



**Measure and Integration** — *On Brooks-Jewett, Vitali-Hahn-Saks and Nikodým convergence theorems for quasi-triangular functions*, by PAOLA CAVALIERE and PAOLO DE LUCIA.

*Dedicated to the memory of Renato Caccioppoli*

ABSTRACT. — Connections are established between Brooks-Jewett, Vitali-Hahn-Saks and Nikodým type convergence theorems for quasi-triangular functions on Boolean rings satisfying the Subsequential Completeness Property and valued into Hausdorff topological spaces.

KEY WORDS: Non-additive functions, Convergence theorems.

MATHEMATICS SUBJECT CLASSIFICATION: 28A12, 28A33.

## 1. INTRODUCTION

In 1928, in dealing with sufficient conditions for interchanging limit and Stieltjes integral, R. Caccioppoli [1] introduced the following definition of uniformly bounded variation for families of countably additive real-valued functions. Let  $\mathcal{A}$  be the  $\sigma$ -algebra of Lebesgue measurable subsets of a rectangle  $[a, b] \subset \mathbb{R}^m$ . A sequence  $(\mu_n)_{n \in \mathbb{N}}$  of countably additive real-valued functions defined on  $\mathcal{A}$  is said to be of uniformly bounded variation if, for every sequence  $(A_i)_{i \in \mathbb{N}}$  in  $\mathcal{A}$  such that

$$(1.1) \quad \bigcap_{k \in \mathbb{N}} \bigcup_{i \geq k} A_i = \emptyset,$$

the real sequence  $(\mu_n(A_i))_{i \in \mathbb{N}}$  converges to zero, uniformly in  $n$ .

In a footnote, Caccioppoli observed that condition (1.1) could be relaxed on requiring that the  $A_i$ 's are pairwise disjoint. However, he did not further investigate on this possibility. The notion of uniformly bounded variation was recovered only sixteen years later by V. M. Dubrovski [14, 15] and exploited by F. Cafiero in [2], [3, Chapter V] and [4], under the name of uniform additivity.

In 1943, in connection with the study of the decomposition of additive set functions, R. Rickart [25] suggested the following definition of exhaustivity for finitely additive vector-valued functions. Let  $\mathcal{A}$  be a  $\sigma$ -algebra of sets and let  $X$  be a normed linear space. A finitely additive function  $\mu : \mathcal{A} \rightarrow X$  is said to be exhaustive if, for every sequence  $(A_i)_{i \in \mathbb{N}}$  of pairwise disjoint elements of  $\mathcal{A}$ , the sequence  $(\mu(A_i))_{i \in \mathbb{N}}$  converges to the zero element of  $X$ .

Plainly, as long as families of countably additive real-valued functions are taken into account, the definition of uniform exhaustivity agrees with that of uniformly bounded variation to which Caccioppoli alluded in the aforementioned footnote (see [10] for more information). Hence, the notion of an exhaustive function may be regarded as an extension of that of finitely additive vector measures of bounded variation and countably additive measures on a  $\sigma$ -algebra, whose additional advantage relies on its versatility. Its applicability extends far beyond the classes of finitely additive functions valued into topological groups or uniform semigroups, for example in the setting of non-additive functions.

On the other hand, recent years have witnessed an increasing interest in the study of *non-additive* functions. Research in this topic is motivated by theoretical and applied problems where even an assumption of finite additivity is too restrictive (see, *e.g.*, [12, 16, 18, 21, 22, 23, 29, 32] and the references therein). As a consequence, the additivity assumption has to be replaced by a considerably weaker assumption, and various additional conditions have to be required in order to introduce special kinds of non-additive measures. The present paper is concerned with the class of quasi-triangular functions—see, *e.g.*, [5, 6, 7, 19, 20, 26, 27, 28]. The importance of this class of functions mainly stems from the following two facts. First, it provides us with a generalization of the class of classical finitely additive functions where the additivity assumption is removed and no algebraic structure is required on the range space as well. Second, it includes various families of non-additive functions extensively studied in the literature, as elucidated by several examples in [6].

In the present paper we continue our investigations, initiated in [5, 6], on the issue of whether classical convergence theorems are available for quasi-triangular functions acting on Boolean rings satisfying the Subsequential Completeness Property. In this framework, we prove a Nikodým type theorem, and establish connections between theorems of Brooks-Jewett [5], Vitali-Hahn-Saks [6] and Nikodým type. This is the content of Theorem 2.1, where these results are shown to be in fact equivalent, provided that the range space is a Tychonoff space. Thus, Theorem 2.1 may be regarded as an extension to the quasi-triangular case of a well-known result of L. Drewnowski [13] (see also [11]) concerning finitely additive functions. Moreover, as a straightforward corollary, we deduce that each one of the aforementioned convergence theorems can be derived as a consequence of Cafiero criterion for uniform exhaustivity [5, 27, 28].

The outline of this note is as follows. Section 2 contains some basic definitions and the statement of our main result. The latter consists of various implications, whose proofs, which require tools of different nature, are given in separate subsections of Section 3.

## 2. STATEMENT OF THE MAIN RESULT

Throughout this note  $\mathcal{R}$  stands for a Boolean ring, whose least point is denoted by  $0_{\mathcal{R}}$  or  $0$ , and  $\mathcal{S} = (S, \tau)$  for a Hausdorff topological space. On  $\mathcal{S}$  we assume that a point  $e \in \mathcal{S}$  is arbitrarily fixed and use the notation  $\tau[e]$  to denote one of its fundamental system of  $\tau$ -neighborhoods.

We denote by  $\mathcal{M}[e]$  the family of all functions  $\varphi : \mathcal{R} \rightarrow \mathcal{S}$  such that  $\varphi(0) = e$ . Recall that if  $\varphi \in \mathcal{M}[e]$ , then

- i)  $\varphi$  is said to be (*e*-)exhaustive if, whenever  $(a_k)_{k \in \mathbb{N}}$  is a sequence of pairwise disjoint elements of  $\mathcal{R}$ , then  $\lim_k \varphi(a_k) = e$ ;
- ii)  $\varphi$  is said to be (*e*-)order continuous if, whenever  $(b_k)_{k \in \mathbb{N}}$  is a monotone decreasing sequence in  $\mathcal{R}$  whose infimum is  $0_{\mathcal{R}}$  (briefly, decreasing to  $0_{\mathcal{R}}$  or  $b_k \searrow 0_{\mathcal{R}}$ ), then  $\lim_k \varphi(b_k) = e$ ;
- iii)  $\varphi$  is said to be quasi-triangular if, for every  $U \in \tau[e]$  there exists  $V = V(U) \in \tau[e]$  such that for each pair of disjoint elements  $a, b$  in  $\mathcal{R}$ , it holds

$$\begin{aligned} \varphi(a) \in V, \varphi(b) \in V &\Rightarrow \varphi(a \vee b) \in U; \\ \varphi(a) \in V, \varphi(a \vee b) \in V &\Rightarrow \varphi(b) \in U. \end{aligned}$$

Note that the uniform version of the above definitions can be easily formulated for any given subcollection  $\Phi := (\varphi_i)_{i \in I}$  of  $\mathcal{M}[e]$ .

In what follows, for any function  $\varphi \in \mathcal{M}[e]$  and each  $a \in \mathcal{R}$  we adopt the notation  $\varphi([0, a])$ , or simply  $\tilde{\varphi}(a)$ , for the set  $\{\varphi(b) : b \in \mathcal{R}, b \leq a\}$ . Moreover, whenever  $\varphi$  is quasi-triangular, we simply denote by  $V^{(0)}$  the intersection  $U \cap V(U)$  of the sets appearing in the definition iii) above and then  $V^{(1)} := V^{(0)} \cap V(V^{(0)})$ .

We will use the following notion, introduced in [6] mymicking [18, 31].

**DEFINITION 2.1.** Let  $(\mathcal{S}_o, \tau_o)$  be a topological space and  $e_o \in \mathcal{S}_o$  be arbitrarily fixed. If  $\psi \in \mathcal{M}[e]$  and  $\nu \in \mathcal{M}[e_o]$ , then  $\psi$  is said to be  $\nu$ -continuous ( $\psi \ll \nu$ , for short) if, and only if, for every  $U \in \tau[e]$  there exists some  $V \in \tau_o[e_o]$  such that  $\nu([0, a]) \subseteq V$  implies that  $\psi([0, a]) \subseteq U$ . Moreover,  $\psi$  is said to be equivalent to  $\nu$ , in symbols  $\psi \asymp \nu$ , whenever both  $\psi \ll \nu$  and  $\nu \ll \psi$ .

Finally let us recall that  $\mathcal{R}$  is said to satisfy the *Subsequential Completeness Property* (briefly *SCP*) (see, e.g., [8, 17, 30] for more details), if for each sequence  $(a_k)_{k \in \mathbb{N}}$  of pairwise disjoint elements in  $\mathcal{R}$  there is an infinite part  $M$  of  $\mathbb{N}$  ( $M \in \mathcal{I}_{\infty}(\mathbb{N})$ ) such that the supremum of the set  $\{a_k : k \in M\}$  exists in  $\mathcal{R}$ , denoted by  $\bigvee_{k \in M} a_k$  as usual.

Now we are in a position to state our main result dealing with connections between generalized versions of three convergence theorems classically known as the Brooks-Jewett Theorem, the Vitali-Hahn-Saks Theorem and the Nikodým Theorem.

**THEOREM 2.1.** *If  $\mathcal{R}$  is a SCP-Boolean ring and  $(\varphi_n)_{n \in \mathbb{N}}$  is a sequence of exhaustive and uniformly quasi-triangular elements of  $\mathcal{M}[e]$  converging pointwise in  $\mathcal{R}$  to an exhaustive element  $\varphi$  of  $\mathcal{M}[e]$ , then the implications*

- (a): (BJ) implies (VHS)
- (b): (VHS) implies (N)
- (c): (BJ) implies (N)
- (d): (N) implies (BJ), when the range space  $\mathcal{S}$  is a Tychonoff space

hold among the following statements:

- (BJ): Then the sequence  $(\varphi_n)_{n \in \mathbb{N}}$  is uniformly exhaustive;
- (VHS): If each  $\varphi_n$  is  $v$ -continuous, where  $v$  belongs to  $\mathcal{M}[e_o]$  and  $e_o$  is an arbitrarily fixed point in some topological space  $(\mathcal{S}_o, \tau_o)$ , then the sequence  $(\varphi_n)_{n \in \mathbb{N}}$  is uniformly  $v$ -continuous;
- (N): If each  $\varphi_n$  is order continuous, then the sequence  $(\varphi_n)_{n \in \mathbb{N}}$  is uniformly order continuous.

Consequently, (BJ), (VHS) and (N) are in fact equivalent when the range space  $\mathcal{S}$  is a Tychonoff space.

Let us mention that implication (a) is proved in [6], whereas the Brooks-Jewett Theorem is established in [5] via a Cafiero criterion of uniform exhaustivity for quasi-triangular functions. So, by implication (b), here we prove a Nikodým type theorem. Moreover our main result yields as a corollary that each one of the above convergence theorems can be deduced via such criterion.

### 3. PROOF OF THE MAIN RESULT

#### 3.1. (VHS) implies (N)

Before going into details, it is worth noting that in the spirit of [13] we employ Fréchet-Nikodým topologies on rings (briefly, *FN*-topologies) as powerful tool in obtaining our conclusions. The reader may refer to [9, 31] for a comprehensive treatment of *FN*-topologies in the additive context and to [6] for a similar analysis in the non-additive setting of quasi-triangular functions. In order to keep this paper self-contained, using Definition 2.1, we gather together the special results from [6] needed hereafter in the following

REMARK 3.1.1. If  $\varphi$  is quasi-triangular, then

- (i)  $\varphi$  generates a *FN*-topology (namely  $\Gamma_\varphi$ ) on the underlying ring  $\mathcal{R}$  having as neighbourhood base at each  $a \in \mathcal{R}$  the family  $\Gamma_\varphi(a) := (\{x \in \mathcal{R} : \tilde{\varphi}(x \triangle a) \subseteq U\})_{U \in \mathcal{B}[e]}$ , where  $\mathcal{B}[e]$  is a neighbourhood base at the point  $e \in \mathcal{S}$ ;
- (ii) there exists a finitely additive function  $\mu$ , acting on the same Boolean ring  $\mathcal{R}$  and valued in some topological Abelian group  $(\mathcal{G}_o, \tau_o)$ , with  $\mu(0_{\mathcal{R}}) = e_o$ , which is equivalent to  $\varphi$ , i.e.  $\mu \asymp \varphi$ . Consequently, the *FN*-topologies generated by  $\varphi$  and  $\mu$ , respectively, coincide, i.e.  $\Gamma_\varphi \equiv \Gamma_\mu$ .

Our first aim is to strengthen conclusion (ii) of the previous remark, proving that the additional assumption of order continuity of  $\varphi$  yields to the countable additivity of the function  $\mu$ . To this end we need the following

LEMMA 3.1.2. *Let  $\varphi$  be quasi-triangular. Then  $\varphi$  is order continuous if, and only if, for every  $U \in \tau[e]$  and every sequence  $(b_k)_{k \in \mathbb{N}}$  in  $\mathcal{R}$  decreasing to  $0_{\mathcal{R}}$  there exists some  $k_o \in \mathbb{N}$  such that  $\varphi([0, b_k]) \subseteq U$  for every  $k \geq k_o$ .*

PROOF. Given  $\varphi \in \mathcal{M}[e]$  quasi-triangular, suppose first that  $\varphi$  is order continuous and, by way of contradiction, the condition is not satisfied. Then there exist some  $U_o \in \tau[e]$  and some sequence  $(b_k)_{k \in \mathbb{N}}$  decreasing to  $0_{\mathcal{R}}$  such that for each  $i \in \mathbb{N}$  there are an index  $k_i > i$  and an element  $y_i \in \mathcal{R}$  such that  $\varphi(y_i \wedge b_{k_i}) \notin U_o$ . In particular, this implies that corresponding to  $V^{(0)} := U_o \cap V(U_o)$  one may construct inductively a strictly increasing sequence  $(i_j)_{j \in \mathbb{N}}$  such that

$$\varphi(y_{i_j} \wedge b_{k_{i_j}}) \notin U_o, \quad \varphi(y_{i_j} \wedge b_{k_{i_{j+1}}}) \in V^{(0)} \quad \forall j \in \mathbb{N}.$$

The quasi-triangularity of  $\varphi$  then tells us that

$$(3.1.1) \quad \varphi(y_{i_j} \wedge (b_{k_{i_j}} \setminus b_{k_{i_{j+1}}})) =: \varphi(c_j \setminus b_{k_{i_{j+1}}}) \notin V^{(0)} \quad \forall j \in \mathbb{N},$$

where  $c_j := (y_{i_j} \wedge b_{k_{i_j}}) \vee b_{k_{i_{j+1}}}$ . Since  $(b_k)_{k \in \mathbb{N}}$  decreases to  $0_{\mathcal{R}}$ , clearly so is  $(c_j)_{j \in \mathbb{N}}$ . So by the order continuity of  $\varphi$ , corresponding to  $V^{(1)}$ , there exists some  $j_o \in \mathbb{N}$  such that

$$\varphi(c_j) \in V^{(1)}, \quad \varphi(b_{k_{i_{j+1}}}) \in V^{(1)} \quad \forall j \geq j_o.$$

This clearly contradicts (3.1.1), because of the quasi-triangularity of  $\varphi$ , ending the proof. The converse implication follows immediately from definition ii) (and clearly works without assuming  $\varphi$  quasi-triangular). □

Now we are able to state

LEMMA 3.1.3. *Let  $\varphi$  be quasi-triangular. Then  $\varphi$  is order continuous if, and only if, there exists a countably additive function  $\mu_\sigma$ , acting on the same Boolean ring  $\mathcal{R}$  and valued in some topological Abelian group  $(\mathcal{G}_o, \tau_o)$ , with  $\mu_\sigma(0_{\mathcal{R}}) = e_o$ , which is equivalent to  $\varphi$ , i.e.  $\mu_\sigma \asymp \varphi$ .*

PROOF. Assume first that  $\varphi$  is quasi-triangular as well as order continuous. According to Remark 3.1.1(i) and Lemma 3.1.2, the *FN*-topology  $\Gamma_\varphi$  generated by  $\varphi$  on  $\mathcal{R}$  is order continuous. Hence the *FN*-topology  $\Gamma_\mu$  generated on  $\mathcal{R}$  by the finitely additive function  $\mu$  equivalent to  $\varphi$  (see Remark 3.1.1(ii)) is order continuous as well. This implies that  $\mu$  is actually countably additive, by [9, (2.4) in Chap. III]. The converse implication follows directly because countable additivity implies order continuity as well as  $\Gamma_{\mu_\sigma} \equiv \Gamma_\varphi$  by assumption. □

PROOF OF THEOREM 2.1(b). Let  $(\varphi_n)_{n \in \mathbb{N}}$  be a sequence of exhaustive, order continuous and uniformly quasi-triangular elements of  $\mathcal{M}[e]$  converging pointwise in  $\mathcal{R}$  to an exhaustive element  $\varphi$  of  $\mathcal{M}[e]$ . By (the proof of) Lemma 3.1.3, for each  $n$  the *FN*-topology  $\Gamma_{\varphi_n}$  generated by  $\varphi_n$  on  $\mathcal{R}$  is order continuous. Hence

$$(3.1.2) \quad \Gamma_{\varphi_n} \subseteq \Gamma_o := \bigvee \{ \Gamma \in FN(\mathcal{R}) : \Gamma \text{ order continuous} \},$$

where  $FN(\mathcal{R})$  denotes the distributive complete lattice (with the inclusion as partial order) of all  $FN$ -topologies on  $\mathcal{R}$  (see [31, Proposition 1.5(a)]). Since  $\Gamma_o$  is order continuous, according to [9, (2.3)–(2.4) in Chap. III], there is a countably additive function  $\mu_\sigma$  defined on  $\mathcal{R}$  and valued into some topological Abelian group  $\mathcal{G}_o$  such that  $\Gamma_o \equiv \Gamma_{\mu_\sigma}$ . Thus, according to (3.1.2), each  $\varphi_n$  is actually  $\mu_\sigma$ -continuous. Now (VHS) tells us that the sequence  $(\varphi_n)_{n \in \mathbb{N}}$  is uniformly  $\mu_\sigma$ -continuous. As the countable additivity of  $\mu_\sigma$  guarantees that, for any sequence  $(b_k)_{k \in \mathbb{N}}$  decreasing to  $0_{\mathcal{R}}$ , then  $\lim_k \mu_\sigma(b_k \wedge x) = e_o$  uniformly respect to  $x \in \mathcal{R}$ , the proof ends.  $\square$

### 3.2. (BJ) implies (N)

PROOF OF THEOREM 2.1(c). Let  $(\varphi_n)_{n \in \mathbb{N}}$  be a sequence of exhaustive, order continuous and uniformly quasi-triangular elements of  $\mathcal{M}[e]$  converging pointwise in  $\mathcal{R}$  to an exhaustive element  $\varphi$  of  $\mathcal{M}[e]$ . Then (BJ) states that the sequence  $(\varphi_n)_{n \in \mathbb{N}}$  is uniformly exhaustive.

Now suppose, by way of contradiction, that  $(\varphi_n)_{n \in \mathbb{N}}$  fails to be uniformly order continuous. Then, passing to a subsequence if needed, there are some  $U_o \in \tau[e]$  and some sequence  $(b_k)_{k \in \mathbb{N}}$  decreasing to  $0_{\mathcal{R}}$  such that  $\varphi_i(b_i) \notin U_o$  for each  $i$ . So, taken  $V^{(0)} := U_o \cap V(U_o)$ , one may construct inductively a strictly increasing sequence  $(i_j)_{j \in \mathbb{N}}$  such that  $\varphi_{i_j}(b_{i_{j+1}}) \in V^{(0)}$  for every  $j$ . The uniform quasi-triangularity of  $(\varphi_n)_{n \in \mathbb{N}}$  then implies that  $\varphi_{i_j}(b_{i_j} \wedge b'_{i_{j+1}}) \notin V^{(0)}$  for each  $j$ . This clearly contradicts the uniform exhaustivity of the sequence  $(\varphi_n)_{n \in \mathbb{N}}$  just stated, since  $(b_{i_j} \wedge b'_{i_{j+1}})_{j \in \mathbb{N}}$  is disjoint in  $\mathcal{R}$ .  $\square$

REMARK 3.2.1. By [24, Theorem 1.3.23], the above result improves the one established in [26].

### 3.3. (N) implies (BJ), If $\mathcal{S}$ is a Tychonoff Space

Our approach to the proof depends mainly on two lemmas, which are closely modeled on [13].

LEMMA 3.3.1. *Let  $\mathcal{R}$  be a SCP-Boolean ring and let  $\eta : \mathcal{R} \rightarrow [0, +\infty[$  be a monotone increasing function vanishing in  $0_{\mathcal{R}}$ . If  $\eta$  is exhaustive, then for each sequence  $(a_k)_{k \in \mathbb{N}}$  of pairwise disjoint elements of  $\mathcal{R}$  there is a subsequence  $(a_{k_n})_{n \in \mathbb{N}}$  so that  $\eta$  is order continuous on the SCP-Boolean ring generated by  $(a_{k_n})_{n \in \mathbb{N}}$ .*

PROOF. We first claim that, given a disjoint sequence  $(a_k)_{k \in \mathbb{N}}$  in  $\mathcal{R}$ , for each  $I \in \mathcal{I}_\infty(\mathbb{N})$  and every  $\varepsilon > 0$  there is some  $M \in \mathcal{I}_\infty(I)$  such that  $\bigvee_{k \in M} a_k$  exists in  $\mathcal{R}$  and  $\eta(\bigvee_{k \in M} a_k) < \varepsilon$ . To see this, partition  $I$  into a pairwise disjoint sequence  $(I_h)_{h \in \mathbb{N}}$  of infinite sets. As  $\mathcal{R}$  satisfies the subsequential completeness property, then for each  $h \in \mathbb{N}$  there is some  $N_h \in \mathcal{I}_\infty(I_h)$  such that  $\bigvee_{k \in N_h} a_k := b_h$  exists in  $\mathcal{R}$ . Clearly,  $(b_h)_{h \in \mathbb{N}}$  is a pairwise disjoint sequence in  $\mathcal{R}$ . By the exhaustivity of  $\eta$ , then there is some positive integer  $h_\varepsilon$  such that  $\eta(\bigvee_{k \in N_h} a_k) < \varepsilon$  for each  $h \geq h_\varepsilon$ . One gets the claim setting  $M := N_{h_\varepsilon}$ .

A recursive application of the claim just stated leads to the existence of a monotone decreasing sequence  $(M_n)_{n \in \mathbb{N}}$  in  $\mathcal{I}_\infty(\mathbb{N})$  satisfying the following conditions:

$$(3.3.1) \quad M_n \subseteq M_{n-1} \setminus \{\min M_{n-1}\}, \quad \exists \bigvee_{k \in M_n} a_k, \quad \eta\left(\bigvee_{k \in M_n} a_k\right) < \frac{1}{n},$$

where  $M_0 := \mathbb{N}$ .

Writing  $k_n := \min M_n$ ,  $n \in \mathbb{N}$ , the subsequence  $(a_{k_n})_{n \in \mathbb{N}}$  is well-defined and fulfils the properties

( $\star$ ) for any  $n \in \mathbb{N}$  and every  $J \in \mathcal{P}(\mathbb{N} \setminus \{1, \dots, n\})$  such that  $\bigvee_{j \in J} a_{k_j}$  exists in  $\mathcal{R}$ , it holds that

$$\bigvee_{j \in J} a_{k_j} \leq \bigvee_{k \in M_n} a_k;$$

moreover  $\eta(\bigvee_{j \in J} a_{k_j}) < \frac{1}{n}$ .

The latter condition follows from the monotonicity of  $\eta$  together with the last estimate in (3.3.1).

Now we claim that  $\eta$  is order continuous on the SCP-Boolean ring  $\mathcal{R}_o$  generated by  $(a_{k_n})_{n \in \mathbb{N}}$ . Indeed, let  $(b_m)_{m \in \mathbb{N}}$  be a sequence in  $\mathcal{R}_o \setminus \{0_{\mathcal{R}}\}$  decreasing to  $0_{\mathcal{R}}$ . This means in particular that for each  $m$  there is some non-empty  $J_m \in \mathcal{P}(\mathbb{N})$  such that  $b_m = \bigvee_{j \in J_m} a_{k_j}$  and, moreover,  $\lim_m \min J_m = +\infty$ . Hence for any  $n \in \mathbb{N}$  then there exists some  $m_o$  such that index sets  $J_m$  belong to  $\mathcal{P}(\mathbb{N} \setminus \{1, \dots, n\})$  for  $m \geq m_o$ . Thus, by ( $\star$ ),  $\eta(b_m) < \frac{1}{n}$  for  $m \geq m_o$ , i.e.  $\eta(b_m) \rightarrow 0$ , as claimed.  $\square$

LEMMA 3.3.2. *Let  $\mathcal{R}$  be a SCP-Boolean ring,  $\mathcal{S} = (S, p)$  a pseudometric space and  $(\varphi_n)_{n \in \mathbb{N}}$  a sequence of exhaustive elements of  $\mathcal{M}[e]$ . Then any sequence  $(a_k)_{k \in \mathbb{N}}$  of pairwise disjoint elements of  $\mathcal{R}$  admits a subsequence  $(a_{k_i})_{i \in \mathbb{N}}$  such that*

- the supremum  $\bigvee_{i \in \mathbb{N}} a_{k_i}$  exists in the SCP-Boolean ring  $\mathcal{R}_p$  generated by it;
- each function  $\varphi_n$  is order continuous on  $\mathcal{R}_p$ .

PROOF. For each  $n \in \mathbb{N}$ , let us define  $\psi_n : \mathcal{R} \rightarrow [0, +\infty]$  by

$$\psi_n(a) := \sup_{\substack{x \in \mathcal{R} \\ x \leq a}} p(e, \varphi_n(x)), \quad a \in \mathcal{R},$$

and then

$$\eta : a \in \mathcal{R} \mapsto \sum_{n \in \mathbb{N}} \frac{1}{2^n} (1 \wedge \psi_n(a)).$$

Clearly each  $\varphi_n$  is  $\eta$ -continuous. Moreover it is easily to check that the function  $\eta$  fulfils the assumptions of Lemma 3.3.1. Therefore for any sequence  $(a_k)_{k \in \mathbb{N}}$  of pairwise disjoint elements of  $\mathcal{R}$  one may extract a subsequence  $(a_{k_i})_{i \in \mathbb{N}}$  such that  $\eta$  is order continuous on the *SCP*-Boolean ring  $\mathcal{R}_p$  generated by it. Without loss of generality one may suppose that  $\bigvee_{i \in \mathbb{N}} a_{k_i}$  belongs to  $\mathcal{R}_p$ . Then, being  $\eta$ -continuous, each  $\varphi_n$  is actually order continuous on the *SCP*-Boolean ring  $\mathcal{R}_p$ .  $\square$

Now we are in position to prove the last implication in Theorem 2.1.

**PROOF OF THEOREM 2.1(d).** Let  $(\varphi_n)_{n \in \mathbb{N}}$  be a sequence of exhaustive and uniformly quasi-triangular elements of  $\mathcal{M}[e]$  converging pointwise in  $\mathcal{R}$  to an exhaustive element  $\varphi$  of  $\mathcal{M}[e]$ .

If (BJ) fails, there are  $U_o \in \tau[e]$ , a disjoint sequence  $(a_k)_{k \in \mathbb{N}}$  in  $\mathcal{R}$  and subsequence  $(\varphi_{n_k})_{k \in \mathbb{N}}$  of  $(\varphi_n)_{n \in \mathbb{N}}$  such that

$$(3.3.2) \quad \varphi_{n_k}(a_k) \notin U_o \quad \text{for each } k.$$

As  $\mathcal{S}$  is a Tychonoff space, there is some uniformity  $\mathcal{U}$  on  $S$  which generates the topology  $\tau$  of  $\mathcal{S}$ . In particular there is some pseudometric  $p$  on  $S$ , belonging to the gage  $D$  of  $\mathcal{U}$ , such that  $U_o \in \tau_p[e]$ , where  $\tau_p$  is the topology generated by  $p$  on  $S$ . Thus Lemma 3.3.2 yields to the existence of a subsequence  $(a_{k_i})_{i \in \mathbb{N}}$  of  $(a_k)_{k \in \mathbb{N}}$  so that each  $\varphi_{n_k}$  is order continuous on the *SCP*-Boolean ring  $\mathcal{R}_p$  generated by it and the supremum  $\bigvee_{i \in \mathbb{N}} a_{k_i}$  belongs  $\mathcal{R}_p$  as well.

Passing to the metric identification of  $(S, p)$  if needed, *i.e.* the metric space of the equivalence classes in  $S$  under the equivalence relation defined by  $x \sim y$  iff  $p(x, y) = 0$ , since  $(\varphi_{n_k})_{k \in \mathbb{N}}$  converges pointwise on  $\mathcal{R}_p$ , (N) guaranties that the sequence  $(\varphi_{n_k})_{k \in \mathbb{N}}$  is actually uniformly order continuous on  $\mathcal{R}_p$ . As  $(\bigvee_{j \geq i} a_{k_j})_{i \in \mathbb{N}}$  is clearly a sequence in  $\mathcal{R}_p$  decreasing to  $0_{\mathcal{R}}$ , a straightforward uniform version of Lemma 3.1.2 implies that there is some  $i_o \in \mathbb{N}$  such that  $\varphi_{n_k}([0, \bigvee_{j \geq i} a_{k_j}]) \subseteq U_o$  for  $i \geq i_o$ , uniformly in  $k$ . Hence, in particular,  $\varphi_{n_{k_i}}(a_{k_i}) \in U_o$  for  $i \geq i_o$ , which contradicts (3.3.2).  $\square$

## REFERENCES

- [1] R. CACCIOPPOLI, *Integrali impropri di Stieltjes. Estensione del teorema di Vitali*. Rend. Acc. Sci. fis. mat. Napoli (3) 35 (1928), 34–40.
- [2] F. CAFIERO, *Sulle famiglie di funzioni additive d'insieme, uniformemente continue*. (Italian) Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 12 (1952), 155–162.
- [3] F. CAFIERO, *Misura ed integrazione*. Consiglio Nazionale delle Ricerche, Monografie Matematiche, No. 5. Edizioni Cremonese, Roma, 1959.
- [4] F. CAFIERO, *Sull' uniforme additività*. (Italian) In: Symposia Mathematica, Vol. II (INDAM, Roma, 1968), pp. 265–280. Academic Press, London, 1969.
- [5] P. CAVALIERE - P. DE LUCIA, *The Cafiero criterion on a Boolean ring*. Rend. Acc. Sci. fis. mat. Napoli 75 (2008), 1–5.
- [6] P. CAVALIERE - P. DE LUCIA, *Some new results in non-additive measure theory*. Comm. Appl. Analysis (in press), pp. 1–11.

- [7] P. CAVALIERE - P. DE LUCIA - F. VENTRIGLIA, *On Drewnowski lemma for non-additive functions and its consequences*. Positivity (in press), pp. 1–15.
- [8] C. CONSTANTINESCU, *Spaces of Measures*. Walter de Gruyter & Co., Berlin, 1984.
- [9] P. DE LUCIA, *Funzioni finitamente additive a valori in un gruppo topologico*. Quaderni dell'Unione Matematica Italiana, vol. 29, Pitagora, Bologna, 1985.
- [10] P. DE LUCIA, *Measure theory in Naples: Renato Caccioppoli*. (Italian) International Symposium in honor of Renato Caccioppoli (Naples, 1989). Ricerche Mat. 40 (1991), suppl., 101–110.
- [11] P. DE LUCIA - P. MORALES, *Equivalence of Brooks-Jewett, Vitali-Hahn-Saks and Nikodým convergence theorems for uniform semigroup-valued additive functions on a Boolean ring*. Ricerche Mat. (1) 35 (1986), 75–87.
- [12] D. DENNEBERG, *Non-additive measure and integral*. Kluwer Academic Publishers Group, Dordrecht, 1994.
- [13] L. DREWNOWSKI, *Equivalence of Brooks-Jewett, Vitali-Hahn-Saks and Nikodým theorems*. Bull. Acad. Polon. Sci. Sr. Sci. Math. Astronom. Phys. 20 (1972), 725–731.
- [14] V. M. DUBROVSKIĬ [V. M. DOUBROVSKY], *On some properties of completely additive set functions and their application to generalization of a theorem of Lebesgue*. (Russian) Rec. Math. [Mat. Sbornik] N.S. (62) 20 (1947), 317–329.
- [15] V. M. DUBROVSKIĬ, *On the basis of a family of completely additive functions of sets and on the properties of uniform additivity and equi-continuity*. (Russian) Doklady Akad. Nauk SSSR (N. S.), 58 (1947), 737–740.
- [16] P. GHIRARDATO, *On independence for non-additive measures, with a Fubini theorem*. J. Econom. Theory (2) 73 (1997), 261–291.
- [17] R. HAYDON, *A nonreflexive Grothendieck space that does not contain  $l_\infty$* . Israel J. Math. (1) 40 (1981), 65–73.
- [18] N. J. KALTON - J. W. ROBERTS, *Uniformly exhaustive submeasures and nearly additive set functions*. Trans. Amer. Math. Soc. (2) 278 (1983), 803–816.
- [19] V. M. KLIMKIN - T. A. SRIBNAYA, *Convergence of a sequence of weakly regular set functions*. Math. Notes (1-2) 62 (1997), 87–92.
- [20] V. M. KLIMKIN - T. A. SRIBNAYA, *Uniform continuity of a family of weakly regular set functions in a topological space*. Math. Notes (1-2) 74 (2003), 56–63.
- [21] M. MARINACCI, *Limit laws for non-additive probabilities and their frequentist interpretation*. J. Econom. Theory (2) 84 (1999), 145–195.
- [22] S. MAASS, *A philosophical foundation of non-additive measure and probability*. Theory and Decision (2-3) 60 (2006), 175–191.
- [23] E. PAP, *Null-additive set functions*. Kluwer Academic Publishers Group, Dordrecht, 1995.
- [24] P. PTÁK - S. PULMANOVÁ, *Orthomodular structures as quantum logics*, Kluwer Academic Publishers Group, Dordrecht, 1991.
- [25] C. E. RICKART, *Decomposition of additive set functions*. Duke Math. J. 10 (1943), 653–665.
- [26] T. A. SRIBNAYA, *On a generalization of the Nikodým convergence theorem*. (Russian) Vestn. Samar. Gos. Univ. Mat. Mekh. Fiz. Khim. Biol. 2 (2000), 69–75.
- [27] T. A. SRIBNAYA, *The Cafiero criterion on an orthomodular partially ordered set*. (Russian) Vestn. Samar. Gos. Univ. Estestvennonauchn. Ser. 4 (2002), 23–30.
- [28] F. VENTRIGLIA, *Cafiero theorem for  $k$ -triangular functions on an orthomodular lattice*. Rend. Acc. Sci. fis. mat. Napoli 75 (2008), 45–52.

- [29] Z. WANG - G. J. KLIR, *Generalized measure theory*. Springer Science + Business Media, New York, 2009.
- [30] H. WEBER, *Compactness in spaces of group-valued contents, the Vitali-Hahn-Saks theorem and Nikodým's boundedness theorem*. Rocky Mountain J. Math. (2) 16 (1986), 253–275.
- [31] H. WEBER, *FN-Topologies and group-valued measures*, in: Handbook of measure theory, vol I, II, pp. 703–743 (Ed: E. Pap), North-Holland, Amsterdam, 2002.
- [32] S. WEBER,  *$\perp$ -decomposable measures and integrals for Archimedean  $t$ -conorms  $\perp$* . J. Math. Anal. Appl. (1) 101 (1984), 114–138.

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