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Irrational l^2-invariants arising from the lamplighter group

Łukasz Grabowski

Abstract. We show that the Novikov–Shubin invariant of an element of the integral group ring of the lamplighter group $\mathbf{Z}_2 \wr \mathbf{Z}$ can be irrational. This disproves a conjecture of Lott and Lück. Furthermore we show that every positive real number is equal to the Novikov–Shubin invariant of some element of the real group ring of $\mathbf{Z}_2 \wr \mathbf{Z}$. Finally we show that the l^2-Betti number of a matrix over the integral group ring of the group $\mathbf{Z}_p \wr \mathbf{Z}$, where p is a natural number greater than 1, can be irrational. As such the groups $\mathbf{Z}_p \wr \mathbf{Z}$ become the simplest known examples which give rise to irrational l^2-Betti numbers.

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Let Γ be a countable finitely generated. A real number r is said to be an l^2-*Betti number arising from* Γ if there is a matrix T with entries in the integral group ring $\mathbb{Z}[\Gamma]$, such that the *von Neumann dimension* of the kernel of T is equal to r.

The motivation for the name is as follows: when r is an l^2-Betti number arising from Γ, then there exists a normal covering M of a finite CW-complex whose deck transformation group is Γ, and such that one of the l^2-Betti numbers of M is equal to r. We refer to the very readable introduction [6] for more details.

The following problem is a fine-grained version of a question asked by Atiyah in [2].

Problem 1 (the Atiyah problem for Γ). *What is the set of l^2-Betti numbers arising from* Γ?

Let us denote this set by $\mathcal{C}(G)$. For a class of groups C define $\mathcal{C}(C) = \bigcup_{\Gamma\in C} \mathcal{C}(\Gamma)$.

So far $\mathcal{C}(\Gamma)$ has been computed only in cases where $\mathcal{C}(\Gamma)$ turns out to be a subset of $\mathbb{Q}$. In fact, the statement known as the *Atiyah conjecture for torsion-free groups* says that $\mathcal{C}(\Gamma) = \mathbb{N}$ for any torsion-free group, and before [5] it was widely conjectured that $\mathcal{C}(\Gamma) \subset \mathbb{Q}$ for every group Γ. However, [5] gives an example of a group ring element T together with an heuristic argument showing why the von

Neumann dimension of $\ker T$ is probably irrational. That example is based on [10], where a weaker form of the Atiyah conjecture was disproved.

Only recently Austin [3] obtained a definite result by proving that $\mathcal{C}$(Finitely generated groups) is uncountable. His results were extended and simplified in [9] and [17], and additional examples were found in [15].

All the groups G for which it was shown $\mathcal{C}(G) \not\subset \mathbb{Q}$ have one of the lamplighter groups $\mathbf{Z}_p \wr \mathbf{Z}$, where p is a natural number greater than 1, as a subgroup, but are substantially more complicated than that.

Our first result is as follows.

Theorem 2. *There is a matrix T with entries in the group ring $\mathbb{Z}[\mathbf{Z}_p \wr \mathbf{Z}]$ such that*

$$\dim_{vN} \ker T = 1344\Big(\frac{4p^3+3p^2+2p-1}{8p^3} + \frac{1}{8p^3}\sum_{k=1}^{\infty}\Big(\frac{p-1}{p}\Big)^{k+2^k}\Big),$$

which is a transcendental number.

In the view of the preceding discussion, the following problem captures the limit of the currently available methods for finding groups Γ such that $\mathcal{C}(\Gamma) \not\subset \mathbb{Q}$.

Problem 3. *Does $\mathcal{C}(\Gamma) \not\subset \mathbb{Q}$ imply $\mathbf{Z}_p \wr \mathbf{Z} \subset \Gamma$ for some p?*

As mentioned above, $\mathcal{C}(\Gamma)$ has been computed only in the cases where in fact $\mathcal{C}(\Gamma) \subset \mathbb{Q}$. Since $\mathbf{Z}_p \wr \mathbf{Z}$ are the simplest groups for which we know $\mathcal{C}(\Gamma) \not\subset \mathbb{Q}$, it is natural to ask the following.

Problem 4. *Is there a description of $\mathcal{C}(\mathbf{Z}_2 \wr \mathbf{Z})$ substantially different from the definition?*

To state a more concrete problem: is $\sqrt{2} \in \mathcal{C}(\mathbf{Z}_2 \wr \mathbf{Z})$?

For our second result let us recall the definition of another spectral invariant associated to an element of a group ring, the *Novikov–Shubin invariant*. It measures the growth of the number of eigenvalues around 0. More precisely, given a self-adjoint $T \in \mathbb{C}[\Gamma]$, the Novikov–Shubin invariant of T is defined as

$$\alpha(T) := \liminf_{\lambda\to 0^+} \frac{\log(\mu_T((0,\lambda]))}{\log(\lambda)}, \tag{1}$$

where μ_T is the spectral measure of T (see [14, Chapter 2] for more details).

Remarks 5. (i) It is irrelevant whether we take $\mu_T((0,\lambda])$ or $\mu_T((0,\lambda))$ in (1). However, it is important that we do not include 0, since otherwise $\alpha(T)$ would be equal to 0 whenever the spectral measure of T has an atom at 0. It is also irrelevant what is the base of the logarithm. It is convenient for us to take the base-2 logarithm.

(ii) Both the numerator and the denominator are negative when λ is sufficiently small, so $\alpha(T) \in [0, \infty]$.

(iii) If for some d and all ε there is a constant $C > 0$ such that for sufficiently small λ we have $\frac{1}{C}\lambda^{d+\varepsilon} < \mu_T((0,\lambda)) < C\lambda^{d-\varepsilon}$ then a short computation shows that $\alpha(T) = d$.

Lott and Lück [12] proposed the following conjecture.

Conjecture 6. *When $T \in \mathbb{Z}[\Gamma]$ then $\alpha(T) > 0$ and $\alpha(T) \in \mathbb{Q}$.*

For partial results and the motivations for Conjecture 6 see [14, Section 2.5]. For counterexamples to the positivity part see [8]. In the present paper we construct $T \in \mathbb{Z}[\mathbf{Z}_2 \wr \mathbf{Z}]$ such that $\alpha(T) \notin \mathbb{Q}$. In fact we show the following.

Theorem 7. *There is a family $T(b) \in \mathbb{R}[\mathbf{Z}_2 \wr \mathbf{Z}]$, $b \in (1, \infty)$ such that for $b \in \mathbb{Q}$ we have $T(b) \in \mathbb{Q}[\mathbf{Z}_2 \wr \mathbf{Z}]$ and $\alpha(T(b)) = \frac{1}{2\log_2(b)}$.*

Note that the Novikov–Shubin invariant of T and kT is the same for $k > 0$, and so we also obtain examples of $T \in \mathbb{Z}[\mathbf{Z}_2 \wr \mathbf{Z}]$ with irrational Novikov–Shubin invariants.

To the author's best knowledge, the counterexamples to the rationality part of Conjecture 6 were not known before even if $\mathbb{Z}[\Gamma]$ is replaced by $\mathbb{R}[\Gamma]$. The family $T(b)$ is a modification of the operator studied by Grigorchuk and Żuk [10].

As in the case of l^2-Betti numbers, when r is a Novikov–Shubin invariant of some $T \in \mathbb{Z}[\Gamma]$, then there exists a normal covering M of a finite CW-complex whose deck transformation group is Γ, and such that one of the Novikov–Shubin invariants of M is equal to r. Conjecture 6 could still be true in the case of a finite *aspherical* CW-complex.

Theorem 7 has an interesting consequence that the set of the Novikov–Shubin invariants of all the elements of $\mathbb{Q}[\mathbf{Z}_2 \wr \mathbf{Z}]$, which is countable, is different than the set of the Novikov–Shubin invariants of all the elements of $\mathbb{R}[\mathbf{Z}_2 \wr \mathbf{Z}]$. The analogous question has been asked among the experts for l^2-Betti numbers, since there are classes of torsion-free groups for which the Atiyah conjecture is known for $\mathbb{Q}[\Gamma]$ but not for $\mathbb{R}[\Gamma]$.

Problem 8. *Is it the case that for every $T \in \mathbb{R}[\Gamma]$ there exists $T' \in \mathbb{Q}[\Gamma]$ such that* $\dim_{vN} \ker T = \dim_{vN} \ker T'$?

Although Theorem 7 shows that the answer is negative when we replace $\dim_{\mathrm{vN}} \ker T = \dim_{\mathrm{vN}} \ker T'$ with $\alpha(T) = \alpha(T')$, the author believes that at least for $\Gamma = \mathbf{Z}_2 \wr \mathbf{Z}$ the answer to Problem 8 is positive.

The structure of the article is as follows. In the next section we describe the computational tool, in a generality which is just enough for the proof of Theorem 7. A general version is presented in [9, Section 2] and we refer there for the proofs. Various variants of it were also used for example in [4], [5], [13], [3], and [17].

In Section 2 we prove Theorem 7. Section 3 presents a slightly different version of the computational tool, which is then used in Section 4 to prove Theorem 2.

Some elementary linear algebra computations are deferred to the appendix.

Notation. The rings of integer, rational, real and complex numbers are $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. The cyclic group of order p is $\mathbf{Z}_p$ and the infinite cyclic group is $\mathbf{Z}$. We fix a generator of $\mathbf{Z}$ and denote it by t. Given an action $\Gamma \curvearrowright X$, the result of the action of $\gamma \in \Gamma$ on $x \in X$ is denoted by $\gamma.x$. For example the translation action of $\mathbf{Z} \curvearrowright \mathbb{Z}$ is, by definition, given by $t.k := k+1$.

Given two groups A and B the wreath product $A \wr B$ is defined to be $B \ltimes \bigoplus_B A$, where the action $B \curvearrowright \bigoplus_B A$ is by shifting the coordinates from the left. However, in the case $B = \mathbf{Z}$, we write $\mathbf{Z} \ltimes \bigoplus_{\mathbb{Z}} A$ because it is easier to refer to the coordinates of an element of $\bigoplus_{\mathbb{Z}} A$ (which are simply integer numbers) than to the coordinates of an element of $\bigoplus_{\mathbf{Z}} A$ (which are powers of t).

The neutral element of a group is denoted by e.

Note on chronology. The first version of this article submitted to arXiv in 2010 contained only Theorem 2. Theorem 7 was added in 2014.

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1. Computational tool in the case of a free action

Assume that (X, μ) is a compact abelian group with the normalized Haar measure which is Pontryagin-dual to a countable discrete abelian group A. Assume furthermore that the action $\Gamma \curvearrowright X$ is by continuous group automorphisms. The Pontryagin duality gives us an embedding $\mathbb{C}[A] \hookrightarrow L^\infty(X)$. The preimage of $f \in L^\infty(X)$ under this embedding, if it exists, is denoted by $\hat{f}$ (see [7, Chapter 4] for more on the Pontryagin duality).

Let $\chi_1, \ldots, \chi_n$ be the indicator functions of subsets $X_1, \ldots, X_n \subset X$ such that all χ_i are in the image of the above embedding. Let $a_1, \ldots, a_n \in \mathbb{C}$ and $\gamma_1, \ldots, \gamma_n \in \Gamma$.

Let $\widehat{T} \in \mathbb{C}[\Gamma \ltimes A]$ be defined as $\widehat{T} := \sum a_i \gamma_i \widehat{\chi}_i$, and let $T \in \Gamma \ltimes L^\infty(X)$ be defined as $T := \sum a_i \gamma_i \chi_i$.

We consider $\mathbb{C}[\Gamma \ltimes A]$ as acting on $l^2(\Gamma \ltimes A)$ by bounded operators. The spectral measures of the elements of $\mathbb{C}[\Gamma \ltimes A]$ are computed with respect to this action and the vector in $l^2(\Gamma \ltimes A)$ which is the indicator function of the neutral element.

Similarly the *group-measure space von Neumann algebra* $\Gamma \ltimes L^\infty(X)$ (see e.g. [14, Chapters 1 and 2]) acts on the direct integral Hilbert space $\int_X^\oplus l^2(\Gamma)\, d\mu(x)$, and the spectral measure is computed with respect to the vector equal to the function which sends all $x \in X$ to the indicator function of the neutral element.

As explained for example in [9, Section 2], we have the following lemma.

Lemma 9. *The spectral measures of $\widehat{T}$ and of T are the same.* □

We will now explain how to compute the spectral measure of T *under the assumption that the action $\Gamma \curvearrowright X$ is essentially free*, i.e. there is a subset $X' \subset X$ of full measure which is Γ-invariant and such that the action of Γ on X' is free.

Consider the oriented edge-labelled graph $\mathcal{G}$ defined as follows. The set of vertices of $\mathcal{G}$ is X, and there is an edge from x_1 to x_2 if for some i we have $x_1 \in X_i$ and $\gamma_i.x_1 = x_2$. On such an edge we set the label to be equal to

$$\sum_{\substack{j:\gamma_j=\gamma_i \\ x_1 \in X_j}} a_j.$$

Let $\mathcal{G}(x)$ be the connected component of x in $\mathcal{G}$. Let $l^2(\mathcal{G}(x))$ be the Hilbert space spanned by the vertices of $\mathcal{G}(x)$. Let $T(x)\colon l^2(\mathcal{G}(x)) \to l^2(\mathcal{G}(x))$ be the adjacency operator on $\mathcal{G}(x)$, i.e. the entry of the matrix of $T(x)$ corresponding to a pair of vertices (v_1, v_2) is equal to the label on the edge from v_1 to v_2, if there is such an edge, and 0 otherwise.

We say T is *self-adjoint* if the set of those x for which the matrix of $T(x)$ is Hermitian is of measure 1. The next proposition follows from [9, Proposition 2.10].

Proposition 10. *Let us assume that T is self-adjoint and that the set of those x for which $\mathcal{G}(x)$ is finite is of measure* 1. *Then for a measurable subset $D \subset \mathbb{R}$ we have*

$$\mu_T(D) = \int_X \frac{\mu_{T(x)}(D)}{|\mathcal{G}(x)|}\, d\mu(x).$$

We will apply this proposition in the next section. Its utility comes from the fact that among the labelled graphs $\mathcal{G}(x)$, $x \in X$, there are only countably many different ones, and they can be computed explicitly. As such the above integral will decompose as an explicit countable sum of spectral measures of *finite-dimensional* matrices.

2. Possible values of the Novikov–Shubin invariants

We need a more quantitative version of [8, Lemma 5]. For $b \in \mathbb{R}$ and $n \in \mathbb{N}$ let $M(b,n)$ be the $n \times n$ matrix

$$\begin{pmatrix} 1 & b & & & & \\ b & b^2+1 & b & & & \\ & b & b^2+1 & & & \\ & & & \cdots & & \\ & & & & b^2+1 & b \\ & & & & b & b^2+1 \end{pmatrix}$$

Lemma 11. *For every ε and $b > 1$ there is N such that for $n > N$ the matrix $M(b,n)$ has an eigenvalue $\lambda_1(b,n)$ such that*

$$\left(\frac{1}{b^2} - \varepsilon\right)^n < \lambda_1(b,n) < \left(\frac{1}{b^2} + \varepsilon\right)^n,$$

and such that all the other eigenvalues are bigger than or equal to $(b-1)^2$.

Proof. Let us fix ε and b. Let $K(b,n) = M(b,n) + \mathrm{Diag}(b^2, 0, 0, \ldots, 0)$, i.e. we replace the anomalous 1 on the diagonal with b^2+1. Let $\kappa_1 \leqslant \kappa_2 \leqslant \ldots \leqslant \kappa_n$ be the eigenvalues of $K(b,n)$ and let $\lambda_1 \leqslant \lambda_2 \leqslant \ldots\ldots \lambda_n$ be the eigenvalues of $M(b,n)$. Note that the norm of the matrix $K(b,n) - (b^2+1)\mathrm{I}_m$ is $2b$, so we have the following claim.

Claim A. *All the eigenvalues of $K(b,n)$ lie between $b^2+1-2b = (b-1)^2$ and $b^2+1+2b = (b+1)^2$.*

Let $D_n = \det(K(b,n))$ and $E_n = \det(M(b,n))$. By expanding both determinants along the final row we see that D_n and E_n fulfil the recurrence relations

$$D_{n+2} = (b^2+1)D_{n+1} - b^2 D_n \quad E_{n+2} = (b^2+1)E_{n+1} - b^2 E_n.$$

Solving the recurrence in the standard way gives us $E_n = 1$ for all n and

$$D_n = \frac{b^2}{b^2-1} b^{2n} - \frac{1}{b^2-1}. \tag{2}$$

Note that for any constant $C > 0$ we have that for sufficiently large n the following holds:

$$(b^2-\varepsilon)^n \leqslant C D_n \leqslant (b^2+\varepsilon)^n \tag{3}$$

Note that the difference $M(b,n) - K(b,n)$ is a rank 1 matrix, so we can use the Weyl inequality for rank 1 perturbations (e.g. [11, Theorem 4.3.4]), which in particular implies that for $i = 2,\dots,n$ we have $\lambda_i \geqslant \kappa_{i-1}$. Since $\kappa_1 \cdot \ldots \cdot \kappa_n = D_n$, it follows that $\lambda_2 \cdot \ldots \cdot \lambda_n \geqslant \frac{D_n}{\kappa_n}$.

Similarly the Weyl inequality implies that for $i = 2,\dots,n-1$ we have $\lambda_i \leqslant \kappa_{i+1}$, so that $\lambda_2 \cdot \ldots \cdot \lambda_n \leqslant \frac{\lambda_n D_n}{\kappa_1 \kappa_2}$.

Note that the norm of $M(b,n) = K(b,n) - \mathrm{Diag}(b^2, 0, 0, \dots, 0)$ is at most $(b+1)^2 + b^2$, so in particular $\lambda_n \leqslant (b+1)^2 + b^2$. This, together with Claim A, shows

$$\frac{D_n}{(b+1)^2} \leqslant \lambda_2 \cdot \ldots \cdot \lambda_n \leqslant \frac{((b+1)^2 + b^2) D_n}{(b-1)^4}.$$

Now (3) implies that for sufficiently large n we have

$$(b^2 - \varepsilon)^n \leqslant (\lambda_2 \cdot \ldots \cdot \lambda_n) \leqslant (b^2 + \varepsilon)^n.$$

Finally since $\lambda_1 \cdot \ldots \cdot \lambda_n = E_n = 1$ we obtain

$$\frac{1}{(b^2+\varepsilon)^n} \leqslant \lambda_1 \leqslant \frac{1}{(b^2-\varepsilon)^n},$$

which implies the statement about λ_1.

As for all the other eigenvalues, by Claim A we have $\kappa_1 \geqslant (b-1)^2$, and for $i \geqslant 2$ we have $\lambda_i \geqslant \lambda_2 \geqslant \kappa_1$ by the Weyl inequality, which finishes the proof. □

We introduce the following notation for the subsets of $\mathbf{Z}_2^{\mathbb{Z}}$. The elements of $\mathbf{Z}_2$ are denoted by 0 and 1. For $\varepsilon_i \in \{0,1\}$ we denote the set

$$\{(m_i) \in \mathbf{Z}_2^{\mathbb{Z}} : m_{-a} = \varepsilon_{-a}, \dots, m_b = \varepsilon_b\} \subset \mathbf{Z}_2^{\mathbb{Z}},$$

by

$$[\varepsilon_{-a}\varepsilon_{-a+1}\dots\varepsilon_{-1}\underline{\varepsilon_0}\varepsilon_1\dots\varepsilon_b],$$

and we let

$$\chi[\varepsilon_{-a}\varepsilon_{-a+1}\dots\varepsilon_{-1}\underline{\varepsilon_0}\varepsilon_1\dots\varepsilon_b, x] \in L^\infty(\mathbf{Z}_2^{\mathbb{Z}})$$

be the corresponding indicator function. Elements from the set above will be denoted with the curly brackets () instead of [].

Recall that t is the generator of the infinite cyclic group $\mathbf{Z}$. For $b \in \mathbb{R}$ let $T(b) \in \mathbf{Z} \ltimes L^\infty(\mathbf{Z}_2^{\mathbb{Z}})$ be defined as

$$T(b) := -b^2 \chi[1\underline{0}] + b(t[\underline{0}] + t^{-1}\chi[0\underline{*}]) + (b^2+1).$$

In this notation, the operator studied in [10] was $t[\underline{0}] + t^{-1}\chi[0\underline{*}]$. Note that the indicator functions in the definition of $T(b)$ are in the image of the Pontryagin duality map $\mathbb{Q}[\bigoplus_{\mathbb{Z}} \mathbf{Z}_2] \hookrightarrow L^\infty(\mathbf{Z}_2^{\mathbb{Z}})$. So, by Lemma 9, the Novikov–Shubin invariant of $T(b)$ is the same as the Novikov–Shubin invariant of the corresponding $\widehat{T(b)} \in \mathbb{R}[\mathbf{Z} \ltimes \bigoplus_{\mathbb{Z}} \mathbf{Z}_2] = \mathbb{R}[\mathbf{Z}_2 \wr \mathbf{Z}]$.

Theorem 12. *For $b > 1$ the Novikov–Shubin invariant of $T(b)$ is equal to $\frac{1}{2\log_2(b)}$.*

Proof. We use Proposition 10 with $X = \mathbf{Z}_2^{\mathbb{Z}}$, $\Gamma = \mathbf{Z}$, $A = \bigoplus_{\mathbb{Z}} \mathbf{Z}_2$. Let us compute two examples of a graph $\mathcal{G}(x)$.

First let $x = (1\underline{1})$. Then $x \notin [1\underline{0}]$, $x \notin [\underline{0}]$ and $x \notin [0\underline{*}]$, so the only outgoing arrow at x is the self-loop with label (b^2+1).

As for the incoming arrows at x, other than the self-loop, we see that $x \notin t.[\underline{0}]$ and $x \notin t^{-1}.[0\underline{*}]$, so there are no incoming arrows. Accordingly $\mathcal{G}(x)$ consists only of the vertex x with a self-loop with label b^2+1.

Now let $x = (1\underline{0}01)$. Since $x \in [\underline{0}]$ there is an outgoing arrow from x to $t.x = (10\underline{0}1)$ with label b. Since $t.x \in [\underline{0}]$, there is an outgoing arrow from $t.x$ to $t^2.x = (100\underline{1})$ with label b. Since $t.x \in [0\underline{*}]$, there is also an outgoing arrow from $t.x$ to x with label b. Similarly $t^2.x \in [0\underline{*}]$ so there is an arrow from $t^2.x$ to $t.x$ with label b.

As for the self-loops , $x \in [1\underline{0}]$, so there is a self-loop at x with label $(b^2+1) - b^2 = 1$. The vertices $t.x$ and $t^2.x$ have self-loops with labels b^2+1.

In analogy with these two examples we see that when $x \in [1\underline{0}0^k1]$ then $\mathcal{G}(x)$ is the graph on Figure 1 with $k+2$ vertices.

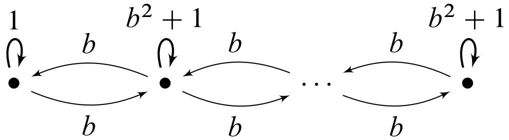

Figure 1

Let us check that, up to a set of measure 0, every point of X is in a connected component of $\mathcal{G}(x)$ for some $x \in [1\underline{0}0^k1]$:

$$\mu([1\underline{1}]) + \sum_{k=0}^{\infty}(k+2)\mu([1\underline{0}0^k1]) = \frac{1}{4} + \sum_{k=0}^{\infty}(k+2)\frac{1}{2^{k+3}} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{k}{2^k} = 1.$$

In particular the subset of those x for which $\mathcal{G}(x)$ is finite is of full measure. Clearly the adjacency operator on the graph with m vertices on Figure 1 is given by the matrix $M(b,m)$. Proposition 10 now shows that

$$\mu_T = \frac{1}{4}\mu_{\mathrm{Diag}(b^2+1)} + \sum_{m=2}^{\infty}\frac{1}{2^{m+1}}\mu_{M(b,m)}.$$

Let us use Lemma 11 to estimate $\mu_T((0,z])$ for small $z > 0$. Let us fix a small ε in Lemma 11. Then for sufficiently small z we have

$$\mu_T((0,z]) = \sum_{m:\lambda_1(b,m)\leqslant z} \frac{1}{2^{m+1}}. \tag{4}$$

By Lemma 11, the smallest m such that $\lambda_1(b,m) \leqslant z$ is between

$$\frac{|\log(z)|}{\left|\log\left(\frac{1}{b^2}+\varepsilon\right)\right|}$$

and

$$\frac{|\log(z)|}{\left|\log\left(\frac{1}{b^2}-\varepsilon\right)\right|}.$$

We estimate $\mu_T((0,z])$ from (i) below and (ii) above by taking in the sum (4) respectively (i) only the smallest m such that $\lambda_1(b,m) \leqslant z$, and (ii) the smallest such m and all the natural numbers larger than m. We obtain that $\mu_T((0,z])$ lies between

$$2^{\frac{\log(z)}{\left|\log\left(\frac{1}{b^2}-\varepsilon\right)\right|}} = z^{\frac{1}{\left|\log\left(\frac{1}{b^2}-\varepsilon\right)\right|}}$$

and

$$2\cdot 2^{\frac{\log(z)}{\left|\log\left(\frac{1}{b^2}+\varepsilon\right)\right|}} = 2z^{\frac{1}{\left|\log\left(\frac{1}{b^2}+\varepsilon\right)\right|}}$$

(in the algebraic manipulations we used that $\log(z)$ is negative for small z).

This shows that the Novikov–Shubin invariant of $T(b)$ lies between $\frac{1}{\left|\log\left(\frac{1}{b^2}-\varepsilon\right)\right|}$ and $\frac{1}{\left|\log\left(\frac{1}{b^2}+\varepsilon\right)\right|}$, for every ε, and so in fact must be equal to $\frac{1}{\left|\log\left(\frac{1}{b^2}\right)\right|} = \frac{1}{2\log(b)}$. □

3. Computational tool in the case of a non-free action

We will now repeat the discussion from Section 1, and add some extra structure in order to deal with a non-free action. For the proofs see [9, Section 2].

Let $\Gamma \curvearrowright X$ be as in Section 1, with the exception that it is not necessarily a free action. Let $T \in \Gamma \ltimes L^\infty(X)$ be defined as

$$T := \sum_{i=1}^{n} a_i \gamma_i \chi_i$$

(with the notation from Section 1).

Consider the oriented graph $\mathcal{G}_\Gamma$ whose set of vertices is X, and with edges labelled by the elements of the set $\{\gamma_1, \dots, \gamma_n\}$, defined as follows. There is an edge with label γ_i from x_1 to x_2 if $x_1 \in X_i$ and $\gamma_i.x_1 = x_2$. Let $\mathcal{G}_\Gamma(x)$ be the connected component of x. We say $\mathcal{G}_\Gamma(x)$ is *simply-connected* if multiplying edge-labels along any closed loop gives the trivial element of Γ (if a loop traverses an edge in the direction opposite to the orientation of the edge, we invert the label).

Let $\mathcal{G}(x)$ be the graph which arises from $\mathcal{G}_\Gamma(x)$ by changing the label γ_i on the edge between x_1 and x_2 as above to the sum

$$\sum_{\substack{j:\gamma_j=\gamma_i \\ x_1\in X_j}} a_j.$$

Finally let

$$T(x)\colon l^2(\mathcal{G}(x)) \longrightarrow l^2(\mathcal{G}(x))$$

be the adjacency operator on $\mathcal{G}(x)$. The next proposition follows from [9, Proposition 2.10].

Proposition 13. *Let us assume that the set of x such that $\mathcal{G}_\Gamma(x)$ is finite and simply-connected is of full measure. Then* $\dim_{vN} \ker T$ *is equal to*

$$\int_X \frac{\dim\ker T(x)}{|\mathcal{G}(x)|}\, d\mu(x).$$

4. Irrational l^2-Betti numbers arising from $\mathbf{Z}_p \wr \mathbf{Z}$

For the rest of the article let X be the compact abelian group $\mathbf{Z}_p^{\mathbb{Z}} \times \mathbf{Z}_2^3$, and $\Gamma = \mathbf{Z} \times \mathrm{Aut}(\mathbf{Z}_2^3)$. The action $\Gamma \curvearrowright X$ is the natural one, i.e. $\mathrm{Aut}(\mathbf{Z}_2^3)$ acts on $\mathbf{Z}_2^3$ and $\mathbf{Z}$ acts on $\mathbf{Z}_p^{\mathbb{Z}}$ by shifting the coordinates.

Note that $\Gamma \ltimes A$ is isomorphic to $(\mathbf{Z}_p \wr \mathbf{Z}) \times (\mathrm{Aut}(\mathbf{Z}_2^3) \ltimes \mathbf{Z}_2^3)$. We will shortly define $T \in \mathbb{Q}[\Gamma \ltimes A]$ such that

$$\dim_{\mathrm{vN}} \ker T = \frac{4p^3+3p^2+2p-1}{8p^3} + \frac{1}{8p^2(p-1)} \sum_{k=1}^{\infty} \Big(\frac{p-1}{p}\Big)^{k+2^{k-1}},$$

The additional factor 1344 in Theorem 2 comes from the fact that $\mathbf{Z}_p \wr \mathbf{Z}$ is a subgroup in $\Gamma \ltimes A$ of index 1344 (see e.g. [9, Lemma 6.2] for more explanation). Furthermore, for $k \neq 0$ the kernels of T and kT are the same, so we will also obtain a matrix over $\mathbb{Z}[\Gamma \ltimes A]$ whose kernel dimension is as above.

Let $A, B, C, D, F, I, U_1, U_2$ (U stands for *unimportant*, F for *final* and I for *initial*) denote the elements of $\mathbf{Z}_2^3$. The only assumption on this labelling is that the first 6 symbols correspond to non-zero elements of $\mathbf{Z}_2^3$.

For every pair (x, y) of different elements from the set $\{A, B, C, D, F, I\}$ we fix an automorphism denoted by $(x \to y) \in \mathrm{Aut}(\mathbf{Z}_2^3)$ which sends x to y, in such a way that

$$(x \longrightarrow y) = (y \longrightarrow x)^{-1} \tag{5}$$

and

$$(C \longrightarrow D)(A \longrightarrow C) = (I \longrightarrow D)(A \longrightarrow I). \tag{6}$$

To treat the case of an arbitrary p, we change our notation in the following way. Let $0 := \{0\} \subset \mathbf{Z}_p$ and $1 := \{1, 2, 3, \dots, p-1\} \subset \mathbf{Z}_p$. Let

$$[\varepsilon_{-a}\varepsilon_{-a+1} \dots \varepsilon_{-1}\underline{\varepsilon_0}\varepsilon_1 \dots \varepsilon_b, x],$$

where $\varepsilon_i \in \{0, 1\}$, denote

$$\{((m_i), y) \in \mathbf{Z}_p^{\mathbb{Z}} \times \mathbf{Z}_2^3 : m_{-a} \in \varepsilon_{-a}, \dots, m_b \in \varepsilon_b, y = x\} \subset X,$$

and let

$$\chi[\varepsilon_{-a}\varepsilon_{-a+1} \dots \varepsilon_{-1}\underline{\varepsilon_0}\varepsilon_1 \dots \varepsilon_b, x] \in L^\infty(X)$$

be the corresponding indicator function.

Let $S \in \mathbb{Q}[\Gamma \ltimes A]$ be represented by the sum of the following terms:

$$(-t(I \to D) + t^{-1}(I \to A)) \cdot \chi[1\underline{0}1, I], \tag{7a}$$
$$(-t^2(A \to C) - 2t^{-1}) \cdot \chi[1\underline{1}01, A], \tag{7b}$$
$$-t^2(A \to C) \cdot \chi[0\underline{1}01, A], \tag{7c}$$
$$-2t^{-1} \cdot \chi[1\underline{1}00, A], \tag{7d}$$
$$0 \cdot \chi[0\underline{1}00, A], \tag{7e}$$
$$-2t^{-1} \cdot \chi[1\underline{1}1, A], \tag{7f}$$
$$-(A \to B) \cdot \chi[0\underline{1}1, A], \tag{7g}$$
$$-t \cdot \chi[\underline{1}1, B], \tag{7h}$$
$$-(B \to A) \cdot \chi[\underline{1}0, B], \tag{7i}$$
$$(-t + (C \to D)) \cdot \chi[\underline{1}1, C], \tag{7j}$$
$$+(C \to D) \cdot \chi[\underline{1}0, C], \tag{7k}$$
$$-t \cdot \chi[\underline{1}1, D], \tag{7l}$$
$$-(D \to F) \cdot \chi[\underline{1}0, D], \tag{7m}$$
$$0 \cdot \chi[\underline{1}0, F], \tag{7n}$$
$$0 \cdot \chi_R, \tag{7o}$$

where χ_R is the indicator function of the set R defined to be “all the rest”, i.e. the complement of the union of the sets $[1\underline{0}1, I]$, $[1\underline{1}01, A]$, $[0\underline{1}01, A]$, $[1\underline{1}00, A]$, $[0\underline{1}00, A]$, $[1\underline{1}1, A]$, $[0\underline{1}1, A]$, $[\underline{1}1, B]$, $[\underline{1}0, B]$, $[\underline{1}1, C]$. $[\underline{1}0, C]$, $[\underline{1}1, D]$, $[\underline{1}0, D]$ and $[\underline{1}0, F]$.

Finally define

$$T := S + 1 - \chi_R - \chi[1\underline{0}1, I] - \chi[\underline{1}0, F]. \tag{8}$$

Remark 14. (i) The reason we explicitly write the terms "$0 \cdot \ldots$" is that this way the right hand sides are indicator functions of disjoint sets whose union is X. This is helpful when checking that the connected components $\mathcal{G}_\Gamma(x)$ are as claimed. To reassure the reader, without any 0-terms it would be the same operator and the same computations would have to be performed.

(ii) The definitions of S and T might seem complicated at first. Let us informally describe how the author came up with them. In the process of finding a group ring element over $\mathbf{Z}_2 \wr \mathbf{Z}$ (or a matrix of group ring elements) whose kernel dimension is irrational, the first step was a realization that any family of *simple-to-describe* graphs can appear as the connected components $\mathcal{G}(x)$. Examples of simple-to-describe graphs are on Figures 3, 5, and 7; one could formalize the notion of being simple-to-describe using regular languages. Then it was necessary to find a simple-to-describe family whose kernel dimensions behave in an irregular way. This was the most difficult step - after trial and error the family from Figure 7 was found. The operator T above is defined in such a way so that that family appears among the connected components $\mathcal{G}(x)$ (two other infinite families, those from Figures 3 and 5 also appear, but their kernel dimensions behave in a regular way, so they do not interfere with the irregularity of the family from Figure 7).

We will now describe the graphs $\mathcal{G}_\Gamma(x)$ and $\mathcal{G}(x)$ for $x \in X$. It is convenient to describe them in four families, which we do in separate subsections.

We will show figures for the graphs, but for clarity we suppress self-loops. Note that the self-loops are given only by the terms in (8), so it is also easy to take them into account.

In all the cases it is somewhat tedious but, using Remark 14, straightforward to check that the graph $\mathcal{G}_\Gamma(x)$ is as claimed for a given $x \in X$.

4.1. Case 1: $x \in R$.

The graph $\mathcal{G}_\Gamma(x)$ consists of just one vertex with no edges. Accordingly, the adjacency operator $T(x)$ is the 0 operator. We clearly deduce the following lemma.

Lemma 15. *We have the following properties*:

(1) $\dim \ker T(x) = 1$;

(2) $\mathcal{G}_\Gamma(x)$ *is simply-connected*;

(3) $\mu(R) = \frac{1}{8}\big(2 + 5\frac{1}{p} + \frac{1}{p^3} + 2\frac{p-1}{p^3} + \frac{p-1}{p} + \big(\frac{p-1}{p}\big)^2\big)$.

Proof. (1) and (2) are clear. As for (3), note that we can explicitly write

$$\begin{aligned} R = {} & [\underline{0}, A] \sqcup [\underline{0}, B] \sqcup [\underline{0}, C] \sqcup [\underline{0}, D] \sqcup [\cdot, U_1] \sqcup [\cdot, U_2] \\ & \sqcup [\underline{0}, F] \sqcup [\underline{1}1, F] \sqcup [\underline{1}, I] \sqcup [1\underline{0}0, I] + \sqcup [0\underline{0}1, I] \sqcup [0\underline{0}0, I]. \end{aligned}$$

Since μ is the product measure, it is easy to compute the measures of the sets above. We start with $\mu([\underline{0}]) = \frac{1}{p}$, $\mu([\underline{1}]) = \frac{p-1}{p}$, and then for example $\mu([0\underline{0}1, I]) = \left(\frac{1}{p}\right)^2 \cdot \frac{p-1}{p} \cdot \frac{1}{8}$. $\square$

4.2. Case 2: $x \in [0\underline{1}1^{k-1}00, A]$. If we denote $x = (0\underline{1}1^{k-1}00, A)$, then the vertices of $\mathcal{G}_\Gamma(x)$ are

$$\begin{array}{llll} (0\underline{1}1^{k-1}00, A), & (01\underline{1}1^{k-2}00, A), & \dots, & (01^{k-1}\underline{1}00, A), \\ (0\underline{1}1^{k-1}00, B), & (01\underline{1}1^{k-2}00, B), & \dots, & (01^{k-1}\underline{1}00, B). \end{array}$$

$\mathcal{G}_\Gamma(x)$ is shown on Figure 2. Each vertex should additionally have a self-loop with label e. To avoid clutter only some vertices are explicitly identified as elements of X.

To facilitate to the reader checking that $\mathcal{G}_\Gamma(x)$ is as claimed we indicate that the corresponding terms in (7) are

$$\begin{array}{lllll} [0\underline{1}1, A], & [1\underline{1}1, A], & \dots, & [1\underline{1}1, A], & [1\underline{1}00, A], \\ [\underline{1}1, B], & [\underline{1}1, B], & \dots, & [\underline{1}1, B], & [\underline{1}0, B]. \end{array}$$

The graphs $\mathcal{G}(x)$ are shown on Figure 3. Each vertex should additionally have a self-loop with label 1.

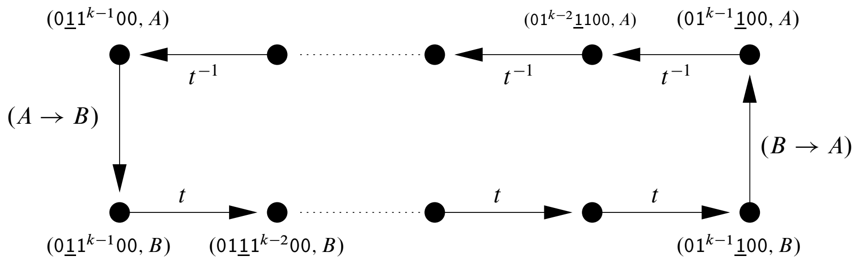

Figure 2. $\mathcal{G}_\Gamma(x)$ without self-loops for $x = (0\underline{1}1^{k-1}00, A)$.

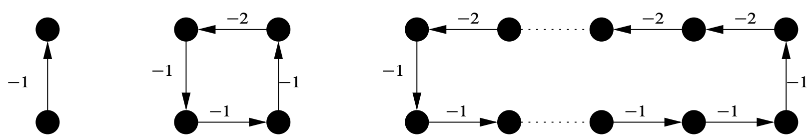

Figure 3. $\mathcal{G}(x)$ without self-loops for $x = (0\underline{1}00, A)$, $x = (0\underline{1}100, A)$, and $x = (0\underline{1}1^{k-1}00, A)$.

Lemma 16. *We have the following properties*:

(1) $\dim\ker T(x) = 0$;

(2) $\mathcal{G}_\Gamma(x)$ *is simply-connected*;

(3) $\mu([0\underline{1}1^{k-1}00, A]) = \frac{1}{8}\cdot\left(\frac{1}{p}\right)^3\cdot\left(\frac{p-1}{p}\right)^k$ *and* $|\mathcal{G}(x)| = 2k$.

Proof. (2) follows easily from Figure 2 and Equation (5). (3) is a direct computation as in Lemma 15. (1) follows from analysing Figure 3, but for completeness we give a proof in the appendix. □

4.3. Case 3: $x \in [00\underline{1}1^{l-1}0, C]$. If we denote $x = (00\underline{1}1^{l-1}0, C)$ then the vertices of $\mathcal{G}_\Gamma(x)$ are

$$\begin{array}{lll} (00\underline{1}1^{l-1}0, C), & \ldots, & (001^{l-1}\underline{1}0, C), \\ (00\underline{1}1^{l-1}0, D,) & \ldots, & (001^{l-1}\underline{1}0, D), \\ & & (001^{l-1}\underline{1}0, F). \end{array}$$

$\mathcal{G}_\Gamma(x)$ is shown on Figure 4. Each vertex except the final one should additionally have a self-loop with label e. To avoid clutter only some vertices are explicitly identified as elements of X.

To facilitate to the reader checking that $\mathcal{G}_\Gamma(x)$ is as claimed we indicate that the corresponding terms in (7) are

$$\begin{array}{llll} [\underline{1}1, C], & \ldots, & [\underline{1}1, C], & [\underline{1}0, C], \\ [\underline{1}1, D], & \ldots, & [\underline{1}1, D], & [\underline{1}0, D], \\ & & & [\underline{1}0, F]. \end{array}$$

The graphs $\mathcal{G}(x)$ are shown on Figure 5. Each vertex except the final one should additionally have a self-loop with label 1.

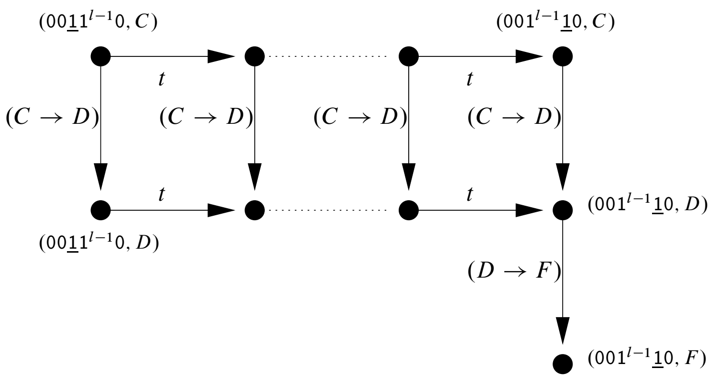

Figure 4. $\mathcal{G}_\Gamma(x)$ without self-loops for $x = (00\,\underline{1}1^{l-1}0, C)$.

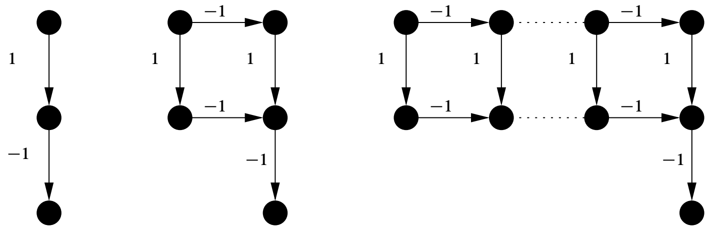

Figure 5. $\mathcal{G}(x)$ without self-loops for $x = (00\,\underline{1}0, C)$, $x = (00\,\underline{1}10, C)$, and $x = (00\,\underline{1}1^{l-1}0, C)$

Lemma 17. *The following properties are true*:

(1) $\dim\ker T(x) = 1$;

(2) $\mathcal{G}_\Gamma(x)$ *is simply-connected*;

(3) $\mu([00\underline{1}1^{l-1}0, C]) = \frac{1}{8} \cdot \left(\frac{1}{p}\right)^3 \cdot \left(\frac{p-1}{p}\right)^l$ *and* $|\mathcal{G}(x)| = 2l + 1$.

Proof. (2) follows easily from Figure 4 and Equation (5). (3) is a direct computation as in Lemma 15. (1) follows from analysing Figure 5, but for completeness we give a proof in the appendix. □

4.4. Case 4: $x \in [0\underline{1}1^{k-1}01^l0, A]$. If we denote $x = (0\underline{1}1^{k-1}01^l0, A)$ then the vertices of $\mathcal{G}_\Gamma(x)$ are

$$\begin{array}{rrcr}
(0\underline{1}1^{k-1}01^l0, A), & (01\underline{1}1^{k-2}01^l0, A), & \ldots, & (01^{k-1}\underline{1}01^l0, A),\\
(0\underline{1}1^{k-1}01^l0, B), & (01\underline{1}1^{k-2}01^l0, B), & \ldots, & (01^{k-1}\underline{1}01^l0, B),\\
 & & & (01^k\underline{0}1^l0, I),\\
 & (01^k0\underline{1}1^{l-1}0, C), & \ldots, & (01^k01^{l-1}\underline{1}0, C),\\
 & (01^k0\underline{1}1^{l-1}0, D), & \ldots, & (01^k01^{l-1}\underline{1}0, D),\\
 & & & (01^k01^{l-1}\underline{1}0, F).
\end{array}$$

$\mathcal{G}_\Gamma(x)$ is shown on Figure 6. Each vertex except the final and the initial ones should additionally have a self-loop with label e. To avoid clutter only some vertices are explicitly identified as elements of X. Because it could be unclear which labels correspond to which vertices, the identified vertices are marked white.

To facilitate to the reader checking that $\mathcal{G}_\Gamma(x)$ is as claimed we indicate that the corresponding terms in (7) are

$$
\begin{array}{ccccc}
[0\underline{1}1, A], & [1\underline{1}1, A], & \ldots, & [1\underline{1}1, A], & [1\underline{1}101, A], \\
[\underline{1}1, B], & [\underline{1}1, B], & \ldots, & [\underline{1}1, B], & [\underline{1}0, B], \\
 & & & & [1\underline{0}1, I], \\
 & [\underline{1}1, C], & \ldots, & [\underline{1}1, C], & [\underline{1}0, C], \\
 & [\underline{1}1, D], & \ldots, & [\underline{1}1, D], & [\underline{1}0, D], \\
 & & & & [\underline{1}0, F].
\end{array}
$$

The graphs $\mathcal{G}(x)$ are shown on Figure 7. Each vertex except the final and the initial ones should additionally have a self-loop with label 1.

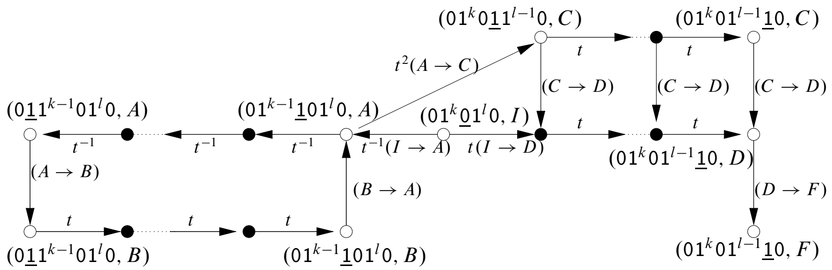

Figure 6. $\mathcal{G}_\Gamma(x)$ without self-loops for $x = (0\underline{1}1^{k-1}01^l0, A)$.

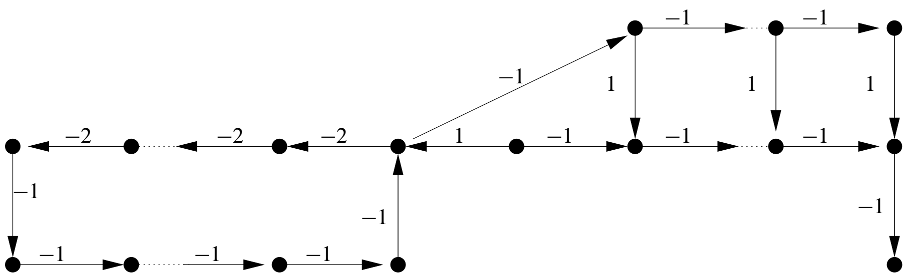

Figure 7. $\mathcal{G}(x)$ without self-loops for $x = (0\underline{1}1^{k-1}01^l0, A)$.

Lemma 18. *The following properties are true*:

(1) $\dim\ker T(x) = \begin{cases} 2 & \text{if } l = 2^{k-1}-1, \\ 1 & \text{otherwise}; \end{cases}$

(2) $\mathcal{G}_\Gamma(x)$ *is simply-connected*;

(3) $\mu([0\underline{1}1^{k-1}01^l0, A]) = \frac{1}{8}\cdot\left(\frac{1}{p}\right)^3\cdot\left(\frac{p-1}{p}\right)^{k+l}$ *and* $|\mathcal{G}(x)| = 2k+2l+2$.

Proof. (2) follows easily from Figure 6 and Equations (5) and (6). (3) is a direct computation as in Lemma 15. (1) follows from analysing Figure 5, but for completeness we give a proof in the appendix. □

4.5. Checking that we have not missed any graphs. We need to check that the graphs $\mathcal{G}(x)$ on Figures 2, 4 and 6, together with the set R cover the whole space X. To this end we compute that the measure of the covered part is 1, by using the formulas in Lemmas 15(3), 16(3), 17(3) and 18(3).

Let $\alpha := \frac{1}{p}$, $\beta := \frac{p-1}{p}$. We need to check that

$$\frac{1}{8}(2+5\alpha+\alpha^3+2\beta\alpha^2+\beta+\beta^2) + \sum_{k=1}^{\infty} 2k\cdot\frac{1}{8}\cdot\alpha^3\cdot\beta^k$$
$$+\sum_{l=1}^{\infty}(2l+1)\cdot\frac{1}{8}\cdot\alpha^3\cdot\beta^l + \sum_{k,l=1}^{\infty}(2k+2l+2)\cdot\frac{1}{8}\cdot\alpha^3\cdot\beta^{k+l} = 1.$$

This is a tedious but elementary exercise in using the formula

$$\sum_{n=1}^{\infty}(n+C)x^n = \frac{x}{(1-x)^2} + \frac{Cx}{1-x},$$

valid for $0 \leqslant x \leqslant 1$.

4.6. The end game. We are now in a position to use Proposition 13. The following corollary, together with the discussion at the beginning of Section 4, proves Theorem 2.

Corollary 19. *We have*

$$\dim_{vN}\ker T = \frac{4p^3+3p^2+2p-1}{8p^3} + \frac{1}{8p^3}\sum_{k=1}^{\infty}\left(\frac{p-1}{p}\right)^{k+2^k},$$

which is a transcendental number.

Proof. Let T_0 be the 0 operator $\mathbb{C} \to \mathbb{C}$, let

$$T_1(k)\colon \mathbb{C}^{2k} \longrightarrow \mathbb{C}^{2k}$$

be the adjacency operator on the graph from Figure 3, let

$$T_2(l)\colon \mathbb{C}^{2l+1} \longrightarrow \mathbb{C}^{2l+1}$$

be the adjacency operator on the graph from Figure 5, and finally let

$$T_3(k,l)\colon \mathbb{C}^{2k+2l+2} \longrightarrow \mathbb{C}^{2k+2l+2}$$

be the adjacency operator on the graph from Figure 7.

By Proposition 13 and the computations in the previous subsections, the left-hand side is equal to the sum of the following terms

$$\frac{1}{8}(2 + 5\alpha + \alpha^3 + 2\beta\alpha^2 + \beta + \beta^2) \cdot \dim\ker T_0,$$

$$\sum_{k=1}^{\infty} \frac{1}{8} \cdot \alpha^3 \cdot \beta^k \cdot \dim\ker T_1(k),$$

$$\sum_{l=1}^{\infty} \frac{1}{8} \cdot \alpha^3 \beta^l \dim\ker T_2(l),$$

$$\sum_{k,l=1}^{\infty} \frac{1}{8} \cdot \alpha^3 \beta^{k+l} \dim\ker T_3(k,l).$$

Substituting the values for the kernel dimensions we get

$$\frac{1}{8}(2 + 5\alpha + \alpha^3 + 2\beta\alpha^2 + \beta + \beta^2) + 0$$
$$+ \sum_{l=1}^{\infty} \frac{1}{8} \cdot \alpha^3 \beta^l + \sum_{k,l=1}^{\infty} \frac{1}{8} \cdot \alpha^3 \beta^{k+l} + \sum_{k=2}^{\infty} \frac{1}{8} \cdot \alpha^3 \beta^{k+2^{k-1}-1}.$$

Noting that

$$\sum_{k,l=1}^{\infty} \beta^{k+l} = \sum_k \beta^k \sum_l \beta^l = \left(\frac{\beta}{\alpha}\right)^2,$$

after a short calculation we obtain

$$\frac{1}{8}(2 + 5\alpha + \alpha^3 + 2\beta\alpha^2 + \beta + \beta^2) + \frac{1}{8}\alpha^2\beta + \frac{1}{8}\alpha\beta^2 + \frac{1}{8}\alpha^3 \sum_{k=1}^{\infty} \beta^{k+2^k},$$

which is equal to the right-hand side.

Transcendence of $\sum_{k=1}^{\infty} \left(\frac{p-1}{p}\right)^{k+2^{k-1}}$ follows from [1, Theorem 1]. Although similar series have been studied already by Mahler [16], the article [1] seems to be the first work which implies the transcendence of $\sum_{k=1}^{\infty} \left(\frac{p-1}{p}\right)^{k+2^{k-1}}$. □

Appendix A: Linear algebra computations

The following obvious lemma will be used many times.

Lemma 20 ("flow lemma at a vertex v"). *Let T be the adjacency operator on an edge-labelled directed graph, let v be a vertex, let $w_1, \ldots, w_n$ be all the vertices for which there are directed edges towards v, and let the corresponding edge labels be $a_1, \ldots, a_n \in \mathbb{C}$. Let $f \in \ker T$. Then*

$$\sum a_i f(w_i) = 0.$$

A.1. $x \in [0\underline{1}1^{k-1}00, A]$. We give the vertices of $\mathcal{G}(x)$ shorthand names as in Figure 8.

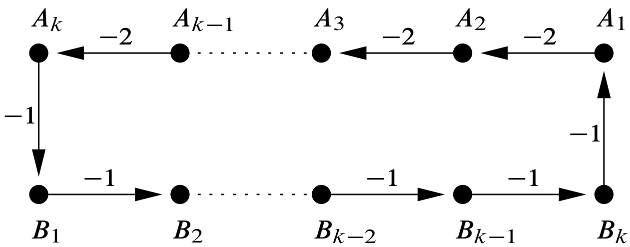

Figure 8

Lemma 21. *We have* $\dim \ker T(x) = 0$.

Proof. A direct check confirms the claim when $k = 1$. For $k > 1$ let $f \in \ker T(x)$. From the flow lemma at A_1 we see that $f(A_1) = f(B_k)$, and inductively we get $f(A_1) = f(B_1) = f(A_k)$.

On the other hand from the flow lemma at A_2 we see $f(A_2) = 2 \cdot f(A_1)$, and inductively $f(A_k) = 2^{k-1} \cdot f(A_1)$. Altogether we get

$$f(A_1) = 2^{k-1} \cdot f(A_1),$$

which is a contradiction. □

A.2. $x \in [00\underline{1}1^{l-1}0, C]$**.** We give the vertices of $\mathcal{G}(x)$ shorthand names as in Figure 9.

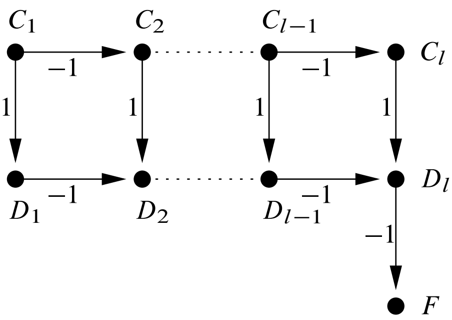

Figure 9

Lemma 22. *We have* $\dim \ker T(x) = 1$.

Proof. The matrix of $T(x)$ in the basis $C_1, \ldots, C_l, D_1, \ldots, D_l, F$ is upper-triangular. The diagonal entries corresponding to C_i and D_i are equal to 1, and the diagonal entry corresponding to F is 0. This shows the lemma. $\square$

A.3. $x \in [0\underline{1}1^{k-1}01^l0, A]$**.** We give the vertices of $\mathcal{G}(x)$ shorthand names as in Figure 10.

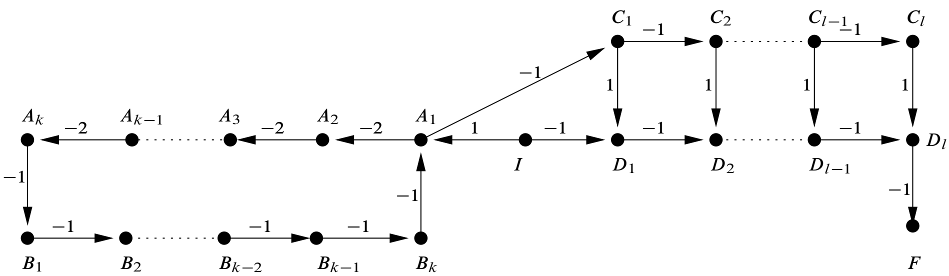

Figure 10

Lemma 23. *If* $l = 2^{k-1} - 1$ *then* $\dim \ker T(x) = 2$*. Otherwise* $\dim \ker T(x) = 1$.

Proof. We will focus on the case $k > 1$. The arguments in the case $k = 1$ are very similar and are left to the reader.

First, assume $l = 2^{k-1} - 1$. The first generator of $\ker T(x)$ is the indicator function of the vertex F. The coefficients of another generator of $\ker T(x)$ are depicted on Figure 11.

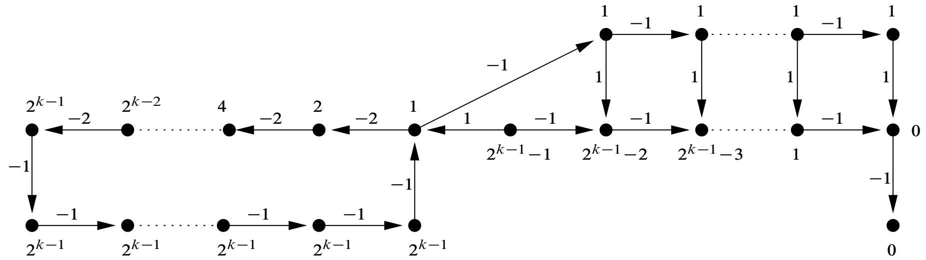

Figure 11. Coefficients of the second generator of $\ker T(x)$ when $l = 2^{k-1} - 1$.

To see that these two vectors generate all of $\ker T(x)$ let us prove the following.

Lemma. *Let $f \in \ker T(x)$ be such that $f(F) = f(A_1) = 0$. Then $f = 0$.*

Proof. From the flow lemma at A_2 we see that $f(A_1) = 0$ implies $f(A_2) = 0$. Similarly we show $f(A_i) = f(B_i) = 0$ for all i. Now the flow lemma at A_1 together with $f(A_1) = f(B_k) = 0$ implies $f(I) = 0$, and the flow lemma at C_1 and $f(A_1) = 0$ imply $f(C_1) = 0$. The flow lemma at D_1 together with $f(I) = f(C_1) = 0$ implies $f(D_1) = 0$.

Now note that the flow lemma at C_{i+1} and $f(C_i) = 0$ imply $f(C_{i+1}) = 0$. Thus we get $f(C_i) = 0$ for all i.

Finally the flow lemma at D_{i+1} and $f(D_i) = f(C_{i+1}) = 0$ imply $f(D_{i+1}) = 0$, and so we also get $f(D_i) = 0$ for all i. Since $f(F) = 0$ by assumption, the claim follows. $\triangle$

Note that the indicator function of the vertex F is in $\ker T(x)$ for arbitrary (k, l). Thus to finish the proof it is enough to show that if $f \in \ker T$ is such that $f(A_1) = 1$ then $l = 2^{k-1} - 1$.

So assume $f(A_1) = 1$. From the flow lemma at A_2 we get $f(A_2) = 2$. Similarly $f(A_i) = 2^{i-1}$ for all i, and in particular $f(A_k) = 2^{k-1}$.

Now from the flow lemma at B_1 we have also $f(B_1) = 2^{k-1}$ and by induction $f(B_k) = 2^{k-1}$.

Since $f(A_1) = 1$ and $f(B_k) = 2^{k-1}$, the flow lemma at A_1 implies $f(I) = 2^{k-1}$. The flow lemma at C_1 together with $f(A_1) = 1$ implies $f(C_1) = 1$, and by induction $f(C_i) = 1$ for all $i's$. Thus by the flow lemma at D_1 we get $f(D_1) = 2^{k-1} - 2$ and inductively $f(D_i) = 2^{k-1} - i - 1$.

This means that $f(D_l) = 0$ only if $0 = 2^{k-1} - l - 1$. Since the flow lemma at F implies $f(D_l) = 0$, this ends the proof. $\square$

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Łukasz Grabowski, Department of Mathematics and Statistics, Lancaster University, Lancaster LA1 4YF, UK

e-mail: graboluk@gmail.com