

Day’s fixed point theorem, group cohomology, and quasi-isometric rigidity

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Abstract. In this note we explain how Day’s fixed point theorem can be used to conjugate certain groups of biLipschitz maps of a metric space into special subgroups like similarity groups. In particular, we use Day’s theorem to establish Tukia-type theorems and to give new proofs of quasi-isometric rigidity results.

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1. Introduction

In [25] Tukia proves that any uniform group of quasiconformal maps of S^n for $n \geq 3$ that acts cocompactly on the space of distinct triples of S^n is conjugate into the conformal group. For $n = 2$ this was already proved by Sullivan in [23] without the added assumption on triples. Since the boundary of the real hyperbolic space \mathbb{H}^n can be identified with S^{n-1} and since quasi-isometries (resp. isometries) of \mathbb{H}^n induce quasiconformal (resp. conformal) boundary maps, Tukia’s theorem is used as a key ingredient in the proof of quasi-isometric rigidity of uniform lattices in the isometry groups of real hyperbolic spaces. (For more details see [2]).

The main goal of this paper is to extend Tukia's result to maps on boundaries of different δ -hyperbolic spaces; namely to boundaries of certain *negatively curved homogeneous spaces* and certain *millefeuille spaces*. We leave the description of the geometry of these negatively curved homogeneous spaces (resp. millefeuille spaces) to Section 5. For the statement of our theorems we only need to know that their visual boundaries can be identified with $N \cup \{\infty\}$ (resp. $(N \times \mathbb{Q}_m) \cup \{\infty\}$) where N is a nilpotent Lie group (and \mathbb{Q}_m is the m -adics). Additionally our maps fix the point at infinity ∞ and so we consider *parabolic visual metrics* on N (resp. $N \times \mathbb{Q}_m$) instead of the standard visual metrics on the whole boundary.

The Tukia style theorems we prove have direct consequences for Eskin–Fisher–Whyte's program on the quasi-isometric rigidity of polycyclic groups that was started in [9, 10, 11]. In particular, since by [20] polycyclic groups are all virtually lattices in solvable Lie groups, Eskin–Fisher–Whyte's program involves first ascertaining the structure of quasi-isometries of solvable Lie groups and then studying the induced boundary maps (which in many cases coincide with the boundaries of negatively curved homogeneous spaces). The second step is to then show that uniform groups of these boundary maps (which are identified as *quasi-similarities*) can be conjugated to groups of maps that correspond to boundary maps of isometries (in this case *similarities*). Our work in this paper involves proving the second step in this program for various classes of solvable Lie groups but the ideas presented here should also be useful in the proof of quasi-isometric rigidity of more general polycyclic groups. More details will be provided in Section 5.

The first boundary we treat is when $N = F^n$ (which has dimension $n + 1$) is a model Filiform group and the parabolic visual metric is the usual Carnot–Carathéodory metric. Recall that a group Γ of quasiconformal maps of a metric space X is called a *uniform quasiconformal group* if there is some $K \geq 1$ such that every element γ of Γ is K -quasiconformal. The following theorem is proved in Section 4.

Theorem 1.1. *Let $\Gamma \subset QC(F^n)$ ($n \geq 3$) be a locally compact uniform quasiconformal group of the n -th model Filiform group. Then there is some $f \in QC(F^n)$ such that $f\Gamma f^{-1}$ is a conformal group.*

The second case we consider is when $N = \mathbb{R}^n$ but where the parabolic visual metric, denoted d_A , is not the usual metric on \mathbb{R}^n but is instead determined by a matrix A that is diagonalizable over complex numbers and whose eigenvalues have positive real parts. This metric is defined in Section 5. In this case, the analogue of quasiconformal maps are quasisimilarity maps. A bijection $F: (X, d) \rightarrow (X, d)$ of a metric space is a (M, C) -quasisimilarity ($M \geq 1, C > 0$) if

$$\frac{C}{M} \cdot d(x_1, x_2) \leq d(F(x_1), F(x_2)) \leq MC \cdot d(x_1, x_2)$$

for all $x_1, x_2 \in X$. When $M = 1$ the map is called a *similarity*. A group Γ of quasisimilarities of a metric space is *uniform* if there is some $M \geq 1$ such that

every element γ of Γ is a (M, C_γ) -quasisimilarity (C_γ may depend on γ). Note that a quasisimilarity map is actually just a biLipschitz map but a uniform group of quasisimilarities is not the same as a uniform group of biLipschitz maps.

Theorem 1.2. *Let A be as above and Γ a locally compact uniform quasisimilarity group of (\mathbb{R}^n, d_A) . Suppose in addition that Γ is amenable and acts cocompactly on the space of distinct pairs of \mathbb{R}^n . Let \bar{A} denote the matrix obtained from the Jordan form of A by replacing the eigenvalues with their real parts. Then there exists a biLipschitz map $f: (\mathbb{R}^n, d_A) \rightarrow (\mathbb{R}^n, d_{\bar{A}})$ such that $f\Gamma f^{-1}$ is a group of similarities of $(\mathbb{R}^n, d_{\bar{A}})$.*

1.1. Remarks (1) As promised above, both Theorems 1.1 and 1.2 can be thought of as *Tukia-type* theorems and like Tukia's original theorem can be used to prove quasi-isometric rigidity results; this time for lattices in certain solvable Lie groups (see Section 5).

(2) Theorem 1.2 extends partial results that can be found in [6].

(3) The two extra assumptions in Theorem 1.2, amenability and cocompact action on pairs, are not an obstacle for the quasi-isometric rigidity results, nevertheless it would be interesting to know if they can be removed in certain cases. For Tukia's original theorem on quasiconformal maps of S^n cocompactness on triples is necessary when $n > 2$ (see for example [26]). We only need to assume cocompactness on distinct pairs since our groups fix the point at infinity. For Theorem 1.1 amenability and cocompactness on distinct pairs are not needed as assumptions because from [28] we know that maps in $QC(F^n)$ have a particularly nice form which allows us to deduce that $QC(F^n)$ is itself solvable (see Claim 4.1).

(4) Theorem 1.1 could have also been stated as a theorem about uniform quasisimilarity groups since in [28] it is proved that all quasiconformal maps are in fact quasisimilarities.

(5) Theorem 1.2 can be extended to uniform quasisimilarity groups of $\mathbb{R}^n \times \mathbb{Q}_m$ but we will leave the statement of this result to Theorem 5.4 in Section 5.

1.2. Day's Theorem and bounded cohomology. One key feature in studying quasiconformal (resp. quasisimilarity) maps of boundaries of negatively curved homogeneous spaces is that often differentiability fails in certain directions. This forces quasiconformal and quasisimilarity maps to preserve certain families of foliations. While it is sometimes possible to prove that these maps can be conjugated to act by similarities along the leaves of these foliations, this does not automatically imply that the map is a global similarity map. To solve this problem in the case of *amenable groups* we appeal to Day's theorem:

Theorem (Day’s fixed point theorem, [5]). *Let K be a compact convex subset of a locally convex topological vector space E and let Γ be a locally compact group that acts on K by affine transformations. If Γ is amenable and the action $\Gamma \times K \rightarrow K, (\gamma, x) \mapsto \gamma \cdot x$, is separately continuous, then the action of Γ has a global fixed point.*

In fact Day’s theorem is often taken as the definition of amenability but in our case the groups we consider will be amenable by virtue of being quasi-isometric to polycyclic (hence amenable) groups. Roughly speaking, the relation between Day’s fixed point theorem and a Tukia-type theorem is the following: on the one hand, finding a conjugating map corresponds to the vanishing of the first (bounded) group cohomology; on the other hand there is a well known connection between the vanishing of the first group cohomology and the existence of a global fixed point for an affine action. This relation is explained in detail in Section 2.

We now give a simple example to illustrate how Day’s theorem can be used to prove a Tukia-type theorem. This example is the simplest case of Theorem 3.3 from Section 3 which is later used to prove Theorems 1.1 and 1.2 in Sections 4 and 5.

Example. Suppose Γ is a locally compact uniform quasimilarity group of \mathbb{R}^2 and each $\gamma \in \Gamma$ has the form

$$\gamma(x, y) = (a_\gamma(x + c_\gamma + h_\gamma(y)), b_\gamma(y + d_\gamma)),$$

where $a_\gamma, b_\gamma \in \mathbb{R} \setminus \{0\}$, $c_\gamma, d_\gamma \in \mathbb{R}$ and $h_\gamma: \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz map that satisfies $h_\gamma(0) = 0$. It is easy to check that each γ is a biLipschitz map of \mathbb{R}^2 .

In the above example, the locally convex topological vector space is

$$E = \{h: \mathbb{R} \rightarrow \mathbb{R} \text{ is Lipschitz and } h(0) = 0\}.$$

Here E is equipped with the topology of pointwise convergence. Similar to the Banach–Alaoglu theorem, the closed bounded subsets are compact in the topology of pointwise convergence. Here the boundedness is with respect to the norm on E given by the Lipschitz constant. The affine action of Γ (actually of the opposite group Γ^* of Γ) on E is given by $\phi(\gamma)h = \pi_\gamma h + h_\gamma$ for $\gamma \in \Gamma$ and $h \in E$, where

$$\pi_\gamma h(y) = a_\gamma^{-1} h(b_\gamma(y + d_\gamma)) - a_\gamma^{-1} h(b_\gamma d_\gamma).$$

The uniformity condition on Γ implies that this action has bounded orbits. The compact convex subset K is the closed convex hull of an orbit.

Of course the claim for the above particular example also follows from Sullivan’s theorem [23]. But the approach in this paper applies in all dimensions and in situations where differentiability of Lipschitz maps fails such as in Theorem 1.2 above. In particular the maps considered in this example should be thought of as toy examples of *almost similarities* which were previously the roadblock to proving complete Tukia-type theorems. (See Theorem 2 of [6] or the proof of Theorem 1.2 in Section 5.1.)

1.3. Applications. As mentioned above, Theorems 1.1 and 1.2 can be used to prove quasi-isometric rigidity results for solvable Lie groups and their lattices. Additionally, they can be used to prove results on *envelopes* of certain finitely generated solvable groups. These results are described in Section 5. As an example, here we mention a quasi-isometric rigidity result for solvable Lie groups. (For a more general version of this theorem that applies also to compactly generated locally compact groups please see Section 5.)

Theorem 5.8. *Let F^n ($n \geq 3$) be the n -th model Filiform group and $S = F^n \rtimes \mathbb{R}$ the semi-direct product, where the action of \mathbb{R} on F^n is by standard Carnot dilation. Then any connected and simply connected solvable Lie group G quasi-isometric to S is isomorphic to S .*

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2. Vanishing of Cohomology and Fixed Points

In this section we recall the connection between vanishing of group cohomology in dimension one and the existence of global fixed points of affine actions. We include it here to make this note more self contained.

Let Γ be a group and E a Γ -module. So E is an abelian group and a homomorphism $\pi: \Gamma \rightarrow \text{Aut}(E)$ is fixed, where $\text{Aut}(E)$ is the group of automorphisms of E . We denote the image of $\gamma \in \Gamma$ under π by π_γ . A map $b: \Gamma \rightarrow E$, $\gamma \mapsto b_\gamma$, is called a 1-cocycle if

$$b_{\gamma_1\gamma_2} = b_{\gamma_1} + \pi_{\gamma_1}b_{\gamma_2} \quad \text{for all } \gamma_1, \gamma_2 \in \Gamma.$$

Here we denote the group operation in E by addition. We only consider 1-cocycles, so we will simply call b a cocycle. The set $Z^1(\Gamma, E)$ of all cocycles is an abelian group under addition given by $(b + \tilde{b})_\gamma = b_\gamma + \tilde{b}_\gamma$. A map $b: \Gamma \rightarrow E$ is called a *coboundary* if there exists $v_0 \in E$ such that

$$b_\gamma = v_0 - \pi_\gamma v_0 \quad \text{for any } \gamma \in \Gamma.$$

It is easy to check that a coboundary is a cocycle and the set $B^1(\Gamma, E)$ of all coboundaries is a subgroup of $Z^1(\Gamma, E)$. The first cohomology of Γ with coefficients in E is

$$H^1(\Gamma, E) = Z^1(\Gamma, E)/B^1(\Gamma, E).$$

When E is also a metric space (for example, a normed vector space) and Γ acts on E by uniformly biLipschitz maps (that is, there exists some $M \geq 1$ such that $d(v_1, v_2)/M \leq d(\pi_\gamma v_1, \pi_\gamma v_2) \leq M \cdot d(v_1, v_2)$ for all $\gamma \in \Gamma$, $v_1, v_2 \in E$), a cocycle is said to be bounded if its image is a bounded subset of E . The set $Z_b^1(\Gamma, E)$ of all bounded cocycles is an abelian group under addition. Notice that since the action of Γ is uniformly biLipschitz, every coboundary is a bounded cocycle so that $B_b^1(\Gamma, E) = B^1(\Gamma, E)$. The first bounded cohomology of Γ with coefficients in E is

$$H_b^1(\Gamma, E) = Z_b^1(\Gamma, E)/B^1(\Gamma, E).$$

A map $f: E \rightarrow E$ is called an *affine map* if there exists some automorphism $A: E \rightarrow E$ and some $v_0 \in E$ such that $f(v) = Av + v_0$ for all $v \in E$. Let $\text{Aff}(E)$ be the group of all affine maps of E . An *affine action* of Γ on E is a homomorphism $\phi: \Gamma \rightarrow \text{Aff}(E)$. Given an affine action $\phi: \Gamma \rightarrow \text{Aff}(E)$, we define a map $\pi: \Gamma \rightarrow \text{Aut}(E)$ by $\pi_\gamma = A$, where $A \in \text{Aut}(E)$ is determined by $\phi(\gamma)(v) = Av + v_0$. It is easy to check that π is a homomorphism. We call π the *linear part* of the affine action ϕ .

Lemma 2.1. *Let E be a Γ -module with the Γ action on E given by $\pi: \Gamma \rightarrow \text{Aut}(E)$.*

- (1) *There is a bijection between $Z^1(\Gamma, E)$ and the set of affine actions of Γ on E with linear part π ;*
- (2) *A cocycle is a coboundary if and only if the corresponding affine action has a global fixed point;*
- (3) *$H^1(\Gamma, E) = 0$ if and only if every affine action with linear part π has a global fixed point.*

Proof. (1) Given a cocycle $b: \Gamma \rightarrow E$, define an affine action $\phi: \Gamma \rightarrow \text{Aff}(E)$ of Γ on E by

$$\phi(\gamma)(v) = \pi_\gamma v + b_\gamma \quad \text{for } \gamma \in \Gamma, v \in E.$$

Indeed, $\phi(\gamma)$ is an affine map and for $\gamma_1, \gamma_2 \in \Gamma$, we have

$$\begin{aligned} \phi(\gamma_1) \circ \phi(\gamma_2)(v) &= \phi(\gamma_1)(\pi_{\gamma_2} v + b_{\gamma_2}) \\ &= \pi_{\gamma_1}(\pi_{\gamma_2} v + b_{\gamma_2}) + b_{\gamma_1} \\ &= \pi_{\gamma_1} \pi_{\gamma_2} v + (b_{\gamma_1} + \pi_{\gamma_1} b_{\gamma_2}) \\ &= \pi_{\gamma_1 \gamma_2} v + b_{\gamma_1 \gamma_2} \\ &= \phi(\gamma_1 \gamma_2)(v). \end{aligned}$$

Conversely, assume $\phi: \Gamma \rightarrow \text{Aff}(E)$ is an affine action of Γ on E with linear part π . For any $\gamma \in \Gamma$, there is some $b_\gamma \in E$ such that for any $v \in E$,

$$\phi(\gamma)(v) = \pi_\gamma v + b_\gamma.$$

For any $\gamma_1, \gamma_2 \in \Gamma$, the condition $\phi(\gamma_1\gamma_2) = \phi(\gamma_1) \circ \phi(\gamma_2)$ implies $b_{\gamma_1\gamma_2} = b_{\gamma_1} + \pi_{\gamma_1} b_{\gamma_2}$. Hence $b: \Gamma \rightarrow E$ is a cocycle. Now it is easy to see that the above correspondence $b \rightarrow \phi$ is a bijection between $Z^1(\Gamma, E)$ and the set of affine actions of Γ on E with linear part π .

(2) First assume that $b: \Gamma \rightarrow E$ is a coboundary. Then there is some $v_0 \in E$ such that $b_\gamma = v_0 - \pi_\gamma v_0$ for all $\gamma \in \Gamma$. Then for any $\gamma \in \Gamma$, we have $\phi(\gamma)(v_0) = \pi_\gamma v_0 + b_\gamma = v_0$. Hence v_0 is a global fixed point of the affine action of Γ corresponding to the cocycle b . Conversely, assume the affine action has a global fixed point v_0 . Then $v_0 = \phi(\gamma)(v_0) = \pi_\gamma v_0 + b_\gamma$ for any $\gamma \in \Gamma$. Hence $b_\gamma = v_0 - \pi_\gamma v_0$ for all $\gamma \in \Gamma$ and b is a coboundary.

(3) follows from (1) and (2). □

Similarly we have

Lemma 2.2. *Let E be a Γ -module with the Γ action on E given by $\pi: \Gamma \rightarrow \text{Aut}(E)$. Assume E is also a metric space and Γ acts on E by uniformly biLipschitz maps.*

- (1) *There is a bijection between $Z_b^1(\Gamma, E)$ and the set of affine actions of Γ on E with linear part π and bounded orbits;*
- (2) *A bounded cocycle is a coboundary if and only if the corresponding affine action has a global fixed point;*
- (3) *$H_b^1(\Gamma, E) = 0$ if and only if every affine action with linear part π and bounded orbits has a global fixed point.*

3. BiLipschitz maps of $\mathbb{R}^n \times Y$

In this section we explain how Day's fixed point theorem can be used to conjugate certain groups of biLipschitz maps of metric spaces of the form $\mathbb{R}^n \times Y$ into groups of *similarities*. Recall that a similarity is a map $f: (X, d) \rightarrow (X, d)$ such that there is a constant $c_f > 0$ for which $d(f(x), f(x')) = c_f d(x, x')$ for all $x, x' \in X$.

Let (Y, d) be a metric space. Let $0 < \beta \leq 1$ and \mathbb{R}^n be equipped with the metric $|p - q|^\beta$ ($p, q \in \mathbb{R}^n$), where $|\cdot|$ is the usual Euclidean norm. Let $y_0 \in Y$ be a fixed base point. Let

$$E = \{h: (Y, d) \rightarrow (\mathbb{R}^n, |\cdot|^\beta) \text{ is Lipschitz and } h(y_0) = 0\}.$$

It is not hard to check that E is a Banach space with the norm

$$\|h\| = \sup_{y_1 \neq y_2 \in Y} \frac{|h(y_1) - h(y_2)|}{d(y_1, y_2)^{\frac{1}{\beta}}}.$$

Let $\mathbb{R}^n \times Y$ be equipped with the metric

$$d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|^\beta, d(y_1, y_2)\}$$

for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^n \times Y$. Suppose that a group Γ acts on $\mathbb{R}^n \times Y$ by biLipschitz maps, and for every $\gamma \in \Gamma$, there exist $a_\gamma \in \mathbb{R}_+$, $A_\gamma \in O(n)$, $x_\gamma \in \mathbb{R}^n$, $h_\gamma \in E$ and a similarity $\sigma_\gamma: Y \rightarrow Y$ of Y such that

$$\gamma(x, y) = (a_\gamma A_\gamma(x + x_\gamma + h_\gamma(y)), \sigma_\gamma(y)) \quad \text{for } (x, y) \in \mathbb{R}^n \times Y.$$

Lemma 3.1. *Suppose Γ acts on $\mathbb{R}^n \times Y$ as described above, and acts as a uniform quasisimilarity group. Then*

(1) *for any $\gamma \in \Gamma$, the similarity constant of σ_γ is a_γ^β ; that is,*

$$d(\sigma_\gamma(y_1), \sigma_\gamma(y_2)) = a_\gamma^\beta \cdot d(y_1, y_2)$$

for any $y_1, y_2 \in Y$;

(2) *there is a constant $C > 0$ such that $h_\gamma: (Y, d) \rightarrow (\mathbb{R}^n, |\cdot|^\beta)$*

is C -Lipschitz for every $\gamma \in \Gamma$.

Proof. (1) Since Γ acts as a uniform quasisimilarity group, there is a constant $M \geq 1$ such that every $\gamma \in \Gamma$ is a (M, C_γ) -quasisimilarity. Denote by \tilde{a}_γ the similarity constant of σ_γ . We shall show $\tilde{a}_\gamma = a_\gamma^\beta$. Fix $x \in \mathbb{R}^n$ and $y_1 \neq y_2 \in Y$, and let $p = (x, y_1)$, $q = (x, y_2)$. Then $d(p, q) = d(y_1, y_2)$. Hence

$$\begin{aligned} \tilde{a}_\gamma \cdot d(y_1, y_2) &= d(\sigma_\gamma(y_1), \sigma_\gamma(y_2)) \\ &\leq d(\gamma(p), \gamma(q)) \\ &\leq MC_\gamma \cdot d(p, q) \\ &= MC_\gamma \cdot d(y_1, y_2). \end{aligned}$$

So $\tilde{a}_\gamma \leq MC_\gamma$. By considering γ^{-1} we obtain

$$\frac{1}{\tilde{a}_\gamma} = \tilde{a}_{\gamma^{-1}} \leq MC_{\gamma^{-1}} = M \cdot \frac{1}{C_\gamma}.$$

Hence

$$\frac{C_\gamma}{M} \leq \tilde{a}_\gamma \leq MC_\gamma \quad \text{for all } \gamma \in \Gamma. \quad (1)$$

Next we fix $y \in Y$ and $x_1 \neq x_2 \in \mathbb{R}^n$, and let $p = (x_1, y)$, $q = (x_2, y)$. Then $d(p, q) = |x_1 - x_2|^\beta$ and $d(\gamma(p), \gamma(q)) = (a_\gamma \cdot |x_1 - x_2|)^\beta$. The quasimilarity condition on γ implies

$$\begin{aligned} \frac{C_\gamma}{M} \cdot |x_1 - x_2|^\beta &= \frac{C_\gamma}{M} \cdot d(p, q) \\ &\leq (a_\gamma \cdot |x_1 - x_2|)^\beta = d(\gamma(p), \gamma(q)) \\ &\leq MC_\gamma \cdot d(p, q) \\ &= MC_\gamma \cdot |x_1 - x_2|^\beta. \end{aligned}$$

Hence

$$\frac{C_\gamma}{M} \leq a_\gamma^\beta \leq MC_\gamma \quad \text{for all } \gamma \in \Gamma. \tag{2}$$

Inequalities (1) and (2) imply

$$\frac{1}{M^2} \leq \frac{\tilde{a}_\gamma}{a_\gamma^\beta} \leq M^2 \quad \text{for all } \gamma \in \Gamma. \tag{3}$$

In particular, (3) holds for γ^m for any $m \in \mathbb{Z}$. Since $a_{\gamma^m} = a_\gamma^m$ and $\tilde{a}_{\gamma^m} = \tilde{a}_\gamma^m$, we must have $\tilde{a}_\gamma = a_\gamma^\beta$.

(2) Let $y_1, y_2 \in Y$ be arbitrary. Fix $x \in \mathbb{R}^n$ and let $p = (x, y_1)$, $q = (x, y_2)$. Then $d(p, q) = d(y_1, y_2)$. The quasimilarity condition on γ implies

$$\begin{aligned} &(a_\gamma \cdot |h_\gamma(y_1) - h_\gamma(y_2)|)^\beta \\ &= |a_\gamma A_\gamma(x + x_\gamma + h_\gamma(y_1)) - a_\gamma A_\gamma(x + x_\gamma + h_\gamma(y_2))|^\beta \\ &\leq d(\gamma(p), \gamma(q)) \\ &\leq MC_\gamma \cdot d(p, q) \\ &= MC_\gamma \cdot d(y_1, y_2). \end{aligned}$$

Now inequality (2) implies $|h_\gamma(y_1) - h_\gamma(y_2)|^\beta \leq M^2 \cdot d(y_1, y_2)$. □

Let Γ^* be the opposite group of Γ . We denote the group operation in Γ^* by $\gamma_1 * \gamma_2$ (and the group operation in Γ by $\gamma_1 \gamma_2$). We define a Γ^* -module structure $\pi: \Gamma^* \rightarrow \text{Aut}(E)$ on E as follows. For $\gamma \in \Gamma^*$, $h \in E$ and $y \in Y$, define

$$(\pi_\gamma h)(y) = a_\gamma^{-1} A_\gamma^{-1} h(\sigma_\gamma(y)) - a_\gamma^{-1} A_\gamma^{-1} h(\sigma_\gamma(y_0)). \tag{4}$$

Using Lemma 3.1 (1) it is not difficult to check that Γ^* acts on E by linear isometries.

Now define a map $b: \Gamma^* \rightarrow E$ by $b\gamma = h_\gamma$. By comparing the two sides of $(\gamma_1 \gamma_2)(x, y) = \gamma_1(\gamma_2(x, y))$, we obtain $h_{\gamma_1 \gamma_2} = h_{\gamma_2} + \pi_{\gamma_2} h_{\gamma_1}$. In other words, $b_{\gamma_2 * \gamma_1} = b_{\gamma_2} + \pi_{\gamma_2} b_{\gamma_1}$. Hence $b: \Gamma^* \rightarrow E$ is a cocycle.

Lemma 3.2. *Let Γ and b be as above. If the cocycle b is a coboundary, then there exists some $h_0 \in E$ such that if we denote by $H_0: \mathbb{R}^n \times Y \rightarrow \mathbb{R}^n \times Y$ the biLipschitz map given by*

$$H_0(x, y) = (x + h_0(y), y),$$

then every element of $H_0\Gamma H_0^{-1} \subset \text{Homeo}(\mathbb{R}^n \times Y)$ is a similarity.

Proof. Since b is a coboundary, there is some $h_0 \in E$ such that $h_\gamma = b_\gamma = h_0 - \pi_\gamma h_0$ for all $\gamma \in \Gamma$. Now for any $\gamma \in \Gamma$, we have

$$\begin{aligned} H_0 \circ \gamma \circ H_0^{-1}(x, y) &= H_0 \circ \gamma(x - h_0(y), y) \\ &= H_0(a_\gamma A_\gamma[x - h_0(y) + x_\gamma + h_\gamma(y)], \sigma_\gamma(y)) \\ &= (a_\gamma A_\gamma[x - h_0(y) + x_\gamma + h_\gamma(y) + a_\gamma^{-1} A_\gamma^{-1} h_0(\sigma_\gamma(y))], \sigma_\gamma(y)) \\ &= (a_\gamma A_\gamma[x - h_0(y) + x_\gamma + h_\gamma(y) + \pi_\gamma h_0(y) + a_\gamma^{-1} A_\gamma^{-1} h_0(\sigma_\gamma(y_0))], \sigma_\gamma(y)) \\ &= (a_\gamma A_\gamma[x + x_\gamma + a_\gamma^{-1} A_\gamma^{-1} h_0(\sigma_\gamma(y_0))], \sigma_\gamma(y)). \end{aligned}$$

Notice that $x_\gamma + a_\gamma^{-1} A_\gamma^{-1} h_0(\sigma_\gamma(y_0))$ is a constant vector in \mathbb{R}^n (independent of (x, y)). So

$$x \mapsto a_\gamma A_\gamma[x + x_\gamma + a_\gamma^{-1} A_\gamma^{-1} h_0(\sigma_\gamma(y_0))]$$

is a similarity of \mathbb{R}^n . Now Lemma 3.1 (1) implies that $H_0 \circ \gamma \circ H_0^{-1}$ is a similarity of $\mathbb{R}^n \times Y$. \square

Recall that our goal is to show that Γ can be conjugated into a group of similarities. By Lemma 2.1 and Lemma 3.2, it now suffices to show that the affine action corresponding to the cocycle b has a global fixed point. This is where Day's fixed point theorem can be useful.

Theorem 3.3. *Let Y be a metric space, $0 < \beta \leq 1$ and $\mathbb{R}^n \times Y$ be equipped with the metric*

$$d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|^\beta, d(y_1, y_2)\}$$

for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^n \times Y$. Suppose that a locally compact group Γ acts continuously on $\mathbb{R}^n \times Y$ by biLipschitz maps, and for every $\gamma \in \Gamma$, there exist $a_\gamma \in \mathbb{R}_+$, $A_\gamma \in \text{O}(n)$, $x_\gamma \in \mathbb{R}^n$, $h_\gamma \in E$ and a similarity $\sigma_\gamma: Y \rightarrow Y$ of Y such that

$$\gamma(x, y) = (a_\gamma A_\gamma(x + x_\gamma + h_\gamma(y)), \sigma_\gamma(y)) \quad \text{for } (x, y) \in \mathbb{R}^n \times Y.$$

If Γ is amenable and is a uniform quasimilarity group of $\mathbb{R}^n \times Y$, then there exists some $h_0 \in E$ such that if we denote by $H_0: \mathbb{R}^n \times Y \rightarrow \mathbb{R}^n \times Y$ the biLipschitz map given by

$$H_0(x, y) = (x + h_0(y), y),$$

then every element of $H_0\Gamma H_0^{-1} \subset \text{Homeo}(\mathbb{R}^n \times Y)$ is a similarity.

Proof. We equip the vector space E with the topology of pointwise convergence. It is easy to check that E is a locally convex topological vector space. Notice that $E \subset (\mathbb{R}^n)^Y$ is a subset of the space of all maps from Y to \mathbb{R}^n and the topology of pointwise convergence on E is the same as the subspace topology $E \subset (\mathbb{R}^n)^Y$, where $(\mathbb{R}^n)^Y$ has the product topology.

We observe that closed balls in E are compact in the topology of pointwise convergence (this is similar to the fact that closed balls in the dual space of a normed vector space is compact in the weak* topology). Indeed, for $r > 0$, the closed ball $B_r = \{h \in E: \|h\| \leq r\}$ is a closed subset of

$$A := \prod_{y \in Y} \bar{B}(0, r \cdot (d(y, y_0))^{\frac{1}{\beta}}) \subset (\mathbb{R}^n)^Y,$$

where $\bar{B}(0, r \cdot (d(y, y_0))^{\frac{1}{\beta}})$ is the closed ball in \mathbb{R}^n . By Tychonoff's theorem, A is compact. As a closed subset of A , the set B_r is also compact. Consequently, all closed bounded subsets of E are compact in the topology of pointwise convergence.

By Lemma 3.1 (2), there is a constant $C > 0$ such that $\|h_\gamma\| \leq C$ for all $\gamma \in \Gamma^*$. So the cocycle b is bounded. By Lemma 2.2, the affine action of Γ^* on E corresponding to b has bounded orbits. This is also easy to see directly since the orbit of $0 \in E$ is $\{h_\gamma: \gamma \in \Gamma^*\} \subset B_C$. Let $K \subset E$ be the closed convex hull of this orbit. By the preceding paragraph, K is a compact convex subset of E . Notice that Γ^* acts on K by isometric affine transformations. Since Γ acts continuously on $\mathbb{R}^n \times Y$, it is easy to check that the action of Γ^* on K is separately continuous. Finally Γ^* is amenable since Γ is. Now all the conditions in Day's theorem are satisfied and so this affine action has a global fixed point. Theorem 3.3 follows from Lemma 2.2 (2) and Lemma 3.2. □

4. Quasiconformal groups of model Filiform groups

In this section we show through an example that Theorem 3.3 is more applicable than it appears. The point is that the space that Γ acts on does not have to be a product like $\mathbb{R}^n \times Y$. Specifically we show how to use Theorem 3.3 to prove Theorem 1.1.

The n -step ($n \geq 2$) model Filiform algebra \mathfrak{f}^n is an $(n + 1)$ -dimensional real Lie algebra. It has a basis $\{e_1, e_2, \dots, e_{n+1}\}$ and the only non-trivial bracket relations are $[e_1, e_j] = e_{j+1}$ for $2 \leq j \leq n$. The Lie algebra \mathfrak{f}^n admits a direct sum decomposition of vector subspaces $\mathfrak{f}^n = V_1 \oplus \dots \oplus V_n$, where V_1 is the linear subspace spanned by e_1, e_2 , and V_j ($2 \leq j \leq n$) is the linear subspace spanned by e_{j+1} . It is easy to check that $[V_1, V_j] = V_{j+1}$ for $1 \leq j \leq n$, where $V_{n+1} = \{0\}$. Hence \mathfrak{f}^n is a stratified Lie algebra. The connected and simply connected Lie group with Lie algebra \mathfrak{f}^n will be denoted by F^n and is called the n -step model Filiform group.

A group Γ of quasiconformal maps of a metric space X is called a uniform quasiconformal group if there is some $K \geq 1$ such that every element γ of Γ is K -quasiconformal. By Theorem 4.7 in [17], every K -quasiconformal map $f: N \rightarrow N$ of a Carnot group (equipped with a left invariant Carnot metric) is η -quasisymmetric, where $\eta: [0, \infty) \rightarrow [0, \infty)$ is a homeomorphism depending only on K and N . By Lemma 3.10 in [28], every η -quasisymmetric map $f: F^n \rightarrow F^n$ for $n \geq 3$ is a (M, C) -quasisimilarity, where M depends only on η . It follows that every uniform quasiconformal group of F^n ($n \geq 3$) is a uniform quasisimilarity group.

Recall that, for a connected and simply connected nilpotent Lie group N with Lie algebra \mathfrak{n} , the exponential map $\exp: \mathfrak{n} \rightarrow N$ is a diffeomorphism. We shall identify \mathfrak{f}^n with F^n via the exponential map. For any $p \in \mathfrak{f}^n$, let $L_p: \mathfrak{f}^n \rightarrow \mathfrak{f}^n$, $L_p(x) = p * x$ be the left translation by p . For $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, let $h_{a_1, a_2}: \mathfrak{f}^n \rightarrow \mathfrak{f}^n$ be the graded automorphism given by

$$h_{a_1, a_2}(e_1) = a_1 e_1,$$

$$h_{a_1, a_2}(e_j) = a_1^{j-2} a_2 e_j \quad \text{for } 2 \leq j \leq n + 1.$$

For any Lipschitz function $h: \mathbb{R} \rightarrow \mathbb{R}$, define $h_j: \mathbb{R} \rightarrow \mathbb{R}$ ($2 \leq j \leq n + 1$) inductively by $h_2 = h$,

$$h_j(x) = - \int_0^x h_{j-1}(s) ds, \quad j = 3, \dots, n + 1.$$

Let $F_h: \mathfrak{f}^n \rightarrow \mathfrak{f}^n$ be given by

$$F_h(x) = x * \sum_{j=2}^{n+1} h_j(x_1) e_j,$$

where $x = \sum_{j=1}^{n+1} x_j e_j$.

Let V_1 be equipped with the inner product with e_1, e_2 as orthonormal basis, and \mathfrak{f}^n be equipped with the Carnot metric determined by this inner product. It is easy to check that $h_{\epsilon_1, \epsilon_2}$ with $\epsilon_1, \epsilon_2 \in \{1, -1\}$ is an isometry of \mathfrak{f}^n . Recall that the standard Carnot dilation $\delta_t: \mathfrak{f}^n \rightarrow \mathfrak{f}^n$ ($t > 0$) is defined by $\delta_t(v) = t^j v$ for $v \in V_j$. They are similarities with respect to the Carnot metric: $d(\delta_t(p), \delta_t(q)) = t \cdot d(p, q)$ for any $p, q \in \mathfrak{f}^n$. Since δ_t ($t > 0$) is a similarity and left translations are isometries of \mathfrak{f}^n , the group Q generated by $h_{\epsilon_1, \epsilon_2}$ ($\epsilon_1, \epsilon_2 \in \{1, -1\}$), δ_t ($t > 0$) and left translations consists of similarities of \mathfrak{f}^n . In fact, it is not hard to see that Q is the group $\text{Sim}(\mathfrak{f}^n)$ of similarities of \mathfrak{f}^n . Notice that the identity component Q_0 of $\text{Sim}(\mathfrak{f}^n)$ consists of maps of the form $L_p \circ \delta_t$ ($p \in \mathfrak{f}^n, t > 0$), and that $\text{Sim}(\mathfrak{f}^n)$ has four connected components $h_{\epsilon_1, \epsilon_2} Q_0$.

Proof of Theorem 1.1. Let $\Gamma \subset QC(\mathfrak{f}^n)$ be a locally compact uniform quasiconformal group. As indicated above, Γ is a uniform quasisimilarity group. By Theorem 1.1 of [28], every quasiconformal map $F: \mathfrak{f}^n \rightarrow \mathfrak{f}^n$ has the form $F = h_{a_1, a_2} \circ L_p \circ F_h$, where $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, $p \in \mathfrak{f}^n$ and $h: \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz function. It follows that every quasiconformal map $F: \mathfrak{f}^n \rightarrow \mathfrak{f}^n$ induces a biLipschitz map f of $V_1 = \mathbb{R}e_2 \times Y$ (with $Y = \mathbb{R}e_1$) of the following form

$$f(x_2e_2 + x_1e_1) = a_2(x_2 + b + h(x_1))e_2 + a_1(x_1 + a)e_1,$$

where $a, b \in \mathbb{R}$, $a_2, a_1 \in \mathbb{R} \setminus \{0\}$ and $h: \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz. By replacing h with $h - h(0)$ and b with $b + h(0)$ we may assume $h(0) = 0$. Furthermore, it is not difficult to show that if F is a (M, C) -quasisimilarity, then so is f . Consequently, the map $\rho: \Gamma \rightarrow \text{Homeo}(\mathbb{R}^2)$, $F \mapsto f$, defines an action of Γ on \mathbb{R}^2 and $\rho(\Gamma)$ is a uniform quasisimilarity group. Lemma 3.1 (1) implies that $|a_1| = |a_2|$.

Claim 4.1. $QC(\mathfrak{f}^n)$ is a solvable group.

We first finish the proof of Theorem 1.1 assuming Claim 4.1. It follows from Claim 4.1 that Γ is solvable and hence is amenable. Now Theorem 3.3 implies that there is some Lipschitz function $h_0: \mathbb{R} \rightarrow \mathbb{R}$ such that if we denote by $H_0: V_1 \rightarrow V_1$ the map given by

$$H_0(x_2e_2 + x_1e_1) = (x_2 + h_0(x_1))e_2 + x_1e_1,$$

then all elements of $H_0\rho(\Gamma)H_0^{-1}$ have the form

$$x_2e_2 + x_1e_1 \mapsto a_2(x_2 + b)e_2 + a_1(x_1 + a)e_1 \quad \text{with } |a_2| = |a_1|. \tag{5}$$

Notice that H_0 is the map on V_1 induced by the biLipschitz map $F_{h_0}: \mathfrak{f}^n \rightarrow \mathfrak{f}^n$. It follows that every element of $F_{h_0}\Gamma F_{h_0}^{-1}$ induces a biLipschitz map of V_1 as in (5). Notice that every map of the form (5) is also induced by a biLipschitz map of \mathfrak{f}^n of the form $h_{\epsilon_1, \epsilon_2} \circ \delta_t \circ L_p$, where $\epsilon_1, \epsilon_2 \in \{1, -1\}$, $p \in \mathfrak{f}^n$ and $t > 0$. If two quasiconformal maps induce the same map on V_1 , then they have the same Pansu differential a.e. By Lemma 2.5 in [28], these two quasiconformal maps differ by a left translation. Hence every element of $F_{h_0}\Gamma F_{h_0}^{-1}$ has the form $L_q \circ h_{\epsilon_1, \epsilon_2} \circ \delta_t \circ L_p$. Since each of $L_q, h_{\epsilon_1, \epsilon_2}, \delta_t, L_p$ is a similarity, the group $F_{h_0}\Gamma F_{h_0}^{-1}$ consists of similarities.

Next we prove the claim. First we define a homomorphism

$$\pi_1: QC(\mathfrak{f}^n) \longrightarrow \mathbb{R}^* \times \mathbb{R}^*$$

by

$$\pi_1(h_{a_1, a_2} \circ L_p \circ F_h) = (a_1, a_2).$$

Since $\mathbb{R}^* \times \mathbb{R}^*$ is abelian, $QC(\mathfrak{f}^n)$ is solvable if the kernel $H_1 = \ker(\pi_1)$ is. Notice H_1 consists of all quasiconformal maps of \mathfrak{f}^n of the form $F = L_p \circ F_h$, where $p \in \mathfrak{f}^n$ and $h: \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz satisfying $h(0) = 0$. Now define a homomorphism $\pi_2: H_1 \rightarrow \mathbb{R}$ by

$$\pi_2(L_{\sum_{j=1}^{n+1} x_j e_j} \circ F_h) = x_1.$$

The kernel $H_2 = \ker(\pi_2)$ of π_2 consists of all maps of the form $L_{\sum_{j=2}^{n+1} x_j e_j} \circ F_h$, where $x_j \in \mathbb{R}$ and $h: \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz satisfying $h(0) = 0$. A direct calculation shows that H_2 is abelian. Since \mathbb{R} is also abelian, H_1 is solvable. Hence $QC(\mathfrak{f}^n)$ is solvable. \square

Here we make some remarks on the connection between Carnot groups (in particular, the model Filiform groups) and homogeneous manifolds with negative curvature. Recall that \mathbb{R}^n (together with ∞) can be identified with the ideal boundary of the $n + 1$ dimensional real hyperbolic space (in the upper half space model). The Euclidean metric on \mathbb{R}^n is a parabolic visual metric associated to ∞ . Similarly, given any Carnot group N with Lie algebra $\mathfrak{n} = V_1 \oplus \cdots \oplus V_r$, let \mathbb{R} act on N by the standard Carnot group dilations δ_{e^t} ($t \in \mathbb{R}$). Let $S = N \rtimes \mathbb{R}$ be the corresponding semi-direct product. Then S is a solvable Lie group. Fix an inner product on the tangent space at the identity element of S such that \mathfrak{n} and \mathbb{R} are perpendicular and that V_i and V_j are perpendicular for $i \neq j$. Then S is negatively curved [16] and the ideal boundary of S can be identified with $N \cup \{\infty\}$. Furthermore, the Carnot metric on N is a parabolic visual metric associated with ∞ .

5. Applications to quasi-isometric rigidity

In this section we show how Theorem 3.3 can be used to prove Tukia-type Theorem 1.2 and Theorem 5.4 below. Additionally we show how Theorem 1.2 can be used to simplify some of the proofs of quasi-isometric rigidity found in [6] and [21, 22]. Then we show how both Theorem 1.2 and Theorem 5.4 can be used to improve results on envelopes of abelian-by-cyclic groups found in [8]. Finally in Theorems 5.8 and 5.9 we prove results on quasi-isometric rigidity of Lie groups and locally compact groups quasi-isometric to certain solvable Lie groups.

5.1. Negatively curved homogeneous spaces. In this subsection we prove Theorem 1.2.

Let A be an $n \times n$ matrix with real entries. Suppose A is diagonalizable over the complex numbers and its eigenvalues have positive real parts. We list the real parts of eigenvalues in increasing order $\alpha_1 < \alpha_2 < \cdots < \alpha_r$. Let \mathbb{R} act on \mathbb{R}^n by the one parameter group e^{tA} ($t \in \mathbb{R}$) and $G_A = \mathbb{R}^n \rtimes_A \mathbb{R}$ the associated semi-direct product. Then G_A is a solvable Lie group. Equip G_A with the left invariant

Riemannian metric that is determined by the standard inner product at the identity element $(0, 0) \in \mathbb{R}^n \times \mathbb{R} = G_A$. Then by [16] we have that G_A is negatively curved. For $x_0 \in \mathbb{R}^n$, the path $c_{x_0}: \mathbb{R} \rightarrow G_A, c_{x_0}(t) = (x_0, t)$, is a geodesic in G_A . We call c_{x_0} a vertical geodesic. All vertical geodesics are asymptotic as $t \rightarrow \infty$, and so they determine a point ∞ in the ideal boundary. If $t \rightarrow -\infty$ all vertical geodesics diverge from one another. We call such geodesics downward oriented. The ideal boundary ∂G_A of G_A is naturally identified with $\mathbb{R}^n \cup \{\infty\}$, where points in \mathbb{R}^n correspond to downward oriented vertical geodesics.

Let \bar{A} be the matrix obtained from the Jordan form of A by replacing the eigenvalues with their real parts. By Proposition 4.1 of [12] and Corollary 3.2 of [27], the ideal boundaries of G_A and $G_{\bar{A}}$ are biLipschitz. It follows that we can assume that A is already diagonal with eigenvalues exactly α_i listed in increasing order. Let V_j ($1 \leq j \leq r$) be the eigenspace associated to α_i . Then $\mathbb{R}^n = V_1 \times \dots \times V_r$. We write a point $x \in \mathbb{R}^n$ as $x = (x_1, \dots, x_r)$ with $x_j \in V_j$. The parabolic visual metric d_A on $\mathbb{R}^n = \partial G_A \setminus \{\infty\}$ associated with ∞ is given by

$$d_A((x_1, \dots, x_r), (x'_1, \dots, x'_r)) = \max_i \{|x_i - x'_i|^{\alpha_1/\alpha_i}\}.$$

The notation used in [6] defined G_A as G_M where $M = e^A$ but we switch to writing G_A instead of G_M to be in line with the notation used in [27].

By Proposition 4 in [6], every biLipschitz (and by [27] any quasi-symmetric) map of (\mathbb{R}^n, d_A) has the form

$$F(x_1, \dots, x_r) = (f_1(x_1, \dots, x_r), f_2(x_2, \dots, x_r), \dots, f_r(x_r)),$$

where f_i is biLipschitz in x_i and α_i/α_j -Hölder in x_j for $j > i$.

We are now able to prove Theorem 1.2.

Proof of Theorem 1.2. By Theorem 2 of [6] we have that Γ can be conjugated to act by maps that are the composition of similarities and maps of the form

$$(x_1, x_2, \dots, x_r) \mapsto (x_1 + h_1(x_2, \dots, x_r), x_2 + h_2(x_3, \dots, x_n), \dots, x_r + h_r)$$

where h_i is α_i/α_j -Hölder in x_j for $j > i$. In other words, after conjugation by a biLipschitz map, any $F \in \Gamma$ has the form

$$F(x_1, \dots, x_r) = (a^{\alpha_1} A_1(x_1 + h_1(x_2, \dots, x_r)), \dots, a^{\alpha_{r-1}} A_{r-1}(x_{r-1} + h_{r-1}(x_r)), a^{\alpha_r} A_r(x_r + h_r))$$

where $a > 0$ and the A_j is an orthogonal transformation of V_j . These maps are referred to as *almost similarities* in [6].

Notice that Γ induces an action on $V_j \times \cdots \times V_r$ for any $1 \leq j \leq r$ and that the induced action of Γ on V_r is by similarities. We will finish the proof by induction. Assume the induced action of Γ on $V_{j+1} \times \cdots \times V_r$ is by similarities. We will show that there is a biLipschitz map $F_0: (\mathbb{R}^n, d_A) \rightarrow (\mathbb{R}^n, d_A)$ such that the induced action of $F_0\Gamma F_0^{-1}$ on $V_j \times \cdots \times V_r$ is by similarities. This will complete the proof.

Set $Y = V_{j+1} \times \cdots \times V_r$. The metric on Y is given by

$$D((x_{j+1}, \dots, x_r), (x'_{j+1}, \dots, x'_r)) = \max_{j+1 \leq k \leq r} |x_k - x'_k|^{\frac{\alpha_1}{\alpha_k}}.$$

Notice that the map $h_j: (Y, D) \rightarrow (V_j, |\cdot|^{\frac{\alpha_1}{\alpha_j}})$ is Lipschitz.

Denote $a_j = h_j(0)$ and let $g_j(y) = h_j(y) - a_j$. Then $g_j: (Y, D) \rightarrow (V_j, |\cdot|^{\frac{\alpha_1}{\alpha_j}})$ is Lipschitz and $g_j(0) = 0$. Also by assumption, the induced action of Γ on Y is by similarities. Now the induced action of Γ on $V_j \times Y$ is by biLipschitz maps of the form

$$(x_j, y) \mapsto (a^{\alpha_j} A_j(x_j + a_j + g_j(y)), \sigma(y)),$$

where $g_j: (Y, D) \rightarrow (V_j, |\cdot|^{\frac{\alpha_1}{\alpha_j}})$ is Lipschitz, $g_j(0) = 0$ and σ is a similarity of Y . Since Γ is a uniform quasimilarity group of (\mathbb{R}^n, d_A) , it is easy to see that the induced action of Γ on $V_j \times Y$ is a uniform quasimilarity action. Hence Theorem 3.3 implies that there is some Lipschitz map $h: (Y, D) \rightarrow (V_j, |\cdot|^{\frac{\alpha_1}{\alpha_j}})$ such that after conjugation by the biLipschitz map $V_j \times Y \rightarrow V_j \times Y, (x_j, y) \mapsto (x_j + h(y), y)$, Γ acts on $V_j \times Y$ by similarities. Let $F_0: (\mathbb{R}^n, d_A) \rightarrow (\mathbb{R}^n, d_A)$ be the map given by

$$F_0(x_1, \dots, x_j, y) = (x_1, \dots, x_j + h(y), y).$$

Then F_0 is biLipschitz. Furthermore, the induced action of $F_0\Gamma F_0^{-1}$ on $V_j \times \cdots \times V_r$ is by similarities. □

5.2. Quasi-isometric rigidity for lattices in certain solvable Lie groups. In this subsection we show how Theorem 1.2 can be used to simplify parts of the proof of quasi-isometric rigidity for lattices in certain solvable Lie groups. The first proof we can simplify is the following theorem for lattices in certain abelian-by-cyclic groups that was announced in [9]:

Theorem 5.1. *Let A be a $n \times n$ real matrix diagonalizable over the complex numbers. Suppose $\text{tr } A = 0$ and that A has no purely imaginary eigenvalues. Let $G_A = \mathbb{R}^n \rtimes_A \mathbb{R}$. If Γ is a finitely generated group quasi-isometric to G_A , then Γ is virtually a lattice in $\mathbb{R}^n \rtimes_B \mathbb{R}$, where B is a matrix that has the same absolute Jordan form as αA for some positive $\alpha \in \mathbb{R}$.*

Here A necessarily has both eigenvalues with positive real part and eigenvalues with negative real part. Theorem 5.1 is proved in [21, 22] as part of the following more general theorem on lattices in abelian-by-abelian solvable Lie groups.

Theorem 5.2 (Peng). *Let $G_\psi = \mathbf{H} \rtimes_\psi \mathbf{A}$ where \mathbf{H}, \mathbf{A} are Euclidean groups and $\psi: \mathbf{A} \rightarrow \text{Aut}(\mathbf{H})$ is such that every (nontrivial) element of $\psi(\mathbf{A})$ is diagonalizable and has at least one eigenvalue whose absolute value is not equal to one. Suppose that Γ is a finitely generated group quasi-isometric to a lattice in $G_\psi = \mathbf{H} \rtimes_\psi \mathbf{A}$. Then Γ is virtually a lattice in $\mathbf{H} \rtimes_{\psi'} \mathbf{A}$ for some $\psi': \mathbf{A} \rightarrow \text{Aut}(\mathbf{H})$.*

We present the outline of the simplified proof of Theorems 5.1 and 5.2 now that we have access to Theorem 1.2.

A key part of the argument is understanding the quasi-isometry group of G_ψ . Let X be a metric space. Two quasi-isometries $f, g: X \rightarrow X$ are equivalent if

$$d(f, g) := \sup\{d(f(x), g(x)) \mid x \in X\} < \infty.$$

Let $\text{QI}(X)$ be the set of equivalence classes $[f]$ of self quasi-isometries of X . The formula $[f] \cdot [g] = [f \circ g]$ defines a group structure on $\text{QI}(X)$. The inverse of $[f]$ is the class represented by a quasi-inverse of f . We call $\text{QI}(X)$ the *quasi-isometry group* of X . A subgroup U of $\text{QI}(X)$ is called *uniform* if there are fixed constants $K \geq 1, C \geq 0$ such that each class in U has at least one representative that is a (K, C) quasi-isometry.

Outline of proof of Theorems 5.1 and 5.2.

- (1) The first ingredient is Peng's theorem on the structure of quasi-isometries $f: G_\psi \rightarrow G_\psi$ (Theorem 5.3.6 in [22]). Peng shows that all such quasi-isometries f are a bounded distance from a map of the form

$$(\mathbf{x}, \mathbf{t}) \mapsto (f_{\mathbf{H}}(\mathbf{x}), f_{\mathbf{A}}(\mathbf{t})),$$

for $(\mathbf{x}, \mathbf{t}) \in \mathbf{H} \rtimes_\psi \mathbf{A}$ such that $f_{\mathbf{A}}: \mathbf{A} \rightarrow \mathbf{A}$ is an affine map.

- (2) Using Peng's theorem one can identify the quasi-isometry group (up to finite index) as a product of groups of biLipschitz maps:

$$\text{QI}(G_\psi) \simeq \prod \text{Bilip}(\mathbb{R}^{n_i}, d_{A_i})$$

where the A_i depend on ψ . In the special case of Theorem 5.1 this becomes

$$\text{QI}(G_A) \simeq \text{Bilip}(\mathbb{R}^{n_\ell}, d_{A_\ell}) \times \text{Bilip}(\mathbb{R}^{n_u}, d_{A_u}),$$

where A_u corresponds to the eigenvalues of A with positive real part and A_ℓ corresponds to the eigenvalues with negative real part, and n_ℓ, n_u are the dimensions of A_ℓ and A_u respectively. (See Sections 2.4 and 2.5 in [6] for the details of the abelian-by-cyclic case and Proposition 5.3.5 (vi) in [22] for details in the general case.)

- (3) Any finitely generated group Γ quasi-isometric to G_ψ is, up to finite kernel, a uniform subgroup of $\text{QI}(G_\psi)$ and hence acts as a uniform quasi-similarity subgroup of each $\text{Bilip}(\mathbb{R}^{n_i}, d_{A_i})$.
- (4) By Theorem 1.2 we have that after conjugation

$$f\Gamma f^{-1} \subset \prod \text{Sim}(\mathbb{R}^{n_i}, d_{\bar{A}_i}).$$

Note that we are allowed to use Theorem 1.2 since Γ being quasi-isometric to an amenable group (a lattice in a solvable Lie group) is itself amenable.

- (5) By uniformity, the similarity constants in each $\text{Sim}(\mathbb{R}^{n_i}, d_{\bar{A}_i})$ factor interact in such a way that $f\Gamma f^{-1}$ can be viewed as a cocompact subgroup of the isometry group of $\mathbf{H} \rtimes_{\bar{\psi}} \mathbf{A}$ for some faithful homomorphism $\bar{\psi}: \mathbf{A} \rightarrow \text{Aut}(\mathbf{H})$ whose image consists of diagonal matrices. (This is discussed in more detail in Section 4 of [6]).
- (6) By Proposition 5.3 below (due to Dave Witte Morris) this implies that Γ is virtually a lattice in $\mathbf{H} \rtimes_{\psi'} \mathbf{A}$ for some ψ' .

Comments. The previous proofs of Theorems 5.1 and 5.2 diverge from this outline at Step 4. Specifically, by Theorem 2 in [6], after conjugation Γ acts by *almost similarities* on each of the $(\mathbb{R}^{n_i}, d_{\bar{A}_i})$ factors (see the proof of Theorem 1.2 for a definition of almost similarity). Without Theorem 1.2 one is forced to study the structure of almost similarities to first show that the group Γ must be virtually polycyclic (and hence virtually a lattice is some solvable Lie group). Further analysis is then used to show that it must be a lattice in $\mathbf{H} \rtimes_{\psi'} \mathbf{A}$ for some $\psi': \mathbf{A} \rightarrow \text{Aut}(\mathbf{H})$. In the special case (i.e. Theorem 5.1), this is done in Section 4 of [6]. For the general case this is discussed in Corollaries 5.3.9 and 5.3.11 in [22]. In both cases it requires substantial additional analysis.

The following proposition is due to Dave Witte Morris.

Proposition 5.3. *Fix a left-invariant metric on the semidirect product $G = \mathbf{H} \rtimes_{\bar{\psi}} \mathbf{A}$, where $\bar{\psi}$ is a faithful homomorphism from \mathbf{A} to the diagonal matrices in $\text{Aut}(\mathbf{H})$. Then any lattice Γ in $\text{Isom}(G)$ has a finite-index subgroup that is isomorphic to a lattice in some semidirect product $\mathbf{H} \rtimes_{\psi'} \mathbf{A}$.*

Proof. For convenience, let $I = \text{Isom}(G)$. From [15, Corollary 1.12 and Theorems 4.2 and 4.3], we have $I = G \rtimes K$, where K is a compact subgroup of $\text{Aut } G$. Since K is compact, it has only finitely many components, so, by passing to a finite-index subgroup, we may assume Γ is contained in the identity component I° of I .

Since $\bar{\psi}$ is faithful, it is easy to see that $\mathbf{H} = \text{nil } G$, so $(\text{Aut } G)^\circ$ centralizes G/\mathbf{H} (and we have $\mathbf{H} \subseteq \text{nil } I^\circ$). Combining this with the observation that K (being compact) acts reductively, and, being a group of automorphisms, acts faithfully,

we conclude that K acts faithfully on \mathbf{H} . This implies that $\mathbf{H} = \text{nil } I^\circ$. Hence, a theorem of Mostow [19, Lem. 3.9] tells us that $\Gamma \cap \mathbf{H}$ is a lattice in \mathbf{H} .

We may assume, after replacing \mathbf{A} by a conjugate, that K° centralizes \mathbf{A} (cf. [1, Proposition 11.23(ii), p.158]). Then $I^\circ = \mathbf{H} \rtimes (\mathbf{A} \times K^\circ)$, so there is a natural projection $\pi: I^\circ \rightarrow \mathbf{A} \times K^\circ$. Since the kernel of π is \mathbf{H} , the conclusion of the preceding paragraph implies that $\pi(\Gamma)$ is a lattice in $\mathbf{A} \times K^\circ$.

After replacing Γ by an appropriate finite-index subgroup, it is not difficult to see that there is a closed subgroup \mathbf{T} of $\mathbf{A} \times K^\circ$, such that $\mathbf{T} \cong \mathbf{A}$ and $\pi(\Gamma)$ is a lattice in \mathbf{T} . Then Γ is a lattice in $\mathbf{H} \rtimes \mathbf{T} \simeq \mathbf{H} \rtimes_{\psi'} \mathbf{A}$. This is the desired conclusion. \square

5.3. Boundaries of amenable hyperbolic locally compact groups. In this subsection we prove a Tukia-type theorem (Theorem 5.4) for boundaries of millefeuille spaces.

We consider locally compact uniform quasisimilarity groups of the metric space $\mathbb{R}^n \times \mathbb{Q}_m$ with the metric $d_{A,m} = \max\{d_A, d_{\mathbb{Q}_m}\}$ where d_A is the metric on \mathbb{R}^n defined in Section 5.1 and $d_{\mathbb{Q}_m}$ is the usual metric on the m -adics \mathbb{Q}_m :

$$d_{\mathbb{Q}_m}\left(\sum a_i m^i, \sum b_i m^i\right) = m^{-(k+1)},$$

where k is the smallest index for which $a_i \neq b_i$. As promised in the introduction we will explain how $(\mathbb{R}^n \times \mathbb{Q}_m, d_{A,m})$ is the boundary of a *millefeuille space* $X_{\varphi,m}$. A millefeuille space is a fibered product of a negatively curved homogeneous space $G_\varphi = N \rtimes_\varphi \mathbb{R}$ and an $m + 1$ valent tree T_{m+1} with respect to height functions on both factors. On G_φ the height function h_φ is given by projecting to the \mathbb{R} coordinate and on the tree h_m is given by fixing a base point in the ideal boundary and orienting all the edges towards this base point. Then $X_{\varphi,m} = \{(g, t) \in G_\varphi \times T_{m+1} \mid h_\varphi(g) = h_m(t)\}$. Alternatively one can construct $X_{m,\varphi}$ inductively by identifying m copies of G_φ above integral heights. Then $X_{\varphi,m}$ is a CAT(-1) space with boundary $(N \times \mathbb{Q}_m) \cup \{\infty\}$ and it is easy to see that $d_{\varphi,m}$ is the resulting parabolic visual metric. For more details see [7] and [4].

The following theorem strengthens part of Theorem 1.6 in [7].

Theorem 5.4. *Let A be a real $n \times n$ matrix diagonalizable over complex numbers and whose eigenvalues have positive real part. Let Γ be a separable locally compact uniform quasisimilarity group of $(\mathbb{R}^n \times \mathbb{Q}_m, d_{A,m})$. Suppose Γ is amenable and acts cocompactly on the space of distinct pairs. Then, there exist some $\lambda > 0$ and some integer $s \geq 1$ and a biLipschitz map $F_0: (\mathbb{R}^n \times \mathbb{Q}_m, d_{A,m}) \rightarrow (\mathbb{R}^n \times \mathbb{Q}_s, d_{\lambda \bar{A},s})$, such that m, s are powers of a common integer, and $\Gamma' := F_0 \Gamma F_0^{-1}$ acts on $\mathbb{R}^n \times \mathbb{Q}_s$ by similarities. Here \bar{A} denotes the matrix obtained from the Jordan form of A by replacing the eigenvalues with their real parts.*

Proof. As with the proof of Theorem 1.2 we start with existing partial results. By Theorem 1.6 in [7] there exist some $\lambda > 0$ and some integer $s \geq 1$, and a biLipschitz map

$$F_0: (\mathbb{R}^n \times \mathbb{Q}_m, d_{A,m}) \longrightarrow (\mathbb{R}^n \times \mathbb{Q}_s, d_{\lambda \bar{A},s})$$

such that m, s are powers of a common integer, and $\Gamma' := F_0 \Gamma F_0^{-1}$ acts on $\mathbb{R}^n \times \mathbb{Q}_s$ by maps that are similarities composed with maps of the form

$$(x_1, \dots, x_r, y) \longmapsto (x_1 + h_1(x_2, \dots, x_r, y), \dots, x_r + h_r(y), \sigma(y)).$$

where σ is an isometry of \mathbb{Q}_s , and

$$h_j: (V_{j+1} \times \dots \times V_r \times \mathbb{Q}_m, D_{j+1}) \longrightarrow (V_j, |\cdot|^{\frac{\alpha_1}{\alpha_j}})$$

is Lipschitz, where the metric D_{j+1} is given by

$$\begin{aligned} D_{j+1}((x_{j+1}, \dots, x_r, u), (x'_{j+1}, \dots, x'_r, u')) \\ = \max\{d_{\mathbb{Q}_s}(u, u'), |x_k - x'_k|^{\frac{\alpha_1}{\alpha_k}}, j + 1 \leq k \leq r\}. \end{aligned}$$

In other words any $F \in \Gamma'$ has the form

$$\begin{aligned} F(x_1, \dots, x_r, y) = (a_F^{\alpha_1} A_{1,F}(x_1 + h_1^F(x_2, \dots, x_r, y)), \dots, \\ a_F^{\alpha_r} A_{r,F}(x_r + h_r^F(y)), \sigma_F(y)) \end{aligned}$$

where $a_F > 0$, the $A_{i,F}$ are orthogonal matrices of the appropriate size, and σ_F is a similarity.

We proceed by induction as in the proof of Theorem 1.2. The action of Γ' on \mathbb{Q}_s is already by similarities. Now assume the induced action of Γ' on $Y := (V_{j+1} \times \dots \times V_r \times \mathbb{Q}_s, D_{j+1})$ is by similarities. We shall find a biLipschitz map H_0 of $(\mathbb{R}^n \times \mathbb{Q}_s, d_{\lambda \bar{A},s})$ such that the induced action of $H_0 \Gamma' H_0^{-1}$ on $(V_j, |\cdot|^{\frac{\alpha_1}{\alpha_j}}) \times Y$ is by similarities. This will complete the proof. For this, we note that the induced action of Γ' on $(V_j, |\cdot|^{\frac{\alpha_1}{\alpha_j}}) \times Y$ is a uniform quasisimilarity action and has the form required by Theorem 3.3. Since Γ' is amenable, Theorem 3.3 implies there is some Lipschitz map $h: Y \rightarrow (V_j, |\cdot|^{\frac{\alpha_1}{\alpha_j}})$ such that if we define $H_0: (\mathbb{R}^n \times \mathbb{Q}_s, d_{\lambda \bar{A},s}) \rightarrow (\mathbb{R}^n \times \mathbb{Q}_s, d_{\lambda \bar{A},s})$ by

$$H_0(x_1, \dots, x_j, y) = (x_1, \dots, x_j + h(y), y) \text{ for } x_i \in V_i, y \in Y,$$

then the induced action of $H_0 \Gamma' H_0^{-1}$ on $(V_j, |\cdot|^{\frac{\alpha_1}{\alpha_j}}) \times Y$ is by similarities. □

Remark. In the above theorem we are sometimes forced to take s different from m : as observed in [18] there are quasisimilarity groups of \mathbb{Q}_m that cannot be conjugated into the group of similarities of \mathbb{Q}_m but can always be conjugated into the similarity group of some \mathbb{Q}_s , where s, m are powers of a common integer. Furthermore, by Corollary 5 in [7] we know that we must have $s = m^\lambda$.

5.4. Envelopes. In this subsection we show how Theorems 1.2 and 5.4 can be used to answer the following problem for certain finitely generated solvable groups.

Problem 5.1. *Given a finitely generated group Γ , classify up to extensions of and by compact groups, all locally compact groups H such that $\Gamma \subset H$ as a lattice.*

Such an H was called an *envelope* of Γ by Furstenberg in [14] where he proposed the study of this problem. In [13], Furman classifies all locally compact envelopes of lattices in semisimple Lie groups and outlines a technique using the quasi-isometry group to solve Problem 5.1 for cocompact lattice embeddings. In [8] the first author adapts this outline for cocompact lattices to solve the envelopes problem for various classes of solvable groups. Below we show how Theorems 1.2 and 5.4 simplify the proofs from [8] in certain cases and extend the results to other groups as well.

The outline for approaching this problem is as follows. If $\Gamma \subset H$ is a cocompact lattice embedding then one can construct a map $\Psi: H \rightarrow \text{QI}(\Gamma)$ where the image of H sits as a uniform subgroup inside $\text{QI}(\Gamma)$ and where $\Psi(\Gamma)$ sits inside $\Psi(H)$ as a subgroup of isometries.

At this point we should remark that this construction is only useful when the kernel of the map $H \rightarrow \text{QI}(\Gamma)$ is compact. This is not true in general but it does hold for so called *QI-tame* spaces. For more details on the construction of Ψ see [13], Section 3, and for the definition of QI-tame see [8], Section 4. If $\text{QI}(\Gamma)$ can be identified with a group of quasiconformal or biLipschitz maps of some sort of boundary then Tukia-type theorems can be used to conjugate the image of H into a subgroup that can be identified with the isometry group of some proper geodesic metric space. The conclusion is then that H must be a cocompact subgroup of the isometry group of this metric space up to possibly some compact kernel. In [13] Furman applies this outline to uniform lattices in the isometry groups of real and complex hyperbolic spaces.

This outline works in more generality. For any locally compact compactly generated group H (with or without lattices) quasi-isometric to a metric space X , one can similarly define $\Psi: H \rightarrow \text{QI}(X)$ so that $\Psi(H)$ is a uniform subgroup of $\text{QI}(X)$. If H (and hence X) are QI-tame then there is a natural topology on uniform subgroups of $\text{QI}(X)$ and the map $\Psi: H \rightarrow \Psi(H) \subset \text{QI}(X)$ is a continuous map with respect to this topology. Again, if it is possible to identify $\text{QI}(X)$ with biLipschitz or quasiconformal maps of appropriate boundaries then Tukia-type theorems can be used to show that $\Psi(H)$ is a subgroup of a similarity group (which can be further identified with the isometry group of X or potentially a related metric space). This fact is used in the proof of Theorems 5.8 and 5.9 in the next section.

5.4.1. Envelopes of abelian-by-cyclic groups. In our case the above outline in conjunction with Theorems 1.2 and 5.4 is used to answer Problem 5.1 for various finitely presented abelian-by-cyclic groups. Finitely presented abelian-by-cyclic groups can be specified by picking $M = (m_{ij}) \in \text{GL}_n(\mathbb{Z})$ and setting

$$\Gamma_M = \langle a, b_1, \dots, b_n \mid ab_i a^{-1} = b_1^{m_{1i}} \cdots b_n^{m_{ni}}, b_i b_j = b_j b_i \rangle.$$

When $M \in \text{SL}_2(\mathbb{Z})$ has eigenvalues off of the unit circle then Γ_M is a lattice in SOL and more generally when $M \in \text{SL}_n(\mathbb{Z})$ these groups are (virtually) lattices in abelian-by-cyclic solvable Lie groups. Alternatively, when M is a one-by-one matrix with entry m then Γ_M is the solvable Baumslag–Solitar group $\text{BS}(1, m)$ but in general their geometry is much more rich. In particular they have model spaces that are fibered products of a solvable Lie group and a tree T_{d+1} of valence $d + 1$ where $d = |\det M|$ with respect to appropriate height functions. (This time the height function on the tree is the negative of the height function we used to construct the millefeuille space in the previous section which results in a space that is not $\text{CAT}(-1)$.) The solvable Lie group in the fibered product can be chosen as follows: up to possibly squaring M (which is equivalent to replacing Γ_M with an index 2 subgroup) we can assume that $M = e^A$ lies on a one parameter subgroup, and then the solvable Lie group can be chosen to be $G_A = \mathbb{R}^n \rtimes_A \mathbb{R}$ where \mathbb{R} acts on \mathbb{R}^n by the one parameter subgroup e^{tA} . (Here A may have both eigenvalues with positive real parts and eigenvalues with negative real parts so G_A is not necessarily negatively curved). Let X_M be the resulting fibered product of G_A and T_{d+1} , where $d = |\det M|$. Then X_M is a model space for Γ_M . Similarly we let $X_{\bar{M}}$ be the fibered product of $G_{\bar{A}}$ and T_{d+1} , where $\bar{M} = e^{\bar{A}}$ and $d = |\det M|$. Note that when $\det M = 1$ then the tree in the fibered product is simply a line and so $X_M \simeq G_A$. For more details on this see Section 6 in [8]. In [8] we prove the following theorem.

Theorem 5.5. *Suppose M has all eigenvalues off of the unit circle and either $\det M = 1$ or all of the eigenvalues have norm greater than one. Then any envelope H of Γ_M is, up to compact groups, a cocompact closed subgroup H' in $\text{Isom}(X_{\bar{M}^k})$ for some $k \in \mathbb{Q}$.*

In this theorem by “up to compact groups” we mean that there is a short exact sequence

$$1 \longrightarrow H' \longrightarrow H/K \longrightarrow K' \longrightarrow 1$$

with K, K' compact. Furthermore, we may choose H' such that Γ_M embeds in H' .

If instead we assume that M (and hence A) is diagonalizable then we can use Theorems 1.2 and 5.4 to simplify the proof and strengthen the statement of Theorem 5.5 and to extend it to more cases.

Theorem 5.6. *Suppose M is diagonalizable over the complex numbers and with all eigenvalues off of the unit circle. Let H be an envelope of Γ_M . Then there is a compact normal subgroup N of H such that H/N is isomorphic to a cocompact subgroup of $\text{Isom}(X_{\bar{M}^k})$ for some $k \in \mathbb{Q}$.*

This theorem is stronger than the previous one in that now we can conclude that H/N is isomorphic to a cocompact subgroup of $\text{Isom}(X_{\bar{M}^k})$.

Outline of proof of Theorem 5.6. Since most of the set up for this theorem is in [8] we will only sketch the argument and refer the reader to [8] for details. Let H be an envelope of Γ_M .

- (1) Following the outline given in the preamble above we can view H (up to compact kernel) as a subgroup of $\text{QI}(\Gamma_M)$ via the map $\Psi: H \rightarrow \text{QI}(\Gamma_M)$.
- (2) From Section 7 in [8] we have the identification

$$\text{QI}(\Gamma_M) \simeq \text{Bilip}(\mathbb{R}^{n_\ell}, d_{A_\ell}) \times \text{Bilip}(\mathbb{R}^{n_u} \times \mathbb{Q}_{\det M}, d_{A_u, \det M}).$$

- (3) Since H is a uniform subgroup of $\text{QI}(\Gamma_M)$ it projects to uniform quasi-similarity subgroups of $\text{Bilip}(\mathbb{R}^{n_\ell}, d_{A_\ell})$ and $\text{Bilip}(\mathbb{R}^{n_u} \times \mathbb{Q}_{\det M}, d_{A_u, \det M})$.
- (4) By Theorems 1.2 and 5.4 we can conjugate these actions to similarity actions

$$fHf^{-1} \subset \text{Sim}(\mathbb{R}^{n_\ell}, d_{\bar{A}_\ell}) \times \text{Sim}(\mathbb{R}^{n_u} \times \mathbb{Q}_{(\det M)^k}, d_{k\bar{A}_u, (\det M)^k}).$$

- (5) Since fHf^{-1} must still be a uniform subgroup of $\text{QI}(\Gamma_M)$ we see that it must actually sit inside $\text{Isom}(X_{\bar{M}^k})$.

Comments. Without Theorems 1.2 and 5.4 the proof in [8] diverges from this proof after Step 3. Instead the partial Tukia-type results of Theorem 2 in [6] and Theorem 1.6 in [7] are used along with substantial additional analysis to find the conjugating map f .

5.4.2. Envelopes of Filiform-by-cyclic groups. Let N be a connected and simply connected nilpotent Lie group with Lie algebra \mathfrak{n} . We call N and \mathfrak{n} *rational* if \mathfrak{n} has a basis with rational structure constants. It is well known that N admits lattices if and only if it is rational. In particular, if e_1, \dots, e_n is a basis of \mathfrak{n} with rational structure constants, then there exists a positive integer K such that the integral linear combinations of Ke_1, \dots, Ke_n is a lattice in N (after identification of \mathfrak{n} and N via the exponential map), see [3], Theorem 5.1.8.

The model Filiform group \mathfrak{f}^n is rational. Let e_1, \dots, e_{n+1} be the standard basis of \mathfrak{f}_n . As indicated above, there exists a positive integer K such that the set L of integral linear combinations of Ke_1, \dots, Ke_{n+1} is a lattice in \mathfrak{f}^n . Let $M = \mathfrak{f}_n/L$ be the quotient. Then $\pi_1(M) = L$.

If $\lambda \geq 1$ is an integer, then the standard Carnot group dilation $\delta_\lambda: \mathfrak{f}^n \rightarrow \mathfrak{f}^n$ maps L into L and $\phi := \delta_\lambda|_L: L \rightarrow L$ is an injective homomorphism. Let $\Gamma = L_\phi$ be the associated HNN extension. Clearly δ_λ projects to a covering map $f: M \rightarrow M$. Furthermore, $f_*: \pi_1(M) \rightarrow \pi_1(M)$ agrees with ϕ after the identification $\pi_1(M) = L$. Let M_f be the mapping torus of f . Then $\pi_1(M_f) = \Gamma$ and M_f is a $K(\pi, 1)$ for Γ .

For any integer $m \geq 1$, let $X_{\mathfrak{f}^n, m} = \mathfrak{f}^n \times T_{m+1}$. The metric on $X_{\mathfrak{f}^n, m}$ is defined as follows. Fix an end ξ_0 of the tree T_{m+1} and let $b: T_{m+1} \rightarrow \mathbb{R}$ be a Busemann function associated to ξ_0 . A complete geodesic in T_{m+1} is vertical if one of its ends is ξ_0 . We identify vertical geodesics with \mathbb{R} using the Busemann function b . In this way, for each vertical geodesic c , we identify $\mathfrak{f}^n \times c$ with the negatively curved solvable Lie group $S = \mathfrak{f}^n \rtimes \mathbb{R}$ defined at the end of Section 4. We equip $X_{\mathfrak{f}^n, m}$ with the path metric. The space $X_{\mathfrak{f}^n, m}$ is similar to the space X_M from subsection 5.4.1 in the case when all eigenvalues of M lie outside the unit ball. When $m = [L : \phi(L)]$, where L is the lattice in \mathfrak{f}^n described above, $X_{\mathfrak{f}^n, m}$ is the universal cover of M_f .

Now the argument in the proof of Theorem 5.6 (using Theorem 1.1 instead of Theorem 1.2) shows the following result:

Theorem 5.7. *Let L be a lattice in \mathfrak{f}^n constructed above and Γ an HNN extension as above. Suppose $n \geq 3$. Then for every envelope H of Γ , there is a compact normal subgroup N of H such that H/N is isomorphic to a cocompact subgroup of $\text{Isom}(X_{\mathfrak{f}^n, s})$ for some integer $s \geq 2$.*

5.5. Lie groups quasi-isometric to certain solvable Lie groups. Tukia-type theorems can be used to characterize Lie groups and locally compact groups quasi-isometric to a given solvable Lie group. In this subsection we illustrate this using Theorem 1.1 and Theorem 1.2.

Notice that any two left invariant Riemannian metrics on a connected Lie group are biLipschitz equivalent. Recall that we identify the model filiform algebra \mathfrak{f}^n and the model filiform group F^n via the exponential map. Also recall that a locally compact group G is compactly generated if there is a compact neighborhood K of the identity element that generates G . Similar to the word metric on a finitely generated group, a word pseudometric on G can be defined using the compact generating set K , and different generating sets result in quasi-isometric word pseudometrics.

Theorem 5.8. *Let $n \geq 3$ and $S = \mathfrak{f}^n \rtimes \mathbb{R}$ be the semidirect product associated to the standard action of \mathbb{R} on the model filiform group \mathfrak{f}^n by dilations.*

- (1) *Let G be a connected and simply connected solvable Lie group. Let S and G be equipped with left invariant Riemannian metrics. If G and S are quasi-isometric, then they are isomorphic.*

(2) *If a compactly generated locally compact group G is quasi-isometric to S , then there is a compact normal subgroup N of G , such that G/N is isomorphic to a cocompact subgroup of $\text{Isom}(S)$.*

Proof. (1) Let $f: G \rightarrow S$ be a quasi-isometry and $\tilde{f}: S \rightarrow G$ a quasi-inverse of f . For any $x \in G$, let $L_x: G \rightarrow G$ be the left translation by x . Since L_x is an isometry, there exist $L \geq 1$ and $A \geq 0$ such that for every $x \in G$, the map

$$T_x := f \circ L_x \circ \tilde{f}: S \longrightarrow S$$

is a (L, A) -quasi-isometry. So T_x induces a boundary map $\phi(x): \partial S \rightarrow \partial S$, which is a quasiconformal map. Here ∂S is equipped with a visual metric. In this way we get a continuous group homomorphism $\phi: G \rightarrow QC(\partial S)$ and the image $\phi(G)$ is a uniform quasiconformal group of ∂S . The kernel $K := \ker(\phi)$ is a closed normal subgroup of G .

For any $s \in S$, consider three geodesic rays starting at s such that the angle between any two of them is at least $\pi/2$. For each $k \in K$, T_k is a (L, A) -quasi-isometry and induces the identity map on the ideal boundary. By using stability of quasigeodesics, it is easy to see that there is a constant C depending only on L, A and the Gromov hyperbolicity constant of S such that $d(s, T_k(s)) \leq C$ for all $s \in S$. Since f is a quasi-isometry, it follows that K is bounded. Hence K is a compact subgroup of G . Since the only compact subgroup of a connected and simply connected solvable Lie group is the trivial subgroup, we have $K = \{e\}$. So ϕ is an embedding.

Recall that ∂S can be identified with $\mathfrak{f}^n \cup \{\infty\}$. By the results in [28], each $F \in QC(\partial S)$ fixes ∞ and $F|_{\mathfrak{f}^n}: \mathfrak{f}^n \rightarrow \mathfrak{f}^n$ is a quasiconformal map with the same quasiconformality constant. So we may assume G is a uniform quasiconformal group of \mathfrak{f}^n . By Theorem 1.1, we may assume that G is a subgroup of the similarity group $\text{Sim}(\mathfrak{f}^n)$ of \mathfrak{f}^n . Recall that $\text{Sim}(\mathfrak{f}^n)$ has 4 connected components and the identity component Q_0 of $\text{Sim}(\mathfrak{f}^n)$ consists of maps of the form $L_p \circ \delta_t$ ($p \in \mathfrak{f}^n$, $t > 0$). Notice that $S = \mathfrak{f}^n \rtimes \mathbb{R}$ is isomorphic to Q_0 via the isomorphism $(p, t) \mapsto L_p \circ \delta_{e^t}$. Since G is connected, we may assume $G \subset Q_0 \cong S$. However, it is easy to see that the only connected subgroup of S that is quasi-isometric to S is S itself.

(2) follows from the above argument. □

It is easy to identify $\text{Isom}(S)$. Let $S = \mathfrak{f}^n \rtimes \mathbb{R}$ be equipped with the left invariant Riemannian metric such that \mathfrak{f}^n and \mathbb{R} are perpendicular to each other and such that e_1, \dots, e_{n+1} are orthonormal. Then $\text{Isom}(S)$ is isomorphic to $\text{Sim}(\mathfrak{f}^n)$, which is described in Section 4 before the proof of Theorem 1.1. The reason is that each isometry of S induces a similarity of \mathfrak{f}^n and each similarity of \mathfrak{f}^n is the boundary map of an isometry of S .

The argument in the proof of Theorem 5.8 also shows the following result:

Theorem 5.9. *Let A and \bar{A} be as in Theorem 1.2.*

- (1) *If a connected and simply connected solvable Lie group is quasi-isometric to G_A , then it is isomorphic to a cocompact subgroup of $\text{Isom}(G_{\bar{A}})$;*
- (2) *If a compactly generated locally compact group G is quasi-isometric to G_A , then there is a compact normal subgroup N of G , such that G/N is isomorphic to a cocompact subgroup of $\text{Isom}(G_{\bar{A}})$.*

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