Groups Geom. Dyn. 10 (2016), 885–931
DOI 10.4171/GGD/370

Groups, Geometry, and Dynamics

Asymptotic geometry in higher products of rank one Hadamard spaces

Gabriele Link

Abstract. Given a product X of locally compact rank one Hadamard spaces, we study asymptotic properties of certain discrete isometry groups Γ of X. First we give a detailed description of the structure of the geometric limit set and relate it to the limit cone; moreover, we show that the action of Γ on a quotient of the regular geometric boundary of X is minimal and proximal. This is completely analogous to the case of Zariski dense discrete subgroups of semi-simple Lie groups acting on the associated symmetric space (compare [5]). In the second part of the paper we study the distribution of Γ-orbit points in X. As a generalization of the critical exponent $\delta(\Gamma)$ of Γ we consider for any $\theta \in \mathbb{R}^r_{\geq 0}$, $\|\theta\| = 1$, the exponential growth rate $\delta_\theta(\Gamma)$ of the number of orbit points in X with prescribed "slope" θ. In analogy to Quint's result in [26] we show that the homogeneous extension Ψ_Γ to $\mathbb{R}^r_{\geq 0}$ of $\delta_\theta(\Gamma)$ as a function of θ is upper semi-continuous, concave and strictly positive in the relative interior of the intersection of the limit cone with the vector subspace of $\mathbb{R}^r$ it spans. This shows in particular that there exists a unique slope θ^* for which $\delta_{\theta^*}(\Gamma)$ is maximal and equal to the critical exponent of Γ.

We notice that an interesting class of product spaces as above comes from the second alternative in the Rank Rigidity Theorem ([12, Theorem A]) for CAT(0)-cube complexes. Given a finite-dimensional CAT(0)-cube complex X and a group Γ of automorphisms without fixed point in the geometric compactification of X, then either Γ contains a rank one isometry or there exists a convex Γ-invariant subcomplex of X which is a product of two unbounded cube subcomplexes; in the latter case one inductively gets a convex Γ-invariant subcomplex of X which can be decomposed into a finite product of rank one Hadamard spaces.

Mathematics Subject Classification (2010). 20F69, 22D40, 22G44 51F99.

Keywords. CAT(0)-spaces, products, cubical complexes, discrete groups, rank one isometries, limit set, limit cone, critical exponent.

Contents

1. Introduction

After the publication of the article “Asymptotic geometry in products of Hadamard spaces with rank one isometries” ([21]) I was asked several times whether the results naturally extend to the setting of more than two factors. Unfortunately this is not the case since the methods of proof used there rely heavily on the possibility to control the position of pairs of points in a quotient of the regular geometric boundary; when more factors are present, the set of pairs of points which are in an uncontrollable position becomes much larger – in the case of symmetric spaces this phenomenon is reflected in the presence of more and higher dimensional “small” Bruhat cells when the rank gets bigger. So one goal of the present article was to give a generalization of the results in the aforementioned paper to products with more than two factors. Apart from that, the article contains a variety of results which were not yet known in the case of two factors. Among these I only want to mention here the construction of freely generated discrete subgroups (Proposition 6.5) with limit cone contained in a prescribed set and the positivity of the critical exponent (Theorem 7.9).

To be more precise, we let (X, d) be a product of r locally compact Hadamard spaces (X_i, d_i) endowed with the ℓ^2-metric, which makes X itself a locally compact Hadamard space, i.e. a locally compact complete simply connected metric space of non-positive Alexandrov curvature. It is well-known that every locally compact Hadamard space X can be compactified by adding its geometric boundary ∂X endowed with the cone topology (see [2, Chapter II]); if X is a product, then the *regular geometric boundary* $\partial X^{\mathrm{reg}}$ of X – which consists of the set of equivalence classes of geodesic rays which do not project to a point in one of the factors – is a dense open subset of ∂X homeomorphic to the Cartesian product of the geometric boundaries ∂X_i of the factors X_i (which we call the *Furstenberg boundary* $\partial^F X$ of X) times a factor $E^+ := \{\theta \in \mathbb{R}^r_{>0} \colon \|\theta\| = 1\}$.

We next let $\Gamma < \mathrm{Is}(X)$ be a group acting properly discontinuously by isometries on X; passing to a subgroup of finite index if necessary we may further assume that Γ preserves the product decomposition ([15, Corollary 1.3]). The *geometric limit set* $L_\Gamma \subset \partial X$ of Γ is defined as the set of accumulation points of a Γ-orbit in X. Unlike in the case of CAT(-1)-spaces this geometric limit set is in general not a minimal set for the action of Γ on the geometric boundary ∂X of X. This is due to the fact that isometries preserving the product decomposition of X cannot change the *slope* (i.e. the projection to E^+) of regular boundary points. So we are also going to consider the projection of the regular geometric limit set $L_\Gamma \cap \partial X^{\mathrm{reg}}$ to the Furstenberg boundary $\partial X_1 \times \partial X_2 \times \cdots \times \partial X_r$, which we will call the *Furstenberg limit set* F_Γ of Γ.

In this note we further restrict our attention to discrete groups Γ as above which contain an element projecting to a rank one isometry in each factor, i.e. Γ contains an element h such that all its projections h_i to $\mathrm{Is}(X_i)$ possess an invariant geodesic which does not bound a flat half-plane in the corresponding factor X_i. This requires in particular that all factors are rank one, i.e. possess a geodesic line which does not bound a flat half-plane. For products of rank one Hadamard spaces the presence of such a *regular axial* isometry is guaranteed in many interesting cases. If for example X is a finite-dimensional CAT(0)-cube complex for which every irreducible factor is non-Euclidean, unbounded and locally compact with a cocompact and essential action of its automorphism group, then by Theorem C in [12] every (possibly non-uniform) lattice $\Gamma < \mathrm{Is}(X)$ contains a regular axial isometry.

We will moreover need a second regular axial isometry $g \in \Gamma$ such that all projections to $\mathrm{Is}(X_i)$ of g and h are independent. This condition is clearly satisfied when Γ contains a (not necessarily regular axial) element γ such that all projections γ_i to $\mathrm{Is}(X_i)$ map the two fixed points of h_i to their complement in ∂X_i; then $g = \gamma h \gamma^{-1} \in \Gamma$ is the desired second regular axial isometry. Another important class of examples satisfying this assumption in the case of only two factors are Kac–Moody groups Γ over a finite field. They act by isometries on a product, the CAT(0)-realization of the associated twin building $\mathcal{B}_+ \times \mathcal{B}_-$, and there exists an element $h = (h_1, h_2) \in \Gamma$ projecting to a rank one element in each factor by Remark 5.4 and the proof of Corollary 1.3 in [10]. Moreover, the action of the Weyl group produces many regular axial isometries $g = (g_1, g_2) \in \Gamma$ with g_1 independent from h_1 and g_2 independent from h_2. Notice that if the order of the ground field is sufficiently large, then Γ is an irreducible lattice (see e.g. [27] and [11]).

Apart from these examples possible factors of X include locally compact Hadamard spaces of strictly negative Alexandrov curvature such as locally finite trees or manifolds with sectional curvature bounded from above by a negative constant as e.g. rank one symmetric spaces of the non-compact type. In this special case every non-elliptic and non-parabolic isometry in one of the factors is already a rank one element. Prominent examples here which are already covered by the results of Y. Benoist and J.-F. Quint are Hilbert modular groups acting as irreducible lattices on a product of hyperbolic planes, and graphs of convex cocompact groups of rank one symmetric spaces (see also [9]). More generally, given a set of locally compact rank one Hadamard spaces $X_1, X_2, \dots, X_r$, and faithful representions $\rho_i\colon G \to \mathrm{Is}(X_i)$ of a group G acting properly discontinuously by isometries on X_i, $i \in \{1, 2, \dots, r\}$, the *graph group* Γ_ρ associated to $\rho = (\rho_1, \rho_2, \dots, \rho_r)$ is defined by

$$\Gamma_\rho = \{(\rho_1(g), \rho_2(g), \dots, \rho_r(g))\colon g \in G\};$$

it clearly satisfies our assumption if G contains two elements g and h such that $\rho_i(g)$ and $\rho_i(h)$ are independent rank one isometries of X_i for all $i \in \{1, 2, \dots, r\}$. Notice that thanks to general arguments given by F. Dal'bo in [13, Proposition 3.4] any discrete group $\Gamma < \mathrm{Is}(X_1)\times\mathrm{Is}(X_2)\times\cdots\times\mathrm{Is}(X_r)$ which acts freely and satisfies $L_\Gamma \subset \partial X^{\mathrm{reg}}$ is a graph group; the converse clearly does not hold.

We will now state our main results. The first one is a strengthening of Theorem A in [21].

Theorem A. *The Furstenberg limit set is a boundary limit set for the action of* Γ *on the Furstenberg boundary* $\partial^F X$.

This implies in particular that the Furstenberg limit set is minimal, i.e. the smallest non-empty, Γ-invariant closed subset of $\partial^F X$. Notice that in the recent article [24] A. Nevo and M. Sageev proved an analogous statement for the Poisson boundary $B(X)$ of a proper cocompact G-action on an essential, strictly non-Euclidean CAT(0)-cube complex X. More precisely, their Theorem 5.8 states that the closure $B(X)$ of the set of non-terminating ultrafilters in the Roller boundary of X is a boundary limit set for the action of G on the collection of all ultrafilters.

In our setting we have – as in the case of symmetric spaces or Bruhat-Tits buildings of higher rank – the following structure theorem.

Theorem B. *The regular geometric limit set splits as a product* $F_\Gamma \times P_\Gamma^{\mathrm{reg}}$, *where* $P_\Gamma^{\mathrm{reg}} \subset E^+$ *denotes the set of slopes of regular limit points of* Γ.

We recall that $\Gamma < \mathrm{Is}(X_1)\times\mathrm{Is}(X_2)\times\cdots\times\mathrm{Is}(X_r)$ is a discrete group containing two regular axial isometries $h = (h_1, h_2, \dots, h_r)$ and $g = (g_1, g_2, \dots, g_r)$ which project to independent rank one elements in each factor. For a rank one isometry γ_i of one of the factors X_i we denote γ_i^+ its attractive, γ_i^- its repulsive fixed point in ∂X_i, and $l_i(\gamma_i) > 0$ its translation length, i.e. the minimum of the displacement function $d_i(\gamma_i)\colon X_i \to \mathbb{R}$, $x_i \mapsto d_i(x_i, \gamma_i x_i)$. So if $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_r)$ is regular axial we canonically get two fixed points in the Furstenberg boundary $\gamma^+ := (\gamma_1^+, \gamma_2^+, \dots, \gamma_r^+)$, $\gamma^- := (\gamma_1^-, \gamma_2^-, \dots, \gamma_r^-)$, and a *translation vector* $L(\gamma) := (l_1(\gamma_1), l_2(\gamma_2), \dots, l_r(\gamma_r)) \in \mathbb{R}^r_{>0}$. With this notation we can state the following theorem.

Theorem C. *The set of pairs of fixed points $(\gamma^+, \gamma^-) \in \partial^F X \times \partial^F X$ of regular axial isometries $\gamma \in \Gamma$ is dense in $\big(F_\Gamma \times F_\Gamma\big) \setminus \Delta$, where Δ denotes the set of pairs of points in F_Γ with a common projection to some ∂X_i.*

We mention that this result can be viewed as a strong topological version of the double ergodicity property of Poisson boundaries due to Burger-Monod ([8]) and Kaimanovich ([18]). We also remark that in the case of only one factor and a non-elementary, but not necessarily discrete isometry group Theorems A and C were proved by U. Hamenstädt ([17, Theorem 1.1]) under the (a priori) weaker hypothesis that the group contains only one rank one isometry; part (4) of the aforementioned theorem then implies the existence of a pair of independent rank-one isometries.

We next define the *limit cone* ℓ_Γ of Γ as the closure in $\mathbb{R}^r_{\geq 0}$ of the set of half-lines spanned by all translation vectors $L(\gamma)$ with $\gamma \in \Gamma$ regular axial. This cone is related to the set of slopes of all (regular and singular) limit points $P_\Gamma \subset E := \overline{E^+} \subset \mathbb{R}^r_{\geq 0}$ as follows.

Theorem D. *The set $P_\Gamma \subset E$ of slopes of all limit points of Γ is equal to $\ell_\Gamma \cap E$. Moreover, the limit cone ℓ_Γ is convex.*

For the formulation of the last main result of the paper we fix a point $x = (x_1, x_2, \ldots, x_r)$ in X, $\theta = (\theta_1, \theta_2, \ldots, \theta_r) \in E$, $\varepsilon > 0$, $n \gg 1$ and consider the cardinality $N_\theta^\varepsilon(n)$ of the set

$$\Big\{\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_r) \in \Gamma\colon 0 < d(x, \gamma x) < n,$$
$$\Big|\frac{d_i(x_i, \gamma_i x_i)}{d(x, \gamma x)} - \theta_i\Big| < \varepsilon \;\text{ for all }\; 1 \leq i \leq r\}\Big\}.$$

This number counts all orbit points γx of distance less than n to the point x which in addition are "close" to a geodesic ray in the class of a boundary point with slope θ. We further consider

$$\delta_\theta^\varepsilon := \limsup_{n \to \infty} \frac{\ln N_\theta^\varepsilon(n)}{n}$$

and finally

$$\delta_\theta(\Gamma) := \lim_{\varepsilon \to 0} \delta_\theta^\varepsilon.$$

$\delta_\theta(\Gamma)$ can be thought of as a function of $\theta \in E$ which describes the exponential growth rate of orbit points converging to limit points of slope θ. It is an invariant of Γ which carries finer information than the critical exponent $\delta(\Gamma)$ which is defined as the exponent of convergence of the Poincaré series and describes the exponential growth rate of a Γ-orbit in X. We will see that the critical exponent is simply the maximum of $\delta_\theta(\Gamma)$ among all $\theta \in E$. As in [26] it will be convenient to study the homogeneous extension

$$\Psi_\Gamma \colon \mathbb{R}^r_{\geq 0} \longrightarrow \mathbb{R}$$

of this function $\theta \mapsto \delta_\theta(\Gamma)$. Our final result appropriately generalizes Theorem 4.2.2 in [26] (which holds for symmetric spaces and Euclidean buildings of higher rank) to our setting.

Theorem E. *Ψ_Γ is upper semi-continuous, concave, and the set*

$$\{L \in \mathbb{R}^r_{\geq 0} \colon \Psi_\Gamma(L) > -\infty\}$$

is precisely the limit cone ℓ_Γ of Γ. Moreover, Ψ_Γ is non-negative on the limit cone ℓ_Γ and strictly positive in the relative interior of its intersection with the vector subspace of $\mathbb{R}^r$ it spans.

An easy corollary of Theorem E is the fact that the critical exponent $\delta(\Gamma)$ of Γ is strictly positive and that there exists a unique $\theta^* \in E$ such that $\delta_{\theta^*}(\Gamma) = \delta(\Gamma)$. With the help of Theorem E it is also possible to construct generalized conformal densities on each Γ-invariant subset of the geometric limit set as performed in [22]; in the setting of higher rank symmetric spaces and Euclidean buildings such densities were already introduced in [20] and [25]. Moreover, Proposition 3.3.1 of [26] allows to deduce the following counting results for Γ.

Theorem F. *There exists $C \geq 1$ such that for all $T > 0$ the estimate*

$$\#\{\gamma \in \Gamma \colon d(o, \gamma o) \leq T\} \leq C \cdot T^{r-1} \mathrm{e}^{\delta(\Gamma) T}$$

holds. Moreover, one has

$$\lim_{T \to \infty} \frac{1}{T} \ln \#\{\gamma \in \Gamma \colon d(o, \gamma o) \leq T\} = \delta(\Gamma).$$

The paper is organized as follows. Section 2 summarizes basic facts about Hadamard spaces and rank one isometries, in Section 3 specific properties of products of Hadamard spaces are collected. Section 4 provides the main tools needed in the proofs of our results. In Section 5 we describe the structure of the limit set and prove Theorems A, B, and C; Section 6 is devoted to the study of the limit cone. Finally, in Section 7 we introduce and study the homogeneous function Ψ_Γ as above and prove Theorems D, E, and F.

Acknowledgements. The author thanks the anonymous referee for his thorough reading of the paper und many valuable remarks and corrections.

2. Preliminaries

The purpose of this section is to introduce terminology and notation and to summarize basic results about Hadamard spaces and rank one isometries. The main references here are [7] and [2] (see also [3], and [1] and [4] in the case of Hadamard manifolds).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ is a map σ from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $\sigma(0) = x$, $\sigma(l) = y$ and $d(\sigma(t), \sigma(t')) = |t - t'|$ for all $t, t' \in [0, l]$. We will denote such a geodesic path $\sigma_{x,y}$. X is called *geodesic* if any two points in X can be connected by a geodesic path, if this path is unique, we say that X is *uniquely geodesic*. In this text X will be a Hadamard space, i.e. a complete geodesic metric space in which all triangles satisfy the CAT(0)-inequality. This implies in particular that X is simply connected and uniquely geodesic. A *geodesic* or *geodesic line* in X is a map $\sigma\colon \mathbb{R} \to X$ such that $d(\sigma(t), \sigma(t')) = |t - t'|$ for all $t, t' \in \mathbb{R}$, a *geodesic ray* is a map $\sigma\colon [0, \infty) \to X$ such that $d(\sigma(t), \sigma(t')) = |t - t'|$ for all $t, t' \in [0, \infty)$. Notice that in the non-Riemannian setting completeness of X does not imply that every geodesic path or ray can be extended to a geodesic, i.e. X need not be geodesically complete.

From here on we will assume that X is a locally compact Hadamard space. The geometric boundary ∂X of X is the set of equivalence classes of asymptotic geodesic rays endowed with the cone topology (see e.g. [2, Chapter II]). The action of the isometry group $\mathrm{Is}(X)$ on X naturally extends to an action by homeomorphisms on the geometric boundary. Moreover, since X is locally compact, this boundary ∂X is compact and the space X is a dense and open subset of the compact space $\overline{X} := X \cup \partial X$. For $x \in X$ and $\xi \in \partial X$ arbitrary there exists a geodesic ray emanating from x which belongs to the class of ξ. We will denote such a ray $\sigma_{x,\xi}$.

We say that two points $\xi, \eta \in \partial X$ can be joined by a geodesic if there exists a geodesic $\sigma\colon \mathbb{R} \to X$ such that $\sigma(-\infty) = \xi$ and $\sigma(\infty) = \eta$. It is well-known that if all triangles in X satisfy the CAT(-1)-inequality, then every pair of distinct points in the geometric boundary can be joined by a geodesic. This is not true in general. For convenience we therefore define the *visibility set at infinity* $\mathrm{Vis}^\infty(\xi)$ of a point $\xi \in \partial X$ as the set of points in the geometric boundary which can be joined to ξ by a geodesic, i.e.

$$\begin{aligned}\mathrm{Vis}^\infty(\xi) := \{\eta \in \partial X\colon &\text{ there exists a geodesic } \sigma \text{ such that} \\ &\sigma(-\infty) = \xi,\ \sigma(\infty) = \eta\}.\end{aligned} \tag{1}$$

A geodesic $\sigma\colon \mathbb{R} \to X$ is said to *bound a flat half-plane* if there exists a closed convex subset $\iota([0, \infty) \times \mathbb{R})$ in X isometric to $[0, \infty) \times \mathbb{R}$ such that $\sigma(t) = \iota(0, t)$ for all $t \in \mathbb{R}$; otherwise σ will be called a *rank one geodesic*.

Notice that the existence of a rank one geodesic imposes severe restrictions on the CAT(0)-space X. For example, X can neither be a symmetric space or Euclidean building of higher rank nor a product of Hadamard spaces.

The following important lemma states that even though we cannot join any two distinct points in the geometric boundary of X, given a rank one geodesic we can at least join points in a neighborhood of its extremities. More precisely, we have the following well-known

Lemma 2.1 ([2], Lemma III.3.1). *Let $\sigma\colon \mathbb{R} \to X$ be a rank one geodesic. Then there exist $c > 0$ and neighborhoods U^-, U^+ of $\sigma(-\infty)$, $\sigma(\infty)$ in $\overline{X}$ such that for any $\xi \in U^-$ and $\eta \in U^+$ there exists a rank one geodesic joining ξ and η. For any such geodesic σ' we have $d(\sigma', \sigma(0)) \le c$.*

Moreover, we will need the following technical lemma which immediately follows from Lemma 4.3 and Lemma 4.4 in [3].

Lemma 2.2. *Let $\sigma\colon \mathbb{R} \to X$ be a rank one geodesic and set $y := \sigma(0)$, $\eta := \sigma(\infty)$. Then for any $T \gg 1$, $\varepsilon > 0$ there exists a neighborhood U^- of $\sigma(-\infty)$ in $\overline{X}$ and a number $R > 0$ such that for any $x \in X$ with $d(x, \sigma) > R$ or $x \in U^-$ we have*

$$d(\sigma_{x,y}(t), \sigma_{x,\eta}(t)) \le \varepsilon \quad \textit{for all } t \in [0, T].$$

The following kind of isometries will play a central role in the sequel.

Definition 2.3. An isometry h of X is called *axial*, if there exists a constant $l = l(h) > 0$ and a geodesic σ such that $h(\sigma(t)) = \sigma(t + l)$ for all $t \in \mathbb{R}$. We call $l(h)$ the *translation length* of h, and σ an *axis* of h. The boundary point $h^+ := \sigma(\infty)$ is called the *attractive fixed point*, and $h^- := \sigma(-\infty)$ the *repulsive fixed point* of h. We further set $\mathrm{Ax}(h) := \{x \in X \colon d(x, hx) = l(h)\}$.

We remark that $\mathrm{Ax}(h)$ consists of the union of parallel geodesics translated by h, and $\overline{\mathrm{Ax}(h)} \cap \partial X$ is exactly the set of fixed points of h. Following the definition in [6] and [10] we will call two axial isometries $g, h \in \mathrm{Is}(X)$ *independent* if for any given $x \in X$ the map

$$\mathbb{Z} \times \mathbb{Z} \longrightarrow [0, \infty), \quad (m, n) \longmapsto d(g^m x, h^m x)$$

is proper.

Definition 2.4. An axial isometry is called *rank one* if it possesses a rank one axis.

Notice that if h is rank one, then h^+ and h^- are the only fixed points of h. Moreover, it is easy to verify that two rank one elements $g, h \in \mathrm{Is}(X)$ are independent if and only if $\{g^+, g^-\} \cap \{h^+, h^-\} = \emptyset$. Let us recall some properties of rank one isometries.

Lemma 2.5. ([2], *Lemma III.3.3*) *Let h be a rank one isometry. Then*

(a) $\mathrm{Vis}^{\infty}(h^+) = \partial X \setminus \{h^+\}$,

(b) *any geodesic joining a point $\xi \in \partial X \setminus \{h^+\}$ to h^+ is rank one,*

(c) *given neighborhoods U^- of h^- and U^+ of h^+ in $\overline{X}$ there exists $N_0 \in \mathbb{N}$ such that $h^{-n}(\overline{X} \setminus U^+) \subset U^-$ and $h^n(\overline{X} \setminus U^-) \subset U^+$ for all $n > N_0$.*

The following lemma allows to find many rank one isometries; it will play an important role in the proof of several results such as Proposition 4.6, which in turn is needed for Proposition 6.3.

Lemma 2.6 ([2], Lemma III.3.2). *Let $\sigma\colon \mathbb{R} \to X$ be a rank one geodesic, and $(\gamma_n) \subset \mathrm{Is}(X)$ a sequence of isometries such that $\gamma_n x \to \sigma(\infty)$ and $\gamma_n^{-1} x \to \sigma(-\infty)$ for one (and hence any) $x \in X$. Then for n sufficiently large, γ_n is axial and possesses an axis σ_n such that $\sigma_n(\infty) \to \sigma(\infty)$ and $\sigma_n(-\infty) \to \sigma(-\infty)$.*

The following proposition is the clue to the proof of all results in Section 5 and 6. It is more general, but similar in spirit to Proposition 2.8 in [21] which was inspired by the proof of Lemma 4.1 in [14] in the easier context of CAT(-1)-spaces. Part (b) in particular gives a relation between the geometric length and the combinatorial length of words in a free group on two generators which will be the key ingredient in the proof of Proposition 6.3. If α, β generate a free group we say that a word $\gamma = s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n}$ with $s_j \in \{\alpha, \alpha^{-1}, \beta, \beta^{-1}\}$ and $k_j \in \mathbb{N}$, $j \in \{1, 2, \dots n\}$ is *cyclically reduced* if $s_{j+1} \notin \{s_j, s_j^{-1}\}$ for $j \in \{1, 2, \dots n-1\}$ and $s_n \neq s_1^{-1}$.

Proposition 2.7. *Suppose g and h are independent rank one elements in* $\mathrm{Is}(X)$. *Then there exist neighborhoods $V(\eta), U(\eta)$ of $\eta \in \{g^-, g^+, h^-, h^+\}$ with $V(\eta) \subset U(\eta) \subset \overline{X}$ and a constant $c > 0$ such that the following holds.*

(a) *Any two points in different sets $U(\eta)$ can be joined by a rank one geodesic $\sigma \subset X$ with $d(o, \sigma) \le c$.*

(b) *Given a pair of rank one isometries $\alpha, \beta \in \mathrm{Is}(X)$ with $\alpha^{\pm} \in V(g^{\pm})$, $\beta^{\pm} \in V(h^{\pm})$ and $N_\alpha, N_\beta \in \mathbb{N}$ sufficiently large, then for all $n \in \mathbb{N}$ and every cyclically reduced word $\gamma = s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n}$ with*

$$s_j \in S := \{\alpha^{N_\alpha}, \alpha^{-N_\alpha}, \beta^{N_\beta}, \beta^{-N_\beta}\}$$

and $k_j \in \mathbb{N}$, $j \in \{1, 2, \dots n\}$,

(i) *we have*

$$\gamma^+ \in \begin{cases} U(g^{\pm}) & \text{if } s_1 = \alpha^{\pm N_\alpha}, \\ U(h^{\pm}) & \text{if } s_1 = \beta^{\pm N_\beta}, \end{cases}$$

and

$$\gamma^- \in \begin{cases} U(g^\pm) & \text{if } s_n^{-1} = \alpha^{\pm N_\alpha}, \\ U(h^\pm) & \text{if } s_n^{-1} = \beta^{\pm N_\beta}; \end{cases}$$

(ii) *we have*

$$\Big| l(\gamma) - \sum_{j=1}^{n} k_j l(s_j) \Big| \le 4c \cdot n$$

and

$$\Big| d(o, \gamma o) - \sum_{j=1}^{n} k_j l(s_j) \Big| \le 4c \cdot n.$$

Proof. We fix a base point $o \in \mathrm{Ax}(h)$. For $\eta \in \{g^-, g^+, h^-, h^+\}$ let $U(\eta) \subset \overline{X}$ be a small neighborhood of η with $o \notin U(\eta)$ such that all $U(\eta)$ are pairwise disjoint, and $c > 0$ a constant such that any pair of points in distinct neighborhoods can be joined by a rank one geodesic σ with $d(o, \sigma) \le c$. This is possible by Lemma 2.1 and proves part (a).

According to Lemma 2.6, for $\eta \in \{g^-, g^+, h^-, h^+\}$ there exist neighborhoods $W(\eta) \subset U(\eta)$ such that every $\gamma \in \Gamma$ with $\gamma o \in W(\eta)$, $\gamma^{-1} o \in W(\zeta)$, $\zeta \ne \eta$, is rank one with $\gamma^+ \in U(\eta)$ and $\gamma^- \in U(\zeta)$. We claim that assertion (b) holds for all neighborhoods $V(\eta) \subset W(\eta) \subset \overline{X}$ of $\eta \in \{g^-, g^+, h^-, h^+\}$ with $c > 0$ as above.

Let $\alpha, \beta \in \mathrm{Is}(X)$ be rank one isometries with $\alpha^\pm \in W(g^\pm)$, $\beta^\pm \in W(h^\pm)$ and set $W(\alpha^\pm) = W(g^\pm)$ and $W(\beta^\pm) = W(h^\pm)$. By Lemma 2.5 (c) there exist $N_\alpha, N_\beta \in \mathbb{N}$ such that

$$\alpha^{\pm N_\alpha}(\overline{X} \setminus W(\alpha^\mp)) \subset W(\alpha^\pm) \quad \text{and} \quad \beta^{\pm N_\beta}(\overline{X} \setminus W(\beta^\mp)) \subset W(\beta^\pm). \tag{2}$$

We set $U(\alpha^\pm) = U(g^\pm)$, $U(\beta^\pm) = U(h^\pm)$, $S := \{\alpha^{N_\alpha}, \alpha^{-N_\alpha}, \beta^{N_\beta}, \beta^{-N_\beta}\}$ and consider a cyclically reduced word $\gamma = s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n}$ with $s_j \in S$ and $k_j \in \mathbb{N}$, $j \in \{1, 2, \ldots n\}$. By choice of N_α, N_β and (2) we have $\gamma o \in W(s_1^+)$ and $\gamma^{-1} o \in W(s_n^-) \ne W(s_1^+)$ since $s_1 \ne s_n^{-1}$. Therefore γ is rank one with $\gamma^+ \in U(s_1^+)$ and $\gamma^- \in U(s_n^-)$, which shows (i).

Choosing a point $x \in \mathrm{Ax}(\gamma)$ with $d(o, x) \le c$ (which is possible according to (a)) we get

$$l(\gamma) \le d(o, \gamma o) \le d(o, x) + d(x, \gamma x) + d(\gamma x, \gamma o) \le l(\gamma) + 2c. \tag{3}$$

Similarly, for all $j \in \{1, 2, \ldots n\}$ we get $l(s_j^{k_j}) \le d(o, s_j^{k_j} o) \le l(s_j^{k_j}) + 2c$.

We fix $j \in \{1, 2, \ldots n\}$ and abbreviate $\gamma_j := s_j^{k_j} s_{j+1}^{k_{j+1}} \cdots s_n^{k_n}$. Then we have $\gamma_2 o \in W(s_2^+)$, $s_1^{-k_1} o \in W(s_1^-) \ne W(s_2^+)$, so there exists a geodesic σ_2 joining $\gamma_2 o$ to $s_1^{-k_1} o$ with $d(o, \sigma_2) \le c$. If y denotes a point on σ_2 with $d(o, y) \le c$ we get

$$d(s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n} o, o) = d(\gamma_2 o, s_1^{-k_1} o) = d(\gamma_2 o, y) + d(y, s_1^{-k_1} o)$$

which proves $|d(\gamma o, o) - d(o, s_1^{k_1} o) - d(o, \gamma_2 o)| \leq 2c$. Applying the same arguments to γ_j for $j \geq 2$ and using the fact that $s_{j+1} \neq s_j^{-1}$ we deduce

$$|d(\gamma_j o, o) - d(o, s_j^{k_j} o) - d(o, \gamma_{j+1} o)| \leq 2c$$

and hence inductively

$$\Big| d(o, \gamma o) - \sum_{j=1}^{n} d(o, s_j^{k_j} o) \Big| \leq 2(n-1)c.$$

Using (3) we conclude

$$\Big| l(\gamma) - \sum_{j=1}^{n} k_j l(s_j) \Big| \leq 4c \cdot n \quad \text{and} \quad \Big| d(o, \gamma o) - \sum_{j=1}^{n} k_j l(s_j) \Big| \leq 4c \cdot n.$$

Therefore part (ii) of (b) is also true. □

3. Products of Hadamard spaces

Now let $(X_1, d_1), (X_2, d_2), \dots, (X_r, d_r)$ be locally compact Hadamard spaces, and $X = X_1 \times X_2 \times \cdots \times X_r$ the product space endowed with the product distance $d = \sqrt{d_1^2 + d_2^2 + \cdots + d_r^2}$. Notice that such a product is again a locally compact Hadamard space.

We denote $\mathbb{R}^r_{\geq 0} := \{(t_1, t_2, \dots, t_r) \in \mathbb{R}^r : t_i \geq 0 \text{ for all } i \in \{1, 2, \dots, r\}\}$ and $\mathbb{R}^r_{>0} := \{(t_1, t_2, \dots, t_r) \in \mathbb{R}^r : t_i > 0 \text{ for all } i \in \{1, 2, \dots, r\}\}$. To any pair of points $x = (x_1, x_2, \dots, x_r)$, $z = (z_1, z_2, \dots, z_r) \in X$ we associate the vector

$$H(x, z) := \begin{pmatrix} d_1(x_1, z_1) \\ d_2(x_2, z_2) \\ \vdots \\ d_r(x_r, z_r) \end{pmatrix} \in \mathbb{R}^r_{\geq 0},$$

which we call the *distance vector* of the pair (x, z). Notice that if $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^r$, we clearly have $\|H(x, z)\| = d(x, z)$. If $z \neq x$ we therefore define the *direction* of z with respect to x by the unit vector

$$\widehat{H}(x, z) := \frac{H(x, z)}{d(x, z)} \in \mathbb{R}^r_{\geq 0}.$$

Denote $p_i: X \to X_i$, $i = 1, 2, \ldots, r$, the natural projections. Every geodesic path $\sigma: [0, l] \to X$ can be written as a product

$$\sigma(t) = (\sigma_1(t \cdot \theta_1), \sigma_2(t \cdot \theta_2), \ldots, \sigma_r(t \cdot \theta_r)),$$

where σ_i are geodesic paths in X_i, $i = 1, 2, \ldots r$, and the $\theta_i \geq 0$ satisfy

$$\sum_{i=1}^{r} \theta_i^2 = 1.$$

The unit vector

$$\mathrm{sl}(\sigma) := \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_r \end{pmatrix} \in E := \{\theta \in \mathbb{R}^r_{\geq 0}: \|\theta\| = 1\}$$

equals the direction of the points $\sigma(t)$, $t \in (0, l]$, with respect to $\sigma(0)$ and is called the *slope of* σ. We say that a geodesic path σ is *regular* if its slope does not possess a coordinate zero, i.e. if

$$\mathrm{sl}(\sigma) \in E^+ := \{\theta \in \mathbb{R}^r_{>0}: \|\theta\| = 1\};$$

otherwise σ is said to be *singular*. In other words, σ is regular if and only if none of the projections $p_i(\sigma([0, l]))$, $i \in \{1, 2, \ldots, r\}$, is a point.

If $x \in X$ and $\sigma: [0, \infty) \to X$ is an arbitrary geodesic ray, then elementary geometric estimates yield the relation

$$\mathrm{sl}(\sigma) = \lim_{t \to \infty} \widehat{H}(x, \sigma(t)) = \lim_{t \to \infty} \frac{H(x, \sigma(t))}{d(x, \sigma(t))}$$

between the slope of σ and the directions of $\sigma(t)$, $t > 0$, with respect to x. Similarly, one can easily show that any two geodesic rays representing the same (possibly singular) point in the geometric boundary necessarily have the same slope. So the slope $\mathrm{sl}(\tilde{\xi}) \in E$ of a point $\tilde{\xi} \in \partial X$ can be defined as the slope of an arbitrary geodesic ray representing $\tilde{\xi}$. The *regular geometric boundary* $\partial X^{\mathrm{reg}}$ and the *singular geometric boundary* $\partial X^{\mathrm{sing}}$ of X are then naturally defined by

$$\partial X^{\mathrm{reg}} := \{\tilde{\xi} \in \partial X: \mathrm{sl}(\tilde{\xi}) \in E^+\}, \quad \partial X^{\mathrm{sing}} := \partial X \setminus \partial X^{\mathrm{sing}};$$

the singular boundary $\partial X^{\mathrm{sing}}$ consists of equivalence classes of geodesic rays in X which project to a point in one of the factors X_i.

We further notice that two regular geodesic rays σ, σ' with the same slope represent the same point in $\partial X^{\mathrm{reg}}$ if and only if $\sigma_i(\infty) = \sigma_i'(\infty)$ for all $i \in \{1, 2, \ldots, r\}$. So $\partial X^{\mathrm{reg}}$ is homeomorphic to $\partial X_1 \times \partial X_2 \times \cdots \times \partial X_r \times E^+$.

If $\gamma \in \mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r)$, then the slope of $\gamma \cdot \tilde{\xi}$ equals the slope of $\tilde{\xi}$. In other words, if ∂X_θ denotes the set of points in the geometric boundary of slope $\theta \in E$, then ∂X_θ is invariant by the action of $\mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r)$.

In analogy to the case of symmetric spaces of higher rank we define the *Furstenberg boundary* $\partial^F X$ of X as the product $\partial X_1 \times \partial X_2 \times \cdots \times \partial X_r$ endowed with the product topology. Since $\partial X^{\mathrm{reg}}$ is homeomorphic to $\partial^F X \times E^+$ we have a natural projection

$$\begin{aligned} \pi^F \colon \partial X^{\mathrm{reg}} &\longrightarrow \partial^F X, \\ (\xi_1, \xi_2, \ldots, \xi_r, \theta) &\longmapsto (\xi_1, \xi_2, \ldots, \xi_r), \end{aligned}$$

and a natural action of the group $\mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r)$ by homeomorphisms on the Furstenberg boundary of X. Clearly $\partial^F X$ is homeomorphic to each of the sets $\partial X_\theta \subset \partial X^{\mathrm{reg}}$ with $\theta \in E^+$. Notice that in the special case $r = 1$ the Furstenberg boundary $\partial^F X$ equals the geometric boundary and the projection π^F is the identity; $E^+ = E$ is simply a point.

The following two elementary lemmata provide important facts concerning the topology of $\overline{X}$.

Lemma 3.1. *Suppose $(y_n) \subset X$ is a sequence converging to a point $\tilde{\eta} \in \partial X_\theta$ for some $\theta \in E$. Then for any $x \in X$ we have $\widehat{H}(x, y_n) \to \theta$ as $n \to \infty$.*

Notice that if $\theta = (\theta_1, \theta_2, \ldots, \theta_r) \in E$ satisfies $\theta_i = 0$ for some $i \in \{1, 2, \ldots, r\}$, then the projections $y_{n,i} = p_i(y_n)$ of y_n to X_i necessarily satisfy

$$\lim_{n \to \infty} \frac{d_i(x_i, y_{n,i})}{d(x, y_n)} = 0.$$

So the sequence $(y_{n,i}) \subset X_i$ can be either bounded or unbounded and may possess more than one accumulation point in $\overline{X}_i$. However, if $\theta \in E^+$ then we have the following

Lemma 3.2. *Suppose $(y_n) \subset X$ is a sequence converging to a regular boundary point $\tilde{\eta} \in \partial X^{\mathrm{reg}}$ with $\pi^F(\tilde{\eta}) = (\eta_1, \eta_2, \ldots, \eta_r) \in \partial^F X$. Then for all $i \in \{1, 2, \ldots, r\}$ the projections $p_i(y_n)$ to X_i converge to η_i as $n \to \infty$.*

Recall the definition of the visibility set at infinity $\mathrm{Vis}^\infty(\tilde{\xi})$ of a point $\tilde{\xi} \in \partial X$ from (1). It is easy to see that a point $\tilde{\eta} \in \partial X$ cannot belong to $\mathrm{Vis}^\infty(\tilde{\xi})$ if the slope of $\tilde{\eta}$ is different from the slope of $\tilde{\xi}$. This motivates the less restrictive definition of the *Furstenberg visibility set* of a point $\xi \in \partial^F X$ which is

$$\mathrm{Vis}^F(\xi) := \pi^F\big(\mathrm{Vis}^\infty(\tilde{\xi})\big), \quad \text{where } \tilde{\xi} \in (\pi^F)^{-1}(\xi) \text{ is arbitrary.}$$

We say that $\xi, \eta \in \partial^F X$ are *opposite* if $\eta \in \mathrm{Vis}^F(\xi)$. Notice that $\xi = (\xi_1, \xi_2, \ldots, \xi_r)$, $\eta = (\eta_1, \eta_2, \ldots, \eta_r) \in \partial^F X$ are opposite if and only if ξ_i and η_i can be joined by a geodesic in X_i for all $i \in \{1, 2, \ldots, r\}$, i.e.

$$\mathrm{Vis}^F(\xi) = \{(\eta_1, \eta_2, \ldots, \eta_r) \in \partial^F X \colon \eta_i \in \mathrm{Vis}^\infty(\xi_i) \text{ for all } i \in \{1, 2, \ldots, r\}\}. \quad (4)$$

We terminate the section with a few definitions concerning isometries of products. By abuse of notation we also denote

$$p_i \colon \mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r) \longrightarrow \mathrm{Is}(X_i), \quad i = 1, 2, \ldots, r,$$

the natural projections.

Definition 3.3. An isometry $h = (h_1, h_2, \ldots, h_r) \in \mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r)$ is called *regular axial*, if $h_i = p_i(h)$ is a rank one isometry of X_i for all $i \in \{1, 2, \ldots, r\}$; we denote $\widetilde{h^+}$ its attractive fixed point in $\partial X^{\mathrm{reg}}$ and set

$$h^+ := \pi^F(\widetilde{h^+}) = (h_1^+, h_2^+, \ldots, h_r^+).$$

Moreover, we denote by $l_i(h)$, $i = 1, 2, \ldots, r$, the *translation length* of the rank one isometry $p_i(h)$ in X_i, and by

$$L(h) := \begin{pmatrix} l_1(h) \\ l_2(h) \\ \vdots \\ l_r(h) \end{pmatrix} \in \mathbb{R}^r_{>0}$$

the *translation vector* of h.

Notice that (4) and Lemma 2.5 (a) imply

$$\mathrm{Vis}^F(h^+) = \{(\xi_1, \xi_2, \ldots, \xi_r) \in \partial^F X \colon \xi_i \neq h_i^+, \text{ for all } i \in \{1, 2, \ldots, r\}\}.$$

Moreover, the translation length $l(h)$ in X of a regular axial isometry is given by $l(h) = \|L(h)\|$; the unit vector

$$\widehat{L}(h) := \frac{L(h)}{l(h)} \in E^+$$

is sometimes called the *translation direction* of h.

Definition 3.4. Two regular axial isometries $h, g \in \mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r)$ are called *independent*, if $p_i(h)$ and $p_i(g)$ are independent for all $i \in \{1, 2, \ldots, r\}$.

In other words, h and g are independent if

$$\{g^+, g^-\} \subset \mathrm{Vis}^F(h^+) \cap \mathrm{Vis}^F(h^-);$$

such pairs of independent regular axial isometries will play a key role throughout the article.

4. Key results on pairs of independent regular axials

Recall that $X = X_1 \times X_2 \times \cdots \times X_r$ is a product of locally compact Hadamard spaces. We fix a regular axial isometry

$$h = (h_1, h_2, \ldots, h_r) \in \mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r)$$

and a base point $o = (o_1, o_2, \ldots, o_r) \in \mathrm{Ax}(h)$; in particular, for each $i \in \{1, 2, \ldots, r\}$ the point $o_i \in X_i$ lies on an invariant geodesic of the rank one isometry $h_i \in \mathrm{Is}(X_i)$.

The results in this section are key ingredients for the proofs of all results. Proposition 4.1, Lemma 4.2, Proposition 4.4 and Corollary 4.5 generalize Proposition 5.1 and its consequences in [21], where the analogous results in the case of two factors were proved by a case-by-case study. When considering more factors this method does not work any more, because the projections of points in $\overline{X}_1 \times \overline{X}_2 \times \cdots \times \overline{X}_r$ to all factors need to be controlled simultanously. The new idea for Proposition 4.1 here is to move the first projection to the desired set without worrying about what happens to the other projections, and then step by step take care of the remaining projections without messing up what was already moved to the right place. This idea can also be used to show that *two* points in the Furstenberg boundary can be moved by the *same* isometry to an arbitrarily small neighborhood of h^+; this is the content of Proposition 4.3.

In Proposition 4.6 we further provide the analogon of Proposition 2.2.7 in [26] (compare also [5]), which turns out to be indispensable for the construction of free semi-groups in a discrete group as performed in Proposition 6.3. This construction in turn plays a central role in the proofs of Proposition 6.4, Proposition 6.5 and finally Theorem 7.9. Finally, Proposition 4.7 states the equivalent of Proposition 2.3.1 in [26] in our setting, which is necessary for the proof of Theorem 7.6. Thanks to Proposition 4.1, the proof of the corresponding Proposition 7.2 in [21] easily extends to more than two factors.

Proposition 4.1. *Assume that $g \in \mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r)$ is regular axial with $\{g^+, g^-\} \subset \mathrm{Vis}^F(h^-)$ and $g^- \in \mathrm{Vis}^F(h^+)$. Given neighborhoods $U_i \subset \overline{X}_i$ of h_i^+, $i = 1, 2, \ldots, r$, there exist $N \in \mathbb{N}$ and a finite set $\Lambda_r \subset \langle g^N, h^N\rangle^+$ in the semi-group generated by g^N and h^N consisting of 2^r words of length at most $2r$ in the generators g^N, h^N such that for any $z = (z_1, z_2, \ldots, z_r) \in \overline{X}_1 \times \overline{X}_2 \times \cdots \times \overline{X}_r$ there exists $\lambda(z) = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \Lambda_r$ with*

$$\lambda(z) \cdot z \in U_1 \times U_2 \times \cdots \times U_r.$$

Moreover, if $z = (z_1, z_2, \ldots, z_r) \in \overline{X}_1 \times \overline{X}_2 \times \cdots \times \overline{X}_r$ and

$$\lambda(z) = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \Lambda_r,$$

then there exist open neighborboods $V_i \subset \overline{X}_i$ of z_i, $i = 1, 2, \ldots r$, with the property

$$\lambda_i \cdot V_i \subset U_i \quad \textit{for all } i \in \{1, 2, \ldots, r\}.$$

Proof. For $i = 1, 2, \ldots, r$ and $\eta_i \in \partial X_i$ a point in the set $\{g_i^-, g_i^+, h_i^-, h_i^+\}$ (which consists of only 3 points if $g_i^+ = h_i^+$) let $W_i(\eta_i) \subset \overline{X}_i$ be an arbitrary sufficiently small neighborhood of $\eta_i \in \partial X_i$ with $o_i \notin W_i(\eta_i)$ such that the closures of all $W_i(\eta_i)$ are pairwise disjoint in $\overline{X}_i$. Making $W_i(h_i^+)$ smaller if necessary we may further assume that $W_i(h_i^+) \subset U_i$ for all $i \in \{1, 2, \ldots, r\}$. According to Lemma 2.5 (c) there exists a constant $N \in \mathbb{N}$ such that for all $i \in \{1, 2, \ldots, r\}$

$$g_i^{\pm N}\big(\overline{X}_i \setminus W_i(g_i^{\mp})\big) \subset W_i(g_i^{\pm}), \quad h_i^{\pm N}\big(\overline{X}_i \setminus W_i(h_i^{\mp})\big) \subset W_i(h_i^{\pm}). \tag{5}$$

We prove the first claim by induction on r. For $r = 1$ we let

$$z = z_1 \in \overline{X}_1 = W_1(h_1^-) \cup \overline{X}_1 \setminus \overline{W_1(h_1^-)}.$$

If $z_1 \in W_1(h_1^-)$, then from $W_1(h_1^-) \subset \overline{X}_1 \setminus W_1(g_1^-)$ and (5) we get $g_1^N z_1 \in W_1(g_1^+) \subset \overline{X}_1 \setminus W_1(h_1^-)$, hence again by (5)

$$h_1^N g_1^N z_1 \in W_1(h_1^+) \subset U_1.$$

If $z_1 \in \overline{X}_1 \setminus \overline{W_1(h_1^-)} \subset \overline{X}_1 \setminus W_1(h_1^-)$, then (5) directly gives $h_1^N z_1 \in W_1(h_1^+) \subset U_1$. In particular this shows that $\Lambda_1 := \{h^N g^N, h^N\} \subset \langle g^N, h^N\rangle^+$ is the desired set consisting of $2 = 2^1$ elements of length $\leq 2 = 2 \cdot 1$.

Moreover, since both $W_1(h_1^-)$ and $\overline{X}_1 \setminus \overline{W_1(h_1^-)}$ are open, there exists an open neighborhood $V_1 \subset \overline{X}_1$ of z_1 such that either

$$h_1^N g_1^N \cdot V_1 \subset U_1 \qquad \text{or} \quad h_1^N \cdot V_1 \subset U_1.$$

Now assume the assertion holds for $r - 1$; we claim that it also holds when r factors are involved. By the induction hypothesis there exists a finite set

$$\begin{aligned}\Lambda_{r-1} \subset \langle (g_1, g_2, \ldots, g_{r-1})^N, (h_1, h_2, \ldots, h_{r-1})^N\rangle^+ \\ < \mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_{r-1})\end{aligned}$$

consisting of 2^{r-1} words of length at most $2(r-1)$ in its generators such that for all $(y_1, y_2, \ldots, y_{r-1}) \in \overline{X}_1 \times \overline{X}_2 \times \cdots \times \overline{X}_{r-1}$ there exists

$$\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{r-1}) \in \Lambda_{r-1}$$

and open neighborhoods

$$V_i \subset \overline{X}_i$$

of y_i with $\lambda'_i \cdot V_i \subset U_i$ for all $i \in \{1, 2, \ldots, r-1\}$. We denote by

$$\Lambda' \subset \langle g^N, h^N\rangle^+$$

the finite set of the same words as in Λ_{r-1}, but now considered as elements in $\mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r)$, and let

$$z = (z_1, z_2, \ldots, z_r) \in \overline{X}_1 \times \overline{X}_2 \times \cdots \times \overline{X}_r$$

arbitrary. By the properties of Λ_{r-1} we know that there exists

$$\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{r-1}, \lambda'_r) \in \Lambda'$$

and open neighborhoods $V_i \subset \overline{X}_i$ of z_i, $1 \le i \le r-1$ such that

$$\lambda'_i \cdot V_i \subset U_i \quad \text{for all} \quad i \in \{1, 2, \ldots, r-1\},$$

but we do not know the position of $\lambda'_r z_r \in \overline{X}_r = W_r(h_r^-) \cup \overline{X}_r \setminus \overline{W_r(h_r^-)}$.

However, as in the case $r = 1$ the north-south dynamics (5) implies

$$h_r^N g_r^N \lambda'_r z_r \in U_r \quad \text{or } h_r^N \lambda'_r z_r \in U_r$$

according to the two cases $\lambda'_r z_r \in W_r(h_r^-)$ or $\lambda'_r z_r \in \overline{X}_r \setminus \overline{W_r(h_r^-)}$. Moreover, there exists an open neighborhood $V_r \subset \overline{X}_r$ of z_r such that $\lambda'_r \cdot V_r \subset W_r(h_r^-)$ or $\lambda'_r \cdot V_r \subset \overline{X}_r \setminus \overline{W_r(h_r^-)}$ and hence

$$h_r^N g_r^N \lambda'_r \cdot V_r \subset U_r \quad \text{or } h_r^N \lambda'_r \cdot V_r \subset U_r.$$

Since for all $i \in \{1, 2, \ldots, r-1\}$ we have

$$h_i^N g_i^N \cdot U_i \subset W_i(h_i^+) \subset U_i \quad \text{and} \quad h_i^N \cdot U_i \subset U_i$$

we conclude that the set Λ_r consisting of all words in g^N, h^N of the form $h^N \lambda'$ or $h^N g^N \lambda'$ with $\lambda' \in \Lambda'$ works. Clearly, all such words have length $\le 2 + 2(r-1) = 2r$ in the generators g^N, h^N and the cardinality of Λ_r is equal to $2 \cdot 2^{r-1} = 2^r$. $\square$

Remark. If g and h are independent, then – replacing h by h^{-1} – an analogous statement holds for neighborhoods U_i of h_i^-, $i = 1, 2, \ldots, r$.

In this case we have the following easy corollary which will be used in the proof of Theorem 5.3.

Lemma 4.2. *Assume that $g \in \mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r)$ is regular axial and g, h are independent. Then for any $\zeta \in \partial^F X$ and all $\eta \in F_\Gamma$ there exists $\alpha \in \Gamma$ such that $\alpha\zeta \in \mathrm{Vis}^F(\eta)$.*

Proof. For $i \in \{1, 2, \ldots, r\}$ and $\eta_i \in \{h_i^+, h_i^-\}$ we let $U_i(\eta_i) \subset \overline{X}_i$ be the neighborhoods satisfying property (a) of Proposition 2.7. According to Proposition 4.1 there exist $\lambda, \mu \in \Gamma$ such that

$$\lambda\zeta \in U_1(h_1^+) \times U_2(h_2^+) \times \cdots \times U_r(h_r^+)$$

and

$$\mu\eta \in U_1(h_1^-) \times U_2(h_2^-) \times \cdots \times U_r(h_r^-),$$

which immediately gives $\lambda\zeta \in \mathrm{Vis}^F(\mu\eta)$ and hence $\mu^{-1}\lambda\zeta \in \mathrm{Vis}^F(\eta)$. $\square$

We next state a stronger version of Proposition 4.1 which will also be needed in the proof of Theorem 5.3.

Proposition 4.3. *Assume that $g \in \mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r)$ is regular axial and g, h are independent. Then for any neighborhood $U \subset \partial^F X$ of h^+ there exists a finite set $\Lambda \subset \langle g, h\rangle$ such that for any two points $\zeta, \eta \in \partial^F X$ there exists $\lambda \in \Lambda$ such that $\{\lambda\zeta, \lambda\eta\} \subset U$.*

Proof. For $i = 1, 2, \ldots, r$ and $\eta_i \in \partial X_i$ a point in the set $\{g_i^-, g_i^+, h_i^-, h_i^+\}$ we let $W_i(\eta_i) \subset \overline{X}_i$ be an arbitrary neighborhood of $\eta_i^+ \in \partial X_i$ with $o_i \notin W_i(\eta_i)$ such that the closures of all $W_i(\eta_i)$ are pairwise disjoint in $\overline{X}_i$. Making $W_i(h_i^+)$ smaller if necessary we may further assume that

$$W_1(h_1^+) \times W_2(h_2^+) \times \cdots \times W_r(h_r^+) \subset U.$$

According to Lemma 2.5 (c) there exists a constant $N \in \mathbb{N}$ such that for all $i \in \{1, 2, \ldots, r\}$

$$g_i^{\pm N}(\overline{X}_i \setminus W_i(g_i^{\mp})) \subset W_i(g_i^{\pm}), \quad h_i^{\pm N}(\overline{X}_i \setminus W_i(h_i^{\mp})) \subset W_i(h_i^{\pm}).$$

We prove the claim by induction on r. For $r = 1$ we let $\zeta = \zeta_1 \in \partial X_1$ and $\eta = \eta_1 \in \partial X_1$ arbitrary. If both ζ and η belong to $W_1(h_1^-)$, then $\lambda = h^N g^N$ is the desired element, if both ζ and η are contained in $\overline{X}_1 \setminus W_1(h_1^-)$, then $\lambda = h^N$ is. So it remains to deal with the case that one of the points, say ζ, belongs to $W_1(h_1^-)$ and the second one does not. If $\eta \in \overline{X}_1 \setminus W_1(g_1^-)$, then $\lambda = h^N g^N$ works again, if $\eta \in W_1(g_1^-)$ we can take $\lambda = h^N g^{-N}$. In particular there exists $\lambda \in \Lambda := \{h^N, h^N g^N, h^N g^{-N}\}$ such that $\{\lambda\zeta, \lambda\eta\} \subset W_1(h_1^+) \subset U$.

Now assume that there exists a finite set

$$\begin{aligned}\Lambda_{r-1} &\subset \langle (g_1, g_2, \ldots, g_{r-1}), (h_1, h_2, \ldots, h_{r-1})\rangle \\ &\subset \mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_{r-1})\end{aligned}$$

such that for any $(\zeta_1, \zeta_2, \ldots, \zeta_{r-1}), (\eta_1, \eta_2, \ldots, \eta_{r-1}) \in \partial X_1 \times \partial X_2 \times \cdots \times \partial X_{r-1}$ there exists $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{r-1}) \in \Lambda_{r-1}$ with the property

$$\{\lambda'_i\zeta_i, \lambda'_i\eta_i\} \subset W_i(h_i^+) \quad \text{for all } i \in \{1, 2, \ldots, r-1\}.$$

We denote by $\Lambda' \subset \langle g, h\rangle$ the finite set of the same words as in Λ_{r-1}, but now considered as elements in $\mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r)$, and let

$$\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{r-1}, \zeta_r),$$

$$\eta = (\eta_1, \eta_2, \ldots, \eta_{r-1}, \eta_r) \in \partial X_1 \times \partial X_2 \times \cdots \times \partial X_{r-1} \times \partial X_r = \partial^F X$$

arbitrary. By the properties of Λ_{r-1} we know that there exists

$$\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{r-1}, \lambda'_r) \in \Lambda'$$

such that for $i \in \{1, 2, \ldots, r-1\}$ we have $\{\lambda'_i \zeta_i, \lambda'_i \eta_i\} \subset W_i(h_i^+)$. For $\lambda'_r \zeta_r$ and $\lambda'_r \eta_r$ there are different possibilities. If both points belong to $W_r(h_r^-)$, then $\lambda := h^N g^N \lambda'$ moves both ζ and η to $W_1(h_1^+) \times W_2(h_2^+) \times \cdots \times W_r(h_r^+) \subset U$, if both points are contained in $\overline{X}_r \setminus W_r(h_r^-)$, then $\lambda := h^N \lambda'$ satisfies

$$\{\lambda\zeta, \lambda\eta\} \subset W_1(h_1^+) \times W_2(h_2^+) \times \cdots \times W_r(h_r^+) \subset U.$$

It finally remains to deal with the case that one of the points, say $\lambda'_r \zeta_r$, belongs to $W_r(h_r^-)$ and the second one does not. If $\lambda'_r \eta_r \in \overline{X}_r \setminus W_r(g_r^-)$, then $\lambda = h^N g^N \lambda'$ works again, if $\lambda'_r \eta_r \in W_r(g_r^-)$ we can take $\lambda = h^N g^{-N} \lambda'$. So we conclude that there exists

$$\lambda \in \Lambda := h^N \Lambda' \cup h^N g^N \Lambda' \cup h^N g^{-N} \Lambda'$$

such that

$$\{\lambda\zeta, \lambda\eta\} \in W_1(h_1^+) \times W_2(h_2^+) \times \cdots \times W_r(h_r^+) \subset U. \qquad \square$$

Proposition 4.4. *Let $\Gamma < \mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r)$ be a discrete group containing h and a second regular axial element g such that g and h are independent. Let $(\gamma_n) = ((\gamma_{n,1}, \gamma_{n,2}, \ldots, \gamma_{n,r})) \subset \Gamma$ be a sequence such that for all $i \in \{1, 2, \ldots, r\}$ the sequences $\gamma_{n,i} o_i$ and $\gamma_{n,i}^{-1} o_i$ converge to points in ∂X_i as $n \to \infty$. Then given arbitrarily small distinct neighborhoods*

$$W_i(h_i^+), W_i(h_i^-) \subset \overline{X}_i$$

of h_i^+, h_i^-, $i = 1, 2, \ldots, r$, there exist $N \in \mathbb{N}$, finite sets $\Lambda^+ \subset \langle h^N, g^N \rangle^+$, $\Lambda^- \subset \langle h^{-N}, g^{-N} \rangle^+$, $\lambda \in \Lambda^+$ and $\mu \in \Lambda^-$ such that $\varphi_n := \lambda \gamma_n \mu^{-1}$ satisfies

$$\varphi_n o \in W_1(h_1^+) \times W_2(h_2^+) \times \cdots \times W_r(h_r^+)$$

and

$$\varphi_n^{-1} o \in W_1(h_1^-) \times W_2(h_2^-) \times \cdots \times W_r(h_r^-)$$

for n sufficiently large.

Proof. For the neighborhoods $W_i(h_i^+), W_i(h_i^-) \subset \overline{X}_i$ of h_i^+, h_i^-, $i \in \{1, 2, \ldots, r\}$, we let $N \in \mathbb{N}$ and $\Lambda^+ \subset \langle g^N, h^N \rangle^+$, $\Lambda^- \subset \langle g^{-N}, h^{-N} \rangle^+$ be the finite sets according to Proposition 4.1. That is for any

$$z = (z_1, z_2, \ldots, z_r) \in \overline{X}_1 \times \overline{X}_2 \times \cdots \times \overline{X}_r$$

there exists $\lambda(z) \in \Lambda^+$ and $\mu(z) \in \Lambda^-$ such that

$$\lambda(z) \cdot z \in W_1(h_1^+) \times W_2(h_2^+) \times \cdots \times W_r(h_r^+)$$

and

$$\mu(z) \cdot z \in W_1(h_1^-) \times W_2(h_2^-) \times \cdots \times W_r(h_r^-).$$

We denote $F \subset X$ the finite set of points $\{\lambda^{-1}o\colon \lambda \in \Lambda^+ \cup \Lambda^-\} \subset X$. For $i \in \{1,2,\ldots,r\}$ we let $\xi_i \in \partial X_i$ be the limit of the sequence $(\gamma_{n,i}o_i) \subset X_i$. By Proposition 4.1 there exist $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \Lambda^+$ and neighborhoods V_i^+ of ξ_i in $\overline{X}_i$ with $\lambda_i \cdot V_i^+ \subset W_i(h_i^+)$ for all $i \in \{1,2,\ldots,r\}$. Since for any $x_i \in X_i$ the sequence $\gamma_{n,i}x_i$ also converges to ξ_i, there exists $N_+ \in \mathbb{N}$ such that for all $n > N_+$ and every $x \in F$ we have $\gamma_n x \in V_1^+ \times V_2^+ \times \cdots \times V_r^+$, and hence

$$\lambda\gamma_n x \in W_1(h_1^+) \times W_2(h_2^+) \times \cdots \times W_r(h_r^+).$$

Similarly, if $\zeta_i \in \partial X_i$ is the limit of the sequence $(\gamma_{n,i}^{-1}o_i) \subset X_i$, $i \in \{1,2,\ldots,r\}$, then there exist $\mu = (\mu_1, \mu_2, \ldots, \mu_r) \in \Lambda^-$ and neighborhoods V_i^- of ζ_i in $\overline{X}_i$ with $\mu_i \cdot V_i^- \subset W_i(h_i^-)$ for all $i \in \{1,2,\ldots,r\}$. As before, there exists $N_- \in \mathbb{N}$ such that for all $n > N_-$ and every $x \in F$ we have $\gamma_n^{-1}x \in V_1^- \times V_2^- \times \cdots \times V_r^-$, and hence

$$\mu\gamma_n^{-1}x \in W_1(h_1^-) \times W_2(h_2^-) \times \cdots \times W_r(h_r^-).$$

Since both $\lambda^{-1}o$ and $\mu^{-1}o$ belong to the finite set F the assertion is true for all $n > \max\{N_+, N_-\}$. □

Remark. The assumption concerning the sequence (γ_n) in Γ is clearly satisfied if $\gamma_n o$ and $\gamma_n^{-1}o$ converge to points in the regular boundary $\partial X^{\mathrm{reg}}$ of X. However, the result is also valid if $\gamma_n o$ and $\gamma_n^{-1}o$ converge to singular boundary points in a way that for all $i \in \{1,2,\ldots,r\}$ the sequences $\gamma_{n,i}o_i$ and $\gamma_{n,i}^{-1}o_i$ converge to points in ∂X_i.

In combination with Lemma 2.6, we get the following useful

Corollary 4.5. *Let $\Gamma < \mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r)$ be a discrete group containing h and a second regular axial element g such that g and h are independent. Let $(\gamma_n) = ((\gamma_{n,1}, \gamma_{n,2}, \ldots, \gamma_{n,r})) \subset \Gamma$ be a sequence such that for all $i \in \{1,2,\ldots,r\}$ the sequences $\gamma_{n,i}o_i$ and $\gamma_{n,i}^{-1}o_i$ converge to points in ∂X_i as $n \to \infty$. Then given arbitrarily small distinct neighborhoods $U_i^+, U_i^- \subset \overline{X}_i$ of h_i^+, h_i^-, $i = 1,2,\ldots,r$, there exist a finite set $\Lambda \subset \langle g,h\rangle$ and $N_0 \in \mathbb{N}$ such that for some fixed $\lambda, \mu \in \Lambda$ and $n > N_0$ the isometries $\varphi_n := \lambda\gamma_n\mu^{-1}$ are all regular axial with attractive and repulsive fixed points $\widetilde{\varphi_n}^+, \widetilde{\varphi_n}^- \in \partial X^{\mathrm{reg}}$ satisfying*

$$\pi^F(\widetilde{\varphi_n}^+) \in U_1^+ \times U_2^+ \times \cdots \times U_r^+,$$

and

$$\pi^F(\widetilde{\varphi_n}^-) \in U_1^- \times U_2^- \times \cdots \times U_r^-.$$

The next result in this section will be the main tool for the construction of certain free subgroups according to Proposition 6.3.

Proposition 4.6. *Assume that* $g \in \mathrm{Is}(X_1)\times\mathrm{Is}(X_2)\times\cdots\times\mathrm{Is}(X_r)$ *is regular axial and* g, h *are independent. Fix a regular axial isometry* $\beta \in \mathrm{Is}(X_1)\times\mathrm{Is}(X_2)\times\cdots\times\mathrm{Is}(X_r)$ *and let* $\mathcal{C} \subset \mathbb{R}^r_{>0}$ *be an open cone containing* $L(\beta)$*. Then for all neighborhoods* V_i^+, $V_i^- \subset \overline{X}_i$ *of* g_i^+, g_i^-, $i = 1, 2, \ldots, r$*, there exists a regular axial isometry* $\alpha \in \langle g, h, \beta\rangle$ *with*

$$L(\alpha) \in \mathcal{C}, \quad \alpha_i^+ \in V_i^+ \quad \textit{and} \quad \alpha_i^- \in V_i^- \quad \textit{for all } i \in \{1, 2, \ldots, r\}.$$

Proof. Fix $i \in \{1, 2, \ldots, r\}$. For $\eta_i \in \{g_i^-, g_i^+, h_i^-, h_i^+\}$ we let

$$V_i(\eta_i) \subset U_i(\eta_i) \subset \overline{X}_i$$

be neighborhoods of η_i and $c_i > 0$ as in Proposition 2.7. Making $V_i(g_i^\pm)$ and $U_i(g_i^\pm)$ smaller if necessary we may further assume that

$$V_i(g_i^+) \subset U_i(g_i^+) \subset V_i^+ \quad \text{and} \quad V_i(g_i^-) \subset U_i(g_i^-) \subset V_i^-.$$

Since the sequences $\beta^n o$ and $(\beta^n)^{-1} o = \beta^{-n} o$ converge to the attractive and repulsive fixed points $\widetilde{\beta^+}$, $\widetilde{\beta^-} \in \partial X^{\mathrm{reg}}$ of β, Corollary 4.5 provides a finite set $\Lambda \subset \langle g, h\rangle$, $\lambda, \mu \in \Lambda$ and $N_0 \in \mathbb{N}$ such that for all $n > N_0$ isometries $\varphi_n := \lambda\beta^n\mu^{-1}$ are regular axial with

$$\pi^F(\widetilde{\varphi_n}^+) \in V_1(h_1^+) \times V_2(h_2^+) \times \cdots \times V_r(h_r^+),$$

and

$$\pi^F(\widetilde{\varphi_n}^-) \in V_1(h_1^-) \times V_2(h_2^-) \times \cdots \times V_r(h_r^-).$$

We set

$$c := \max\{c_i : i \in \{1, 2, \ldots, r\}\},$$

$$b := \max\{d_i(o_i, \mathrm{Ax}(\beta_i)) : i \in \{1, 2, \ldots, r\}\},$$

and

$$d := \max\{d_i(o_i, \lambda_i o_i) : i \in \{1, 2, \ldots, r\}, \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \Lambda\}.$$

Then we get for $n > N_0$, $k \in \mathbb{N}$ and $i \in \{1, 2, \ldots, r\}$

$$l_i(\varphi_n^k) \le d_i(o_i, \varphi_{n,i}^k o_i) \le l_i(\varphi_n^k) + 2c,$$

$$l_i(\beta^n) \le d_i(o_i, \beta_i^n o_i) \le l_i(\beta^n) + 2b,$$

and

$$|d_i(o_i, \varphi_{n,i} o_i) - d_i(o_i, \beta_i^n o_i)| \le d_i(o_i, \lambda_i o_i) + d_i(o_i, \mu_i^{-1} o_i) \le 2d,$$

which gives (in the special case $k = 1$)

$$|l_i(\varphi_n) - l_i(\beta^n)| \leq 2b + 2c + 2d. \tag{6}$$

We now fix $n > N_0$ and write

$$\varphi_n = (\varphi_{n,1}, \varphi_{n,2}, \dots, \varphi_{n,r}).$$

Since for $i \in \{1, 2, \dots, r\}$ we have $\varphi_{n,i}^{\pm} \in V_i(h_i^{\pm})$, Proposition 2.7 (b) implies the existence of $N, N_n \in \mathbb{N}$ such that the isometry

$$\gamma_n := g^N \varphi_n^{N_n} g^N$$

satisfies

$$|l_i(\gamma_n) - 2l_i(g^N) - l_i(\varphi_n^{N_n})| \leq 4c \cdot 3 = 12c$$

for all $i \in \{1, 2, \dots, r\}$. Using (6), $l_i(\varphi_n^{N_n}) = N_n \cdot l_i(\varphi_n)$ and $l_i(\beta^n) = n \cdot l_i(\beta)$ we get

$$|l_i(\gamma_n) - 2l_i(g^N) - nN_n \cdot l_i(\beta)| \leq 2N_n(b + c + d) + 12c,$$

which implies

$$\lim_{n\to\infty} \frac{l_i(\gamma_n)}{nN_n} = l_i(\beta).$$

This shows that for n sufficiently large we have $L(\gamma_n) \in \mathcal{C}$. $\square$

For the last result in this section we assume that

$$\Gamma < \mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r)$$

is a discrete group which contains a pair of independent regular axial isometries $g = (g_1, g_2, \dots, g_r)$ and $h = (h_1, h_2, \dots, h_r)$. As before we fix a base point $o = (o_1, o_2, \dots, o_r) \in \mathrm{Ax}(h)$. We further recall the definition of the distance vector from the beginning of Section 3; for an element $\gamma \in \Gamma$ we will use the abbreviation $H(\gamma)$ for the distance vector $H(o, \gamma o) \in \mathbb{R}^r_{\geq 0}$ of the pair $(o, \gamma o)$.

We are going to construct a generic product for Γ as in [26], Proposition 2.3.1, which is the essential tool in the proof of Theorem 7.6. The idea behind is to find a finite set in $\Gamma \times \Gamma$ which maps pairs of orbit points $(\alpha o, \beta^{-1} o)$ close to a set $\mathrm{Ax}(g)$ or $\mathrm{Ax}(h)$. Unfortunately, unlike in the case of symmetric spaces, we do not dispose of an equivalent of the result of Abels-Margulis-Soifer ([26, Proposition 2.3.4]) which plays a crucial role in the article by Quint. Instead, as in Section 7 of [21] we will exploit the dynamics of a free subgroup in $\langle g, h\rangle < \Gamma$.

Proposition 4.7. *There exists a map $\pi\colon \Gamma \times \Gamma \to \Gamma$ with the following properties.*

(a) *There exists $\kappa \geq 0$ such that for all $\alpha, \beta \in \Gamma$ we have*

$$\|H\big(\pi(\alpha, \beta)\big) - H(\alpha) - H(\beta)\| \leq \kappa.$$

(b) *For any $t > 0$ there exists a finite set $\Lambda \subset \Gamma$ such that for all $\alpha, \beta, \hat{\alpha}, \hat{\beta} \in \Gamma$ with $\|H(\alpha) - H(\hat{\alpha})\| \leq t$, $\|H(\beta) - H(\hat{\beta})\| \leq t$ we have*

$$\pi(\alpha, \beta) = \pi(\hat{\alpha}, \hat{\beta}) \iff \hat{\alpha} \in \alpha\Lambda \;\text{ and }\; \hat{\beta} \in \Lambda\beta.$$

Proof. In order to construct a map satisfying property (a) we let

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r), \beta = (\beta_1, \beta_2, \ldots, \beta_r) \in \Gamma$$

arbitrary.

For $i \in \{1, 2, \ldots, r\}$ and $\eta_i \in \{g_i^-, g_i^+, h_i^-, h_i^+\}$ we let $U_i(\eta_i) \subset \overline{X}_i$ be the neighborhoods of η_i and $c_i > 0$ the constant provided by Proposition 2.7. According to Proposition 4.1 there exist a finite set $\Lambda \subset \Gamma$ and $\mu = \mu(\alpha)$, $\lambda = \lambda(\beta) \in \Lambda$ such that

$$\mu\alpha^{-1}o \in U_1(h_1^-) \times U_2(h_2^-) \times \cdots \times U_r(h_r^-)$$

and

$$\lambda\beta o \in U_1(h_1^+) \times U_2(h_2^+) \times \cdots \times U_r(h_r^+).$$

We next set $c := \max\{c_i \colon i \in \{1, 2, \ldots, r\}\}$,

$$d := \max\{d_i(o_i, \lambda_i o_i) \colon i \in \{1, 2, \ldots, r\}, \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \Lambda\}$$

and fix $i \in \{1, 2, \ldots, r\}$. Since $d_i(\alpha_i\mu_i^{-1}\lambda_i\beta_i o_i, o_i) = d_i(\lambda_i\beta_i o_i, \mu_i\alpha_i^{-1}o_i)$, Proposition 2.7 (a) implies

$$|d_i(\alpha_i\mu_i^{-1}\lambda_i\beta_i o_i, o_i) - d_i(\lambda_i\beta_i o_i, o_i) - d_i(o_i, \mu_i\alpha_i^{-1}o_i)| \leq 2c;$$

using $d_i(\lambda_i\beta_i o_i, o_i) = d_i(\beta_i o_i, \lambda_i^{-1}o_i)$ and $d_i(o_i, \mu_i\alpha_i^{-1}o_i) = d_i(\mu_i^{-1}o_i, \alpha_i^{-1}o_i)$ we conclude

$$|d_i(\alpha_i\mu_i^{-1}\lambda_i\beta_i o_i, o_i) - d_i(\beta_i o_i, o_i) - d_i(o_i, \alpha_i^{-1}o_i)| \leq 2c + 2d.$$

This implies

$$\|H(\alpha\mu^{-1}\lambda\beta) - H(\beta) - H(\alpha)\| \leq \sqrt{r}(2c + 2d) =: \kappa,$$

hence the assignment $\pi(\alpha, \beta) := \alpha\mu(\alpha)^{-1}\lambda(\beta)\beta$ satisfies property (a).

It remains to prove that the map π from above also satisfies property (b). Suppose there exists $t > 0$ such that for any finite set $\Lambda_n \subset \Gamma$ there exist $\alpha_n, \beta_n, \hat{\alpha}_n, \hat{\beta}_n \in \Gamma$ with

$$\|H(\alpha_n) - H(\hat{\alpha}_n)\| \le t, \quad \|H(\beta_n) - H(\hat{\beta}_n)\| \le t$$

and

$$\pi(\alpha_n, \beta_n) = \pi(\hat{\alpha}_n, \hat{\beta}_n), \quad \text{but } \alpha_n^{-1}\hat{\alpha}_n \notin \Lambda_n \text{ or } \hat{\beta}_n\beta_n^{-1} \notin \Lambda_n. \tag{$*$}$$

For $n \in \mathbb{N}$ we will work here with the finite set

$$\Lambda_n := \{\gamma \in \Gamma : d(o, \gamma o) \le n\}$$

and fix

$$\alpha_n = (\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,r}), \quad \hat{\alpha}_n = (\hat{\alpha}_{n,1}, \hat{\alpha}_{n,2}, \ldots, \hat{\alpha}_{n,r}),$$

$$\beta_n = (\beta_{n,1}, \beta_{n,2}, \ldots, \beta_{n,r}), \quad \hat{\beta}_n = (\hat{\beta}_{n,1}, \hat{\beta}_{n,2}, \ldots, \hat{\beta}_{n,r}),$$

in Γ such that ($*$) is satisfied.

Passing to subsequences if necessary we assume that for all $i \in \{1, 2, \ldots, r\}$ the sequences $(\alpha_{n,i}^{-1} o_i)$, $(\hat{\alpha}_{n,i}^{-1} o_i)$, $(\beta_{n,i} o_i)$, $(\hat{\beta}_{n,i} o_i) \subset X_i$ converge. Notice that the limit can be a point in X_i or in the geometric boundary ∂X_i. In any case Proposition 4.1 shows that there exist a finite set $\Lambda \subset \Gamma$ and μ, $\hat{\mu}$, λ, $\hat{\lambda} \in \Lambda$ such that for all $i \in \{1, 2, \ldots, r\}$ and $n \in \mathbb{N}$ sufficiently large

$$\mu_i \alpha_{n,i}^{-1} o_i,\ \hat{\mu}_i \hat{\alpha}_{n,i}^{-1} o_i \in U_i(h_i^-) \quad \text{and} \quad \lambda_i \beta_{n,i} o_i,\ \hat{\lambda}_i \hat{\beta}_{n,i} o_i \in U_i(h_i^+). \tag{7}$$

For $n \in \mathbb{N}$ and $i = 1, 2, \ldots, r$ we denote $x_{n,i}$ a point on the geodesic path from $\mu_i \alpha_{n,i}^{-1} o_i$ to $\lambda_i \beta_{n,i} o_i$, and $\hat{x}_{n,i}$ a point on the geodesic path from $\hat{\mu}_i \hat{\alpha}_{n,i}^{-1} o_i$ to $\hat{\lambda}_i \hat{\beta}_{n,i} o_i$ such that $d_i(o_i, x_{n,i}) \le c$ and $d_i(o_i, \hat{x}_{n,i}) \le c$. Furthermore, setting $\gamma_n := \alpha_n \mu^{-1} \lambda \beta_n = \hat{\alpha}_n \hat{\mu}^{-1} \hat{\lambda} \hat{\beta}_n$ and denoting $\sigma_{n,i}$, $i = 1, 2, \ldots, r$, the geodesic path $\sigma_{o_i, \gamma_{n,i} o_i}$ there exist $t_i, \hat{t}_i > 0$ such that

$$\begin{aligned} d_i(\alpha_{n,i}\mu_i^{-1} o_i, \sigma_{n,i}(t_i)) &= d_i(\alpha_{n,i}\mu_i^{-1} o_i, \sigma_{n,i}) \\ &= d_i(o_i, \mu_i \alpha_{n,i}^{-1} \sigma_{n,i}) \\ &\le d_i(o_i, x_{n,i}) \\ &\le c \end{aligned}$$

and

$$\begin{aligned} d_i(\hat{\alpha}_{n,i}\hat{\mu}_i^{-1} o_i, \sigma_{n,i}(\hat{t}_i)) &= d_i(\hat{\alpha}_{n,i}\hat{\mu}_i^{-1} o_i, \sigma_{n,i}) \\ &= d_i(o_i, \hat{\mu}_i \hat{\alpha}_{n,i}^{-1} \sigma_{n,i}) \\ &\le d_i(o_i, \hat{x}_{n,i}) \\ &\le c, \end{aligned}$$

by (7) and Proposition 2.7 (a). Hence using again

$$d = \max\{d_i(o_i, \lambda_i o_i) \colon i \in \{1, 2, \ldots, r\}, \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \Lambda\}$$

we estimate

$$d_i(\alpha_{n,i} o_i, \sigma_{n,i}) \le d_i(\alpha_{n,i} o_i, \alpha_{n,i}\mu_i^{-1} o_i) + d_i(\alpha_{n,i}\mu_i^{-1} o_i, \sigma_{n,i}(t_i)) \le d + c$$

and

$$d_i(\hat\alpha_{n,i} o_i, \sigma_{n,i}) \le d_i(\hat\alpha_{n,i} o_i, \hat\alpha_{n,i}\hat\mu_i^{-1} o_i) + d_i(\hat\alpha_{n,i}\hat\mu_i^{-1} o_i, \sigma_{n,i}(\hat t_i)) \le d + c.$$

For $n \in \mathbb{N}$ and $i = 1, 2, \ldots, r$ we let $y_{n,i}, \hat y_{n,i} \in X_i$ be the points on the geodesic path $\sigma_{n,i}$ such that $d_i(o_i, y_{n,i}) = d_i(o_i, \alpha_{n,i} o_i)$ and $d_i(o_i, \hat y_{n,i}) = d_i(o_i, \hat\alpha_{n,i} o_i)$. Since $\|H(\alpha_n) - H(\hat\alpha_n)\| \le t$ we have $d_i(y_{n,i}, \hat y_{n,i}) \le t$, and, by elementary geometric estimates,

$$d_i(\alpha_{n,i} o_i, y_{n,i}) \le 2(d + c) \quad \text{and} \quad d_i(\hat\alpha_{n,i} o_i, \hat y_{n,i}) \le 2(d + c).$$

We summarize

$$\begin{aligned} d_i(o_i, \alpha_{n,i}^{-1}\hat\alpha_{n,i} o_i) &\le d_i(\alpha_{n,i} o_i, y_{n,i}) + d(y_{n,i}, \hat y_{n,i}) + d_i(\hat y_{n,i}, \hat\alpha_{n,i} o_i) \\ &\le 4(d + c) + t, \end{aligned}$$

hence

$$d(o, \alpha_n^{-1}\hat\alpha_n o) \le \sqrt{r}(4d + 4c + t) =: R.$$

In particular, for $n > R$ we have $\alpha_n^{-1}\hat\alpha_n \in \Lambda_n$, and, in order to obtain the desired contradiction, it remains to prove that $\hat\beta_n\beta_n^{-1} \in \Lambda_n$ for n sufficiently large.

Notice that

$$\hat\beta_n = \hat\lambda^{-1}\hat\mu\hat\alpha_n^{-1}\gamma_n = \hat\lambda^{-1}\hat\mu\hat\alpha_n^{-1}\alpha_n\mu^{-1}\lambda\beta_n,$$

hence

$$\begin{aligned} d(o, \hat\beta_n\beta_n^{-1} o) &= d(o, \hat\lambda^{-1}\hat\mu\hat\alpha_n^{-1}\alpha_n\mu^{-1}\lambda o) \\ &\le \overbrace{d(o, \hat\lambda^{-1} o)}^{\le\sqrt{r}d} + \overbrace{d(\hat\lambda^{-1} o, \hat\lambda^{-1}\hat\mu o)}^{\le\sqrt{r}d} + d(\hat\lambda^{-1}\hat\mu o, \hat\lambda^{-1}\hat\mu\hat\alpha_n^{-1}\alpha_n o) \\ &\qquad + d(\hat\alpha_n^{-1}\alpha_n o, \hat\alpha_n^{-1}\alpha_n\mu^{-1} o) + d(\mu^{-1} o, \mu^{-1}\lambda o) \\ &\le d(o, \alpha_n^{-1}\hat\alpha_n o) + 4\sqrt{r}d \le R + 4\sqrt{r}d. \end{aligned}$$

This finishes the proof. □

5. The structure of the limit set

The *geometric limit set* of a discrete group Γ acting by isometries on a locally compact Hadamard space is defined by $L_\Gamma := \overline{\Gamma\cdot x} \cap \partial X$, where $x \in X$ is arbitrary. In this section we are going to describe the structure of the geometric limit set for certain groups $\Gamma < \mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r) < \mathrm{Is}(X)$ acting properly discontinuously on the product X of r locally compact Hadamard spaces $X_1, X_2, \ldots, X_r$. For convenience the *Furstenberg limit set* of Γ is defined by $F_\Gamma := \pi^F(L_\Gamma \cap \partial X^{\mathrm{reg}})$. Moreover, we let

$$P_\Gamma := \{\theta \in E \colon L_\Gamma \cap \partial X_\theta \neq \emptyset\} \subset E$$

be the set of all slopes of geometric limit points, and $P_\Gamma^{\mathrm{reg}} = P_\Gamma \cap E^+$ the set of slopes of regular limit points.

In [21] – when dealing with only two factors – we were able to prove Theorems 5.1 and 5.2 in the more general context of discrete isometry groups containing a regular axial isometry with projections which do not globally fix a point in the geometric boundary of the corresponding factor and which possess infinitely many limit points. Unfortunately, the methods used there and in particular Lemma 4.1 of [21] are not available in the setting of more factors under the above weak assumption.

From here on we therefore assume – as in the second part of the aforementioned article – that $\Gamma < \mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r)$ *acts properly discontinuously* on X and *possesses two independent regular axial isometries*. This requires in particular that all factors of X are rank one spaces as for example universal covers of geometric rank one manifolds and CAT(−1)-spaces such as locally finite trees or manifolds of pinched negative curvature. Moreover – as already mentioned in the introduction – every finite-dimensional unbounded locally compact CAT(0)-cube complex with an essential and cocompact action of its automorphism group can be decomposed into irreducible factors which are either rank one or Euclidean (compare also [24, Corollary 2.6]); hence such CAT(0)-cube complexes without Euclidean factors constitute interesting examples for our setting.

As in the previous section we let

$$h = (h_1, h_2, \ldots, h_r) \quad \text{and} \quad g = (g_1, g_2, \ldots, g_r) \in \Gamma$$

be independent regular axial elements of $\mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r)$ and fix a base point $o = (o_1, o_2, \ldots, o_r) \in \mathrm{Ax}(h)$. Recall that $\widetilde{h^+}, \widetilde{h^-}, \widetilde{h^+}, \widetilde{h^-} \in \partial X^{\mathrm{reg}}$ are the attractive and repulsive fixed points, and $h^+, h^-, g^+, g^- \in \partial^F X$ their images by the Furstenberg projection π^F.

The following important theorem implies that F_Γ can be covered by finitely many Γ-translates of an appropriate open set in $\partial^F X$.

Theorem 5.1. *The Furstenberg limit set is minimal, i.e. F_Γ is the smallest non-empty, Γ-invariant closed subset of $\partial^F X$.*

Proof. We first show that every non-empty, Γ-invariant closed subset $A \subset \partial^F X$ contains $h^+ = (h_1^+, h_2^+, \dots, h_r^+)$. Indeed, if $\xi = (\xi_1, \xi_2, \dots, \xi_r) \in A$ is arbitrary, then according to Proposition 4.1 there exists $\lambda \in \Gamma$ such that $\lambda\xi \in \mathrm{Vis}^F(h^-)$. So $h^n\lambda\xi$ converges to h^+ as $n \to \infty$ and – since A is Γ-invariant and closed – h^+ belongs to A.

It remains to prove that $F_\Gamma = \overline{\Gamma \cdot h^+}$, so we let $\eta = (\eta_1, \eta_2, \dots, \eta_r) \in F_\Gamma$ arbitrary. Since $\eta \in F_\Gamma$, there exists a sequence $(\gamma_n) \subset \Gamma$ such that $\gamma_n o$ converges to a point $\tilde\eta \in L_\Gamma \cap \partial X^{\mathrm{reg}}$ with $\pi^F(\tilde\eta) = \eta$. Passing to a subsequence if necessary, we may assume that $\gamma_n^{-1} o$ converges to a point $\tilde\zeta \in L_\Gamma \cap \partial X^{\mathrm{reg}}$ and we set $\zeta := \pi^F(\tilde\zeta) \in F_\Gamma$.

We first treat the case $\zeta \in \mathrm{Vis}^F(h^+)$. Let $T \gg 1$ and $\varepsilon > 0$ be arbitrary. Then Lemma 2.2 implies the existence of $N \in \mathbb{N}$ such that for $i \in \{1, 2, \dots, r\}$ and all $n \geq N$ and $t \in [0, T]$ we have

$$d_i(\sigma_{o_i,\gamma_{n,i}o_i}(t), \sigma_{o_i,\gamma_{n,i}h_i^+}(t)) = d_i(\sigma_{\gamma_{n,i}^{-1}o_i,o_i}(t), \sigma_{\gamma_{n,i}^{-1}o_i,h_i^+}(t)) \leq \frac{\varepsilon}{2}.$$

Moreover, according to Lemma 3.2 the sequences $\gamma_{n,i}o_i$ converges to η_i for all $i \in \{1, 2, \dots, r\}$, so we also have

$$d_i(\sigma_{o_i,\gamma_{n,i}o_i}(t), \sigma_{o_i,\eta_i}(t)) \leq \frac{\varepsilon}{2}$$

for $t \in [0, T]$ and n sufficiently large. Hence we conclude that as $n \to \infty$ we have $\gamma_{n,i}h_i^+ \to \eta_i$, and therefore $\eta \in \overline{\Gamma \cdot h^+}$.

It remains to deal with the case $\zeta \notin \mathrm{Vis}^F(h^+)$. Applying Proposition 4.1 with h replaced by h^{-1} there exists $\mu \in \Gamma$ such that $\mu\zeta \in \mathrm{Vis}^F(h^+)$. Since $\gamma_n\mu^{-1}o$ still converges to $\tilde\eta$, and

$$(\gamma_n\mu^{-1})^{-1}o = \mu\gamma_n^{-1}o \longrightarrow \mu\zeta \in \mathrm{Vis}^F(h^+) \quad \text{as } n \to \infty,$$

after replacing the sequence γ_n by $\gamma_n\mu^{-1}$ we are in the first case. □

Theorem 5.2. *The regular geometric limit set $L_\Gamma \cap \partial X^{\mathrm{reg}}$ is isomorphic to the product $F_\Gamma \times P_\Gamma^{\mathrm{reg}}$.*

Proof. If $\tilde\xi \in L_\Gamma \cap \partial X^{\mathrm{reg}}$, then $\pi^F(\tilde\xi) \in F_\Gamma$, and by definition of P_Γ^{reg} the slope of $\tilde\xi$ belongs to P_Γ^{reg}.

Conversely, let us take $\eta = (\eta_1, \eta_2, \dots, \eta_r) \in F_\Gamma$ and $\theta \in P_\Gamma^{\mathrm{reg}}$. We have to show that there exists a limit point $\tilde\eta \in L_\Gamma \cap \partial X_\theta$ of slope θ with $\pi^F(\tilde\eta) = (\eta_1, \eta_2, \dots, \eta_r)$. By definition of P_Γ^{reg} and Lemma 3.1 there exists a sequence $(\gamma_n) \subset \Gamma$ such that $\theta^{(n)} := \widehat{H}(o, \gamma_n o)$ converges to θ as $n \to \infty$.

Moreover, by compactness of $\partial^F X = \partial X_1 \times \partial X_2 \times \cdots \times \partial X_r$ a subsequence of $\gamma_n o$ converges to a point $\tilde{\xi} \in L_\Gamma \cap \partial X_\theta$. We set $\xi := \pi^F(\tilde{\xi})$ and notice that $\tilde{\eta} \in \partial X^{\mathrm{reg}}$ is the unique point in $(\pi^F)^{-1}(\eta)$ of slope θ.

By Theorem 5.1 we have $\overline{\Gamma \cdot \xi} = F_\Gamma$ and therefore

$$\eta \in \overline{\Gamma \cdot \xi} = \pi^F(\overline{\Gamma \cdot \tilde{\xi}}).$$

Since the action of Γ on the geometric boundary does not change the slope of a point, we conclude that the closure of $\Gamma \cdot \tilde{\xi}$ contains $\tilde{\eta}$. In particular we get $\tilde{\eta} \in \overline{\Gamma \cdot \tilde{\xi}} \subset L_\Gamma$. □

Remark. Theorems 5.1 and 5.2 remain true under the weaker assumption that Γ contains a regular axial isometry h and that for any $\xi \in F_\Gamma$ and $\eta \in \{h^+, h^-\}$ there exists $\lambda \in \Gamma$ such that $\lambda\xi \in \mathrm{Vis}^F(\eta)$.

When only two factors are present, Lemma 4.1 in [21] shows that this condition is satisfied if Γ contains a regular axial isometry and if the projections of Γ to $\mathrm{Is}(X_1)$ and $\mathrm{Is}(X_2)$ do not globally fix a point in ∂X_1, ∂X_2 and possess infinitely many limit points.

In the sequel we will establish an important property of the action of Γ on the whole Furstenberg boundary $\partial^F X$, namely the fact that the Furstenberg limit set F_Γ is a so-called *boundary limit set* for the action of Γ on $\partial^F X$ (see [24, Definition 4.2] and also [16]): (1) clearly implies minimality of F_Γ and therefore Theorem 5.1; with the notions from [23, Chapter VI, (1.2)], (2) says that $\partial^F X$ and the action of Γ on $\partial^F X$ are *proximal*, and (3) states that every open set in $\partial^F X$ is *contractible* to a point in F_Γ.

Theorem 5.3. *The Γ-invariant subset $F_\Gamma \subset \partial^F X$ satisfies the following*:

(*1*) *for all $\zeta \in \partial^F X$ and every open subset $U \subset \partial^F X$ with $U \cap F_\Gamma \neq \emptyset$ there exists $\gamma \in \Gamma$ such that $\gamma\zeta \in U$*;

(*2*) *for all $\zeta, \eta \in \partial^F X$ there exists $\xi \in F_\Gamma$ such that for any neighborhood U of ξ there exists $\gamma \in \Gamma$ with $\{\gamma\zeta, \gamma\eta\} \subset U$*;

(*3*) *for all $\zeta \in \partial^F X$ there exists a neighborhood V of ζ and a point $\xi \in F_\Gamma$ such that for any neighborhood U of ξ there exists $\gamma \in \Gamma$ with $\gamma V \subset U$.*

Proof. In order to prove (1) we let $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_r) \in \partial^F X$ arbitrary and choose an open subset $U \subset \partial^F X$ with $U \cap F_\Gamma \neq \emptyset$. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_r) \in U \cap F_\Gamma$, and $U_i \subset \partial X_i$ open neighborhoods of ξ_i, $i = 1, 2, \ldots, r$, such that

$$U_1 \times U_2 \times \cdots \times U_r \subset U.$$

Since $\xi \in F_\Gamma$, there exists a sequence $(\gamma_n) \subset \Gamma$ such that $\gamma_n o$ converges to a point $\tilde{\xi} \in \partial X^{\mathrm{reg}}$ with $\pi^F(\tilde{\xi}) = \xi$. Moreover, passing to a subsequence if necessary we can assume that $\gamma_n^{-1}o$ converges to a point $\tilde\eta \in \partial X^{\mathrm{reg}}$, and we set $\eta = \pi^F(\tilde\eta)$. If $\eta \notin \mathrm{Vis}^F(h^+)$, Proposition 4.1 with h replaced by h^{-1} provides an element $\mu \in \Gamma$ such that $\mu\cdot\eta \in \mathrm{Vis}^F(h^+)$. Replacing the sequence γ_n by $\gamma_n\mu^{-1}$ if necessary we can therefore assume that $\eta \in \mathrm{Vis}^F(h^+)$. Now we conclude as in the proof of Theorem 5.1 that for all $i \in \{1,2,\ldots,r\}$ the sequence $\gamma_{n,i}h_i^+$ converges to ξ_i as $n \to \infty$; in particular, for some fixed and sufficiently large $n \in \mathbb{N}$ the regular axial isometry $\varphi = (\varphi_1,\varphi_2,\ldots,\varphi_n) := \gamma_n h\gamma_n^{-1}$ satisfies $\varphi^+ \in U_1\times U_2\times\cdots\times U_r \subset U$. According to Lemma 4.2 there exists $\alpha \in \Gamma$ such that $\alpha\zeta \in \mathrm{Vis}^F(\varphi^-)$. So $\varphi^n\alpha\zeta$ converges to φ^+ and hence belongs to U for all sufficiently large n.

Proposition 4.3 shows that (2) holds with $\xi = h^+$ for all $\zeta,\eta \in \partial^F X$; (3) follows from Proposition 4.1 (again with $\xi = h^+$). $\square$

The following theorem can be viewed as a strong topological version of the double ergodicity property of Poisson boundaries due to Burger-Monod ([8]) and Kaimanovich ([18]). For its proof we will need an important definition as a substitute for the more familiar notion of Γ-duality used e.g. in [3] and [10] when dealing with only one rank one Hadamard space.

Definition 5.4. Two points $\xi = (\xi_1,\xi_2,\ldots,\xi_r)$, $\eta = (\eta_1,\eta_2,\ldots,\eta_r) \in \partial^F X$ are called Γ-*related* if for all $i \in \{1,2,\ldots,r\}$ and all neighborhoods U_i of ξ_i and V_i of η_i in $\overline{X}_i$ there exists $\gamma = (\gamma_1,\gamma_2,\ldots,\gamma_r) \in \Gamma$ such that

$$\gamma_i(\overline{X}_i\setminus U_i) \subset V_i \quad \text{and } \gamma_i^{-1}(\overline{X}_i\setminus V_i) \subset U_i \quad \text{for all } i \in \{1,2,\ldots,r\}.$$

We will denote $\mathrm{Rel}_\Gamma(\xi)$ the set of points in $\partial^F X$ which are Γ-related to ξ.

Notice that for any $\xi \in \partial^F X$ the set $\mathrm{Rel}_\Gamma(\xi)$ is closed with respect to the topology of $\partial^F X$. Moreover, if $\eta \in \mathrm{Rel}_\Gamma(\xi)$, then η_i is Γ_i-dual to ξ_i for all $i \in \{1,2,\ldots,r\}$. The converse clearly does not hold in general.

The importance of the notion lies in the following. If $\widetilde{h^+}$, $\widetilde{h^-}$ denote the attractive and repulsive fixed point of a regular axial isometry $h \in \Gamma$, then by Lemma 2.5 (c) the points $h^+ = \pi^F(\widetilde{h^+})$ and $h^- = \pi^F(\widetilde{h^-})$ are Γ-related. Conversely, if two points $\xi = (\xi_1,\xi_2,\ldots,\xi_r)$, $\eta = (\eta_1,\eta_2,\ldots,\eta_r) \in \partial^F X$ are Γ-related, then by definition there exists a sequence

$$(\gamma_n) = ((\gamma_{n,1},\gamma_{n,2},\ldots,\gamma_{n,r})) \subset \Gamma$$

such that for all $i \in \{1,2,\ldots,r\}$ we have $\gamma_{n,i}o_i \to \eta_i$ and $\gamma_{n,i}^{-1}o_i \to \xi_i$ as $n \to \infty$. Hence if for $i \in \{1,2,\ldots,r\}$ the points ξ_i, $\eta_i \in \partial X_i$ can be joined by a rank one geodesic, then in view of Lemma 2.6 $\gamma_{n,i}$ is rank one for n sufficiently large and satisfies

$$\gamma_{n,i}^+ \longrightarrow \eta_i \quad \text{and} \quad \gamma_{n,i}^- \longrightarrow \xi_i \quad \text{as } n \to \infty.$$

For the sequel we denote $\Delta \subset \partial^F X \times \partial^F X$ the generalized diagonal

$$\Delta := \{(\xi, \eta) \in \partial^F X \times \partial^F X : \xi_i = \eta_i \text{ for some } i \in \{1, 2, \ldots, r\}\}.$$

With this notation we have the following

Theorem 5.5. *The set of pairs of fixed points $(\gamma^+, \gamma^-) \subset \partial^F X \times \partial^F X$ of regular axial isometries $\gamma \in \Gamma$ is dense in $(F_\Gamma \times F_\Gamma) \setminus \Delta$.*

Proof. Recall that $g = (g_1, g_2, \ldots, g_r)$, $h = (h_1, h_2, \ldots, h_r) \in \Gamma$ are two independent regular axial isometries. In view of the paragraph preceding the theorem we first prove that any two distinct points in $\{g^-, g^+, h^-, h^+\}$ are Γ-related.

For $i \in \{1, 2, \ldots, r\}$ and $\eta_i \in \{g_i^-, g_i^+, h_i^-, h_i^+\}$ we let $U_i(\eta_i) \subset \overline{X}_i$ be an arbitrary, sufficiently small neighborhood of η_i with $o_i \notin U_i(\eta_i)$ such that all $U_i(\eta_i)$ are pairwise disjoint. According to Lemma 2.5 (c) there exists a constant $N \in \mathbb{N}$ such that for all $i \in \{1, 2, \ldots, r\}$

$$g_i^{\pm N}(\overline{X}_i \setminus U_i(g_i^{\mp})) \subset U_i(g_i^{\pm}) \quad \text{and} \quad h_i^{\pm N}(\overline{X}_i \setminus U_i(h_i^{\mp})) \subset U_i(h_i^{\pm}). \tag{8}$$

Let $\gamma, \varphi \in \{g, g^{-1}, h, h^{-1}\}$, $\varphi \neq \gamma$. Using the fact that either $\varphi = \gamma^{-1}$ or γ_i, φ_i are independent for $i \in \{1, 2, \ldots, r\}$ property (8) implies

$$\gamma_i^N \varphi_i^{-N}(\overline{X}_i \setminus U_i(\varphi_i^+)) \subset U_i(\gamma_i^+) \quad \text{and} \quad (\gamma_i^N \varphi_i^{-N})^{-1}(\overline{X}_i \setminus U_i(\gamma_i^+)) \subset U_i(\varphi_i^+)$$

for $i \in \{1, 2, \ldots, r\}$. Hence $\varphi^+ \in \mathrm{Rel}_\Gamma(\gamma^+)$.

Next we show that any $\xi = (\xi_1, \xi_2, \ldots, \xi_r) \in F_\Gamma$ with $\xi_i \notin \{g_i^-, g_i^+, h_i^-, h_i^+\}$ for all $i \in \{1, 2, \ldots, r\}$ is Γ-related to an arbitrary point in $\{g^-, g^+, h^-, h^+\}$. For $i \in \{1, 2, \ldots, r\}$ and $\zeta_i \in \{\xi_i, g_i^-, g_i^+, h_i^-, h_i^+\}$ we let $U_i(\zeta_i) \subset \overline{X}_i$ be a sufficiently small neighborhood of ζ_i with $o_i \notin U_i(\zeta_i)$ such that all $U_i(\zeta_i)$ are pairwise disjoint. By Lemma 2.6 there exist neighborhoods

$$W_i(\zeta_i) \subset U_i(\zeta_i), \quad \zeta_i \in \{\xi_i, g_i^-, g_i^+, h_i^-, h_i^+\},$$

such that every $\gamma_i \in \Gamma_i$ with

$$\gamma_i o_i \in W_i(\zeta_i), \quad \gamma_i^{-1} o_i \in W_i(\eta_i), \quad \eta_i \in \{\xi_i, g_i^-, g_i^+, h_i^-, h_i^+\} \setminus \{\zeta_i\},$$

is rank one with $\gamma_i^+ \in U_i(\zeta_i)$ and $\gamma_i^- \in U_i(\eta_i)$.

Since $\xi \in F_\Gamma$, there exists a sequence

$$(\gamma_n) = ((\gamma_{n,1}, \gamma_{n,2}, \ldots, \gamma_{n,r})) \subset \Gamma$$

such that $\gamma_{n,i} o_i \to \xi_i$ for all $i \in \{1, 2, \ldots, r\}$. Upon passing to a subsequence if necessary we may assume that $\gamma_{n,i}^{-1} o_i$ converges to a point in ∂X_i for all $i \in \{1, 2, \ldots, r\}$. By Proposition 4.1 there exist a finite set $\Lambda \subset \Gamma$ and $\mu \in \Lambda$ such that for all n sufficiently large we have

$$\mu \gamma_n^{-1} o \in W_1(h_1^-) \times W_2(h_2^-) \times \cdots \times W_r(h_r^-).$$

Moreover, since for $i \in \{1, 2, \ldots, r\}$ and $x_i \in X_i$ the sequence $(\gamma_{n,i} x_i)$ converges to ξ_i, we also have

$$\gamma_n \mu^{-1} o \in W_1(\xi_1) \times W_2(\xi_2) \times \cdots \times W_r(\xi_r)$$

for n sufficiently large. By Lemma 2.6 we conclude that for n sufficiently large the isometry $\gamma_n \mu^{-1}$ is regular axial with

$$(\gamma_n \mu^{-1})^+ \in U_1(\xi_1) \times U_2(\xi_2) \times \cdots \times U_r(\xi_r)$$

and

$$(\gamma_n \mu^{-1})^- \in U_1(h_1^-) \times U_2(h_2^-) \times \cdots \times U_r(h_r^-).$$

This implies that $\xi \in \mathrm{Rel}_\Gamma(h^-)$ and by symmetry

$$\xi \in \mathrm{Rel}_\Gamma(g^-) \cap \mathrm{Rel}_\Gamma(g^+) \cap \mathrm{Rel}_\Gamma(h^-) \cap \mathrm{Rel}_\Gamma(h^+). \tag{9}$$

Next we let $\xi = (\xi_1, \xi_2, \ldots, \xi_r)$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_r) \in F_\Gamma$ such that for all $i \in \{1, 2, \ldots, r\}$ we have $\{\xi_i, \eta_i\} \cap \{g_i^-, g_i^+, h_i^-, h_i^+\} = \emptyset$ and $\xi_i \neq \eta_i$. As above, for $\zeta_i \in \{\xi_i, \eta_i, h_i^-\}$ we let $U_i(\zeta_i) \subset \overline{X}_i$ be a small neighborhood of ζ_i with $o_i \notin U_i(\zeta_i)$ such that all $U_i(\zeta_i)$ are pairwise disjoint. By the arguments in the previous paragraph there exists a regular axial isometry $\varphi \in \Gamma$ with

$$\varphi^+ \in U_1(\xi_1) \times U_2(\xi_2) \times \cdots \times U_r(\xi_r)$$

and

$$\varphi^- \in U_1(h_1^-) \times U_2(h_2^-) \times \cdots \times U_r(h_r^-).$$

In particular, φ_i and g_i are independent for $i = 1, 2, \ldots, r$. Replacing h by φ in (9) we know that $\eta \in \mathrm{Rel}_\Gamma(g^-) \cap \mathrm{Rel}_\Gamma(g^+) \cap \mathrm{Rel}_\Gamma(\varphi^-) \cap \mathrm{Rel}_\Gamma(\varphi^+)$, in particular $\eta \in \mathrm{Rel}_\Gamma(\varphi^+)$. So using the fact that η_i can be joined to φ_i^+ by a rank one geodesic in X_i for all $i \in \{1, 2, \ldots, r\}$, given small neighborhoods $U_i(\varphi_i^+) \subset U_i(\xi_i)$ there exists $\gamma \in \Gamma$ regular axial with

$$\gamma^+ \in U_1(\varphi_1^+) \times U_2(\varphi_2^+) \times \cdots \times U_r(\varphi_r^+) \subset U_1(\xi_1) \times U_2(\xi_2) \times \cdots \times U_r(\xi_r)$$

and

$$\gamma^- \in U_1(\eta_1) \times U_2(\eta_2) \times \cdots \times U_r(\eta_r). \qquad \square$$

6. The limit cone

Given a discrete group $\Gamma < \mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r)$, the *limit cone* ℓ_Γ of Γ is defined as the closure of the set of half-lines in $\mathbb{R}^r_{\geq 0}$ spanned by the set of vectors

$$\{L(\gamma) \in \mathbb{R}^r_{>0} : \gamma \in \Gamma \ \text{ regular axial}\}.$$

Notice that this definition differs from the one given by Y. Benoist in [5], where the translation vectors of *all* elements in Γ are considered. However, Y. Benoist showed that in the case of reductive groups one can equivalently use only the translation vectors of $\mathbb{R}$-regular elements in the definition of the limit cone; so our definition can be viewed as an appropriate analogous one.

As before we assume that Γ contains a pair of independent regular axial isometries $g = (g_1, g_2, \dots, g_r), h = (h_1, h_2, \dots, h_r)$ and fix a base point $o = (o_1, o_2, \dots, o_r) \in \mathrm{Ax}(h)$. The following theorem shows that the limit cone is closely related to the set P_Γ introduced at the beginning of Section 5.

Proposition 6.1. *We have the follwing inclusions*:

$$\ell_\Gamma \cap E \subset P_\Gamma \quad \textit{and} \quad P_\Gamma^{\mathrm{reg}} \subset \ell_\Gamma \cap E^+.$$

Proof. We first show $\ell_\Gamma \cap E \subset P_\Gamma$. If $(\gamma_n) = ((\gamma_{n,1}, \gamma_{n,2}, \dots, \gamma_{n,r}))$ is a sequence of regular axial isometries such that $\widehat{L}(\gamma_n)$ converges to $\theta = (\theta_1, \theta_2, \dots, \theta_r) \in E$, we choose

$$k_n \geq 2n \max\left\{\frac{d(o, \mathrm{Ax}(\gamma_n))}{l(\gamma_n)}, \frac{d_i(o_i, \mathrm{Ax}(\gamma_{n,i}))}{l_i(\gamma_n)} : i \in \{1, 2, \dots, r\}\right\}$$

and set $\varphi_n := \gamma_n^{k_n}$. From

$$\begin{aligned} k_n l_i(\gamma_n) &\leq d_i(o_i, \varphi_{n,i} o_i) \\ &\leq k_n l_i(\gamma_n) + 2 d_i(o_i, \mathrm{Ax}(\gamma_{n,i})) \\ &\leq k_n l_i(\gamma_n)\Big(1 + \frac{1}{n}\Big) \end{aligned}$$

for $i \in \{1, 2, \dots, r\}$ and

$$k_n l(\gamma_n) \leq d(o, \varphi_n o) \leq k_n l(\gamma_n) + 2d(o, \mathrm{Ax}(\gamma_n)) \leq k_n l(\gamma_n)\Big(1 + \frac{1}{n}\Big),$$

we conclude that (by definition of $\widehat{L}(\gamma_n)$)

$$\theta_i = \lim_{n\to\infty}\left(\frac{l_i(\gamma_n)}{l(\gamma_n)}\right) = \lim_{n\to\infty}\left(\frac{l_i(\gamma_n)\cdot\big(1 + \frac{1}{n}\big)}{l(\gamma_n)}\right) \geq \lim_{n\to\infty}\frac{d_i(o_i, \varphi_{n,i} o_i)}{d(o, \varphi_n o)}$$

and

$$\theta_i = \lim_{n\to\infty}\left(\frac{l_i(\gamma_n)}{l(\gamma_n)\cdot\big(1 + \frac{1}{n}\big)}\right) \leq \lim_{n\to\infty}\frac{d_i(o_i, \varphi_{n,i} o_i)}{d(o, \varphi_n o)}, \quad i = 1, 2, \dots, r,$$

which shows that $\widehat{H}(o, \varphi_n o)$ converges to θ as $n \to \infty$.

Next we prove the inclusion $P_\Gamma^{\mathrm{reg}} \subset \ell_\Gamma \cap E^+$. If $\theta \in P_\Gamma^{\mathrm{reg}}$, then by definition of P_Γ^{reg} there exists a point $\tilde{\xi} \in L_\Gamma \cap \partial X_\theta \subset \partial X^{\mathrm{reg}}$; in particular, there exists a sequence $(\gamma_n) = ((\gamma_{n,1}, \gamma_{n_2}, \ldots, \gamma_{n,r})) \subset \Gamma$ such that $\gamma_n o$ converges to $\tilde{\xi}$ and hence necessarily the sequence of directions $\widehat{H}(o, \gamma_n o)$ converges to $\theta = (\theta_1, \theta_2, \ldots, \theta_r)$ as $n \to \infty$. Passing to a subsequence if necessary we can assume that $\gamma_n^{-1} o$ converges to a point $\tilde{\zeta} \in \partial X$ (which necessarily also belongs to $\partial X_\theta \subset \partial X^{\mathrm{reg}}$) as $n \to \infty$. For $i \in \{1, 2, \ldots, r\}$ we let $V_i(h_i^\pm) \subset \overline{X}_i$ be neighborhoods of $h_i^\pm$ and $c_i > 0$ such that the assertion of Proposition 2.7 holds in the factor X_i. By Corollary 4.5 there exist a finite set $\Lambda \subset \Gamma$, λ, $\mu \in \Lambda$ and $N_0 \in \mathbb{N}$ such that for all $n > N_0$ the isometries

$$\varphi_n := \lambda \gamma_n \mu^{-1}$$

are regular axial with

$$\varphi_n^\pm \in V_1(h_1^\pm) \times V_2(h_2^\pm) \times \cdots \times V_r(h_r^\pm).$$

Put $c := \max\{c_i : i \in \{1, 2, \ldots, r\}\}$,

$$d := \max\{d_i(o_i, \lambda_i o_i) : \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \Lambda,\ i \in \{1, 2, \ldots, r\}\}$$

and fix $n > N_0$. Writing $\varphi_n = (\varphi_{n,1}, \varphi_{n_2}, \ldots, \varphi_{n,r})$ the triangle inequality implies

$$|d_i(o_i, \varphi_{n,i} o_i) - d_i(o_i, \gamma_{n,i} o_i)| \le 2d \tag{10}$$

for all $i \in \{1, 2, \ldots, r\}$; clearly this also gives

$$|d(o, \varphi_n o) - d(o, \gamma_n o)| \le 2d\sqrt{r}.$$

Moreover, by Proposition 2.7 (a) we have $d_i(o_i, \mathrm{Ax}(\varphi_{n,i})) \le c$, hence

$$l_i(\varphi_n) \le d_i(o_i, \varphi_{n,i} o_i) \le l_i(\varphi_n) + 2c \quad \text{for } i \in \{1, 2, \ldots, r\}$$

and

$$l(\varphi_n) \le d(o, \varphi_n o) \le l(\varphi_n) + 2c\sqrt{r}.$$

In combination with (10) we get

$$\theta_i = \lim_{n \to \infty} \frac{l_i(\varphi_n)}{l(\varphi_n)} \quad \text{for } i \in \{1, 2, \ldots, r\},$$

and therefore

$$\theta = \lim_{n \to \infty} \widehat{L}(\varphi_n). \qquad \square$$

Notice that Proposition 6.1 in particular implies $P_\Gamma^{\mathrm{reg}} = \ell_\Gamma \cap E^+$, which was proved in [21, Theorem 5.2] for the special case of only two factors. We also want to make the following

Remark. Our proof does not give the stronger statement $P_\Gamma = \ell_\Gamma \cap E$ of Theorem D. This is due to the fact that a singular limit point can be approached by orbit points $\gamma_n o$ for which the projections to one of the factors X_i remain at bounded distance of o_i. However, if $\theta \in P_\Gamma \setminus P_\Gamma^{\mathrm{reg}}$ and at least *one* point $\tilde\xi \in \partial X_\theta \subset \partial X^{\mathrm{sing}}$ is the limit of a sequence $\gamma_n o$ for which the projections to *all* factors leave every bounded set, then our proof of the second inclusion together with Corollary 4.5 shows that $\tilde\xi$ is also the limit of a sequence of regular axial elements of Γ and hence $\theta \in \ell_\Gamma$. This is already remarkable because in the original sequence (γ_n) the projections could be parabolic or elliptic of infinite order. Using the exponent of growth in Section 7 we will be able to complete the proof of the full statement of Theorem D.

Our next goal is to describe the limit cone more precisely. The following easy lemma will be useful for proving convexity as stated in Proposition 6.4.

Lemma 6.2. *If $\alpha, \beta \in \Gamma$ are independent regular axial elements with $L(\alpha)$, $L(\beta) \in \ell_\Gamma$, then the convex hull of the half-lines determined by $L(\alpha)$ and $L(\beta)$ is contained in ℓ_Γ.*

Proof. Since $\alpha, \beta \in \Gamma$ are independent regular axial isometries, Propositon 2.7 (b) ensures the existence of $c > 0$ and $N \in \mathbb{N}$ such that for all $i \in \{1, 2, \dots, r\}$ and $k, m \in \mathbb{N}$ we have

$$|l_i(\alpha^{kN}\beta^{mN}) - kNl_i(\alpha) - mNl_i(\beta)| \le 4c \cdot 2 = 8c,$$

which immediately implies

$$\lim_{n\to\infty} \frac{L(\alpha^{knN}\beta^{mnN})}{nN} = kL(\alpha) + mL(\beta).$$

Since ℓ_Γ is closed and $L(\alpha^{knN}\beta^{mnN}) \in \ell_\Gamma$ for all $k, m, n \in \mathbb{N}$ we conclude that for any positive rational number $q \in \mathbb{Q}$ the half-line determined by $L(\alpha) + qL(\beta)$ belongs to ℓ_Γ. So the convex hull of the half-lines determined by $L(\alpha)$ and $L(\beta)$ is included in ℓ_Γ, which we wanted to prove. □

The following proposition is the appropriate analogon of Proposition 2.2.7 in [26] (compare also Proposition 5.1 in [5]) for our setting.

Proposition 6.3. *There is a constant $\kappa > 0$ such that for every open cone $\mathcal{C} \subset \mathbb{R}^r_{>0}$ with $\mathcal{C} \cap \ell_\Gamma \neq \emptyset$ there exist independent regular axial isometries $\alpha, \beta \in \Gamma$ with $L(\alpha)$, $L(\beta) \in \mathcal{C}$ such that the semi-group $\langle \alpha, \beta \rangle^+ \subset \Gamma$ is free.*

Moreover, if $\Phi: \langle\alpha,\beta\rangle^+ \to \mathbb{R}^r$ *is the unique homomorphism of semi-groups sending* α *to* $L(\alpha)$ *and* β *to* $L(\beta)$*, then for any word* $\gamma \in \langle\alpha,\beta\rangle^+$ *of length* $n \geq 1$ *in the generators* α, β *one has*

$$H(o,\gamma o) \in \mathcal{C} \quad \textit{and} \quad \|H(o,\gamma o) - \Phi(\gamma)\| \leq \kappa \cdot n.$$

Proof. For $i \in \{1,2,\dots,r\}$ we fix neighborhoods $V_i(g_i^\pm)$, $V_i(h_i^\pm) \subset \overline{X}_i$ and $c_i > 0$ such that the assertion of Proposition 2.7 holds in the factor X_i. We further set $c := \max\{c_i : i \in \{1,2,\dots,r\}\}$.

Since $\mathcal{C} \cap \ell_\Gamma \neq \emptyset$ there exist $\alpha', \beta' \in \Gamma$ regular axial with $L(\alpha'), L(\beta') \in \mathcal{C}$. Proposition 4.6 implies that there exist regular axial isometries

$$\alpha = (\alpha_1,\alpha_2,\dots,\alpha_r),\ \beta = (\beta_1,\beta_2,\dots,\beta_r) \in \Gamma$$

with $L(\alpha), L(\beta) \in \mathcal{C}$,

$$\alpha_i^\pm \in V_i(g_i^\pm) \quad \text{and} \quad \beta_i^\pm \in V_i(h_i^\pm) \quad \text{for all } i \in \{1,2,\dots,r\}.$$

Obviously every non-trivial linear combination of $L(\alpha)$, $L(\beta) \in \mathbb{R}^r_{>0}$ with non-negative coefficients is included in the sector $\mathcal{S} \subset \mathbb{R}^r_{>0}$ spanned by $L(\alpha)$ and $L(\beta)$; in particular, for any $\gamma \in \langle\alpha,\beta\rangle^+ \setminus \{\mathrm{id}\}$ we have $\Phi(\gamma) \in \mathcal{S}$. Since $\mathcal{C}$ is an open cone containing $L(\alpha)$ and $L(\beta)$, the whole sector $\mathcal{S}$ is included in $\mathcal{C}$ and there exists $\varepsilon > 0$ such that every unit vector $\widehat{H} \in \mathbb{R}^r_{>0}$ with $\|\widehat{H} - \widehat{L}\| < \varepsilon$ for some unit vector $\widehat{L} \in \mathcal{S}$ is contained in $\mathcal{C}$.

Replacing α and β by a sufficiently high power if necessary we can assume that

$$\min\{d(o,\alpha o), d(o,\beta o)\} > 8c\sqrt{r}\Big(1 + \frac{1}{\varepsilon}\Big) \tag{11}$$

and that the assertion of Proposition 2.7 holds in each factor X_i, $i \in \{1,2,\dots,r\}$, with $N_{\alpha_i} = N_{\beta_i} = 1$. So let $\gamma = s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n}$ be a word in the semi-group $\langle\alpha,\beta\rangle^+$ with $s_j \in \{\alpha,\beta\}$ and $k_j \in \mathbb{N}$, $j \in \{1,2,\dots,n\}$; the word length of γ then clearly satisfies

$$\sum_{j=1}^n k_j \geq n$$

and we have

$$\Phi(\gamma) = \sum_{j=1}^n k_j L(s_j).$$

Proposition 2.7 (b) further shows that

$$\Big| d_i(o_i,\gamma_i o_i) - \sum_{j=1}^n k_j l_i(s_j) \Big| \leq 4c \cdot n$$

for all $i \in \{1, 2, \ldots, r\}$, hence

$$\|H(\gamma) - \Phi(\gamma)\|^2 = \sum_{i=1}^{r} \Big(\Big| d_i(o_i, \gamma_i o_i) - \sum_{j=1}^{n} k_j l_i(s_j) \Big|^2 \Big) \leq r \cdot (4c \cdot n)^2.$$

So the second assertion is true with $\kappa := 4c\sqrt{r}$.

Concerning the first assertion we remark that the proof of Proposition 2.7 (b) implies that

$$d(o, \gamma o) \geq \sum_{j=1}^{n} d(o, s_j^{k_j} o) - 2c\sqrt{r} \cdot n,$$

hence by (11)

$$d(o, \gamma o) \geq n \cdot 8c\sqrt{r}\Big(1 + \frac{1}{\varepsilon}\Big) - 2c\sqrt{r} \cdot n > \frac{8c\sqrt{r}}{\varepsilon} \cdot n = \frac{2\kappa \cdot n}{\varepsilon}.$$

So we estimate

$$\begin{aligned}
\Big\| \frac{H(o, \gamma o)}{d(o, \gamma o)} - \frac{\Phi(\gamma)}{\|\Phi(\gamma)\|} \Big\| &\leq \frac{1}{d(o, \gamma o)} \|H(o, \gamma o) - \Phi(\gamma)\| + \Big\| \frac{\Phi(\gamma)}{d(o, \gamma o)} - \frac{\Phi(\gamma)}{\|\Phi(\gamma)\|} \Big\| \\
&\leq \frac{1}{d(o, \gamma o)} \cdot \kappa \cdot n + \|\Phi(\gamma)\| \cdot \Big| \frac{1}{d(o, \gamma o)} - \frac{1}{\|\Phi(\gamma)\|} \Big| \\
&\leq \frac{\kappa \cdot n}{d(o, \gamma o)} + \frac{1}{d(o, \gamma o)} \big| \|\Phi(\gamma)\| - d(o, \gamma o) \big| \\
&\leq \frac{2\kappa \cdot n}{d(o, \gamma o)} \\
&< \varepsilon,
\end{aligned}$$

where we used the inverse triangle inequality

$$\big| \|\Phi(\gamma)\| - \|H(o, \gamma o)\| \big| \leq \|\Phi(\gamma) - H(o, \gamma o)\| \leq \kappa \cdot n.$$

So $\Phi(\gamma) \in \mathcal{S}$ and the choice of $\varepsilon > 0$ imply that $H(o, \gamma o) \in \mathcal{C}$. □

Proposition 6.4. *The limit cone ℓ_Γ is convex.*

Proof. Let $L, L' \in \ell_\Gamma$. Using Proposition 4.6 and arguments as in the proof of the previous proposition there exist two independent regular axial isometries $\alpha, \beta \in \Gamma$ with $L(\alpha)$ and $L(\beta)$ arbitrarily close to the half-lines determined by L and L'.

From Lemma 6.2 we know that the convex hull of the half-lines determined by $L(\alpha)$ and $L(\beta)$ is contained in ℓ_Γ. Since ℓ_Γ is closed, the same is true for the convex hull of L and L', which finishes the proof. □

We finally state a result concerning free subgroups of Γ which is a corollary of Proposition 4.6 and the proof of Proposition 6.3.

Proposition 6.5. *For every open cone $\mathcal{C} \subset \mathbb{R}^r_{>0}$ with $\mathcal{C} \cap \ell_\Gamma \neq \emptyset$ there exists a free subgroup $\Gamma' < \Gamma$ such that*

$$\ell_{\Gamma'} \subset \mathcal{C}.$$

Proof. As in the proof of Proposition 6.3 and due to Proposition 2.7 (b) there exist independent regular axial isometries $\alpha, \beta \in \Gamma$ with $L(\alpha), L(\beta) \in \mathcal{C}$, and a constant $c > 0$ such that for every cyclically reduced word $\gamma = s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n}$ with $s_j \in \{\alpha, \alpha^{-1}, \beta, \beta^{-1}\}$ and $k_j \in \mathbb{N}$, $j \in \{1, 2, \ldots, n\}$ we have

$$\Big\| L(\gamma) - \sum_{j=1}^{n} k_j L(s_j) \Big\| \leq 4c\sqrt{r} \cdot n.$$

Since $L(\alpha^{-1}) = L(\alpha)$ and $L(\beta^{-1}) = L(\beta)$, this shows that $L(\gamma)$ is at distance $\leq 4c\sqrt{r} \cdot n$ of a non-trivial linear combination of $L(\alpha)$, $L(\beta) \in \mathbb{R}^r_{>0}$ with non-negative coefficients (which is included in the sector $\mathcal{S} \subset \mathcal{C}$ spanned by $L(\alpha)$ and $L(\beta)$). Passing to powers of α and β if necessary we can arrange (as in the proof of Proposition 6.3) that for every cyclically reduced word $\gamma \in \langle \alpha, \beta \rangle$ the translation vector $L(\gamma)$ is arbitrarily close to $\mathcal{S}$ and hence also included in $\mathcal{C}$. This finishes the proof, because every element in $\Gamma' := \langle \alpha, \beta \rangle$ is conjugated to a cyclically reduced element as above, and the translation vector is invariant by conjugation. $\square$

7. The exponent of growth for a given slope

For the remainder of the article we let

$$\Gamma < \mathrm{Is}(X_1) \times \mathrm{Is}(X_2) \times \cdots \times \mathrm{Is}(X_r)$$

be a group acting properly discontinuously by isometries on a product

$$X = X_1 \times X_2 \times \cdots \times X_r$$

of r locally compact Hadamard spaces which contains two independent regular axial isometries $h = (h_1, h_2, \ldots, h_r)$ and $g = (g_1, g_2, \ldots, g_r)$. We also fix a base point $o = (o_1, o_2, \ldots, o_r) \in \mathrm{Ax}(h)$.

Let $x, y \in X$ be arbitrary. The *critical exponent* $\delta(\Gamma)$ of Γ is defined as the exponent of convergence of the Poincaré series

$$P^s(x, y) = \sum_{\gamma \in \Gamma} \mathrm{e}^{-sd(x, \gamma y)},$$

i.e. the unique real number such that $P^s(x,y)$ converges for $s > \delta(\Gamma)$ and diverges for $s < \delta(\Gamma)$. By the triangle inequality for the distance function this number is independent of x and y; it is an important invariant of the group Γ. One goal of this section is to get a refinement of this invariant in the particular case of a product space.

Recall the notation introduced in Section 3; in particular, we denote $E \subset \mathbb{R}^r$ the set of unit vectors in $\mathbb{R}^r_{\geq 0}$. The desired refinement of the critical exponent $\delta(\Gamma)$ consists of a map which assigns to each vector $\theta \in E$ the exponential growth rate $\delta_\theta(\Gamma)$ of orbit points of Γ in X with prescribed slope θ. We now fix $\theta \in E$; in order to define $\delta_\theta(\Gamma)$ we first set for $x, y \in X$ and $\varepsilon > 0$

$$\Gamma(x, y; \theta, \varepsilon) := \{\gamma \in \Gamma : \gamma y \neq x \text{ and } \|\widehat{H}(x, \gamma y) - \theta\| < \varepsilon\}.$$

We then introduce a partial sum of the Poincaré series for Γ via

$$Q_\theta^{s,\varepsilon}(x,y) = \sum_{\gamma \in \Gamma(x,y;\theta,\varepsilon)} \mathrm{e}^{-sd(x,\gamma y)}$$

and denote $\delta_\theta^\varepsilon(x,y)$ its exponent of convergence. If $Q_\theta^{s,\varepsilon}(x,y)$ converges for all $s \in \mathbb{R}$, we set $\delta_\theta^\varepsilon(x,y) = -\infty$. Unfortunately, since the summation is only over a subset of Γ, this number may depend on x and y. However, it follows from $Q_\theta^{s,\varepsilon}(x,y) \leq P^s(x,y)$ that $\delta_\theta^\varepsilon(x,y) \leq \delta(\Gamma)$ for all $\varepsilon > 0$. If $\varepsilon > \sqrt{2}$, then $\Gamma(x, y; \theta, \varepsilon) = \{\gamma \in \Gamma : \gamma y \neq x\}$; by discreteness of Γ we have $\gamma y \neq x$ for only finitely many $\gamma \in \Gamma$, hence in this case $\delta_\theta^\varepsilon(x,y) = \delta(\Gamma)$.

An easy calculation shows that using for $n \gg 1$ the definitions

$$N_\theta^\varepsilon(x, y; n) := \#\{\gamma \in \Gamma : \gamma y \neq x,\ d(x,\gamma y) < n,\ \|\widehat{H}(x,\gamma y) - \theta\| < \varepsilon\}$$

and

$$\Delta N_\theta^\varepsilon(x, y; n) := \#\{\gamma \in \Gamma : n-1 \leq d(x,\gamma y) < n,\ \|\widehat{H}(x,\gamma y) - \theta\| < \varepsilon\}$$

we have

$$\delta_\theta^\varepsilon(x,y) = \limsup_{n\to\infty} \frac{\ln N_\theta^\varepsilon(x,y;n)}{n} = \limsup_{n\to\infty} \frac{\ln \Delta N_\theta^\varepsilon(x,y;n)}{n}. \tag{12}$$

Definition 7.1. The number $\delta_\theta(\Gamma) := \lim_{\varepsilon\to 0} \delta_\theta^\varepsilon(o,o)$ is called the *exponent of growth of* Γ *of slope* θ.

Notice that the exponent of growth does not depend on the choice of arguments of $\delta_\theta^\varepsilon$ by elementary geometric estimates. Moreover, at first sight this definition for $\delta_\theta(\Gamma)$ seems to be different from the one given in the introduction using $N_\theta^\varepsilon(n)$; however, since for any unit vector $\theta \in \mathbb{R}^r_{\geq 0}$, all $\varepsilon > 0$ and $n \in \mathbb{N}$ one has

$$N_\theta^\varepsilon(x,x;n) \leq N_\theta^\varepsilon(n) \leq N_\theta^{\varepsilon\sqrt{r}}(x,x;n)$$

the definitions obviously are equivalent.

Before we state properties of the exponent of growth, we illustrate the notion by means of an important example.

Example. Suppose X is a product $X = X_1 \times X_2 \times \cdots \times X_r$ of Hadamard manifolds with pinched negative curvature, and $\Gamma_i < \mathrm{Is}(X_i)$ is a convex cocompact group with critical exponent $\delta_i > 0$, $i \in \{1, 2, \ldots, r\}$. The product group $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_r$ then acts on the product manifold X, and for any unit vector $\theta \in \mathbb{R}^r_{\geq 0}$ with coordinates $\theta_1, \theta_2, \ldots, \theta_r \geq 0$ we have

$$\delta_\theta(\Gamma) = \sum_{i=1}^{r} \delta_i \theta_i .$$

Proof. By Theorem 6.2.5 in [28] (compare also [19]) there exists constants $c_1, c_2, \ldots, c_r > 0$ such that for all $i \in \{1, 2, \ldots, r\}$ one has

$$\#\{\gamma_i \in \Gamma_i : d_i(o_i, \gamma_i o_i) < R\} \sim c_i \mathrm{e}^{\delta_i R}. \tag{13}$$

Given $\theta \in E$, we estimate for $\varepsilon > 0$ sufficiently small and $n > \sqrt{r}/\varepsilon$ the number of orbit points

$$\begin{aligned}\Delta N^\varepsilon_\theta(o, o; n) = \#\{\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_r) \in \Gamma \colon \|\widehat{H}(o, \gamma o) - \theta\| < \varepsilon, \\ n - 1 \leq \mathfrak{Q} < n\},\end{aligned}$$

where

$$\mathfrak{Q} = \sqrt{d_1(o_1, \gamma_1 o_1)^2 + d_2(o_2, \gamma_2 o_2)^2 + \cdots + d_r(o_r, \gamma_r o_r)^2}.$$

We first notice that if $\theta_i(o, \gamma o) \in [0, 1]$, $i \in \{1, 2, \ldots, r\}$, denote the coordinates of $\widehat{H}(o, \gamma o)$, then

$$d_i(o_i, \gamma_i o_i) = d(o, \gamma o) \cdot \theta_i(o, \gamma o) \quad \text{for all } i \in \{1, 2, \ldots, r\}.$$

Moreover, $\|\widehat{H}(o, \gamma o) - \theta\| < \varepsilon$ implies $|\theta_i(o, \gamma o) - \theta_i| < \varepsilon$ for all $i \in \{1, 2, \ldots, r\}$. So in particular

$$\begin{aligned}\Delta N^\varepsilon_\theta(o, o; n) &\leq \#\{\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_r) \in \Gamma \colon |\theta_i(o, \gamma o) - \theta_i| < \varepsilon, \\ &\qquad d_i(o_i, \gamma_i o_i) < n \cdot \theta_i(o, \gamma o), \\ &\qquad \text{for all } i \in \{1, 2, \ldots, r\}\} \\ &\leq \prod_{i=1}^{r} \#\{\gamma_i \in \Gamma_i : d_i(o_i, \gamma_i o_i) < n(\theta_i + \varepsilon)\}.\end{aligned}$$

Setting $C := 2 \max\{c_1, c_2, \ldots, c_r\}$ we conclude from (13)

$$\Delta N^\varepsilon_\theta(o, o; n) \leq C^r \mathrm{e}^{n((\theta_1+\varepsilon)\delta_1 + (\theta_2+\varepsilon)\delta_2 + \cdots + (\theta_r+\varepsilon)\delta_r)}$$

for n sufficiently large.

For the lower bound we first denote $I^+ \subset \{1, 2, \dots, r\}$ the set of indices i such that $\theta_i > 0$, and $I^0 = \{1, 2, \dots, r\} \backslash I^+$. Note that if $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_r) \in \Gamma$ satisfies

$$n - 1 \le \frac{d_i(o_i, \gamma_i o_i)}{\theta_i} < n \quad \text{for all } i \in I^+$$

and $\gamma_i = \mathrm{id}$ for $i \in I^0$, then $n - 1 \le d(o, \gamma o) < n$ and – since $\theta_i \in (0, 1]$ for all $i \in I^+$ – we get

$$\theta_i - \frac{1}{n} \le \theta_i \cdot \frac{n-1}{n} < \frac{d_i(o_i, \gamma_i o_i)}{d(o, \gamma o)} < \theta_i \cdot \frac{n}{n-1} \le \theta_i + \frac{1}{n-1} \quad \text{for all } i \in I^+.$$

If $n > \sqrt{r}/\varepsilon$ this implies

$$|\theta_i(o, \gamma o) - \theta_i| < \frac{1}{n} < \frac{\varepsilon}{\sqrt{r}} \quad \text{for all } i \in I^+;$$

since $d_i(o_i, \gamma_i o_i) = 0 = \theta_i$ for all $i \in I^0$ we obtain $\|\widehat{H}(o, \gamma o) - \theta\| < \varepsilon$. So we conclude that for $n > \sqrt{r}/\varepsilon$ sufficiently large

$$\begin{aligned} \Delta N_\theta^\varepsilon(o, o; n) &\ge \#\{(\gamma_1, \gamma_2, \dots, \gamma_r) \in \Gamma \colon n - 1 \le \frac{d_i(o_i, \gamma_i o_i)}{\theta_i} < n \text{ for } i \in I^+, \\ &\qquad\qquad \gamma_i = \mathrm{id} \ \text{ for } i \in I^0\} \\ &= \prod_{i \in I^+} \#\{\gamma_i \in \Gamma_i \colon (n-1)\theta_i \le d_i(o_i, \gamma_i o_i) < n\theta_i\}. \end{aligned}$$

For $i \in I^+$ and $n \in \mathbb{N}$ we abbreviate

$$N_i(n) := \#\{\gamma_i \in \Gamma_i \colon d_i(o_i, \gamma_i o_i) < n\theta_i\}$$

and

$$\Delta N_i(n) := \#\{\gamma_i \in \Gamma_i \colon (n-1)\theta_i \le d_i(o_i, \gamma_i o_i) < n\theta_i\} = N_i(n) - N_i(n-1).$$

From (13) we get for $i \in I^+$

$$\begin{aligned} \lim_{n \to \infty} \Delta N_i(n) \mathrm{e}^{-n\theta_i \delta_i} &= \lim_{n \to \infty} N_i(n) \mathrm{e}^{-n\theta_i \delta_i} - \lim_{n \to \infty} N_i(n-1) \mathrm{e}^{-(n-1)\theta_i \delta_i} \mathrm{e}^{-\theta_i \delta_i} \\ &= c_i - c_i \mathrm{e}^{-\theta_i \delta_i} \\ &= c_i (1 - \mathrm{e}^{-\theta_i \delta_i}) \\ &> 0 \end{aligned}$$

since $\theta_i \delta_i > 0$. Setting $D := \min\{1, c_i(1 - \mathrm{e}^{-\theta_i \delta_i})/2 \colon i \in I^+\}$ we finally obtain

$$\begin{aligned} \Delta N_\theta^\varepsilon(o, o; n) &\ge \prod_{i \in I^+} \#\{\gamma_i \in \Gamma_i \colon (n-1)\theta_i \le d_i(o_i, \gamma_i o_i) < n\theta_i\} \\ &\ge D^r \cdot \mathrm{e}^{n \sum_{i \in I^+} \delta_i \theta_i} \\ &= D^r \cdot \mathrm{e}^{n(\delta_1 \theta_1 + \delta_2 \theta_2 + \cdots + \delta_r \theta_r)} \end{aligned}$$

for n sufficiently large.

So from (12) we first get

$$\begin{aligned}\delta_1\theta_1+\delta_2\theta_2+\cdots+\delta_r\theta_r &\le \delta_\theta^\varepsilon(o,o)\\ &\le (\theta_1+\varepsilon)\delta_1+(\theta_2+\varepsilon)\delta_2+\cdots+(\theta_r+\varepsilon)\delta_r\end{aligned}$$

and hence

$$\delta_\theta(\Gamma)=\lim_{\varepsilon\to 0}\delta_\theta^\varepsilon(o,o)=\delta_1\theta_1+\delta_2\theta_2+\cdots+\delta_r\theta_r. \qquad \square$$

The first easy property of the exponent of growth in the general case is

Lemma 7.2. *For $\theta\in E$ we have*

$$\delta_\theta(\Gamma)\ge 0 \qquad \text{if } L_\Gamma\cap\partial X_\theta\neq\emptyset$$

and

$$\delta_\theta(\Gamma)=-\infty \quad \text{if } L_\Gamma\cap\partial X_\theta=\emptyset.$$

In particular, $P_\Gamma=\{\theta\in E\colon \delta_\theta(\Gamma)\ge 0\}$.

Proof. Suppose $L_\Gamma\cap\partial X_\theta\neq\emptyset$. Then by Lemma 3.1 for any $\varepsilon>0$ there exist infinitely many $\gamma\in\Gamma$ such that $\|\widehat{H}(o,\gamma o)-\theta\|<\varepsilon$. In particular

$$\sum_{\gamma\in\Gamma(o,o;\theta,\varepsilon)}1=Q_\theta^{0,\varepsilon}(o,o) \text{ diverges,}$$

hence $\delta_\theta^\varepsilon(o,o)\ge 0$. We conclude $\delta_\theta(\Gamma)=\lim_{\varepsilon\to 0}\delta_\theta^\varepsilon(o,o)\ge 0$.

If $L_\Gamma\cap\partial X_\theta=\emptyset$, then for some $\varepsilon>0$ sufficiently small the number of elements $\gamma\in\Gamma$ with $\|\widehat{H}(o,\gamma o)-\theta\|<\varepsilon$ is finite; otherwise there would be an accumulation point in ∂X_θ. In particular, we have

$$Q_\theta^{0,\varepsilon}(o,o)=\sum_{\gamma\in\Gamma(o,o;\theta,\varepsilon)}1=\#\Gamma(o,o;\theta,\varepsilon)<\infty, \quad \text{i.e.} \quad \delta_\theta^\varepsilon(o,o)\le 0.$$

Moreover, if $s\le 0$ and $d:=\max\{d(o,\gamma o)\colon \gamma\in\Gamma(o,o;\theta,\varepsilon)\}$, then

$$\begin{aligned}Q_\theta^{s,\varepsilon}(o,o)&=\sum_{\gamma\in\Gamma(o,o;\theta,\varepsilon)}\mathrm{e}^{|s|d(o,\gamma o)}\\ &\le\sum_{\gamma\in\Gamma(o,o;\theta,\varepsilon)}\mathrm{e}^{|s|\cdot d}\\ &\le \mathrm{e}^{|s|\cdot d}\cdot\#\Gamma(o,o;\theta,\varepsilon)\\ &<\infty.\end{aligned}$$

We conclude $\delta_\theta^\varepsilon(o,o)=-\infty$ and therefore $\delta_\theta(\Gamma)=\lim_{\varepsilon\to 0}\delta_\theta^\varepsilon(o,o)=-\infty.$ $\square$

The following proposition states that the map

$$E \longrightarrow \mathbb{R}, \quad \theta \longmapsto \delta_\theta(\Gamma)$$

is upper semi-continuous.

Proposition 7.3. *Let $(\theta^{(j)}) \subset E$ be a sequence converging to $\theta \in E$. Then*

$$\limsup_{j\to\infty} \delta_{\theta^{(j)}}(\Gamma) \le \delta_\theta(\Gamma).$$

Proof. Let $\varepsilon_0 \in (0,1)$. Then $\theta^{(j)} \to \theta$ implies $\|\theta^{(j)} - \theta\| < \varepsilon_0/2$ for j sufficiently large. Let $\varepsilon \in (0, \varepsilon_0/2)$ and $\gamma \in \Gamma(o, o; \theta^{(j)}, \varepsilon)$. Then

$$\|\widehat{H}(o, \gamma o) - \theta\| < \varepsilon + \varepsilon_0/2 < \varepsilon_0,$$

hence for j sufficiently large $\Gamma(o, o; \theta^{(j)}, \varepsilon) \subset \Gamma(o, o; \theta, \varepsilon_0)$. This shows

$$\delta^{\varepsilon}_{\theta^{(j)}}(o, o) \le \delta^{\varepsilon_0}_\theta(o, o)$$

and therefore

$$\delta_{\theta^{(j)}}(\Gamma) = \lim_{\varepsilon\to 0} \delta^{\varepsilon}_{\theta^{(j)}}(o, o) \le \delta^{\varepsilon_0}_\theta(o, o).$$

We conclude

$$\limsup_{j\to\infty} \delta_{\theta^{(j)}}(\Gamma) \le \delta^{\varepsilon_0}_\theta(o, o),$$

hence

$$\limsup_{j\to\infty} \delta_{\theta^{(j)}}(\Gamma) = \lim_{\varepsilon_0\to 0} (\limsup_{j\to\infty} \delta_{\theta^{(j)}}(\Gamma)) \le \lim_{\varepsilon_0\to 0} \delta^{\varepsilon_0}_\theta(o, o) = \delta_\theta(\Gamma). \qquad \square$$

For convenience, we will now consider the homogeneous extension of the exponent of growth to a map $\Psi_\Gamma \colon \mathbb{R}^r_{\ge 0} \to \mathbb{R}$. Using a special case of a theorem due to J.-F. Quint, we will prove that this homogeneous extension Ψ_Γ is concave, i.e. for any $p, q \in \mathbb{R}^r_{\ge 0}$ and $t \in [0, 1]$ one has

$$\Psi_\Gamma(tp + (1-t)q) \ge t\Psi_\Gamma(p) + (1-t)\Psi_\Gamma(q).$$

In order to state Quint's Theorem, we recall that in a metric space the ball of radius $t \ge 0$ centered at p is denoted $B(p, t)$. Moreover, we let D denote the Dirac measure and $\nu_\Gamma := \sum_{\gamma\in\Gamma} D_{H(o,\gamma o)}$ the counting measure on $\mathbb{R}^r_{\ge 0}$, i.e. the image of the counting measure $\sum_{\gamma\in\Gamma} D_\gamma$ of Γ by the map $\Gamma \to \mathbb{R}^r_{\ge 0}$, $\gamma \mapsto H(o, \gamma o)$.

Theorem 7.4 ([26], Theorem 3.2.1). *If there exist $s, t, c > 0$ such that for any $p, q \in \mathbb{R}^r$ the inequality*

$$\nu_\Gamma(B(p + q, s)) \ge c \cdot \nu_\Gamma(B(p, t)) \cdot \nu_\Gamma(B(q, t)) \tag{14}$$

holds, then Ψ_Γ is concave.

So we only have to prove inequality (14) which will be done with the help of the generic product Proposition 4.7.

Lemma 7.5. *There exist $s, t, c > 0$ such that for any $p, q \in \mathbb{R}^r$ we have*

$$\nu_\Gamma(B(p+q,s)) \geq c \cdot \nu_\Gamma(B(p,t)) \cdot \nu_\Gamma(B(q,t)).$$

Proof. Notice that $\nu_\Gamma(B(p,t)) = \#\{\gamma \in \Gamma \colon \|H(\gamma) - p\| < t\}$. Fix $t > 0$, set $s = \kappa + 2t$ with $\kappa \geq 0$ from Proposition 4.7 (a) and denote $C > 0$ the inverse of the cardinality of the set $\Lambda \times \Lambda$ from Proposition 4.7 (b). We set

$$P(\Gamma) := \{(\alpha, \beta) \in \Gamma \times \Gamma \colon \|H(\alpha) - p\| < t,\ \|H(\beta) - q\| < t\}$$

and will show that for all $p, q \in \mathbb{R}^r$

$$\#\{\gamma \in \Gamma \colon \|H(\alpha) - p - q\| < s\} \geq C \cdot \#P(\Gamma).$$

Let $(\alpha, \beta) \in P(\Gamma)$. Then $\gamma := \pi(\alpha, \beta) \in \Gamma$ satisfies

$$\begin{aligned}\|H(\gamma) - p - q\| &\leq \|H(\gamma) - H(\alpha) - H(\beta)\| + \|H(\alpha) - p\| + \|H(\beta) - q\| \\ &\leq \kappa + t + t = s.\end{aligned}$$

Moreover, Proposition 4.7 (b) implies that the number of different elements in $P(\Gamma)$ which can yield the same element in $\{\gamma \in \Gamma \colon \|H(\gamma) - p - q\| < s\}$ is bounded by $\#(\Lambda \times \Lambda)$. □

So Theorem 7.4 gives

Theorem 7.6. *The function Ψ_Γ is concave.*

This finally allows to complete the proof of the first statement in Theorem D.

Theorem 7.7. *The set of slopes of limit points of Γ satisfies $P_\Gamma = \ell_\Gamma \cap E$.*

Proof. For convenience we denote $\mathcal{P}_\Gamma$ the set of half-lines in $\mathbb{R}^r_{\geq 0}$ spanned by all vectors $\theta \in P_\Gamma$. By Lemma 7.2 we have $\mathcal{P}_\Gamma = \{H \in \mathbb{R}^r_{\geq 0} \colon \Psi_\Gamma(H) \geq 0\}$, and concavity of Ψ_Γ immediately implies that the cone $\mathcal{P}_\Gamma$ is convex. Moreover, by Proposition 6.1 we have $P_\Gamma^{\mathrm{reg}} = \mathcal{P}_\Gamma \cap E^+ = \ell_\Gamma \cap E^+$, and hence $\mathcal{P}_\Gamma \cap \mathbb{R}^r_{>0} = \ell_\Gamma \cap \mathbb{R}^r_{>0}$. Since both $\mathcal{P}_\Gamma$ and ℓ_Γ are closed this gives $\mathcal{P}_\Gamma = \ell_\Gamma$ and the claim follows from $P_\Gamma = \mathcal{P}_\Gamma \cap E$. □

With the notation introduced in the proof of the previous theorem we further remark that Lemma 7.2 implies

$$\mathcal{P}_\Gamma = \{H \in \mathbb{R}^r_{\geq 0} \colon \Psi_\Gamma(H) \geq 0\} = \{H \in \mathbb{R}^r_{\geq 0} \colon \Psi_\Gamma(H) > -\infty\};$$

so the fact that $\mathcal{P}_\Gamma = \ell_\Gamma$ terminates the proof of the first statement in Theorem E of the introduction. In order to show the second statement we need to apply the following special case of Lemma 4.1.5 in [26].

Lemma 7.8. ([26], *Lemma 4.1.5*) *Let $\alpha, \beta \in \Gamma$ be independent regular axial isometries and $\phi_{u,v}: \langle\alpha, \beta\rangle^+ \to \mathbb{R}$ the unique homomorphism of semi-groups sending α to u and β to v in the additive group $(\mathbb{R}, +)$. Then for all $u, v > 0$ the Dirichlet series*

$$\sum_{\omega\in\langle\alpha,\beta\rangle^+} \mathrm{e}^{-s\phi_{u,v}(\omega)}$$

has exponent of convergence $\delta > 0$.

With the help of this lemma we can finally deduce

Theorem 7.9. *Ψ_Γ is strictly positive in the relative interior of the intersection of ℓ_Γ with the vector subspace of $\mathbb{R}^r$ it spans.*

Proof. As a first step we will show that $\delta(\Gamma) > 0$. Concavity of Ψ_Γ then implies that there exists $\theta^* \in P_\Gamma$ such that $\delta_{\theta^*}(\Gamma) = \delta(\Gamma) > 0$. Moreover, concavity and upper-semicontinuity of Ψ_Γ imply continuity of Ψ_Γ on the closed convex cone

$$\{H \in \mathbb{R}^r_{\geq 0}: \Psi_\Gamma(H) \geq 0\}$$

which is equal to ℓ_Γ according to Theorem 7.7. We conclude that Ψ_Γ is strictly positive in the relative interior of the intersection of ℓ_Γ with the vector subspace of $\mathbb{R}^r$ it spans.

Instead of only proving $\delta(\Gamma) > 0$ we next show the stronger statement that for any $\theta \in \ell_\Gamma \cap E$ and $\varepsilon > 0$ we have

$$\delta^\varepsilon_\theta(o, o) > 0;$$

notice that this also implies $\delta_\theta(\Gamma) \geq 0$ and hence $\Psi_\Gamma(L) \geq 0$ for all $L \in \ell_\Gamma$. According to Proposition 6.3 there exist independent regular axial isometries $\alpha, \beta \in \Gamma$ with $\|\widehat{L}(\alpha) - \theta\| < \varepsilon$, $\|\widehat{L}(\beta) - \theta\| < \varepsilon$ such that every word $\omega \in \langle\alpha, \beta\rangle^+$ of length $|\omega|$ in the generators α, β satisfies

$$\|\widehat{H}(o, \omega o) - \theta\| < \varepsilon \quad \text{and} \quad \|H(o, \omega o) - \Phi(\omega)\| \leq \kappa \cdot |\omega|,$$

where $\Phi: \langle\alpha, \beta\rangle^+ \to \mathbb{R}^r$ is the unique homomorphism of semi-groups sending α to $L(\alpha)$ and β to $L(\beta)$. In particular, using the notation introduced in Lemma 7.8, we get from the inverse triangle inequality

$$\big|\|H(o, \omega o)\| - \|\Phi(\omega)\|\big| \leq \kappa \cdot |\omega| = \phi_{\kappa,\kappa}(\omega);$$

from $\|\Phi(\omega)\| \leq \phi_{l(\alpha),l(\beta)}(\omega)$ we further obtain

$$d(o, \omega o) = \|H(o, \omega o)\| \leq \phi_{l(\alpha),l(\beta)}(\omega) + \phi_{\kappa,\kappa}(\omega) = \phi_{l(\alpha)+\kappa,l(\beta)+\kappa}(\omega)$$

and therefore

$$\begin{aligned} Q_\theta^{s,\varepsilon}(o,o) &= \sum_{\gamma\in\Gamma(o,o;\theta,\varepsilon)} \mathrm{e}^{-sd(o,\gamma o)} \\ &\geq \sum_{\omega\in\langle\alpha,\beta\rangle^+} \mathrm{e}^{-sd(o,\omega o)} \\ &\geq \sum_{\omega\in\langle\alpha,\beta\rangle^+} \mathrm{e}^{-s\phi_{l(\alpha)+\kappa,l(\beta)+\kappa}(\omega)}. \end{aligned}$$

Since all constants $l(\alpha)$, $l(\beta)$ and κ are positive, Lemma 7.8 implies that $Q_\theta^{s,\varepsilon}(o,o)$ diverges for some $s > 0$ and hence $\delta_\theta^\varepsilon(o,o) > 0$. As a corollary we obtain that

$$\sum_{\gamma\in\Gamma} \mathrm{e}^{-sd(o,\gamma o)}$$

has exponent of convergence $\delta(\Gamma) > 0$. $\square$

Moreover, as a corollary of the general Proposition 3.3.1 in [26] (applied to the measure $\nu_\Gamma = \sum_{\gamma\in\Gamma} D_{H(o,\gamma o)}$ on $\mathbb{R}^r$ which satisfies (14) and $\tau_{\nu_\Gamma}^{\|\cdot\|} = \delta(\Gamma) > 0$) we get the following counting results for Γ.

Theorem 7.10. *There exists $C \geq 1$ such that for all $T > 0$ the estimate*

$$\#\{\gamma \in \Gamma : d(o,\gamma o) \leq T\} \leq C \cdot T^{r-1}\mathrm{e}^{\delta(\Gamma)T}$$

holds. Moreover, one has

$$\lim_{T\to\infty} \frac{1}{T} \ln \#\{\gamma \in \Gamma : d(o,\gamma o) \leq T\} = \delta(\Gamma).$$

References

[1] W Ballmann, Axial isometries of manifolds of non-positive curvature. *Math. Ann.* **259** (1982), no. 1, 131–144. Zbl 0487.53039 MR 0656659

[2] W. Ballmann, *Lectures on spaces of nonpositive curvature.* With an appendix by M. Brin. DMV Seminar, 25. Birkhäuser Verlag, Basel, 1995. Zbl 0834.53003 MR 1377265

[3] W. Ballmann and M. Brin, Orbihedra of nonpositive curvature. *Inst. Hautes Études Sci. Publ. Math.* **82** (1995), 169–209. Zbl 0866.53029 MR 1383216

[4] W. Ballmann, M. Gromov, and V. Schroeder, *Manifolds of nonpositive curvature.* Progress in Mathematics, 61. Birkhäuser Boston, Boston, MA, 1985. Zbl 0591.53001 MR 0823981

[5] Y. Benoist, Propriétés asymptotiques des groupes linéaires. *Geom. Funct. Anal.* **7** (1997), no. 1, 1–47. Zbl 0947.22003 MR 1437472

[6] M. Bestvina and K. Fujiwara, A characterization of higher rank symmetric spaces via bounded cohomology. *Geom. Funct. Anal.* **19** (2009), no. 1, 11–40. Zbl 1203.53041 MR 2507218

[7] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature.* Grundlehren der Mathematischen Wissenschaften, 319. Springer-Verlag, Berlin, 1999. Zbl 0988.53001 MR 1744486

[8] M. Burger and N. Monod, Continuous bounded cohomology and applications to rigidity theory. *Geom. Funct. Anal.* **12** (2002), no. 2, 219–280. Zbl 1006.22010 MR 1911660

[9] M. Burger, Intersection, the Manhattan curve, and Patterson–Sullivan theory in rank 2. *Internat. Math. Res. Notices* **1993**, no. 7, 217–225. Zbl 0829.57023 MR 1230298

[10] P.-E. Caprace and K. Fujiwara, Rank-one isometries of buildings and quasi-morphisms of Kac–Moody groups. *Geom. Funct. Anal.* **19** (2010), no. 5, 1296–1319. Zbl 1206.20046 MR 2585575

[11] P.-E. Caprace and B. Rémy, Simplicity and superrigidity of twin building lattices. *Invent. Math.* **176** (2009), no. 1, 169–221. Zbl 1173.22007 MR 2485882

[12] P.-E. Caprace and M. Sageev, Rank rigidity for CAT(0) cube complexes. *Geom. Funct. Anal.* **21** (2011), no. 4, 851–891. Zbl 1266.20054 MR 2827012

[13] F. Dal'bo, Géométrie d'une famille de groupes agissant sur le produit de deux variétés d'Hadamard. In *Séminaire de Théorie Spectrale et Géométrie, No. 15, Année 1996–1997.* Séminaire de Théorie Spectrale et Géométrie, 15. Université de Grenoble I, Institut Fourier, Saint-Martin-d'Hères, 1996, 85–98. Zbl 0898.53027 MR 1604251

[14] F. Dal'bo, Remarques sur le spectre des longueurs d'une surface et comptages. *Bol. Soc. Brasil. Mat. (N.S.)* **30** (1999), no. 2, 199–221. Zbl 1058.53063 MR 1703039

[15] Th. Foertsch and A. Lytchak, The de Rham decomposition theorem for metric spaces. *Geom. Funct. Anal.* **18** (2008), no. 1, 120–143. Zbl 1159.53026 MR 2399098

[16] Y. Guivarc'h and É. Le Page, Simplicité de spectres de Lyapounov et propriété d'isolation spectrale pour une famille d'opérateurs de transfert sur l'espace projectif. In V. A. Kaimanovich (ed.) in collaboration with K. Schmidt and W. Woess, *Random walks and geometry.* (Vienna, 2001) Walter de Gruyter GmbH & Co. KG, Berlin, 2004, 181–259. Zbl 1069.60005 MR 2087783

[17] U. Hamenstädt, Rank-one isometries of proper CAT(0)-spaces. In K. Dekimpe, P. Igodt and A. Valette (eds.), *Discrete groups and geometric structures.* (Kortrijk, May 26–30, 2008) American Mathematical Society, Providence, R.I., 2009, *Discrete groups and geometric structures*, 43–59. Zbl 1220.20038 MR 2581914

[18] V. A. Kaimanovich, Double ergodicity of the Poisson boundary and applications to bounded cohomology. *Geom. Funct. Anal.* **13** (2003), no. 4, 852–861. Zbl 1027.60038 MR 2006560

[19] S. P. Lalley, Renewal theorems in symbolic dynamics, with applications to geodesic flows, non-Euclidean tessellations and their fractal limits. *Acta Math.* **163** (1989), no. 1-2, 1–55. Zbl 0701.58021 MR 1007619

[20] G. Link, Hausdorff dimension of limit sets of discrete subgroups of higher rank Lie groups. *Geom. Funct. Anal.* **14** (2004), no. 2, 400–432. Zbl 1058.22010 MR 2062761

[21] G. Link, Asymptotic geometry in products of Hadamard spaces with rank one isometries. *Geom. Topol.* **14** (2010), no. 2, 1063–1094. Zbl 1273.20040 MR 2629900

[22] G. Link, Generalized conformal densities for higher products of rank one Hadamard spaces. *Geom. Dedicata* **178** (2015), 351–387. Zbl 1334.20039 MR 3397499

[23] G. A. Margulis, *Discrete subgroups of semisimple Lie groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 17. Springer-Verlag, Berlin, 1991. Zbl 0732.22008 MR 1090825

[24] A. Nevo and M. Sageev, The Poisson boundary of CAT(0) cube complex groups. *Groups Geom. Dyn.* **7** (2013), no. 3, 653–695. Zbl 06220443 MR 3095714

[25] J.-F. Quint, Mesures de Patterson–Sullivan en rang supérieur. *Geom. Funct. Anal.* **12** (2002), no. 4, 776–809. Zbl 1169.22300 MR 1935549

[26] J.-F. Quint, Divergence exponentielle des sous-groupes discrets en rang supérieur. *Comment. Math. Helv.* **77** (2002), no. 3, 563–608. Zbl 1010.22018 MR 1933790

[27] B. Rémy, Construction de réseaux en théorie de Kac–Moody. *C. R. Acad. Sci. Paris Sér.* I *Math.* **329** (1999), no. 6, 475–478. Zbl 0933.22029 MR 1715140

[28] C. Yue, The ergodic theory of discrete isometry groups on manifolds of variable negative curvature. *Trans. Amer. Math. Soc.* **348** (1996), no. 12, 4965–5005. Zbl 0864.58047 MR 1348871

Received February 11, 2015

Gabriele Link, Karlsruher Institut für Technologie (KIT), Kaiserstraße 12, 76128 Karlsruhe, Germany

e-mail: gabriele.link@kit.edu