# Tarski numbers of group actions

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**Abstract.** The Tarski number of a group action $G \curvearrowright X$ is the minimal number of pieces in a paradoxical decomposition of it. In this paper we solve the problem of describing the set of Tarski numbers of group actions. Namely, for any $k \geq 4$ we construct a faithful transitive action of a free group with Tarski number $k$. We also construct a group action $G \curvearrowright X$ with Tarski number 6 such that the Tarski numbers of restrictions of this action to finite index subgroups of $G$ are arbitrarily large.

## 1. Introduction

Let $G \curvearrowright X$ be a group action. We will always assume that groups are acting from the right.

**Definition 1.1.** The group action $G \curvearrowright X$ admits a *paradoxical decomposition* if there exist positive integers $m$ and $n$, disjoint subsets $P_1, \ldots, P_m, Q_1, \ldots, Q_n$ of $X$ and subsets $S_1 = \{g_1, \ldots, g_m\}$, $S_2 = \{h_1, \ldots, h_n\}$ of $G$ such that

$$X = \bigcup_{i=1}^{m} P_i g_i = \bigcup_{j=1}^{n} Q_j h_j. \tag{1.1}$$

The sets $S_1$, $S_2$ are called the *translating sets* of the paradoxical decomposition.

It is well known [9] that $G \curvearrowright X$ admits a paradoxical decomposition if and only if it is non-amenable. The minimal possible value of $m + n$ in a paradoxical decomposition of $G \curvearrowright X$ is called the *Tarski number* of the action and denoted by $\mathcal{T}(G \curvearrowright X)$. If $G$ acts on itself by right multiplication, the Tarski number of the action is called the Tarski number of $G$ and denoted by $\mathcal{T}(G)$.

Clearly, $m, n \geq 2$ in any paradoxical decomposition. Thus, the Tarski number of any group action cannot be smaller than 4. By a result of Jónsson (see, for example, [8, Theorem 5.8.38]) the Tarski number of a group is 4 if and only if it contains a non-abelian free subgroup.

The problem of describing the set of Tarski numbers of groups was posed in [2]. Recently, together with M. Ershov and M. Sapir [3], we constructed the first examples of groups with Tarski numbers 5 and 6. Note, that no integer $\geq 7$ is known to be the Tarski number of a group.

A similar situation existed for the Tarski numbers of group actions. A result of Dekker characterizes all group actions with Tarski number 4.

**Theorem 1.2** ([9, Theorems 4.5 and 4.8]). *Let $G \curvearrowright X$ be a group action. Then, the Tarski number of the action is 4 if and only if $G$ contains a non-abelian free subgroup $F$ such that the stabilizers of points from $X$ in $F$ are cyclic.*

Groups with Tarski numbers 5 and 6 clearly admit actions with Tarski numbers 5 and 6 respectively. No actions with Tarski number > 6 were known.

In this paper we completely solve the problem of describing Tarski numbers of group actions.

**Theorem 1.** *Every integer $k \geq 4$ is the Tarski number of a faithful transitive action of a finitely generated free group.*

From Theorem 1.2, it follows that if $F$ is a non-abelian free group and the action $F \curvearrowright X$ has cyclic point stabilizers then $\mathcal{T}(F \curvearrowright X) = \mathcal{T}(F)$. Part (2) of the following theorem generalizes this result. Note that this theorem, is the group action analogue of parts (a) and (c) of [3, Theorem 1]. Parts (b) and (d) can be extended to group actions as well.

**Theorem 1.3.** *Let $G \curvearrowright X$ be a group action.*

(1) *Let $H \leq G$ be a finite index subgroup and $H \curvearrowright X$ the action of $G$ restricted to $H$. Then,*

$$\mathcal{T}(H \curvearrowright X) - 2 \leq [G : H](\mathcal{T}(G \curvearrowright X) - 2).$$

(2) *If $G \curvearrowright X$ has amenable point stabilizers then $\mathcal{T}(G \curvearrowright X) = \mathcal{T}(G)$.*

In [3] we observed that there exists $t$ such that the property of having Tarski number $t$ is not invariant under quasi isometry. Indeed, a construction from [4] yields a non-amenable group $G$ with finite index subgroups with arbitrarily large Tarski numbers. The only estimate of the value of $t$ bounds it from above by $10^{10^8}$. Note that the case $t = 4$ is the famous open problem by Benson Farb. So far we are not able to lower $t$, but for group actions we prove the following.

**Theorem 2.** *Let $F$ be a free group of rank* 3. *There exists a faithful transitive action $F \curvearrowright X$ such that $\mathcal{T}(F \curvearrowright X) = 6$ and restrictions of the action to finite index subgroups of $F$ have arbitrarily large Tarski numbers.*

Note that by Theorem 1.2, the number 6 cannot be replaced by the number 4 in Theorem 2. We do not know if it can be replaced by 5.

**Organization.** Section 2 contains background information about Tarski numbers of group actions and the proof of Theorem 1.3. Section 3 contains preliminary information about subgroups of free groups and their Stallings cores. Section 4 contains the proof of Theorem 1 and Section 5 contains the proof of Theorem 2.

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## 2. Tarski numbers of group actions

**Lemma 2.1.** [9, Proposition 1.10] *Let $G \curvearrowright X$ be a free action. Then, if $G$ has a paradoxical decomposition with translating sets $S_1, S_2$, then $G \curvearrowright X$ has a paradoxical decomposition with the same translating sets.*

**Corollary 2.2.** *If the free group $\langle x, y \rangle$ acts freely on $X$, then the action has a paradoxical decomposition with translating sets $\{1, x\}$ and $\{1, y\}$.*

*Proof.* The free group $\langle x, y \rangle$ has a paradoxical decomposition with these translating sets [9, Theorem 1.2]. $\square$

**Lemma 2.3.** *Let $G \curvearrowright X$ be a group action.*
(1) *If $H \leq G$ is a subgroup of $G$ and $H \curvearrowright X$ is the action of $G$ restricted to $H$ then $\mathcal{T}(G \curvearrowright X) \leq \mathcal{T}(H \curvearrowright X)$.*
(2) *Let $G \curvearrowright Y$ be another $G$-action and $f : X \to Y$ be a $G$-equivariant map. If $S_1, S_2$ are translating sets of a paradoxical decomposition of $G \curvearrowright Y$ then they are also translating sets of a paradoxical decomposition of $G \curvearrowright X$.*

*Proof.* (1) Every paradoxical decomposition with translating elements from $H$ is in particular a paradoxical decomposition with translating elements from $G$.

(2) Let $P_1, \ldots, P_m, Q_1, \ldots, Q_n \subseteq Y$ be a paradoxical decomposition of $G \curvearrowright Y$ with translating sets $S_1 = \{g_1, \ldots, g_m\}$ and $S_2 = \{h_1, \ldots, h_n\}$. Then the inverse images $f^{-1}(P_1), \ldots, f^{-1}(P_m), f^{-1}(Q_1), \ldots, f^{-1}(Q_n)$ form a paradoxical decomposition of $G \curvearrowright X$ with the same translating sets. $\square$

**Lemma 2.4.** *Let $H \lhd G$ be a normal subgroup. Then if $G \curvearrowright G/H$ is non-amenable so is the group $G/H$.*

*Proof.* Every translating element from $G$ can be replaced by its image in $G/H$. □

If $G \curvearrowright X$ is a group action and $x \in X$, we let $\mathrm{Stab}_G(x) = \{g \in G : xg = x\}$ be the stabilizer of $x$ in $G$. All quotient sets we encounter below, are sets of right cosets.

**Remark 2.5.** Let $G \curvearrowright X$ be a transitive action and let $x \in X$. Then $G \curvearrowright X$ has a paradoxical decomposition with translating sets $S_1, S_2$ if and only if so does the action $G \curvearrowright G/\mathrm{Stab}_G(x)$.

**Remark 2.6.** Let $G \curvearrowright X$ be a group action. Let $\{X_\alpha\}_{\alpha \in I}$ be a partition of $X$ in which every set is closed under the action of $G$. Then $G \curvearrowright X$ has a paradoxical decomposition with translating sets $S_1, S_2$ if and only if for every $\alpha$, the action $G \curvearrowright X_\alpha$ has a paradoxical decomposition with translating sets $S_1, S_2$.

Combining Remark 2.6 (for the partition of $X$ into $G$-orbits) and Remark 2.5 we get the following.

**Corollary 2.7.** *Let $G \curvearrowright X$ be a group action. It has a paradoxical decomposition with translating sets $S_1, S_2$ if and only if for every $x \in X$, the action $G \curvearrowright G/\mathrm{Stab}_G(x)$ has a paradoxical decomposition with these sets as translating sets.*

The following are the analogues for group actions of results of [3], proved originally for groups. Remark 2.8 is the equivalent of [3, Remark 2.2]. Theorem 2.9 follows from [3, Lemma 2.5] and [3, Theorem 2.6]. Theorem 2.10 is a reformulation of [3, Lemma 5.3(c)].

**Remark 2.8.** If $G \curvearrowright X$ has a paradoxical decomposition with translating sets $S_1, S_2$, then $G \curvearrowright X$ also has a paradoxical decomposition with translating sets $S_1 g_1, S_2 g_2$ for any given $g_1, g_2 \in G$. In particular, we can always assume that $1 \in S_1, S_2$.

**Theorem 2.9.** *Let $G \curvearrowright X$ be a group action. Let $S_1, S_2$ be finite subsets of $G$. Then, the following assertions are equivalent.*

(1) *$G \curvearrowright X$ has a paradoxical decomposition with translating sets $S_1, S_2$.*

(2) *For any pair of finite subsets $A_1, A_2 \subseteq X$, $|A_1 S_1^{-1} \cup A_2 S_2^{-1}| \geq |A_1| + |A_2|$.*

**Theorem 2.10.** *Let $G \curvearrowright X$ be a group action and $S = \{a, b, c\} \subseteq G$. Assume that for any finite $A \subseteq X$ we have $|AS^{-1} \cup A| \geq 2|A|$. Then $\mathcal{T}(G \curvearrowright X) \leq 6$.*

We finish this section with a proof of Theorem 1.3 from the introduction. The proof is an adaptation of the proof of [3, Theorem 1(a),(c)] for the group action case. For the proof of Theorem 1.3(1) we will need the following lemma.

**Lemma 2.11.** *Let $G \curvearrowright X$ be a group action which has a paradoxical decomposition with translating sets $S_1$ and $S_2$. Let $H \leq G$ be a finite index subgroup and $H \curvearrowright X$ be the action of $G$ restricted to $H$. Let $T$ be a right transversal of $H$ in $G$, that is, a subset of $G$ which contains precisely one element from each right coset of $H$. For $i = 1, 2$, let $R_i = TS_iT^{-1} \cap H$. Then, the action $H \curvearrowright X$ has a paradoxical decomposition with translating sets $R_1$ and $R_2$.*

*Proof.* Let $x \in X$ , let $K_H = \mathrm{Stab}_H(x)$ and let $K_G = \mathrm{Stab}_G(x)$. By Corollary 2.7, the action $G \curvearrowright G/K_G$ has a paradoxical decomposition with translating sets $S_1, S_2$. Since $K_H \leq K_G$, the same is true for the action $G \curvearrowright G/K_H$, by Lemma 2.3(2).

To prove the lemma, by Corollary 2.7, it suffices to prove that $H \curvearrowright H/K_H$ has a paradoxical decomposition with translating sets $R_1, R_2$. Let $B_1, B_2 \subseteq H/K_H$ be finite subsets. By Theorem 2.9, it suffices to prove that

$$|B_1 R_1^{-1} \cup B_2 R_2^{-1}| \geq |B_1| + |B_2|.$$

Thus, the following calculation completes the proof of the lemma. The steps of the calculation are explained below it.

$$
\begin{aligned}
|B_1 R_1^{-1} \cup B_2 R_2^{-1}| &= |B_1(TS_1^{-1}T^{-1} \cap H) \cup B_2(TS_2^{-1}T^{-1} \cap H)| \\
&\overset{(*)}{=} |((B_1 TS_1^{-1}T^{-1}) \cap H/K_H) \cup ((B_2 TS_2^{-1}T^{-1}) \cap H/K_H)| \\
&= |((B_1 TS_1^{-1} \cup B_2 TS_2^{-1})T^{-1}) \cap H/K_H| \\
&\overset{(**)}{\geq} \frac{|B_1 TS_1^{-1} \cup B_2 TS_2^{-1}|}{|T|} \\
&\overset{(***)}{\geq} \frac{|B_1 T| + |B_2 T|}{|T|} \\
&\overset{(****)}{=} \frac{|B_1||T| + |B_2||T|}{|T|} \\
&= |B_1| + |B_2|.
\end{aligned}
$$

(*) holds since $B_1, B_2 \subseteq H/K_H$. Indeed, for any $A \subseteq H/K_H$ and any $S \subseteq G$ we have $A(S \cap H) = (AS) \cap H/K_H$.

(**) holds since for any $A \subseteq G/K_H$ we have $|(AT^{-1}) \cap H/K_H| \geq \frac{|A|}{|T|}$. Indeed, since $T$ is a right transversal of $H$ in $G$, for any coset $(K_H)g \in G/K_H$ there exists a unique $t \in T$ such that $(K_H)gt^{-1} \in H/K_H$. Thus, we can define a

map $f: A \to (AT^{-1}) \cap H/K_H$ which sends each $a \in A$ to the unique element $at^{-1} \in H/K_H$ for some $t \in T$. Since the preimage of any element in the range is a set of $|T|$ elements at most, we get the result.

(***) holds by Theorem 2.9 since $S_1, S_2$ are translating sets of a paradoxical decomposition of $G \curvearrowright G/K_H$.

(****) holds since $T$ is a transversal of $H$ in $G$ and $B_1, B_2 \subseteq H/K_H$.  □

For the proof of Theorem 1.3(2) we will need the following lemma.

**Lemma 2.12.** *Let $G$ be a group. Let $U_1, U_2 \subseteq G$ be finite subsets such that for every pair of finite subsets $A_1, A_2 \subseteq G$ we have $|A_1 U_1 \cup A_2 U_2| \geq |A_1| + |A_2|$. Let $H \leq G$ be an amenable subgroup. Then for every pair of finite subsets $A_1', A_2' \subseteq G/H$ we have*

$$|A_1' U_1 \cup A_2' U_2| \geq |A_1'| + |A_2'|.$$

*Proof.* Let $\rho: G \to G/H$ be the natural projection. Let $T$ be a right transversal of $H$ in $G$. Thus, there exist unique maps $\pi_H: G \to H$ and $\pi_T: G \to T$ such that $g = \pi_H(g)\pi_T(g)$ for all $g \in G$. Let $\psi: G/H \to T$ be the bijection taking a coset of $H$ to its representative in $T$. Note that $\rho\psi(Hg) = Hg$ and $\psi\rho(g) = \pi_T(g)$ for all $g \in G$.

Let $U = U_1 \cup U_2$ and fix $\epsilon > 0$. Let $A_1', A_2' \subseteq G/H$ be finite sets, let $A_i'' = \psi(A_i')$ and $A'' = A_1'' \cup A_2''$. Let $U_H = \pi_H(A''U)$. Since $U_H \subseteq H$ is a finite subset of the amenable group $H$, by Følner's criterion [9], there exists a finite set $A_H \subseteq H$ such that $|A_H U_H| < (1 + \epsilon)|A_H|$. Define $A_i = A_H A_i'' \subseteq G$. Since $A_H \subseteq H$ and $A_i'' \subseteq T$, we have $|A_i| = |A_H||A_i''| = |A_H||A_i'|$.

By assumption,

$$
\begin{aligned}
|A_H|(|A_1'| + |A_2'|) = |A_1| + |A_2| \\
\leq |A_1 U_1 \cup A_2 U_2| = |A_H A_1'' U_1 \cup A_H A_2'' U_2| \\
\leq |A_H \pi_H(A_1'' U_1 \cup A_2'' U_2)\pi_T(A_1'' U_1 \cup A_2'' U_2)| \\
\leq |A_H U_H \pi_T(A_1'' U_1 \cup A_2'' U_2)| \\
\leq |A_H U_H||\pi_T(A_1'' U_1 \cup A_2'' U_2)| \\
< (1 + \epsilon)|A_H||\pi_T(A_1'' U_1 \cup A_2'' U_2)|.
\end{aligned}
$$

Letting $\epsilon$ tend to 0 yields, $|A_1'| + |A_2'| \leq |\pi_T(A_1'' U_1 \cup A_2'' U_2)|$.

Note that,

$$
\begin{aligned}
\pi_T(A_1'' U_1 \cup A_2'' U_2) &= \psi\rho(A_1'' U_1 \cup A_2'' U_2) \\
&= \psi\rho(\psi(A_1')U_1 \cup \psi(A_2')U_2) \\
&= \psi(\rho(\psi(A_1'))U_1 \cup \rho(\psi(A_2'))U_2) \\
&= \psi(A_1' U_1 \cup A_2' U_2).
\end{aligned}
$$

Since $\psi$ is a bijection we have that $|\pi_T(A_1'' U_1 \cup A_2'' U_2)| = |A_1' U_1 \cup A_2' U_2|$ which implies that $|A_1'| + |A_2'| \leq |A_1' U_1 \cup A_2' U_2|$ as required.  □

Recall the statement of Theorem 1.3.

**Theorem 1.3.** *Let $G \curvearrowright X$ be a group action.*

(1) *Let $H \leq G$ be a finite index subgroup and $H \curvearrowright X$ the action of $G$ restricted to $H$. Then,*

$$\mathcal{T}(H \curvearrowright X) - 2 \leq [G : H](\mathcal{T}(G \curvearrowright X) - 2).$$

(2) *If $G \curvearrowright X$ has amenable point stabilizers then $\mathcal{T}(G \curvearrowright X) = \mathcal{T}(G)$.*

*Proof.* (1) Choose a paradoxical decomposition of $G \curvearrowright X$ with translating sets $S_1$ and $S_2$ such that $|S_1| + |S_2| = \mathcal{T}(G \curvearrowright X)$ and $1 \in S_1 \cap S_2$ (this is possible by Remark 2.8). Let $T$ be a right transversal of $H$ in $G$ and let $R_i = T S_i T^{-1} \cap H$ for $i = 1, 2$. By Lemma 2.11, $H \curvearrowright X$ has a paradoxical decomposition with translating sets $R_1$ and $R_2$. Note that for each $t' \in T$ and $g \in G$ there exists a unique $t \in T$ such that $t' g t^{-1} \in H$. Moreover, if $g = 1$, then $t = t'$ and therefore $t' g t^{-1} = 1$. Hence for $i = 1, 2$ we have

$$|R_i| = |T S_i T^{-1} \cap H| \leq |T|(|S_i| - 1) + 1$$

(here we use the fact that each $S_i$ contains 1). Hence

$$
\begin{aligned}
\mathcal{T}(H \curvearrowright X) &\leq |R_1| + |R_2| \\
&\leq [G : H](|S_1| + |S_2| - 2) + 2 \\
&= [G : H](\mathcal{T}(G \curvearrowright X) - 2) + 2.
\end{aligned}
$$

(2) By Lemma 2.3(2), $\mathcal{T}(G) \leq \mathcal{T}(G \curvearrowright X)$. Indeed, there exists a $G$-equivariant map from $G$ to $X$. For the other direction, choose a paradoxical decomposition of $G$ with translating sets $S_1$ and $S_2$ such that $|S_1| + |S_2| = \mathcal{T}(G)$. Let $x \in X$ and $K = \operatorname{Stab}_G(x)$. By assumption $K$ is amenable. By Theorem 2.9, for every pair of finite subsets $A_1, A_2 \subseteq G$ we have

$$|A_1 S_1^{-1} \cup A_2 S_2^{-1}| \geq |A_1| + |A_2|.$$

By Lemma 2.12, the same is true for any pair of finite subsets $A_1', A_2' \subseteq G/K$. Thus, again by Theorem 2.9, $G \curvearrowright G/K$ has a paradoxical decomposition with translating sets $S_1$ and $S_2$. Since this is true for all stabilizers of elements of $X$, by Corollary 2.7 $G \curvearrowright X$ has a paradoxical decomposition with $S_1$ and $S_2$ as translating sets. Thus $\mathcal{T}(G \curvearrowright X) \leq \mathcal{T}(G)$. $\square$

## 3. Schreier graphs and Stallings cores

The definitions in this section follow [1, 6, 5]. All the material before Proposition 3.2 is well known.

Given a free group $F = \langle x_1, x_2, \ldots x_m \rangle$ and a subgroup $H \leq F$, let $\mathcal{G}$ denote the oriented Cayley graph of the action $F \curvearrowright F/H$ with respect to the symmetric set $S = \{x_1^{\pm 1}, \ldots, x_m^{\pm 1}\}$. This graph is called the *Schreier graph* of the subgroup $H$. By definition every vertex in the graph has exactly $2m$ outgoing edges, each labeled by a different element of $S$. For every (directed) edge $e$, the vertices $e_-$ and $e_+$ will be the initial and terminal vertex of $e$ respectively. We will say that $e$ is *incident* both to $e_-$ and to $e_+$. Note that every edge $e$ has an inverse edge $f$ such that $e_- = f_+, e_+ = f_-$ and the labels of $e$ and $f$ are inverses of each other. Sometimes we will refer to $e$ and its inverse as a single *geometric* unoriented edge labeled by a letter $c^{\pm 1}$. A path in $\mathcal{G}$ is a sequence of directed edges $e_1, \ldots, e_n$ where for $i < n$ the terminal vertex of $e_i$ is the initial vertex of $e_{i+1}$. It is said to be reduced if $e_{i+1} \neq e_i^{-1}$ for all $i < n$. A cycle $e_1, \ldots, e_n$ is called reduced if it is reduced as a path. That is, $e_n$ might be equal to $e_1^{-1}$ in a reduced cycle.

Let $o$ be the vertex of the Schreier graph corresponding to the group $H$ and $\mathcal{C}$ the minimal subgraph of $\mathcal{G}$ containing $o$ and all reduced cycles from it to itself. $\mathcal{C}$ is called the *Stallings core* of $H$ or simply the *core* of $H$. The vertex $o$ is called the *origin* of the core $\mathcal{C}$. Note that the elements of $H$ are exactly those words which in reduced form can be read on a cycle in $\mathcal{C}$ from $o$ to itself. Also, if for some reduced word $w \in F$, the coset $Hw$ belongs to the core of $H$, then there exists $w' \in F$ such that $ww'$ is reduced and $ww' \in H$. Given the core $\mathcal{C}$ of $H$, it is possible to construct from it the Schreier graph of $H$ by attaching appropriate trees at each vertex of $\mathcal{C}$ with less than $2m$ outgoing edges. In this setting a *tree* is a connected graph with no reduced nontrivial cycles, such that for every edge in the tree the inverse edge also belongs to the tree. In other words, if we replace each pair of inverse edges in a tree by a single geometric edge we get a tree in the usual (unoriented) sense.

For a finitely generated subgroup $H \leq F$ there is a simple procedure to construct the core $\mathcal{C}$ of $H$. (We consider the finitely generated case for the procedure to be finite). If $H$ is generated by the elements $p_1, p_2, \ldots, p_n \in F$, the first step of the procedure consists of attaching $n$ cycles to the origin $o$. For each $i = 1, \ldots, n$ we divide the $i^{th}$ cycle into $|p_i|$ edges (and inverse edges) where $|p_i|$ stands for the length of $p_i$ as a word in $\{x_1^{\pm 1}, \ldots, x_m^{\pm 1}\}$. We label the edges of the $i^{th}$ cycle by the letters of $p_i$ (and the inverse edges accordingly) so that going from the origin $o$ to itself along the cycle, the word $p_i$ is read. The second step, consists of *foldings* of edges. If a vertex $v$ has two outgoing edges with the same label $e$ and $e'$ we identify the edges $e$ and $e'$, the inverse edges $e^{-1}$ and $(e')^{-1}$ and the terminal vertices $e_+$ and $e'_+$. This identification means that all the outgoing and incoming edges of $e_+$ and $e'_+$ are now attached to the single identified vertex. If $o$ is identified with some vertex we consider the resulting vertex as the origin.

The final step of the procedure consists of deleting every vertex of degree one in $\mathcal{C}$, other than the origin. A vertex of degree one is a vertex incident to only one geometric edge. We delete it together with the pair of inverse edges attached to it. For further details and examples of the construction, see [6, 5].

Once the core $\mathcal{C}$ is given, it is possible to erase a finite number of edges and get a spanning tree $T$. We assume that if an edge $e$ is erased, the inverse edge is erased as well. If $k$ pairs of edges were erased, then $H$ is free of rank $k$. Indeed, if the erased edges are $e_1^{\pm 1}, \ldots, e_k^{\pm 1}$ one can construct a free basis of $H$ as follows (see [5, Lemma 3.3]). For each $i \in \{1, \ldots, k\}$, let $a_i$ and $b_i$ be the unique reduced paths in $T$ from $o$ to $(e_i)_-$ and $(e_i)_+$ respectively. For a path $t$ in $\mathcal{C}$ the label $\ell(t)$ is defined as the product of labels of the edges along the path. For all $i \in \{1, \ldots, k\}$ let $\ell_i = \ell(a_i)\ell(e_i)\ell(b_i)^{-1}$, then $\{\ell_1, \ldots, \ell_k\}$ is a free basis of the subgroup $H$. In particular, we have that $k \leq n$. For further details see [5].

Parts (1) and (2) of the following lemma follow from [5, Proposition 6.7]. Part (3) follows from [5, Proposition 8.3]. We include a proof for the convenience of the reader. Here and below, an *n-generated* group is a group of rank $\leq n$.

**Lemma 3.1.** *Let $F = \langle x_1, \ldots, x_m \rangle$ be a free group of rank $m$ and let $H \leq F$ be an n-generated subgroup.*

(1) *Let $\mathcal{A}$ be the core corresponding to the subgroup $H$. Then, the origin $o$ has at most $2n$ incoming edges.*

(2) *Let $K \leq H$ be a (not necessarily finitely generated) subgroup and let $\mathcal{A}'$ be the core of $K$. Then, the origin $o'$ of $\mathcal{A}'$ has at most $2n$ incoming edges.*

(3) *Let $M \leq F$ be any finitely generated subgroup of infinite index and $\mathcal{B}$ the corresponding core. Then, there exists a vertex $v$ in $\mathcal{B}$ such that $v$ has less than $2m$ incoming edges.*

*Proof.* (1) Let $N = \{p_1, \ldots, p_k\}$ be a Nielsen-reduced free basis of $H$. In particular $k \leq n$ and $N$ freely generates $H$. Thus, every element $w \in H$ has a unique presentation as a word in the elements of $N$ and their inverses. Also, if $p_i^\epsilon$ for $\epsilon = \pm 1$ is the last element in the presentation of $w \in H$ then, as a word in the generators of $F$, the last letters of $w$ and $p_i^\epsilon$ coincide. Thus, there are at most $2k \leq 2n$ possibilities for the last letter of a reduced word in $H$. In particular, the origin of $\mathcal{A}$ has at most $2n$ distinct incoming edges.

(2) If the letter $c$ labels an incoming edge of $o'$ in $\mathcal{A}'$ then there is a reduced word $w = w'c$ in $K$ ending with $c$. Since $K \leq H$, the word $w \in H$. Thus $c$ labels an incoming edge of $o$ in $\mathcal{A}$ and the result follows from part (1).

(3) Assume by contradiction that every vertex in $\mathcal{B}$ is of degree $2m$. Then $\mathcal{B}$ is the Schreier graph of the action $F \curvearrowright F/M$. Since $M$ is finitely generated, the set of vertices of $\mathcal{B}$ is finite (indeed, see the construction of the core of a finitely generated subgroup above). Thus, $M$ has finite index in $F$, a contradiction. $\qquad \square$

**Proposition 3.2.** *Let $G_n = \langle x, y_1, \ldots, y_n, z \rangle$ be an $(n+2)$-generated free group. Let $H_n \leq G_n$ be an $n$-generated subgroup. Then, there exists $j \in \{1, \ldots, n\}$ such that for all $g \in G_n$ we have that $[H_n, H_n] \cap \langle x, y_j \rangle^g = \{1\}$.*

*Proof.* The proof is by induction on $n$. For $n = 1$, since $H_1$ is cyclic, the derived subgroup $[H_1, H_1] = \{1\}$ and the proposition holds. Assume the proposition holds for $n$ but not for $n + 1$. Let $H_{n+1} \leq G_{n+1}$ be an $(n+1)$-generated subgroup for which the proposition fails. In particular, for $j = n + 1$ there exist $g \in G_{n+1}$ and an element $h \neq 1$ such that

$$h \in [H_{n+1}, H_{n+1}] \cap \langle x, y_{n+1} \rangle^g.$$

Let $\pi : G_{n+1} \to G_n$ be the epimorphism taking $y_{n+1}$ to $1$ and any other generator of $G_{n+1}$ to its copy in $G_n$. Then

$$\pi(h) \in [\pi(H_{n+1}), \pi(H_{n+1})] \cap \langle \pi(x) \rangle^{\pi(g)} \subseteq [G_{n+1}, G_{n+1}] \cap \langle x \rangle^{\pi(g)} = \{1\}.$$

Since $\pi$ is not injective on $H_{n+1}$, the rank of the image $\pi(H_{n+1})$ is at most $n$. In particular $\pi(H_{n+1})$ is $n$-generated. Let $H_n = \pi(H_{n+1}) \leq G_n$. We will show that the proposition fails for it, which would yield the required contradiction. Indeed, if the proposition holds for $H_n$, then there exists some $j \in \{1, \ldots, n\}$ such that for all $b \in G_n$ we have

$$(*) \quad [H_n, H_n] \cap \langle x, y_j \rangle^b = \{1\}.$$

By assumption, for the same $j$, the proposition fails for $H_{n+1} \leq G_{n+1}$. Therefore, there exist $g' \in G_{n+1}$ and an element $h' \neq 1$ such that

$$h' \in [H_{n+1}, H_{n+1}] \cap \langle x, y_j \rangle^{g'}.$$

Applying the projection $\pi$ we get that

$$\pi(h') \in [H_n, H_n] \cap \langle x, y_j \rangle^{\pi(g')}$$

where here, $x$ and $y_j$ are considered as elements of $G_n$. Since $\pi$ is injective on $\langle x, y_j \rangle \leq G_{n+1}$ and $h'$ is a conjugate of a nontrivial element in $\langle x, y_j \rangle$, its image $\pi(h')$ is a conjugate of a nontrivial element of $\langle x, y_j \rangle \leq G_n$. In particular, it is nontrivial which contradicts $(*)$ for $b = \pi(g')$. $\qquad\square$

**Corollary 3.3.** *Let $G_n = \langle x, y_1, \ldots, y_n, z \rangle$ be a free group of rank $n + 2$ and let $H \leq G_n$ be an $n$-generated subgroup. Let $\mathcal{A}$ be the core of the derived subgroup $[H, H]$. Then there exists $j \in \{1, \ldots, n\}$ such that there are no reduced nontrivial cycles in $\mathcal{A}$ labeled by elements of $\langle x, y_j \rangle$.*

*Proof.* Let $j \in \{1, \ldots, n\}$ be an index for which the conclusion of Proposition 3.2 is satisfied for $H$. Assume by contradiction that $s$ is a reduced nontrivial cycle in

$\mathcal{A}$ labeled by a word in $\langle x, y_j \rangle$ and let $v$ be the initial (and terminal) vertex of $s$. There exists $g \in G$ such that $v$ represents the coset $[H, H]g$. Thus, if $w$ is the label of $s$ we have that $[H, H]gw = [H, H]g$ which implies that $w \in [H, H]^g \cap \langle x, y_j \rangle$. Then

$$w^{g^{-1}} \in [H, H] \cap \langle x, y_j \rangle^{g^{-1}}$$

is a nontrivial element, by contradiction to the choice of $j$. □

## 4. Construction of group actions with a given Tarski number

In this section we prove Theorem 1. Let $F = G_n = \langle x, y_1, \ldots, y_n, z \rangle$ be an $(n+2)$-generated free group for $n \in \mathbb{N}$. We will construct a subgroup $H$ for which the action $F \curvearrowright F/H$ is faithful and has Tarski number $n + 3$. $H$ will be defined by means of its core. In the construction, whenever an edge labeled by a letter $c$ is attached, we assume that the inverse edge, labeled by $c^{-1}$, is attached as well, even when it is not mentioned explicitly.

Let $\{H_i\}_{i \in \mathbb{N}}$ be an enumeration of all the $n$-generated subgroups of $F$. For each $i$ let $\mathcal{A}_i$ be the core representing the derived subgroup $[H_i, H_i]$ and let $o_i$ be its origin. By Lemma 3.1(2), $o_i$ has at most $2n$ incoming edges. Thus, there exists a letter $c_i \notin \{z, z^{-1}\}$ different than the labels of all incoming edges of $o_i$.

We define the core $\mathcal{C}$ of $H$ in the following way (for an illustration, see Figure 1). Let $o$ be the origin of $\mathcal{C}$ and $e_1, e_2, \ldots$ an infinite sequence of edges, all labeled by $z$, such that $(e_1)_- = o$ and for all $i$ we have $(e_i)_+ = (e_{i+1})_-$. Since the letters $c_i \notin \{z, z^{-1}\}$, for each $i$ it is possible to attach to $(e_i)_+$ an outgoing edge labeled by $c_i$. To its head vertex one can attach the core $\mathcal{A}_i$ by identifying $o_i$ with the vertex in question. Indeed, the choice of letters $c_i$ guarantees that no vertex in $\mathcal{C}$ has two outgoing or two incoming edges labeled by the same letter.
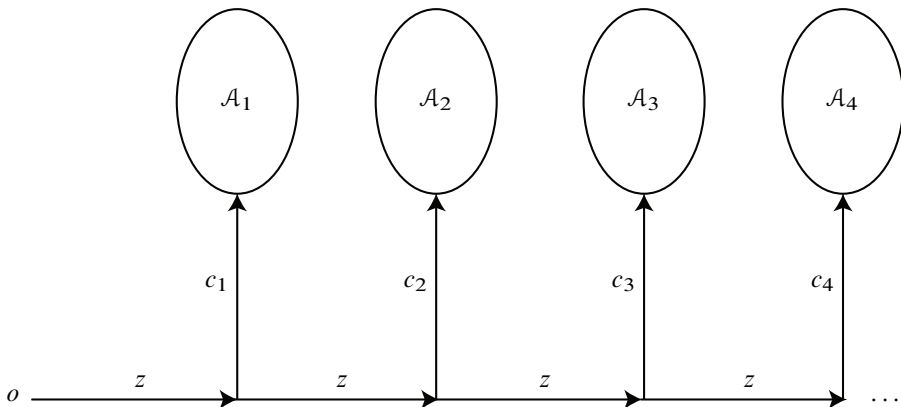


Figure 1. The core of $H$

Clearly, if $H$ is the group represented by $\mathcal{C}$, then $H$ is generated by the union of $[H_i, H_i]^{(z^i c_i)^{-1}}$ for all $i \in \mathbb{N}$. By construction, the origin $o$ has degree 1 in $\mathcal{C}$. Since $1 < 2(n + 2)$ it follows from [5, Theorem 8.14] that $H$ does not contain a nontrivial normal subgroup of $F$. Indeed, to ensure that $H$ does not contain a nontrivial normal subgroup of $F$ it suffices that some vertex of $\mathcal{C}$ has less than $2(n + 2)$ outgoing edges.

Let $\mathcal{G}$ be the Schreier graph of the action $F \curvearrowright F/H$. The graph $\mathcal{G}$ can be obtained from $\mathcal{C}$ by attaching trees to every vertex of $\mathcal{C}$ with less than $2(n + 2)$ outgoing edges.

**Lemma 4.1.** *Let $v$ be a vertex of $\mathcal{G}$. There is at most one core $\mathcal{A}_m$ to which one can get from $v$ via a path whose label does not include the letter $z^{\pm 1}$.*

*Proof.* Clearly, there is no path between two different cores $\mathcal{A}_l$ and $\mathcal{A}_r$ which does not cross an edge labeled by $z^{\pm 1}$. Assume that $t_1, t_2$ are paths from $v$ to two distinct cores $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$, such that both $t_1$ and $t_2$ do not cross any edge labeled by $z^{\pm 1}$. Then the path $t_1^{-1} t_2$ connects $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ and does not contain the letter $z^{\pm 1}$. $\square$

Let $\{X_j\}_{j=1}^n$ be a partition of the set of cores $\{\mathcal{A}_i\}_{i \in \mathbb{N}}$, where $\mathcal{A}_i \in X_j$ if and only if $j$ is the smallest index which satisfies the conclusion of Corollary 3.3 for the core $\mathcal{A}_i$. By Lemma 4.1, for each vertex $v$ of $\mathcal{G}$ there exists at most one core $\mathcal{A}_i$ to which it is possible to get via a path not including the letter $z^{\pm 1}$. Thus, it is possible to define a partition of the vertex set of $\mathcal{G}$ to $n$ sets $\{Y_j\}_{j=1}^n$ in the following way. For a vertex $v$, if $\mathcal{A}_m$ is a core reachable from $v$ via a path not containing the letter $z^{\pm 1}$ and $\mathcal{A}_m$ belongs to $X_j$ for some $j$, then $v$ will belong to $Y_j$ for the same $j$. If no core $\mathcal{A}_i$ is reachable from $v$ via such a path, $v$ will belong to $Y_1$. Note, that each of the sets in the partition is closed under the action of $\langle x, y_1, \ldots, y_n \rangle$.

**Lemma 4.2.** *For $j = 1, \ldots, n$ the group $\langle x, y_j \rangle$ acts freely on $Y_j$.*

*Proof.* Let $j \in \{1, \ldots, n\}$ and $v$ be a vertex of $Y_j$. Assume by contradiction that $w \in \langle x, y_j \rangle$ is a reduced nontrivial word stabilizing $v$. Then $w$ labels a reduced nontrivial cycle $s$ from $v$ to itself in $\mathcal{G}$. Since $s$ is nontrivial, it must contain as a subpath a reduced nontrivial cycle $s'$ through some core $\mathcal{A}_m$. Note that $\mathcal{A}_m$ is reachable from $v$ via a subpath of $s$, which by definition does not contain the letter $z^{\pm 1}$. Therefore, $v \in Y_j$ implies that $\mathcal{A}_m \in X_j$ and thus contains no reduced nontrivial cycle labeled by a word in $\langle x, y_j \rangle$, a contradiction. $\square$

**Lemma 4.3.** *The Tarski number of the action of $F$ on $\mathcal{G}$ is at least $n + 3$.*

*Proof.* Assume by contradiction that the action has Tarski number at most $n + 2$ and let $S_1, S_2$ be translating sets of a paradoxical decomposition with $|S_1| + |S_2| \le n + 2$. By Remark 2.8, we can assume that $1 \in S_1 \cap S_2$. Then, $S = (S_1 \cup S_2) \setminus \{1\}$ is a set of $n$ elements at most. Let $K$ be the subgroup it

generates. Then $K \curvearrowright \mathcal{G}$ has a paradoxical decomposition with translating sets $S_1$ and $S_2$. Since $K$ is $n$-generated, $K = H_m$ for some $m \in \mathbb{N}$ where $H_m$ is one of the $n$-generated subgroups enumerated above. Let $o_m$ be the origin of the core $\mathcal{A}_m$. By Corollary 2.7, $K \curvearrowright K/\operatorname{Stab}_K(o_m)$ has a paradoxical decomposition with translating sets $S_1, S_2$. Since $[K, K] \subseteq \operatorname{Stab}_K(o_m)$, Lemma 2.3(2) implies that the same is true for the action $K \curvearrowright K/[K, K]$. In particular, this action is non-amenable. By Lemma 2.4, the group $K/[K, K]$ is non-amenable, in contradiction to it being abelian.                                                                    $\square$

**Lemma 4.4.** *Let* $F' = \langle x, y_1, \ldots, y_n \rangle$. *Then* $F' \curvearrowright \mathcal{G}$ *has a paradoxical decomposition with translating sets* $S_1 = \{1, x\}$, $S_2 = \{1, y_1, \ldots, y_n\}$. *In particular,* $\mathcal{T}(F' \curvearrowright \mathcal{G}) \leq n + 3$.

*Proof.* The vertex set $V(\mathcal{G})$ of the graph $\mathcal{G}$ is the disjoint union of the sets $Y_j$ for $j = 1, \ldots, n$ where each of the sets is closed under the action of $F'$. By Lemma 4.2, for each $j$, the action of $\langle x, y_j \rangle$ on $Y_j$ is free. Thus by Corollary 2.2, $Y_j$ has a paradoxical decomposition with translating sets $\{1, x\}$ and $\{1, y_j\}$. By adding empty sets to the decomposition, we get that every $Y_j$ has a paradoxical decomposition with translating sets $S_1$ and $S_2$. Thus Remark 2.6 yields the result.                                                                               $\square$

By Lemma 2.3(1), $\mathcal{T}(F \curvearrowright \mathcal{G}) \leq \mathcal{T}(F' \curvearrowright \mathcal{G}) \leq n + 3$. Thus, by Lemma 4.3,

$$\mathcal{T}(F \curvearrowright F/H) = \mathcal{T}(F \curvearrowright \mathcal{G}) = n + 3.$$

**Remark 4.5.** For every $k, l \in \mathbb{N}$ such that $k + l = n + 1$ it is possible to rename the first $n + 1$ generators $x, y_1, \ldots y_n$ of $F = G_n$ by $x_1, \ldots, x_k, y_1, \ldots, y_l$. Then, for the subgroup $H$ constructed above, $F \curvearrowright F/H$ has a paradoxical decomposition with translating sets $S_1 = \{1, x_1, \ldots, x_k\}$ and $S_2 = \{1, y_1, \ldots, y_l\}$. Indeed, the only necessary change is to Proposition 3.2.

**Proposition 4.6.** *Let* $k, l \in \mathbb{N}$ *and* $G_{k,l} = \langle x_1, \ldots, x_k, y_1, \ldots, y_l, z \rangle$ *be a* $k + l + 1$ *generated free group. Let* $H \leq G_{k,l}$ *be a* $(k + l - 1)$-*generated subgroup. Then, there exist* $i \in \{1, \ldots, k\}$ *and* $j \in \{1, \ldots, l\}$ *such that for all* $g \in G_{k,l}$ *we have* $[H, H] \cap \langle x_i, y_j \rangle^g = \{1\}$.

*Proof.* By induction on $k$. The case $k = 1$ is Proposition 3.2. Assume the proposition holds for $k$ (and every $l$) but not for $k + 1$. Then there exists $l \in \mathbb{N}$ such that the proposition fails for $G_{k+1,l}$. The reduction to the case $G_{k,l}$ follows the same argument as that in Proposition 3.2. Here the homomorphism $\pi \colon G_{k+1,l} \to G_{k,l}$ maps $x_{k+1}$ to the identity and any other generator to its copy.                                                                          $\square$

**Corollary 4.7.** *Let $k \geq 4$ and let $F$ be the free group of rank $k - 1$. Then, $F$ has a faithful transitive action $F \curvearrowright X$, such that $\mathcal{T}(F \curvearrowright X) = k$ and for all $m, n \geq 2$ such that $m + n = k$ the action $F \curvearrowright X$ has a paradoxical decomposition with translating sets $S_1, S_2$ such that $|S_1| = m$ and $|S_2| = n$.*

Note that nothing similar is known for groups. Indeed, we don't have an example of a group with Tarski number $k$ which has two paradoxical decompositions, one with translating sets of size $m_1$ and $n_1$ and the other with translating sets of size $m_2$ and $n_2$, such that for $i = 1, 2$ we have $m_i + n_i = k$ and $\{m_1, n_1\} \neq \{m_2, n_2\}$.

## 5. Unbounded Tarski numbers

In what follows, $p$ will be a fixed prime number. Let $F$ be a finitely generated non-abelian free group. Let $\{\omega_n F\}_{n \in \mathbb{N}}$ be the *Zassenhaus $p$-filtration* of $F$ defined by $\omega_n F = \prod_{i \cdot p^j \geq n} (\gamma_i F)^{p^j}$. It is easy to see that $\{\omega_n F\}$ is a descending chain of normal subgroups of $p$-power index in $F$. Moreover, $\{\omega_n F\}$ is a base for the pro-$p$ topology on $F$, so in particular, $F$ being residually-$p$ implies that $\cap \omega_n F = \{1\}$. It follows that for any $n \in \mathbb{N}$ there exists $m(n) \in \mathbb{N}$ such that the reduced form of any element of $\omega_{m(n)} F$ is of length $\geq 12n$. Clearly, the index $[F : \omega_{m(n)} F] > n$. Thus, by the Schreier index formula, $\omega_{m(n)} F$ is free of rank $> n$. In particular, every $n$-generated subgroup of $\omega_{m(n)} F$ generates a subgroup of infinite index inside $\omega_{m(n)} F$ and thus inside $F$.

Theorem 2 is a straightforward corollary of the following theorem.

**Theorem 5.1.** *Let $F$ be the free group $\langle x, y, z \rangle$ and for each $n \in \mathbb{N}$ let $m(n)$ be as described above. There exists $H \leq F$ with the following properties.*

(1) *$H$ does not contain a nontrivial normal subgroup of $F$.*

(2) *For each $n \in \mathbb{N}$, $\mathcal{T}(\omega_{m(n)} F \curvearrowright F/H) \geq n + 3$.*

(3) *$\mathcal{T}(F \curvearrowright F/H) = 6$.*

*Proof.* The construction of the core $\mathcal{C}$ of $H$ will be similar to the construction used in Section 4. Here as well, whenever an edge labeled by a letter $c$ is attached, it is implicitly assumed that an inverse edge labeled by $c^{-1}$ is attached as well.

For each $n \in \mathbb{N}$, let $\{H_i^n\}_{i \in \mathbb{N}}$ be an enumeration of all the $n$-generated subgroups of $\omega_{m(n)} F$. For each $n, i \in \mathbb{N}$ let $\mathcal{A}_{(n,i)}$ be the core corresponding to the subgroup $H_i^n$. By Lemma 3.1(3) there exists a vertex $o'_{(n,i)}$ in $\mathcal{A}_{(n,i)}$ with less than 6 incoming edges. Let $c_{(n,i)}$ be a letter distinct from the labels of all the incoming edges of $o'_{(n,i)}$. Let $\alpha(k)$ for $k = 1, 2, \ldots$ be an enumeration of all the pairs $(n, i) \in \mathbb{N} \times \mathbb{N}$.

To construct the core $\mathcal{C}$ of $H$, let $o$ be the origin and $e_1, e_2, \ldots$ be an infinite sequence of edges such that $(e_1)_- = o$ and for all $k$ we have $(e_k)_+ = (e_{k+1})_-$. It is possible to label the edges $e_k$ inductively as follows. We choose the label $\ell(e_1)$ so that $\ell(e_1) \neq c_{\alpha(1)}^{-1}$. Assuming that the label of $e_k$ for $k \geq 1$ is already chosen, we choose the label $\ell(e_{k+1})$ so that $\ell(e_{k+1}) \notin \{\ell(e_k)^{-1}, c_{\alpha(k)}, c_{\alpha(k+1)}^{-1}\}$. The choice of the labels of $\{e_k\}_{k \in \mathbb{N}}$ means that for all $k$, one can attach to $(e_k)_+$ an outgoing edge labeled by $c_{\alpha(k)}$. To its head vertex, it is possible to attach the core $\mathcal{A}_{\alpha(k)}$ by identifying $o'_{\alpha(k)}$ with the vertex in question. Indeed, the choice of letters $c_{\alpha(k)}$ and the labels $\ell(e_k)$ guarantees than there is no vertex in $\mathcal{C}$ with two outgoing or two incoming edges labeled by the same label. As in the previous section, the origin $o$ having degree 1 implies that $H$ does not contain a nontrivial normal subgroup of $F$. We denote by $\mathcal{G}$ the Schreier graph of the group $H$ represented by $\mathcal{C}$.

**Lemma 5.2.** *For each $n \in \mathbb{N}$ we have $\mathcal{T}(\omega_{m(n)} F \curvearrowright \mathcal{G}) \geq n + 3$.*

*Proof.* Similar to the proof of Lemma 4.3. If $K$ is an $n$-generated subgroup of $\omega_{m(n)} F$, it fixes a point of $\mathcal{G}$. In particular, the action $K \curvearrowright \mathcal{G}$ is amenable.      $\triangle$

**Lemma 5.3.** *Let $n \in \mathbb{N}$. Let $K \leq \omega_{m(n)} F$ be an $n$-generated subgroup and let $\mathcal{A}$ be the core corresponding to the group $K$. Then, there exists a spanning tree $T$ in $\mathcal{A}$ such that every vertex in $\mathcal{A}$ loses at most one of its incoming edges in the transition from $\mathcal{A}$ to $T$.*

*Proof.* We always assume that when an edge is erased in the transition from $\mathcal{A}$ to $T$ its inverse edge is erased as well. Thus, for a vertex to lose an incoming edge is equivalent to losing an incident pair of inverse edges.

As mentioned in Section 3, in order to construct a spanning tree of $\mathcal{A}$ we have to erase at most $n$ pairs of inverse edges from $\mathcal{A}$. Assume that $i$ pairs of edges, $i \in \{0, \ldots, n-1\}$, were already erased and no vertex has lost more than one of its incoming edges. If the resulting graph is a tree, we are done. Otherwise, let $e$ be an edge whose removal (together with $e^{-1}$) would not affect the connectivity of the graph. Let $s$ be a reduced cycle from $e_-$ to itself which starts with the edge $e$ and does not visit any vertex other than $e_-$ twice. In particular, the cycle $s$ does not contain a pair of opposite edges. The removal of any edge of $s$, together with its inverse, would not affect the connectivity of $\mathcal{A}$. If the vertex $e_-$ corresponds to the coset $Kg$ and $w$ is the label of the cycle $s$, then $w \in K^g \subseteq \omega_{m(n)} F$. As such, the length of $w$, and of the cycle $s$, is at least $12n$. Until now, at most $n-1$ pairs of edges have been erased. Each pair is incident to at most 2 vertices. Each of the $2(n-1)$ vertices in question is incident to at most 6 pairs of edges. Thus there are at most $12(n-1)$ pairs of edges incident to vertices which have already lost an incoming edge. As such, at least one edge on the cycle $s$ does not belong to one of these $12(n-1)$ pairs and one can erase it (together with its inverse) to complete the induction.      $\triangle$

**Lemma 5.4.** *Let $S = \{x, y, z\}$. Then for any finite set $A$ of vertices of $\mathcal{G}$, we have $|AS^{-1} \cup A| \geq 2|A|$. In particular, by Theorem 2.10, $\mathcal{T}(F \curvearrowright \mathcal{G}) \leq 6$.*

*Proof.* Consider the graph $\mathcal{G}$. From each of the cores $\mathcal{A}_{(n,i)}$ attached during the construction of the core $\mathcal{C}$, it is possible to erase at most $n$ pairs of edges such that the resulting spanning tree of the core $\mathcal{A}_{(n,i)}$ satisfies the conclusion of Lemma 5.3. Let $\mathcal{T}$ be the graph obtained in this way from the graph $\mathcal{G}$. Clearly, $\mathcal{T}$ is a tree. Lemma 5.3 implies that every vertex of $\mathcal{T}$ has at least 5 incoming edges. Thus, every vertex of $T$ has at least two incoming edges labeled by elements of $S = \{x, y, z\}$.

Let $A$ be a finite set of vertices of $\mathcal{G}$. Let $E$ be the set of all oriented edges $e = (as^{-1}, a)$ such that $a \in A$, $s \in S$ and the edge $(as^{-1}, a)$ lies in $\mathcal{T}$. From the above, $E$ contains at least $2|A|$ edges. Clearly, $E$ does not contain a pair of opposite edges and the endpoints of edges in $E$ lie in the set $A \cup AS^{-1}$. Let $\Lambda$ be the unoriented graph with vertex set $A \cup AS^{-1}$ and edge set $E$ with forgotten orientation. Then $\Lambda$ is a finite unoriented forest. Hence, if $V(\Lambda)$ and $E(\Lambda)$ denote the sets of vertices and edges of $\Lambda$, respectively, then

$$|A \cup AS^{-1}| = |V(\Lambda)| > |E(\Lambda)| = |E| \geq 2|A|,$$

as required.                                                                                       △

**Lemma 5.5.** $\mathcal{T}(F \curvearrowright \mathcal{G}) = 6$.

*Proof.* It is easy to see that Theorem 1.2 implies that $\mathcal{T}(F \curvearrowright \mathcal{G}) \neq 4$. Thus, it suffices to prove that $\mathcal{T}(F \curvearrowright \mathcal{G}) \neq 5$. By contradiction, let $S_1 = \{1, a\}$, $S_2 = \{1, b, c\}$ be translating sets of a paradoxical decomposition of $F \curvearrowright \mathcal{G}$. For $r = p^{m(3)}$, let $p_1 = a^r$, $p_2 = (a^b)^r$ and $p_3 = (a^c)^r$. Then $p_1, p_2, p_3 \in \omega_{m(3)}F$. Let $K \leq F$ be the subgroup generated by $p_1, p_2, p_3$. Let $\mathcal{A}$ be the core of $K$ and let $o_\mathcal{A}$ be its origin. The core $\mathcal{A}$ was attached to the core of $H$ by some vertex of $\mathcal{A}$. Let $A_1, A_2$ be finite sets of vertices of $\mathcal{G}$ defined as follows. $A_1 = o_\mathcal{A} \cdot \{a^j, b^{-1}a^j, c^{-1}a^j : 0 \leq j \leq r - 1\}$ and $A_2 = \{o_\mathcal{A}\}$. A simple calculation shows that $A_1 S_1^{-1} = A_1\{1, a^{-1}\} = A_1$ (for a visual illustration, see Figure 2). Clearly, $A_2 S_2^{-1} \subseteq A_1$. Thus,

$$|A_1 S_1^{-1} \cup A_2 S_2^{-1}| = |A_1| < |A_1| + |A_2|.$$

This contradicts the implication (1) $\Rightarrow$ (2) of Theorem 2.9.                                △

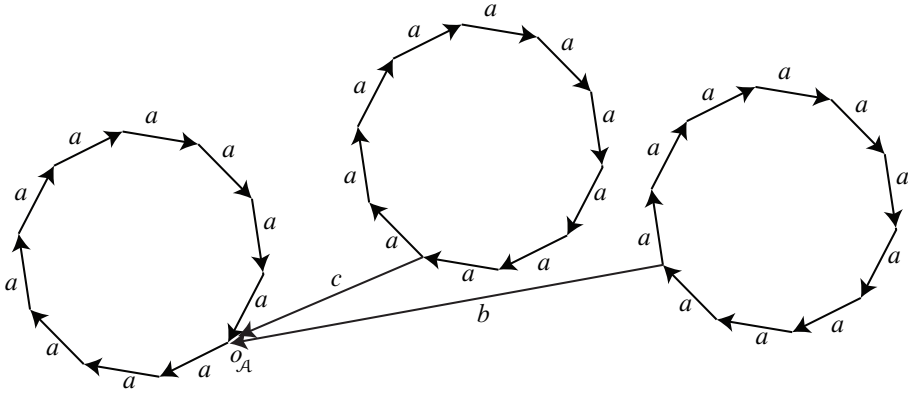This concludes the proof of Theorem 5.1.                                                        □

Figure 2. The core $\mathcal{A}$. The set $A_1$ is the set of all the vertices in the figure.

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