is at least* 2*. Then for all $k \geq 1$ we have*

$$L\left(\mathcal{N}_k(\Sigma)\right) > \max\left(0.197, c \cdot \ln\left(\frac{k+3}{2}\right) - \ln(2)\right),$$

where $c = \frac{\ln(28/25)}{\ln(4)}$.

Remark. In [4, §4], Farb, Leininger, and Margalit prove that $L(\mathcal{N}_k(\Sigma))$ is bounded above by a function which is exponential in k (independent of the genus). Letting F be a free group of rank 2, the key input to their construction is a sequence of nontrivial elements $w_k \in \gamma_k(F)$ whose word lengths are exponential in k. In fact, there are nontrivial elements of $\gamma_k(F)$ whose word lengths are quadratic in k (see the remarks after [12, Theorem 2]). Plugging these into their construction, one can deduce that $L(\mathcal{N}_k(\Sigma))$ is bounded above by a function which is quadratic in k (independent of the genus).

Curves on a surface.   If α and β are simple closed curves on Σ, then denote by $i(\alpha, \beta)$ the *geometric intersection number* of α and β, that is, the minimum size of $\alpha' \cap \beta'$ as α' and β' range over curves homotopic to α and β, respectively. One of the key insights of [4] is that one can obtain estimates on $\lambda(f)$ by bounding the size of $i(f(\delta), \delta)$ from below for all nonnullhomotopic simple closed curves δ. Indeed, [4, Proposition 2.7] says that if $f \in \mathrm{Mod}(\Sigma)$ is pseudo-Anosov and if $i(f(\delta), \delta) \geq n \geq 3$ for all nonnullhomotopic simple closed curves δ, then $\lambda(f) > n/2$.

Recall that there is a bijection between free homotopy classes of oriented closed curves on Σ and conjugacy classes in π_1. If $f \in \mathcal{N}_k(\Sigma)$ and δ is an oriented simple closed curve, then by definition the conjugacy classes in π_1 associated to $f(\delta)$ and δ project to the same conjugacy class in $\pi_1/\gamma_{k+1}(\pi_1)$. The following theorem (applied with $d = k + 1$) therefore provides a lower bound on $i(f(\delta), \delta)$.

Theorem B. *Let Σ be a closed surface whose genus is at least 2 and let α and β be nonisotopic oriented simple closed curves on Σ. Assume that for some $d \geq 3$ the conjugacy classes in $\pi_1(\Sigma)/\gamma_d(\pi_1(\Sigma))$ associated to α and β are the same. Then*

$$i(\alpha, \beta) \geq \Big(\frac{d+2}{2}\Big)^c,$$

where $c = \frac{\ln(28/25)}{\ln(4)}$.

Remark. In Appendix A, we show that the conclusion of Theorem B also holds for compact surfaces with boundary.

Deriving Theorem A from Theorem B.   We now discuss how to derive Theorem A from Theorem B. The bound in Theorem B is greater than 2 starting at $d = 9623$. Therefore, Theorem B together with the aforementioned result [4, Proposition 2.7] implies that for $k \geq 9622$ we have

$$L(\mathcal{N}_k(\Sigma)) \geq \ln\Big(\frac{1}{2}\Big(\frac{k+3}{2}\Big)^c\Big) = c \cdot \ln\Big(\Big(\frac{k+3}{2}\Big)\Big) - \ln(2), \tag{1}$$

where $c = \frac{\ln(28/25)}{\ln(4)}$. Farb, Leininger, and Margalit also proved that $L(\mathcal{N}_1(\Sigma)) > 0.197$ (see [4, Theorem 1.1]), which clearly implies that $L(\mathcal{N}_k(\Sigma)) > 0.197$ for all $k \geq 1$. The bound in (1) is less than 0.197 for $k < 9622$, so Theorem A follows.

Proof techniques for Theorem B.   The heart of our proof of Theorem B is the construction of a suitable nilpotent cover of Σ which "resolves" the intersections of α and β. This nilpotent cover is obtained by constructing a sequence of 2-fold covers each of which resolves at minimum some constant fraction of the intersections. It is easy to construct covers that resolve a single intersection at a time, but that only leads to a logarithmic bound in Theorem B. To get our stronger bound, we need to resolve many intersections at once. It seems difficult to explicitly construct these 2-fold covers, but we show that they exist using probabilistic arguments. Namely, we prove that if the covers are chosen at random in an appropriate way, then the expected value of the number of resolved intersections is larger than some constant fraction of the intersections, and thus that there must exist *some* cover that resolves enough intersections. See the proofs of the key Lemmas 4.1–4.2 below.

Subtleties.   We close this introduction by discussing some difficulties that arise in proving Theorem B. Though in the end we will not phrase it that way, one can view our proof as deriving a contradiction from the existence of nonisotopic simple closed curves α and β on Σ that induce the same conjugacy classes in $\pi_1(\Sigma)/\gamma_d(\pi_1(\Sigma))$ and that satisfy $i(\alpha, \beta) < \left(\frac{d+2}{2}\right)^c$. One difficulty that must be overcome is that given two such curves α and β and a basepoint $v \in \Sigma$, it seems hard to find based representatives $\hat{\alpha}, \hat{\beta} \in \pi_1(\Sigma, v)$ of α and β with the following two properties.

- $\hat{\alpha}$ and $\hat{\beta}$ intersect $i(\alpha, \beta)$ times (perhaps up to a constant factor), and
- $\hat{\alpha}\hat{\beta}^{-1} \in \gamma_d(\pi_1(\Sigma, v))$.

In other words, the algebraic and topological conditions on α and β do not interact very well.

If such $\hat{\alpha}$ and $\hat{\beta}$ existed, then the self-intersection number of $\hat{\alpha}\hat{\beta}^{-1}$ would be at most $i(\alpha, \beta)$ (up to a constant factor) and one could appeal to the paper [12] of the authors, which bounds from below the self-intersection number of nontrivial elements of $\gamma_d(\pi_1(\Sigma, v))$. Our proof does share some ideas with [12], but substantial new ideas were needed to overcome the difficulties just discussed.

Remark.   We do not know any examples of curves α and β as above that cannot be realized by based curves with the above properties, but we conjecture that they exist.

2. Initial reduction

We will prove Theorem B by constructing a certain finite cover of Σ that "resolves" the intersections of α and β in an appropriate way. In this section, we will state the lemma that gives this cover and then show how to derive Theorem B.

Curve-arc triples. We begin with some necessary definitions. Fix a closed surface Σ whose genus is at least 2, and choose a hyperbolic metric on Σ. A *curve-arc triple* on Σ is a tuple $((\alpha, v), \tau, (\beta, w))$ as follows.

- The element α is a simple closed oriented geodesic on Σ which is based at $v \in \Sigma$.

- The element β is a simple closed oriented geodesic on Σ which is based at $w \in \Sigma$.

- α and β are nonisotopic.

- The element τ is a path (possibly with self-intersections) from v to w.

If $((\alpha, v), \tau, (\beta, w))$ is a curve-arc triple and $(\widetilde{\Sigma}, \tilde{v}) \to (\Sigma, v)$ is a (based) cover, then we can *lift* the triple $((\alpha, v), \tau, (\beta, w))$ to $(\widetilde{\Sigma}, \tilde{v})$ as follows. First, endow $\widetilde{\Sigma}$ with the hyperbolic metric that makes the covering map a local isometry. The arc τ lifts to an arc $\tilde{\tau}$ starting at $\tilde{v}$; let $\widetilde{w}$ be the endpoint of $\tilde{\tau}$. Then α lifts to a geodesic arc $\tilde{\alpha}$ starting at $\tilde{v}$ and β lifts to a geodesic arc $\tilde{\beta}$ starting at $\widetilde{w}$. We will say that $((\tilde{\alpha}, \tilde{v}), \tilde{\tau}, (\tilde{\beta}, \widetilde{w}))$ is a *closed lift* of $((\alpha, v), \tau, (\beta, w))$ if $\tilde{\alpha}$ and $\tilde{\beta}$ are simple closed curves (and thus $((\tilde{\alpha}, \tilde{v}), \tilde{\tau}, (\tilde{\beta}, \widetilde{w}))$ is a curve-arc triple on $\widetilde{\Sigma}$). We will say that $((\alpha, v), \tau, (\beta, w))$ has *only a partially closed lift* if one of $\tilde{\alpha}$ and $\tilde{\beta}$ is closed and the other is not closed. Finally, we will say that $((\alpha, v), \tau, (\beta, w))$ has a *nonclosed lift* if neither $\tilde{\alpha}$ nor $\tilde{\beta}$ is closed.

Remark. The reason we require our curves to be geodesics with respect to some hyperbolic metric is that this implies that they intersect minimally (see [5, Corollary 1.9]). Moreover, this persists when we pass to finite covers.

The key lemma. With these definitions in hand, we can state the key lemma of this paper.

Lemma 2.1. *Let Σ be a closed surface whose genus is at least* 2*. Fix a hyperbolic metric on Σ and let $((\alpha, v), \tau, (\beta, w))$ be a curve-arc triple on Σ. Set $n = i(\alpha, \beta)$. Then for some k satisfying*

$$k \leq \begin{cases} 2\log_{28/25}(n) + 1 & \text{if } n \geq 2, \\ 3 & \text{if } 0 \leq n \leq 1, \end{cases}$$

there exists a tower

$$(\Sigma_k, v_k) \longrightarrow (\Sigma_{k-1}, v_{k-1}) \longrightarrow \cdots \longrightarrow (\Sigma_0, v_0) = (\Sigma, v)$$

of regular degree 2 based covers such that $((\alpha, v), \tau, (\beta, w))$ *has only a partially closed lift to* (Σ_k, v_k).

Remark. It is important that the lift is partially closed – the proof below will not work if it is either closed or nonclosed.

The proof of Lemma 2.1 is contained in §3.

The main theorem.   Before deducing Theorem B from Lemma 2.1, we need the following two known lemmas. Recall that a group G is *at most k-step nilpotent* if $\gamma_{k+1}(G) = 0$.

Lemma 2.2 ([3, §6.1 Proposition 2]). *Let G be a finite group of order 2^k for some $k \geq 2$. Then G is at most $(k-1)$-step nilpotent, and hence $\gamma_k(G) = 0$.*

Lemma 2.3 ([12, Lemma 2.3]). *Let*

$$\Gamma_k \lhd \Gamma_{k-1} \lhd \cdots \lhd \Gamma_0$$

be groups satisfying $[\Gamma_i : \Gamma_{i+1}] = 2$ for $0 \leq i < k$. Then there exists a subgroup $\Gamma' \lhd \Gamma_k$ such that $\Gamma' \lhd \Gamma_0$ and $[\Gamma_0 : \Gamma'] = 2^\ell$ with $\ell \leq 2^k - 1$.

Proof of Theorem B.   As in the statement of the theorem, let Σ be a closed surface whose genus is at least 2 and let α and β be nonisotopic oriented simple closed curves on Σ such that for some $d \geq 3$ the conjugacy classes in $\pi_1(\Sigma)/\gamma_d(\pi_1(\Sigma))$ associated to α and β are the same. Set $n = i(\alpha, \beta)$. Our goal is to show that

$$n \geq \left(\frac{d+2}{2}\right)^c$$

with $c = \frac{\ln(28/25)}{\ln(4)}$.

We first claim that $n \geq 2$. If $n = 1$, then α and β have algebraic intersection number ± 1, which is impossible since they are homologous. Assume now that $n = 0$, i.e. that α and β are disjoint. We divide this into two cases.

- If α and β are nonseparating, then [11, Theorem 1.1.2] implies that $T_\alpha T_\beta^{-1} \in \mathcal{N}_{d-1}(\Sigma)$. However, $T_\alpha T_\beta^{-1} \notin \mathcal{N}_2(\Sigma) \supset \mathcal{N}_{d-1}(\Sigma)$; see [9].

- If α and β are separating, then [11, Theorem 1.1.2] implies that $T_\alpha T_\beta^{-1} \in \mathcal{N}_d(\Sigma)$. However, $T_\alpha T_\beta^{-1} \notin \mathcal{N}_3 \supset \mathcal{N}_d(\Sigma)$; see [2, Appendix A].

Remark. As stated, [11, Theorem 1.1.2] concerns surfaces with one boundary component. However, the desired result for closed surfaces easily follows from this via the map on mapping class groups obtained by gluing a disc to the boundary component.

Fix a hyperbolic metric on Σ and isotope α and β such that they are geodesics. Fix basepoints v and w for α and β, respectively, and regard them as based curves. Choose an arc τ' from v to w. By assumption, $\alpha \in \pi_1(\Sigma, v)$ and $\tau' \cdot \beta \cdot (\tau')^{-1} \in \pi_1(\Sigma, v)$ map to conjugate elements in $\pi_1(\Sigma, v)/\gamma_d(\pi_1(\Sigma, v))$. There thus exists some $\tau'' \in \pi_1(\Sigma, v)$ such that letting $\tau = \tau'' \cdot \tau'$, we have $\tau \cdot \beta^{-1} \cdot \tau^{-1} \cdot \alpha \in \gamma_d(\pi_1(\Sigma, v))$. The triple $((\alpha, v), \tau, (\beta, w))$ is a curve-arc triple.

By Lemma 2.1, for some $k \leq 2\log_{28/25}(n) + 1$, there exists a tower

$$(\Sigma_k, v_k) \longrightarrow (\Sigma_{k-1}, v_{k-1}) \longrightarrow \cdots \longrightarrow (\Sigma_0, v_0) = (\Sigma, v)$$

of regular degree 2 based covers such that $((\alpha, v), \tau, (\beta, w))$ has only a partially closed lift to (Σ_k, v_k). For $0 \leq i \leq k$, let $\Gamma_i \leq \pi_1(\Sigma, v)$ be the subgroup associated to (Σ_i, v_i), so we have a sequence

$$\Gamma_k \lhd \Gamma_{k-1} \lhd \cdots \lhd \Gamma_0$$

of groups. Lemma 2.3 says that there exists a subgroup $\Gamma' \lhd \Gamma_k$ such that $\Gamma' \lhd \Gamma_0$ and $[\Gamma_0 : \Gamma'] = 2^\ell$ with $\ell \leq 2^k - 1$. Since $((\alpha, v), \tau, (\beta, w))$ has only a partially closed lift to (Σ_k, v_k), one element in $\{\alpha, \tau \cdot \beta \cdot \tau^{-1}\}$ lies in Γ_k and the other does not lie in Γ_k. We deduce that $\tau \cdot \beta^{-1} \cdot \tau^{-1} \cdot \alpha \notin \Gamma_k$, and hence $\tau \cdot \beta^{-1} \cdot \tau^{-1} \cdot \alpha \notin \Gamma'$.

Lemma 2.2 implies that Γ_0/Γ' is at most $(\ell-1)$-step nilpotent, so $\gamma_\ell(\Gamma_0) \subset \Gamma'$. Since $\tau \cdot \beta^{-1} \cdot \tau^{-1} \cdot \alpha \notin \Gamma'$ and $\tau \cdot \beta^{-1} \cdot \tau^{-1} \cdot \alpha \in \gamma_d(\Gamma_0)$, we conclude that $d < \ell$, i.e. that

$$d \leq 2^k - 2 \leq 2^{2\log_{28/25}(n)+1} - 2.$$

A little elementary algebra then shows that

$$n \geq \left(\frac{d+2}{2}\right)^c$$

with $c = \frac{\ln(28/25)}{\ln(4)}$, as desired. $\qquad\square$

3. Building a tower

This section is devoted to the proof of Lemma 2.1. We begin by presenting several basic constructions used in the proof in §3.1. We then prove the lemma in §3.2.

3.1. Basic constructions. To simplify our notation, from this point onward we will write $\mathbb{F}_2$ instead of $\mathbb{Z}/2$. For an oriented closed curve δ in Σ, the associated element of $H_1(\Sigma; \mathbb{F}_2)$ is denoted by $[\delta]$.

Lemma 3.1. *Let Σ be a closed surface whose genus is at least 2. Fix a hyperbolic metric on Σ and let $((\alpha, v), \tau, (\beta, w))$ be a curve-arc triple on Σ. Assume that $[\alpha] \neq [\beta]$. Then there exists a regular degree 2 cover $(\tilde{\Sigma}, \tilde{v}) \to (\Sigma, v)$ such that $((\alpha, v), \tau, (\beta, w))$ has only a partially closed lift to $(\tilde{\Sigma}, \tilde{v})$.*

Proof. Pick a homomorphism $\phi \colon H_1(\Sigma; \mathbb{F}_2) \to \mathbb{F}_2$ such that $\phi([\alpha]) \neq \phi([\beta])$, and let $(\tilde{\Sigma}, \tilde{v}) \to (\Sigma, v)$ be the 2-fold regular cover associated to ϕ. Let the lift of $((\alpha, v), \tau, (\beta, w))$ to $(\tilde{\Sigma}, \tilde{v})$ be $((\tilde{\alpha}, \tilde{v}), \tilde{\tau}, (\tilde{\beta}, \tilde{w}))$. Clearly $\tilde{\alpha}$ is closed (resp. $\tilde{\beta}$ is closed) if and only if $\phi([\tilde{\alpha}]) = 0$ (resp. $\phi([\tilde{\beta}]) = 0$). Since $\phi([\alpha]) \neq \phi([\beta])$, we deduce that $((\alpha, v), \tau, (\beta, w))$ has only a partially closed lift to $(\tilde{\Sigma}, \tilde{v})$, as desired. $\square$

Next, we need the following.

Lemma 3.2. *Let Σ be a closed surface whose genus is at least 2. Fix a hyperbolic metric on Σ and let $((\alpha, v), \tau, (\beta, w))$ be a curve-arc triple on Σ. Assume that α and β are disjoint and that $[\alpha] = [\beta] \neq 0$. Then there exists a regular degree 2 cover $(\tilde{\Sigma}, \tilde{v}) \to (\Sigma, v)$ such that $((\alpha, v), \tau, (\beta, w))$ has a closed lift $((\tilde{\alpha}, \tilde{v}), \tilde{\tau}, (\tilde{\beta}, \tilde{w}))$ to $(\tilde{\Sigma}, \tilde{v})$ satisfying $[\tilde{\alpha}] \neq [\tilde{\beta}]$.*

Proof. The curves α and β are disjoint and homologous over $\mathbb{F}_2$. It is easy to see that this implies that they are actually homologous over $\mathbb{Z}$, i.e. that they bound an embedded subsurface of Σ. As is shown in Figure 1, there exists a regular degree 2 cover $(\tilde{\Sigma}, \tilde{v}) \to (\Sigma, v)$ with the following properties.
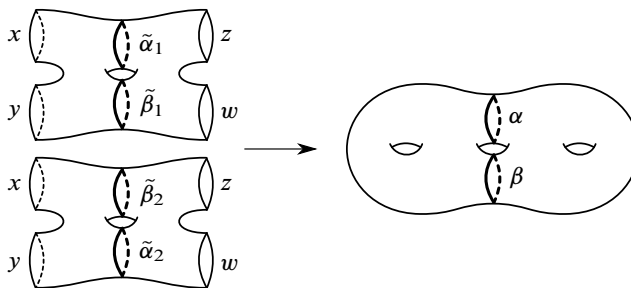

Figure 1. In the left hand surface, the boundary components are glued in pairs to form a closed connected surface $\tilde{\Sigma}$. This has a regular degree 2 covering map to the right hand surface Σ; the deck group exchanges the top and bottom piece while flipping them. The preimage of α is $\tilde{\alpha}_1 \sqcup \tilde{\alpha}_2$ and the preimage of β is $\tilde{\beta}_1 \sqcup \tilde{\beta}_2$.

- The preimage of α in $\widetilde{\Sigma}$ consists of two disjoint simple closed curves, and similarly for β.

- If $\tilde{\alpha}$ and $\tilde{\beta}$ are any components of the preimage in $\widetilde{\Sigma}$ of α and β, respectively, then $[\tilde{\alpha}] \neq [\tilde{\beta}]$.

Clearly $(\widetilde{\Sigma}, \tilde{v}) \to (\Sigma, v)$ is the desired cover. □

A similar idea will yield the following.

Lemma 3.3. *Let Σ be a closed surface whose genus is at least 2. Let δ be a nonnullhomotopic oriented simple closed curve on Σ such that $[\delta] = 0$. Then there exists a regular degree 2 cover $\widetilde{\Sigma} \to \Sigma$ such that the preimage of δ in $\widetilde{\Sigma}$ has two components $\tilde{\delta}_1$ and $\tilde{\delta}_2$ satisfying $[\tilde{\delta}_i] \neq 0$ for $i = 1, 2$.*

Proof. Using the fact that δ is a simple closed curve which is nullhomologous over $\mathbb{F}_2$, it is easy to see that δ is actually nullhomologous over $\mathbb{Z}$, i.e. that δ separates Σ. The needed cover is as depicted in Figure 2. □
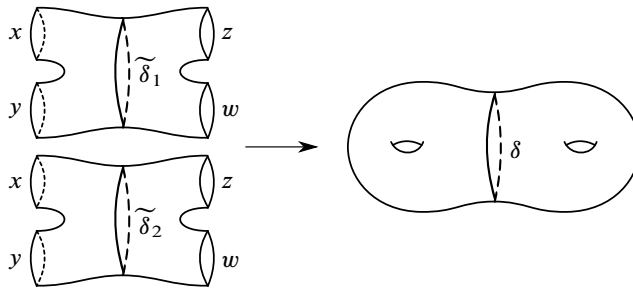

Figure 2. In the left hand surface, the boundary components are glued in pairs to form a closed connected surface $\widetilde{\Sigma}$. This has a regular degree 2 covering map to the right hand surface Σ; the deck group exchanges the top and bottom piece while flipping them. The preimage of δ is $\tilde{\delta}_1 \sqcup \tilde{\delta}_2$.

Finally, the most important construction for the proof of Lemma 2.1 is the following.

Lemma 3.4. *Let Σ_0 be a closed surface whose genus is at least 2. Fix a hyperbolic metric on Σ_0 and let $((\alpha_0, v_0), \tau_0, (\beta_0, w_0))$ be a curve-arc triple on Σ_0. Set $n_0 = i(\alpha_0, \beta_0)$. Assume that $n_0 \geq 2$ and that neither $[\alpha_0]$ nor $[\beta_0]$ vanishes. Then for some q satisfying $1 \leq q \leq 2$, there exists a tower*

$$(\Sigma_q, v_q) \longrightarrow \cdots \longrightarrow (\Sigma_0, v_0)$$

of regular degree 2 based covers such that one of the following holds.

- *The curve-arc triple $((\alpha_0, v_0), \tau_0, (\beta_0, w_0))$ has only a partially closed lift to (Σ_q, v_q), or*

- *The curve-arc triple $((\alpha_0, v_0), \tau_0, (\beta_0, w_0))$ has a closed lift*

$$((\alpha_q, v_q), \tau_q, (\beta_q, w_q))$$

to (Σ_q, v_q) such that $i(\alpha_q, \beta_q) \leq \frac{25}{28} n_0$.

The proof of Lemma 3.4 is difficult and will be postponed until §4.

3.2. The proof of Lemma 2.1. All the pieces are now in place for the proof of Lemma 2.1.

Proof of Lemma 2.1. We first recall the statement of Lemma 2.1. Let Σ be a closed surface whose genus is at least 2. Fix a hyperbolic metric on Σ and let $((\alpha, v), \tau, (\beta, w))$ be a curve-arc triple on Σ. Set $n = i(\alpha, \beta)$. Our goal is to show that for some k satisfying

$$k \leq \begin{cases} 2\log_{28/25}(n) + 1 & \text{if } n \geq 2, \\ 3 & \text{if } 0 \leq n \leq 1, \end{cases}$$

there exists a tower

$$(\Sigma_k, v_k) \longrightarrow (\Sigma_{k-1}, v_{k-1}) \longrightarrow \cdots \longrightarrow (\Sigma_0, v_0) = (\Sigma, v) \tag{2}$$

of regular degree 2 based covers such that $((\alpha, v), \tau, (\beta, w))$ has only a partially closed lift to (Σ_k, v_k). The proof is divided into two steps.

Step 1. *Assume that neither $[\alpha]$ nor $[\beta]$ vanishes. Then we can find a tower as in (2) with*

$$k \leq \begin{cases} 2\log_{28/25}(n) & \text{if } n \geq 2, \\ 2 & \text{if } 0 \leq n \leq 1. \end{cases}$$

The proof is by induction on n. Set $(\Sigma_0, v_0) = (\Sigma, v)$ and $((\alpha_0, v_0), \tau_0, (\beta_0, w_0)) = ((\alpha, v), \tau, (\beta, w))$.

Base cases. Suppose $n = 0$ or $n = 1$. We divide the verification of these base cases into three separate cases.

- $n = 0$ and $[\alpha_0] \neq [\beta_0]$. Lemma 3.1 implies that there exists a regular degree 2 cover $(\Sigma_1, v_1) \rightarrow (\Sigma_0, v_0)$ such that $((\alpha_0, v_0), \tau_0, (\beta_0, w_0))$ has only a partially closed lift to (Σ_1, v_1).

- $n = 0$ and $[\alpha] = [\beta]$. We first apply Lemma 3.2 to obtain a regular degree 2 cover $(\Sigma_1, v_1) \to (\Sigma_0, v_0)$ such that $((\alpha_0, v_0), \tau_0, (\beta_0, w_0))$ has a closed lift $((\alpha_1, v_1), \tau_1, (\beta_1, w_1))$ to (Σ_1, v_1) satisfying $[\alpha_1] \neq [\beta_1]$. Lemma 3.1 then implies that there is a regular degree 2 cover $(\Sigma_2, v_2) \to (\Sigma_1, v_1)$ such that $((\alpha_1, v_1), \tau_1, (\beta_1, w_1))$ has only a partially closed lift to (Σ_2, v_2).

- $n = 1$. The $\mathbb{F}_2$-algebraic intersection numbers of $[\alpha_0]$ and $[\beta_0]$ must be 1, so in particular we must have $[\alpha_0] \neq [\beta_0]$. Lemma 3.1 then implies that there exists a regular degree 2 cover $(\Sigma_1, v_1) \to (\Sigma_0, v_0)$ such that $((\alpha_0, v_0), \tau_0, (\beta_0, w_0))$ has only a partially closed lift to (Σ_1, v_1).

Inductive step. Now assume that $n \geq 2$ and that the claim is true for all smaller values of n. Let

$$(\Sigma_\ell, v_\ell) \longrightarrow \cdots \longrightarrow (\Sigma_0, v_0)$$

be the tower of regular degree 2 covers provided by Lemma 3.4, so $1 \leq \ell \leq 2$. If $((\alpha_0, v_0), \tau_0, (\beta_0, w_0))$ has only a partially closed lift to (Σ_ℓ, v_ℓ), then we are done. Otherwise, let $((\alpha_\ell, v_\ell), \tau_\ell, (\beta_\ell, w_\ell))$ be the lift of $((\alpha_0, v_0), \tau_0, (\beta_0, w_0))$ to (Σ_ℓ, v_ℓ). Set $n_\ell = i(\alpha_\ell, \beta_\ell)$, so $n_\ell \leq \frac{25}{28}n$. By induction, for some $k \geq \ell$ satisfying

$$k - \ell \leq \begin{cases} 2\log_{28/25}(n_\ell) & \text{if } n_\ell \geq 2, \\ 2 & \text{if } 0 \leq n_\ell \leq 1, \end{cases}$$

there exists a tower of regular degree 2 covers

$$(\Sigma_k, v_k) \longrightarrow \cdots \longrightarrow (\Sigma_\ell, v_\ell)$$

such that $((\alpha_\ell, v_\ell), \tau_\ell, (\beta_\ell, w_\ell))$ has only a partially closed lift to (Σ_k, v_k).

We claim that

$$(\Sigma_k, v_k) \longrightarrow \cdots \longrightarrow (\Sigma_0, v_0)$$

is the desired tower of regular degree 2 covers. The only thing that needs verification is the bound on k. There are two cases. If $n_\ell \leq 1$, then $k - \ell \leq 2$, and thus

$$k \leq \ell + 2 \leq 2 + 2 \leq 2\log_{28/25}(2) \leq 2\log_{28/25}(n),$$

as desired. Otherwise, $n_\ell \geq 2$. Hence $k - \ell \leq 2\log_{28/25}(n_\ell)$ and

$$k \leq \ell + 2\log_{28/25}(n_\ell) \leq 2 + 2\log_{28/25}\left(\frac{25}{28}n\right) = 2\log_{28/25}(n),$$

as desired.

Step 2. *Assume that at least one of $[\alpha]$ and $[\beta]$ vanishes. Then we can find a tower as in* (2) *with*

$$k \leq \begin{cases} 2\log_{28/25}(n) + 1 & \text{if } n \geq 2, \\ 3 & \text{if } 0 \leq n \leq 1. \end{cases}$$

We will give the details for the case where $[\alpha] = 0$; the case $[\beta] = 0$ is similar. Lemma 3.3 implies that there exists a regular degree 2 cover $(\Sigma_1, v_1) \to (\Sigma_0, v_0)$ such that the based lift (α_1, v_1) is a closed curve satisfying $[a_1] \neq 0$. Let $((\alpha_1, v_1), \tau_1, (\beta_1, w_1))$ be the lift of $((\alpha_0, v_0), \tau_0, (\beta_0, w_0)$ to (Σ_1, v_1). If β_1 is not a closed curve, then $((\alpha_0, v_0), \tau_0, (\beta_0, w_0)$ has only a partially closed lift to (Σ_1, v_1) and we are done. We can therefore assume that β_1 is closed. If $[\beta_1] = 0$, then we can apply Lemma 3.1 to obtain a regular degree 2 cover $(\Sigma_2, v_2) \to (\Sigma_1, v_1)$ such that $((\alpha_1, v_1), \tau_1, (\beta_1, w_1))$ has only a partially closed lift to (Σ_2, v_2), and we are done. We can assume therefore that $[\beta_1] \neq 0$. The desired tower of covers is then obtained by applying Step 1 to $((\alpha_1, v_1), \tau_1, (\beta_1, w_1))$ and (Σ_1, v_1).   □

4. Eliminating 3/28 of the intersections

This section is devoted to the proof of Lemma 3.4. The skeleton of the proof is in §4.1. This skeleton depends on two lemmas which are proven in §4.2 and §4.3.

4.1. Skeleton of the proof of Lemma 3.4.
We begin with some definitions. Fix a closed surface Σ and equip Σ with a hyperbolic metric. Let $((\alpha, v), \tau, (\beta, w))$ be a curve-arc triple on Σ. A β-arc of α is a subarc of α both of whose endpoints lie in β. A β-arc μ of α defines in a natural way an element $[\mu]_\beta$ in the relative homology group $H_1(\Sigma, \beta; \mathbb{F}_2)$; we will call μ a *good β-arc* if $[\mu]_\beta \neq 0$ and a *bad β-arc* if $[\mu]_\beta = 0$. A set $\mathcal{A}$ of β-arcs of α will be called a *set of disjoint β-arcs* of α if for all distinct $\mu, \mu' \in \mathcal{A}$, we have $\mu \cap \mu' = \emptyset$. If $\mathcal{A}$ is a set of disjoint β-arcs of α, then define

$$\mathcal{A}^g = \{\mu \in \mathcal{A} \mid \mu \text{ is good}\} \quad \text{and} \quad \mathcal{A}^b = \{\mu \in \mathcal{A} \mid \mu \text{ is bad}\},$$

so $\mathcal{A} = \mathcal{A}^g \sqcup \mathcal{A}^b$.

The following lemma shows that we can eliminate intersections using half of the good β-arcs of α. The reason for our notation will become clear during the proof of Lemma 3.4 below.

Lemma 4.1. *Let Σ_1 be a closed surface equipped with a hyperbolic metric and let $((\alpha_1, v_1), \tau_1, (\beta_1, w_1))$ be a curve-arc triple on Σ_1. Assume that $[\alpha_1] = [\beta_1] \neq 0$. Set $n_1 = i(\alpha_1, \beta_1)$. Let $\mathcal{A}_1$ be a set of disjoint β_1-arcs of α_1. Then there exists a regular degree 2 cover $(\Sigma_2, v_2) \to (\Sigma_1, v_1)$ such that $((\alpha_1, v_1), \tau_1, (\beta_1, w_1))$ has a closed lift $((\alpha_2, v_2), \tau_2, (\beta_2, w_2))$ with the following property.*

- *Set $n_2 = i(\alpha_2, \beta_2)$. Then $n_2 \leq n_1 - \frac{1}{2}|\mathcal{A}_1^g|$.*

The proof of Lemma 4.1 is in §4.2. It will turn out that an appropriate "random" cover will do the job.

Unfortunately, it is possible for most of the arcs in a set of disjoint β-arcs of α to be bad. The following lemma shows that we can pass to a cover that makes at least some of the arcs good (or, even better, eliminates some of the intersections). See the remark following the lemma for an explanation of the inequalities in its conclusion.

Lemma 4.2. *Let Σ_0 be a closed surface equipped with a hyperbolic metric and let $((\alpha_0, v_0), \tau_0, (\beta_0, w_0))$ be a curve-arc triple on Σ_0. Assume that $[\alpha_0] = [\beta_0] \neq 0$. Set $n_0 = i(\alpha_0, \beta_0)$. Let $\mathcal{A}_0$ be a set of disjoint β_0-arcs of α_0. Then there exists a regular degree 2 cover $(\Sigma_1, v_1) \to (\Sigma_0, v_0)$ such that $((\alpha_0, v_0), \tau_0, (\beta_0, w_0))$ has a closed lift $((\alpha_1, v_1), \tau_1, (\beta_1, w_1))$ with the following property.*

- *Set $n_1 = i(\alpha_1, \beta_1)$. Then there exists a set $\mathcal{A}_1$ of disjoint β_1-arcs of α_1 and some $r \geq 0$ such that*

$$|\mathcal{A}_1^g| \geq |\mathcal{A}_0^g| + \frac{3}{7}|\mathcal{A}_0^b| - r \quad and \quad n_1 \leq n_0 - r. \tag{3}$$

The proof of Lemma 4.2 is in §4.3. Again, we will see that a suitable "random" cover has the property we seek.

Remark. The meaning of the inequalities in (3) is that $\mathcal{A}_1^g$ is made up of pieces corresponding to $\mathcal{A}_0^g$ (the old good arcs) and $\frac{3}{7}$ of $\mathcal{A}_0^b$ (the old bad arcs), minus some number r of arcs that correspond to eliminated intersections.

We now show how to derive Lemma 3.4 from Lemmas 4.1–4.2.

Proof of Lemma 3.4. We begin by recalling the statement. We are given a closed surface Σ_0 whose genus is at least 2 which is equipped with a hyperbolic metric. Also, we are given a curve-arc triple $((\alpha_0, v_0), \tau_0, (\beta_0, w_0))$ on Σ_0. Setting $n_0 = i(\alpha_0, \beta_0)$, we are given that $n_0 \geq 2$ and that neither $[\alpha_0]$ nor $[\beta_0]$ vanishes. Our goal is to prove that for some ℓ satisfying $1 \leq \ell \leq 2$, there exists a tower

$$(\Sigma_\ell, v_\ell) \longrightarrow \cdots \longrightarrow (\Sigma_0, v_0)$$

of regular degree 2 based covers such that one of the following holds.

- The curve-arc triple $((\alpha_0, v_0), \tau_0, (\beta_0, w_0))$ has only a partially closed lift to (Σ_ℓ, v_ℓ), or

- The curve-arc triple $((\alpha_0, v_0), \tau_0, (\beta_0, w_0))$ has a closed lift

$$((\alpha_\ell, v_\ell), \tau_\ell, (\beta_\ell, w_\ell))$$

to (Σ_ℓ, v_ℓ) such that $i(\alpha_\ell, \beta_\ell) \leq \frac{25}{28}n_0$.

First, if $[\alpha_0] \neq [\beta_0]$, then Lemma 3.1 says that there exists a regular degree 2 cover $(\Sigma_1, v_1) \to (\Sigma_0, v_0)$ such that $((\alpha_0, v_0), \tau_0, (\beta_0, w_0))$ has only a partially closed lift to (Σ_1, v_1), and we are done. We can assume therefore that $[\alpha_0] = [\beta_0]$. This implies that $n_0 = i(\alpha_0, \beta_0) \geq 2$ must be an even number. We can therefore find a set $\mathcal{A}_0$ of disjoint β_0-arcs of α_0 such that $|\mathcal{A}_0| = n_0/2$. We then apply Lemma 4.2 to get a regular degree 2 cover $(\Sigma_1, v_1) \to (\Sigma_0, v_0)$ such that $((\alpha_0, v_0), \tau_0, (\beta_0, w_0))$ has a closed lift $((\alpha_1, v_1), \tau_1, (\beta_1, w_1))$. The lemma also gives a set $\mathcal{A}_1$ of disjoint β_1-arcs of α_1 such that for some $r \geq 0$,

$$|\mathcal{A}_1^g| \geq |\mathcal{A}_0^g| + \frac{3}{7}|\mathcal{A}_0^b| - r \quad \text{and} \quad n_1 \leq n_0 - r,$$

where $n_1 = i(\alpha_1, \beta_1)$.

Again, if $[\alpha_1] \neq [\beta_1]$, then Lemma 3.1 says that there exists a regular degree 2 cover $(\Sigma_2, v_2) \to (\Sigma_1, v_1)$ such that $((\alpha_1, v_1), \tau_1, (\beta_1, w_1))$ has only a partially closed lift to (Σ_2, v_2), and we are done. We can assume therefore that $[\alpha_1] = [\beta_1]$. Lemma 4.1 thus gives a regular degree 2 cover $(\Sigma_2, v_2) \to (\Sigma_1, v_1)$ such that $((\alpha_1, v_1), \tau_1, (\beta_1, w_1))$ has a closed lift $((\alpha_2, v_2), \tau_2, (\beta_2, w_2))$ satisfying

$$
\begin{aligned}
i(\alpha_2, \beta_2) &\leq n_1 - \frac{1}{2}|\mathcal{A}_1^g| \leq (n_0 - r) - \frac{1}{2}\left(|\mathcal{A}_0^g| + \frac{3}{7}|\mathcal{A}_0^b| - r\right) \\
&\leq n_0 - \frac{1}{2}\left(|\mathcal{A}_0^g| + \frac{3}{7}|\mathcal{A}_0^b|\right) \\
&\leq n_0 - \frac{1}{2}\left(\frac{3}{7}|\mathcal{A}_0|\right) \\
&= n_0 - \frac{1}{2}\left(\frac{3}{7}\left(\frac{1}{2}n_0\right)\right) \\
&= \frac{25}{28}n_0,
\end{aligned}
$$

as desired. $\square$

4.2. Resolving intersections using good arcs.

To prove Lemma 4.1, we need the following lemma.

Lemma 4.3. *Let $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{F}_2^n$ be nonzero vectors* (*not necessarily distinct*). *Then there exists a linear map $f: \mathbb{F}_2^n \to \mathbb{F}_2$ such that $f(\vec{v}_i) = 1$ for at least half of the $\vec{v}_i$, i.e. such that $\{i \mid 1 \leq i \leq m, \ f(\vec{v}_i) = 1\}$ has cardinality at least $m/2$.*

Proof. Let Ω be the probability space consisting of all linear maps $\mathbb{F}_2^n \to \mathbb{F}_2$, each given equal probability. Let $\mathcal{X}: \Omega \to \mathbb{R}$ be the random variable that takes $f \in \Omega$ to the cardinality of the set $\{i \mid 1 \leq i \leq m, \ f(\vec{v}_i) = 1\}$. We will prove that the expected value $E(\mathcal{X})$ of $\mathcal{X}$ is $m/2$, which clearly implies that there exists *some* element $f \in \Omega$ such that $\mathcal{X}(f) \geq m/2$.

To prove the desired claim, for $1 \leq i \leq m$ let $\mathcal{X}_i \colon \Omega \to \mathbb{R}$ be the random variable that takes $f \in \Omega$ to 1 if $f(\vec{v}_i) = 1$ and to 0 if $f(\vec{v}_i) = 0$. Viewing $\vec{v}_i$ as an element of the double dual $(\mathbb{F}_2^n)^{**}$, the kernel of $\vec{v}_i$ consists of exactly half of the elements of $(\mathbb{F}_2^n)^*$. This implies $E(\mathcal{X}_i) = 1/2$. Using linearity of expectation (which, recall, does not require that the random variables be independent), we get that

$$E(\mathcal{X}) = E\left(\sum_{i=1}^{m} \mathcal{X}_i\right) = \sum_{i=1}^{m} E(\mathcal{X}_i) = \sum_{i=1}^{m} \frac{1}{2} = \frac{m}{2},$$

as desired. $\qquad\square$

Remark. One can give a (somewhat more complicated) non-probabilistic proof of Lemma 4.3; however, we do not know a non-probabilistic proof of Lemma 4.6 below, which is needed for the proof of Lemma 4.2.

Proof of Lemma 4.1. We start by recalling the setup. We are given a closed surface Σ_1 equipped with a hyperbolic metric and a curve-arc triple $((\alpha_1, v_1), \tau_1, (\beta_1, w_1))$ on Σ_1 such that $[\alpha_1] = [\beta_1] \neq 0$. We are also given a set $\mathcal{A}_1$ of disjoint β_1-arcs of α_1. Letting $n_1 = i(\alpha_1, \beta_1)$, our goal is to construct a regular degree 2 cover $(\Sigma_2, v_2) \to (\Sigma_1, v_1)$ such that $((\alpha_1, v_1), \tau_1, (\beta_1, w_1))$ has a closed lift $((\alpha_2, v_2), \tau_2, (\beta_2, w_2))$ satisfying

$$i(\alpha_2, \beta_2) \leq n_1 - \frac{1}{2}|\mathcal{A}_1^g|.$$

Lemma 4.3 implies that there exists a linear map

$$\phi \colon \mathrm{H}_1(\Sigma_1, \beta_1; \mathbb{F}_2) \longrightarrow \mathbb{F}_2$$

such that $\{\mu \in \mathcal{A}_1^g \mid \phi([\mu]_{\beta_1}) = 1\}$ has cardinality at least $\frac{1}{2}|\mathcal{A}_1^g|$. Let

$$\psi \colon \mathrm{H}_1(\Sigma_1; \mathbb{F}_2) \longrightarrow \mathbb{F}_2$$

be the composition of ϕ with the natural map $\mathrm{H}_1(\Sigma_1; \mathbb{F}_2) \to \mathrm{H}_1(\Sigma_1, \beta_1; \mathbb{F}_2)$ and let $(\Sigma_2, v_2) \to (\Sigma_1, v_1)$ be the regular 2-fold cover associated to ψ. Since $\psi([\beta_1]) = 0$ and $[\alpha_1] = [\beta_1]$, the curve-arc triple $((\alpha_1, v_1), \tau_1, (\beta_1, w_1))$ has a closed lift $((\alpha_2, v_2), \tau_2, (\beta_2, w_2))$ to (Σ_2, v_2).

It remains to prove that $i(\alpha_2, \beta_2) \leq n_1 - \frac{1}{2}|\mathcal{A}_1^g|$. Regarding S^1 as the unit circle in $\mathbb{C}$, parametrize α_1 via a continuous map $f_1 \colon (S^1, 1) \to (\Sigma_1, v_1)$. Define $I_1 = \{\theta \in S^1 \mid f_1(\theta) \in \beta_1\}$. Since α_1 and β_1 are hyperbolic geodesics, we have $n_1 = |I_1|$. Lift f_1 to a map $f_2 \colon (S^1, 1) \to (\Sigma_2, v_2)$ whose image is α_2 and define $I_2 = \{\theta \in S^1 \mid f_2(\theta) \in \beta_2\}$. Again, since α_2 and β_2 are hyperbolic geodesics, we must have $i(\alpha_2, \beta_2) = |I_2|$. Also, by construction we have $I_2 \subset I_1$. It is enough, therefore, to prove that $|I_1 \setminus I_2| \geq \frac{1}{2}|\mathcal{A}_1^g|$.

To do this, it is enough to prove that

$$|I_1 \setminus I_2| \geq |\{\mu \in \mathcal{A}_1^g \mid \phi([\mu]_{\beta_1}) = 1\}|. \tag{4}$$

Consider $\mu \in \mathcal{A}_1^g$ such that $\phi([\mu]_{\beta_1}) = 1$. There exist $\theta, \theta' \in I_1$ such that μ begins at $f_1(\theta)$ and ends at $f_1(\theta')$. To prove (4), it is enough to prove that at most one of θ and θ' lie in I_2. Assume otherwise, so $\theta, \theta' \in I_2$. Let $\tilde{\mu}$ be the oriented arc of α_2 beginning at $f_2(\theta)$ and ending at $f_2(\theta')$ and covering μ. Also, let $\tilde{\eta}$ be one of the two oriented arcs of β_2 beginning at $f_2(\theta')$ and ending at $f_2(\theta)$. Then $\tilde{\eta}$ covers an oriented arc η of β_1 beginning at $f_1(\theta')$ and ending at $f_1(\theta)$. The closed loop $\mu \cdot \eta$ on Σ_1 lifts to the closed loop $\tilde{\mu} \cdot \tilde{\eta}$ on Σ_2, so $\psi([\mu \cdot \eta]) = 0$. However, we also have

$$\psi([\mu \cdot \eta]) = \phi([\mu]_{\beta_1}) = 1,$$

a contradiction.   □

4.3. Lifting bad arcs to good arcs.

We begin by clarifying the topological nature of bad arcs.

Lemma 4.4. *Let Σ be a closed surface equipped with a hyperbolic metric and let $((\alpha, v), \tau, (\beta, w))$ be a curve-arc triple on Σ. Assume that $[\alpha] = [\beta] \neq 0$. Also, let μ be a bad β-arc of α which goes from $p_1 \in \alpha \cap \beta$ to $p_2 \in \alpha \cap \beta$. Then there exists an oriented arc η of β which goes from p_2 to p_1 with the following properties.*

- *The closed curve $\mu \cdot \eta$ is a separating simple closed curve which is not nullhomotopic.*

- *The curve $\mu \cdot \eta$ is isotopic to a curve which is disjoint from β.*

Proof. There are two arcs η and η' of β going from p_2 to p_1. The elements $[\mu \cdot \eta], [\mu \cdot \eta'] \in H_1(\Sigma; \mathbb{F}_2)$ both map to $[\mu]_\beta \in H_1(\Sigma, \beta; \mathbb{F}_2)$, which vanishes by assumption. It follows that $[\mu \cdot \eta]$ and $[\mu \cdot \eta']$ both lie in the kernel of the map $H_1(\Sigma; \mathbb{F}_2) \to H_1(\Sigma, \beta; \mathbb{F}_2)$, which consists of 0 and $[\beta]$. Letting $\bar{\eta}'$ be η' traversed in the opposite direction, we have

$$[\mu \cdot \eta] - [\mu \cdot \eta'] = [\eta \cdot \bar{\eta}'] = [\beta].$$

We conclude that either $[\mu \cdot \eta]$ or $[\mu \cdot \eta']$ must be 0; relabeling, we can assume that $[\mu \cdot \eta] = 0$. The curve $\mu \cdot \eta$ is a simple closed curve which is nullhomologous over $\mathbb{F}_2$. It is not hard to see that this implies that $\mu \cdot \eta$ is in fact nullhomologous over $\mathbb{Z}$, i.e. that $\mu \cdot \eta$ is a separating simple closed curve (the key point here being that $\mu \cdot \eta$ is simple). If $\mu \cdot \eta$ were nullhomotopic, then it would bound a disc. This would imply that α and β could be homotoped so as to intersect fewer times, which is impossible since they are hyperbolic geodesics.

It remains to prove that $\mu \cdot \eta$ is isotopic to a curve which is disjoint from β. The algebraic intersection number of $\mu \cdot \eta$ and β is 0 since $\mu \cdot \eta$ is nullhomologous. If the signs of the intersections of α and β at p_1 and p_2 were the same, then $\mu \cdot \eta$ would have algebraic intersection number 1 with β (see Figure 3.a), so those signs must be opposite. The desired result now follows from Figure 3.b. □

Figure 3. a. On the left is a local picture of μ and β when the signs of the intersections of α and β are the same at p_1 and p_2. On the right is shown an isotopy of $\mu \cdot \eta$ indicating the algebraic intersection number is 1 between $\mu \cdot \eta$ and β.   b. On the left is a local picture of μ and β when the signs of the intersections of α and β are different at p_1 and p_2. On the right is shown an isotopy of $\mu \cdot \eta$ indicating that $\mu \cdot \eta$ can be made disjoint from β.

To "resolve" bad arcs into good arcs, we will need to lift separating curves to nonseparating curves as in the following lemma. Note that if δ is a simple closed separating curve on Σ which divides Σ into subsurfaces S and S', then $H_1(\Sigma; \mathbb{F}_2) = H_1(S; \mathbb{F}_2) \oplus H_1(S'; \mathbb{F}_2)$.

Lemma 4.5. *Let Σ be a closed surface and let δ be a simple closed separating curve on Σ which divides Σ into subsurfaces S and S'. Let $\phi \colon H_1(\Sigma; \mathbb{F}_2) \to \mathbb{F}_2$ be a linear map such that $\phi|_{H_1(S;\mathbb{F}_2)} \neq 0$ and $\phi|_{H_1(S';\mathbb{F}_2)} \neq 0$. Let $\widetilde{\Sigma} \to \Sigma$ be the regular 2-fold cover associated to ϕ. Finally, let $\widetilde{\delta}$ be a component of the preimage of δ in $\widetilde{\Sigma}$. Then $[\widetilde{\delta}] \neq 0$.*

Proof. Let $\widetilde{S}$ and $\widetilde{S}'$ be the preimages in $\widetilde{\Sigma}$ of S and S', respectively. The maps $\widetilde{S} \to S$ and $\widetilde{S}' \to S'$ are 2-fold covering maps. Moreover (and this is the key observation), the assumption that $\phi|_{H_1(S;\mathbb{F}_2)} \neq 0$ and $\phi|_{H_1(S';\mathbb{F}_2)} \neq 0$ implies that both $\widetilde{S}$ and $\widetilde{S}'$ are connected. Since δ is a separating curve, we have $[\delta] = 0$ and thus $\phi([\delta]) = 0$, which implies that the preimage of δ in $\widetilde{\Sigma}$ has two components. These two components are the boundary components of $\widetilde{S}$ and $\widetilde{S}'$, so we deduce that both $\widetilde{S}$ and $\widetilde{S}'$ have two boundary components. The surface $\widetilde{\Sigma}$ is obtained by gluing the boundary components of $\widetilde{S}$ to the boundary components of $\widetilde{S}'$. Consequently, $\widetilde{\delta}$ (which is one of those boundary components) does not separate $\widetilde{\Sigma}$, so $[\widetilde{\delta}] \neq 0$. □

We thus need to construct linear maps $H_1(\Sigma; \mathbb{F}_2) \to \mathbb{F}_2$ which are nontrivial on a large number of splittings of $H_1(\Sigma; \mathbb{F}_2)$. We will do this with the following lemma, which plays the same role in the proof of Lemma 4.2 that Lemma 4.3 played in the proof of Lemma 4.1.

Lemma 4.6. *Fix some $n \geq 3$. For $1 \leq i \leq m$, let $V_i, W_i \subset \mathbb{F}_2^n$ be nontrivial subspaces such that $\mathbb{F}_2^n = V_i \oplus W_i$. Then there exists a linear map $f : \mathbb{F}_2^n \to \mathbb{F}_2$ such that $\{i \mid 1 \leq i \leq m, \ f|_{V_i} \neq 0, \ f|_{W_i} \neq 0\}$ has cardinality at least $\frac{3}{7}m$.*

Proof. Let Ω be the probability space consisting of all *nonzero* linear maps $\mathbb{F}_2^n \to \mathbb{F}_2$, each given equal probability. Let $\mathfrak{X} : \Omega \to \mathbb{R}$ be the random variable that takes $f \in \Omega$ to the cardinality of the set $\{i \mid 1 \leq i \leq m, \ f|_{V_i} \neq 0, \ f|_{W_i} \neq 0\}$. We will prove that the expected value $E(\mathfrak{X})$ of $\mathfrak{X}$ is at least $\frac{3}{7}m$, which clearly implies that there exists *some* element $f \in \Omega$ such that $\mathfrak{X}(f) \geq \frac{3}{7}m$.

To prove the desired claim, for $1 \leq i \leq m$ let $\mathfrak{X}_i : \Omega \to \mathbb{R}$ be the random variable that takes $f \in \Omega$ to 1 if $f|_{V_i}, f|_{W_i} \neq 0$ and to 0 if $f|_{V_i} = 0$ or $f|_{W_i} = 0$. Using linearity of expectation, we have

$$E(\mathfrak{X}) = E\left(\sum_{i=1}^{m} \mathfrak{X}_i\right) = \sum_{i=1}^{m} E(\mathfrak{X}_i).$$

Fixing some $1 \leq i \leq m$, it is therefore enough to prove that $E(\mathfrak{X}_i) \geq 3/7$, i.e. to show that the probability that a random nonzero linear map $f : \mathbb{F}_2^n \to \mathbb{F}_2$ satisfies $f|_{V_i}, f|_{W_i} \neq 0$ is at least $3/7$.

Set $a = \dim(V_i)$, so $1 \leq a \leq n - 1$ and $\dim(W_i) = n - a$. There are $2^a - 1$ (resp. $2^{n-a} - 1$) nonzero linear maps $V_i \to \mathbb{F}_2$ (resp. $W_i \to \mathbb{F}_2$). This implies that there are $(2^a - 1)(2^{n-a} - 1)$ linear maps $f : \mathbb{F}_2^n \to \mathbb{F}_2$ such that $f|_{V_i}, f|_{W_i} \neq 0$. Since there are $2^n - 1$ nonzero linear maps $\mathbb{F}_2^n \to \mathbb{F}_2$, we deduce that the probability that a random nonzero linear map $f : \mathbb{F}_2^n \to \mathbb{F}_2$ satisfies $f|_{V_i}, f|_{W_i} \neq 0$ is $\frac{(2^a-1)(2^{n-a}-1)}{2^n-1}$. Lemma 4.7 below says that this is at least $3/7$, as desired. $\square$

Lemma 4.7. *Set $D = \{(a, n) \mid n \geq 3 \text{ and } 1 \leq a \leq n - 1\}$, and define*

$$\zeta : D \longrightarrow \mathbb{R}$$

via the formula

$$\zeta(a, n) = \frac{(2^a - 1)(2^{n-a} - 1)}{2^n - 1}.$$

Then ζ is bounded below by $\zeta(1, 3) = 3/7$.

Proof. For a fixed $n_0 \geq 3$, it is easy to see that the function $a \mapsto \zeta(a, n_0)$ defined on the domain $1 \leq a \leq n_0 - 1$ attains its global minima at the boundary points $a = 1$ and $a = n_0 - 1$. Moreover, $\zeta(1, n_0) = \zeta(n_0 - 1, n_0)$. Finally, the function $n \mapsto \zeta(1, n)$ defined on the domain $n \geq 3$ achieves its global minimum at the point $n = 3$. The lemma follows. $\qquad\square$

We are finally in a position to prove Lemma 4.2.

Proof of Lemma 4.2. We begin by recalling the setup. We are given a closed surface Σ_0 equipped with both a hyperbolic metric and a curve-arc triple $((\alpha_0, v_0), \tau_0, (\beta_0, w_0))$ on Σ_0 such that $[\alpha_0] = [\beta_0] \neq 0$. We are also given a set $\mathcal{A}_0$ of disjoint β_0-arcs of α_0. Our goal is to construct a regular degree 2 cover $(\Sigma_1, v_1) \to (\Sigma_0, v_0)$ such that $((\alpha_0, v_0), \tau_0, (\beta_0, w_0))$ has a closed lift $((\alpha_1, v_1), \tau_1, (\beta_1, w_1))$ and there exists a set $\mathcal{A}_1$ of disjoint β_1-arcs of α_1 and some $r \geq 0$ satisfying

$$|\mathcal{A}_1^g| \geq |\mathcal{A}_0^g| + \frac{3}{7}|\mathcal{A}_0^b| - r \quad \text{and} \quad n_1 \leq n_0 - r,$$

where $n_1 = i(\alpha_1, \beta_1)$.

Write $\mathcal{A}_0^b = \{\mu_1, \ldots, \mu_{|\mathcal{A}_0^b|}\}$. Using Lemma 4.4, we obtain oriented arcs $\eta_1, \ldots, \eta_{|\mathcal{A}_0^b|}$ of β_0 such that the following hold for $1 \leq i \leq |\mathcal{A}_0^b|$.

- The arc η_i begins at the endpoint of μ_i and ends at the beginning point of μ_i.
- The curve $\mu_i \cdot \eta_i$ is a simple closed separating curve isotopic to a curve disjoint from β_0.

For $1 \leq i \leq |\mathcal{A}_0^b|$, let S_i and S_i' be the two subsurfaces into which Σ_0 is divided by $\mu_i \cdot \eta_i$, ordered so that β_0 is isotopic into S_i'. We thus have $H_1(\Sigma_0; \mathbb{F}_2) = H_1(S_i; \mathbb{F}_2) \oplus H_1(S_i'; \mathbb{F}_2)$ and $[\beta_0] \in H_1(S_i'; \mathbb{F}_2)$. Let X be the quotient of $H_1(\Sigma_0; \mathbb{F}_2)$ by the span of $[\beta_0]$, so $X \cong H_1(\Sigma_0, \beta; \mathbb{F}_2)$. Also, let V_i and W_i be the projections to X of $H_1(S_i; \mathbb{F}_2)$ and $H_1(S_i'; \mathbb{F}_2)$, respectively. Thus $X = V_i \oplus W_i$. Lemma 4.6 implies that there exists a linear map $\phi : X \to \mathbb{F}_2$ such that the set $\{i \mid 1 \leq i \leq |\mathcal{A}_0^b|, \phi|_{V_i} \neq 0, \phi|_{W_i} \neq 0\}$ has cardinality at least $\frac{3}{7}|\mathcal{A}_0^b|$.

Let $\psi : H_1(\Sigma_0; \mathbb{F}_2) \to \mathbb{F}_2$ be the composition of ϕ with the projection $H_1(\Sigma_0; \mathbb{F}_2) \to X$ and let $(\Sigma_1, v_1) \to (\Sigma_0, v_0)$ be the regular 2-fold cover associated to ψ. Since $\psi([\beta_0]) = 0$ and $[\alpha_0] = [\beta_0]$, the curve-arc triple $((\alpha_0, v_0), \tau_0, (\beta_0, w_0))$ has a closed lift $((\alpha_1, v_1), \tau_1, (\beta_1, w_1))$ to (Σ_1, v_1). Define $n_1 = i(\alpha_1, \beta_1)$. Each arc in $\mathcal{A}_0$ lifts to an arc of α_1; let $\mathcal{B}_1$ be the set of these arcs. Not all of the elements of $\mathcal{B}_1$ are β_1-arcs. Let $\mathcal{A}_1 = \{\mu \in \mathcal{B}_1 \mid \mu \text{ is a } \beta_1\text{-arc}\}$. Finally, define $r = |\mathcal{B}_1 \setminus \mathcal{A}_1|$.

We have constructed all of the objects claimed by the lemma. It remains to verify the inequalities from its conclusion. Regarding S^1 as the unit circle in $\mathbb{C}$, parametrize α_0 via a continuous map $f_0 : (S^1, 1) \to (\Sigma_0, v_0)$. Define $I_0 = \{\theta \in S^1 \mid f_0(\theta) \in \beta_0\}$. Since α_0 and β_0 are hyperbolic geodesics, we have

$n_0 = |I_0|$. Lift f_0 to a map $f_1 : (S^1, 1) \to (\Sigma_1, v_1)$ whose image is α_1 and define $I_1 = \{\theta \in S^1 \mid f_1(\theta) \in \beta_1\}$. Again, since α_1 and β_1 are hyperbolic geodesics, we must have $n_1 = |I_1|$. Also, by construction we have $I_1 \subset I_0$. The desired inequality $n_1 \leq n_0 - r$ is thus equivalent to the assertion that $|I_0 \setminus I_1| \geq r$. This follows immediately from the fact that the elements of $\mathcal{B}_1 \setminus \mathcal{A}_1$ are exactly the arcs of $\mathcal{B}_1$ at least one of whose endpoints is $f_1(\theta)$ for some $\theta \in I_0 \setminus I_1$.

All that is left to do is to prove that

$$|\mathcal{A}_1^g| \geq |\mathcal{A}_0^g| + \frac{3}{7}|\mathcal{A}_0^b| - r. \tag{5}$$

We first define $\mathcal{B}_1^g$ and $\mathcal{B}_1^b$ to be the subsets of $\mathcal{B}_1$ that are lifts of elements of $\mathcal{A}_0^g$ and $\mathcal{A}_0^b$, respectively. Next, define $r_g = |\mathcal{B}_1^g \setminus \mathcal{A}_1|$ and $r_b = |\mathcal{B}_1^b \setminus \mathcal{A}_1|$, so $r = r_g + r_b$. We have

$$|\mathcal{A}_1^g| = |\mathcal{B}_1^g \cap \mathcal{A}_1^g| + |\mathcal{B}_1^b \cap \mathcal{A}_1^g|. \tag{6}$$

Moreover, it is clear from the definitions that $\mathcal{B}_1^g \cap \mathcal{A}_1^g = \mathcal{B}_1^g \cap \mathcal{A}_1$, so

$$|\mathcal{B}_1^g \cap \mathcal{A}_1^g| = |\mathcal{B}_1^g \cap \mathcal{A}_1| = |\mathcal{A}_0^g| - r_g. \tag{7}$$

Combining (6) and (7), we see that (5) is equivalent to the assertion that

$$|\mathcal{B}_1^b \cap \mathcal{A}_1^g| \geq \frac{3}{7}|\mathcal{A}_0^b| - r_b, \tag{8}$$

which we will now prove.

Recall that we have enumerated $\mathcal{A}_0^b$ as $\{\mu_1, \ldots, \mu_{|\mathcal{A}_0^b|}\}$. For $1 \leq i \leq |\mathcal{A}_0^b|$, let the lift of μ_i to α_1 be $\tilde{\mu}_i \in \mathcal{B}_1^b$. We know that exactly r_b elements of $\{\tilde{\mu}_1, \ldots, \tilde{\mu}_{|\mathcal{A}_0^b|}\}$ do *not* lie in $\mathcal{A}_1$. Also, we know that the set $\{i \mid 1 \leq i \leq |\mathcal{A}_0^b|, \phi|_{V_i} \neq 0, \phi|_{W_i} \neq 0\}$ has cardinality at least $\frac{3}{7}|\mathcal{A}_0^b|$. We conclude that the set

$$\mathcal{C} := \{\tilde{\mu}_i \mid 1 \leq i \leq |\mathcal{A}_0^b|, \phi|_{V_i} \neq 0, \phi|_{W_i} \neq 0, \tilde{\mu}_i \in \mathcal{A}_1\} \subset \mathcal{B}_1^b$$

has cardinality at least $\frac{3}{7}|\mathcal{A}_0^b| - r_b$. To prove (8), therefore, it is enough to prove that $\mathcal{C} \subset \mathcal{A}_1^g$.

Consider some $\tilde{\mu}_i \in \mathcal{C}$. The endpoints of $\tilde{\mu}_i$ lie in β_1 by construction. Let $\tilde{\eta}_i$ be the lift of η_i to β_1 where η_i is the curve constructed in the second paragraph. Lemma 4.5 implies that $[\tilde{\mu}_i \cdot \tilde{\eta}_i] \neq 0$. Also, we know that $[\tilde{\mu}_i \cdot \tilde{\eta}_i]$ projects to $[\mu_i \cdot \eta_i] = 0$ in $H_1(\Sigma_0; \mathbb{F}_2)$, and $[\beta_1]$ projects to $[\beta_0] \neq 0$. We deduce that $[\tilde{\mu}_i \cdot \tilde{\eta}_i] \neq [\beta_1]$. Since $[\tilde{\mu}_i]_{\beta_1}$ is the projection of $[\tilde{\mu}_i \cdot \tilde{\eta}_i]$ to $H_1(\Sigma_1, \beta_1; \mathbb{F}_2)$, we conclude that $[\tilde{\mu}_i]_{\beta_1} \neq 0$, i.e. that $\tilde{\mu}_i$ is a good β_1-arc of α_1 and thus an element of $\mathcal{A}_1^g$, as desired. $\qquad \square$

Appendix A. Surfaces with boundary

In this appendix, we prove the following theorem. It generalizes Theorem B to surfaces with boundary.

Theorem C. *Let Σ be a compact orientable surface of genus g with $b \geq 1$ boundary components. Assume that $2g + b \geq 3$. Let α and β be nonisotopic oriented simple closed curves on Σ. Assume that for some $d \geq 7$ the conjugacy classes in $\pi_1(\Sigma)/\gamma_d(\pi_1(\Sigma))$ associated to α and β are the same. Then*

$$i(\alpha, \beta) \geq \Big(\frac{d + 2}{2}\Big)^c,$$

where $c = \frac{\ln(28/25)}{\ln(4)}$.

Remark. In the exceptional cases where $2g + b < 3$, the theorem would be vacuous.

We begin with an elementary lemma.

Lemma A.1. *Let Σ be an orientable compact surface of genus $g \geq 0$ with $b \geq 1$ boundary components. Assume that $2g + b \geq 3$. Let α and β be nonisotopic simple closed curves on Σ. Then there is a closed surface Σ' of genus at least 2 and an embedding $f : \Sigma \hookrightarrow \Sigma'$ such that $f(\alpha)$ and $f(\beta)$ are nonisotopic on Σ'.*

Proof. Let Σ' be the double of Σ, i.e. let $\Sigma' = (\Sigma \times \{0\} \cup \Sigma \times \{1\})/\sim$, where we identify $(x, 0) \sim (x, 1)$ for all $x \in \partial\Sigma$. Then Σ' is a closed orientable surface of genus at least 2. Moreover, there is a retraction $r : \Sigma' \to \Sigma$ defined by $r(\overline{(x, i)}) = x$.

Suppose for the sake of contradiction that $f(\alpha)$ and $f(\beta)$ were isotopic curves on Σ'. Then $\alpha = r(f(\alpha))$ is homotopic to $\beta = r(f(\beta))$ on Σ. Since homotopic simple closed curves on a surface are isotopic (see, e.g., [5, Proposition 1.10]), this is a contradiction. $\square$

Proof of Theorem C. Let Σ' and f be as in Lemma A.1 and let f_* be the induced homomorphism on π_1. Then $f_*(\gamma_d(\pi_1(\Sigma))) \subseteq \gamma_d(\pi_1(\Sigma'))$, and so there is an induced map $\pi_1(\Sigma)/\gamma_d(\pi_1(\Sigma)) \to \pi_1(\Sigma')/\gamma_d(\pi_1(\Sigma'))$. This implies that $f(\alpha)$ and $f(\beta)$ have the same conjugacy class in $\pi_1(\Sigma')/\gamma_d(\pi_1(\Sigma'))$. Moreover, it is clear that $i(\alpha, \beta) \geq i(f(\alpha), f(\beta))$, and from the lemma $f(\alpha)$ and $f(\beta)$ are nonisotopic. By Theorem B,

$$i(\alpha, \beta) \geq i(f(\alpha), f(\beta)) \geq \Big(\frac{d + 2}{2}\Big)^c. \qquad \square$$

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