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Groups, Geometry, and Dynamics

A quantitative bounded distance theorem and a Margulis' lemma for $\mathbb{Z}^n$-actions, with applications to homology

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Abstract. We consider the stable norm associated to a discrete, torsionless abelian group of isometries $\Gamma \cong \mathbb{Z}^n$ of a geodesic space (X, d). We show that the difference between the stable norm $\| \ \|_{st}$ and the distance d is bounded by a constant only depending on the rank n and on upper bounds for the diameter of $\bar{X} = \Gamma \backslash X$ and the asymptotic volume $\omega(\Gamma, d)$. We also prove that the upper bound on the asymptotic volume is equivalent to a lower bound for the stable systole of the action of Γ on (X, d); for this, we establish a lemma *à la* Margulis for $\mathbb{Z}^n$-actions, which gives optimal estimates of $\omega(\Gamma, d)$ in terms of $\mathrm{stsys}(\Gamma, d)$, and vice versa, and characterize the cases of equality. Moreover, we show that all the parameters n, $\mathrm{diam}(\bar{X})$ and $\omega(\Gamma, d)$ (or $\mathrm{stsys}(\Gamma, d)$) are necessary to bound the difference $d - \| \ \|_{st}$, by providing explicit counterexamples for each case.

As an application in Riemannian geometry, we prove that the number of connected components of any optimal, integral 1-cycle in a closed Riemannian manifold $\bar{X}$ either is bounded by an explicit function of the first Betti number, $\mathrm{diam}(\bar{X})$ and $\omega(H_1(\bar{X}, \mathbb{Z}))$, or is a sublinear function of the mass.

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1. Introduction

Consider a geodesic metric space (X, d) with a $\mathbb{Z}^n$*-periodic metric*, i.e. admitting a discrete, torsionless abelian group of isometries Γ of rank n acting properly discontinuously:[1] we mainly think of the Cayley graph of a word metric on $\mathbb{Z}^n$, or to a $\mathbb{Z}^n$-covering of a compact Riemannian or Finsler manifold. A motivating example is the torsion free homology covering X of any compact manifold $\bar{X}$ with nontrivial first Betti number, which has automorphism group $\Gamma = H_1(\bar{X}, \mathbb{Z})/\mathrm{tor}$.

[1] That is, each $x \in X$ has an open neighbourhood U such that $\{\gamma \in \Gamma \mid \gamma.U \cap U \neq \varnothing\}$ is finite.

The associated *stable norm* on Γ is defined as

$$\| \gamma \|_{\mathrm{st}} = \lim_{k\to\infty} \frac{1}{k} d(x_0, \gamma^k.x_0)$$

and clearly it does not depend on the choice of $x_0 \in X$, by the triangular inequality. An isomorphism $\Gamma \cong \mathbb{Z}^n$ being chosen, this yields a well-defined norm[2] on $\mathbb{R}^n$, extending the definition by homogeneity to $\mathbb{Q}^n$ first, and then to real coefficients by uniform continuity. For instance, when $\Gamma = H_1(\bar{X}, \mathbb{Z})/\mathrm{tor}$ is the automorphism group of the torsion free homology covering of a compact Riemannian manifold $\bar{X}$, the stable norm coincides with the norm induced by the Riemannian length in the homology with real coefficients, that is (see [7] and [11] Chapter 4, §C):

$$\| \gamma \|_{\mathrm{st}} = \inf \Big\{ \sum_k |a_k| \ell(\gamma_k) \colon a_k \in \mathbb{R}, \gamma_k \text{ Lipschitz 1-cycles}, \\ \gamma = \sum_k a_k \gamma_k \text{ in } H_1(\bar{X}, \mathbb{R}) \Big\}.$$

It is folklore (*bounded distance theorem*, cp. [4], [5], and [11]) that, when Γ acts *cocompactly* by isometries, then (X, d) is almost isometric to $(\mathbb{Z}^n, \| \|_{\mathrm{st}})$: namely, for every $x_0 \in X$, there exists a constant C such that

$$| d(x_0, \gamma.x_0) - \| \gamma \|_{\mathrm{st}} | < C \quad \text{for all } \gamma \in \Gamma.$$

This fact was originally proved D. Burago for periodic metrics on $\mathbb{R}^n$ (see [2] and [10]); however, we were not able to find a complete proof of the general case in literature. The first purpose of this note is to investigate to what extent the constant C depends on the basic geometric invariants of X, i.e. to estimate how far a space admitting an abelian action is from a normed vector space. We prove:

Theorem 1.1 (quantitative bounded distance theorem). *Let $\Gamma = \mathbb{Z}^n$ act freely and properly discontinuously by isometries on a length space (X, d), with compact quotient. There exists a constant $c = c(n, D, \Omega)$ such that for all $x_0 \in X$*

$$| d(x_0, \gamma x_0) - \| \gamma \|_{\mathrm{st}} | < c(n, D, \Omega), \tag{1}$$

where D and Ω are, respectively, upper bounds for the codiameter and the asymptotic volume of Γ with respect to d.

We call *co-diameter* of Γ the diameter of the quotient $\bar{X} = \Gamma \backslash X$.

The *asymptotic volume* of a group $\Gamma \cong \mathbb{Z}^n$, endowed with a Γ-invariant metric d, is the asymptotic invariant defined as (cp. [14])

$$\omega(\Gamma, d) = \lim_{R\to\infty} \frac{\sharp\{\gamma \colon d(x, \gamma.x) < R\}}{R^n}.$$

[2] Since it can be bounded from below by a multiple of a word metric, cp. [5].

For Γ acting on (X, d) as above, for any choice of a base point $x_0 \in X$, the group Γ inherits from X a left-invariant distance d_{x_0}, by identification with the orbit Γx_0; the asymptotic volume $\omega(\Gamma, d_{x_0})$ clearly does not depend on x_0, and we shall simply write $\omega(\Gamma, d)$. For Γ acting on (X, d) as above, any choice of a base point $x_0 \in X$ yields a left-invariant distance d_{x_0} on Γ, by identification with the orbit $\Gamma.x_0$; clearly, the asymptotic volume $\omega(\Gamma, d_{x_0})$ does not depend on x_0, and we shall simply write $\omega(\Gamma, d)$. Moreover, given any Γ-invariant measure μ on X, it is easy to see, by a packing argument, that it equals the usual asymptotic volume of the measure metric space (X, d, μ) divided by the measure of the quotient, i.e.,

$$\omega_\mu(X, d) = \lim_{R\to\infty} \frac{\mu(B_{(X,d)}(x_0, R))}{R^n} = \mu(\bar{X}) \cdot \omega(\Gamma, d).$$

As a consequence of the QBD theorem 1.1, we have an explicit control of the growth function of balls and annuli in (Γ, d) (cp. Proposition 5.1), and of the Gromov-Hausdorff distance between $(X, \lambda d)$ and its asymptotic cone $(\mathbb{R}^n, \| \ \|_{\mathrm{st}})$ in terms of n, D, Ω:

$$d_{GH}((X, \lambda d),\ (\mathbb{R}^n, \| \ \|_{\mathrm{st}})) \le \lambda \cdot (c + 2D)$$

(notice that, for abelian groups endowed with a word metric, the linearity of the rate of convergence of $(X, \lambda d)$ to the asymptotic cone was already known, cp. [3]).

The QBD theorem 1.1 is obtained combining Burago's original idea with a careful control of the dilatation of "natural" maps $(\mathbb{R}^n, \mathrm{euc}) \leftrightarrows (X, d)$ quasi-inverse one to each other.[3] More precisely, the maps are induced from the identification of $\mathbb{Z}^n$ with a finite index subgroup $\mathcal{Z}$ of Γ generated by a set Σ_n of n linearly independent vectors (γ_k). The bounds on the codiameter and the asymptotic volume are then needed to control the index $[\Gamma : \mathcal{Z}]$ and the relative variation of d/d_{Σ_n} on $\mathcal{Z}$. For this, we prove in Section §3 a lemma *à la* Margulis[4] for abelian groups, which gives an estimate from below of the minimal displacement of Γ in terms of an upper bound on the asymptotic volume. Namely, let $\Gamma^* = \Gamma \setminus \{e\}$ and define, respectively, the systole and the stable systole of the action of Γ on (X, d) as

$$\mathrm{sys}(\Gamma, d) = \inf_{x\in X} \inf_{\gamma\in\Gamma^*} d(x, \gamma.x),$$

$$\mathrm{stsys}(\Gamma, d) = \inf_{\gamma\in\Gamma^*} \| \gamma \|_{\mathrm{st}}.$$

Then clearly $\mathrm{stsys}(\Gamma, d) \le \mathrm{sys}(\Gamma, d)$, and we prove:

[3] This difficulty does not emerge in [2], where $\Gamma \curvearrowright \mathbb{R}^n$, since in that case we have two metrics on a torus, which are obviously bi-Lipschitz via the identity map.

[4] The classical Margulis' lemma, in negative curvature, gives an estimate (at some point x_0) of the minimal displacement of a group Γ acting on a Cartan–Hadamard manifold X, under a lower bound on the curvature of X. It has been extended in several directions, in particular with a bound on the volume entropy replacing the bound on curvature, cp. [1] and [6].

Lemma 1.2 (abelian Margulis' lemma). *Let $\Gamma = \mathbb{Z}^n$ act freely and properly discontinuously by isometries on a length space (X, d), with cocompact quotient. Then*

$$\frac{2}{n!} \cdot \frac{1}{\mathrm{codiam}(\Gamma, d)^{n-1} \cdot \omega(\Gamma, d)} \leq \mathrm{stsys}(\Gamma, d) \leq \frac{2}{\omega(\Gamma, d)^{1/n}} \tag{2}$$

Moreover, these inequalities are optimal and the equality cases characterize, up to almost-isometric equivalence, the action of specific lattices of $\mathbb{R}^n$, endowed with particular polyhedral norms. Namely, if $\mathrm{codiam}(\Gamma, d) = D$, $\mathrm{stsys}(\Gamma, d) = \sigma$ *and* $\omega(\Gamma, d) \leq \Omega$, *then there exists a constant $C = C(n, D, \Omega)$ such that*

- *the equality holds in the left-hand side if and only if there is an equivariant, C-almost isometry $f: (X, d) \to (\mathbb{R}^n, \| \ \|_1)$, with respect to the action by translations of the lattice $\Gamma_0 = \sigma \cdot \mathbb{Z} \times 2D \cdot \mathbb{Z}^{n-1} \cong \Gamma$ on $\mathbb{R}^n$;*
- *the equality holds in the right-hand side if and only if there is an equivariant, C-almost isometry $f : (X, d) \to (\mathbb{R}^n, \| \ \|_h)$, with respect to the canonical action of Γ on $\mathbb{R}^n$ and where $\| \ \|_h$ is a **parallelohedral norm**;*[5] *that is, a norm whose unit ball is a **Γ-parallelohedron** (a convex polyhedron which tiles $\mathbb{R}^n$ under the action by translations of Γ, i.e. whose Γ-translates cover $\mathbb{R}^n$ and have disjoint interiors).*

The left-hand side of (2) shows that, provided that the co-diameter is bounded, an upper bound of the asymptotic volume is equivalent to a lower bound of the stable systole. Therefore, the constant $c(n, D, \Omega)$ in theorem 1.1 can as well be expressed in terms of rank, co-diameter and of a lower bound $\mathrm{stsys}(\Gamma, d) \geq \sigma$, instead of an upper bound $\omega(\Gamma, d) \leq \Omega$.

Notice that $\mathrm{stsys}(\Gamma, d)$ cannot be bounded below uniquely in terms of n and $\omega(\Gamma, d)$: any flat Riemannian torus (T, euc) with unitary volume has fundamental group $\Gamma = \pi_1(T)$ with asymptotic volume equal to the volume ω_n of the unit ball in $\mathbb{E}^n$, but the systole of Γ can be arbitrarily small (provided that the diameter of T is sufficiently large).

It is natural to ask whether one can drop the dependence of the constant c in the QBD theorem on any of the parameters n, D, Ω or σ, and possibly replace the dependence on the stable systole by a lower bound on the systole. In Section §4 we give counterexamples ruling out each of these possibilities. In particular,

[5] Examples of parallelohedral norms are

(i) in dimension $n = 2$, all norms whose unit ball is either a parallelogram or a convex hexagon with congruent opposite sides (these are the only convex polygons which tasselate $\mathbb{R}^2$ under the action by translations of a 2-dimensional lattice, cp. [8], [12], and [13]);

(ii) in dimension $n = 3$, there are precisely 37 types of parallelohedra, cp. [9], including for instance the standard n-cube (which gives rise to the sup-norm) or those obtained from $\mathbb{Z}^n$-tessellation by prisms with 2-dimensional base as in (i).

The complete classification of parallelohedral norms is a particular case of Hilbert's eighteenth problem (tiling the Euclidean space by congruent polyhedra) and will be not pursued further here.

one cannot generally bound $d - \| \ \|_{st}$ only in terms of rank, co-diameter and systole: there exists a sequence of actions of $\mathbb{Z}^n$ on length spaces (X_k, d_k) with $\mathrm{sys}(\mathbb{Z}^n, d_k) \geq 1$ and $\mathrm{codiam}(\mathbb{Z}^n, d_k) \leq 1$ such that the difference between d_k and the corresponding stable norms $| \ |_{st,k}$ is arbitrarily large, cp. Example 4.1. The same example also shows that a lower bound of the systole does not imply any upper bound for the asymptotic volume, i.e. the right-hand side of (2) does not hold with the stable systole replaced by the systole.

Finally, in Section §5 we use the QBD theorem to address the following basic problem on a closed Riemannian manifold $\bar{X}$: given an integral homology class $\gamma \in H_1(\bar{X}, \mathbb{Z})$, what is the minimal number $\#_{CC}$ of connected components of an optimal cycle in γ? Namely, we want to estimate the number

$$N(\gamma) = \min\{\#_{CC}(\mathsf{c}) \mid \mathsf{c} \in Z_1(\bar{X}, \mathbb{Z}),\ [\mathsf{c}] = \gamma,\ \ell(\mathsf{c}) = |\gamma|_{H_1}\},$$

where $|\gamma|_{H_1}$ is the *mass* in homology, i.e. the total length of a shortest,[6] possibly disconnected, collection of closed curves representing γ. We call *optimal* a cycle $\mathsf{c} \in [\gamma]$ which is length-minimizing in its class and having precisely the minimum number $N(\gamma)$ of connected components.

Recall that the *homological systole* $\mathrm{sys}_{H_1}(\bar{X})$ of $\bar{X}$ is the length of the shortest closed geodesic which is non-trivial in homology; if $\Gamma = H_1(\bar{X}, \mathbb{Z})$ and (X, d) is the Riemannian homology covering of $\bar{X}$, we clearly have $\mathrm{sys}_{H_1}(\bar{X}) = \mathrm{sys}(\Gamma, d)$. Notice that a lower bound of the homological systole $\mathrm{sys}_{H_1}(\bar{X}) \geq \sigma_1$ (as given for instance, in the torsionless case, by the left-hand side of the abelian Margulis' lemma 1.2) readily implies an estimate $N(\gamma) \leq \sigma_1^{-1}|\gamma|_{H_1}$. However, as an application of the QBD theorem, we actually show that $N(\gamma)$ is sublinear in $|\gamma|_{H_1}$.

Theorem 1.3. *Assume that $\bar{X}$ has first Betti number $b_1(\bar{X}) = n$,* $\mathrm{diam}(\bar{X}) < D$ *and $\omega(H_1(\bar{X}, \mathbb{Z})) < \Omega$. Then, for any torsionless homology class $\gamma \in H_1(\bar{X}, \mathbb{Z})$,*

(i) *either $N(\gamma)$ is bounded by an explicit, universal function $N(n, D, \Omega)$,*

(ii) *or*

$$N(\gamma) \leq 2 \cdot 3^{2n} \cdot \Omega^{\frac{1}{n+1}} \cdot |\gamma|_{H_1}^{\frac{n}{n+1}}.$$

We will see that one can take

$$N(n, D, \Omega) = 2^{18n^3} \cdot n^{2n} \cdot (n!)^{n(n+2)} \cdot (\Omega D^n + 1)^{6n^2}.$$

It is noticeable that the bound (ii) does not even depend on the diameter of $\bar{X}$.

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[6] Notice that a 1-cycle of minimal length in $\bar{X}$ always exists, and is given by a finite collection of closed geodesics, by general representation results of minimizers in homology by currents, and by regularity of rectifiable 1-currents.

2. QBD theorem

Let $\Gamma \cong \mathbb{Z}^n$ act freely, properly discontinuously and cocompactly by isometries on a length space (X, d). For any given $x_0 \in X$ we consider the left invariant metric on Γ given by $d_{x_0}(\gamma_1, \gamma_2) = d(\gamma_1.x_0, \gamma_2.x_0)$. We will write d_S for the word metric relative to a generating set S of a group, and also use the abridged notations $|\gamma|_{x_0} = d(x_0, \gamma.x_0)$, $|\gamma|_S = d_S(e, \gamma)$.

Assume that $\mathrm{diam}(\Gamma\backslash X) \leq D$, $\omega(\Gamma, d) \leq \Omega$ and $\mathrm{sys}(\Gamma, d) \geq \sigma$. We consider the generating set[7] $\Sigma_D = \{\gamma \in \Gamma^* \mid d(\gamma x_0, x_0) \leq 3D\}$, and we extract from Σ_D a set $\Sigma_n = \{\gamma_1, \dots, \gamma_n\}$ of n linearly independent vectors which generate a finite index subgroup $\mathcal{Z} = \langle \Sigma_n \rangle$, again isomorphic to $\mathbb{Z}^n$. Then, fix once and for all a set of representatives $S = \{s_0 = e, s_1, \dots, s_d\}$ for $\Gamma/\mathcal{Z}$ which are *minimal* for the word metric d_{Σ_D} associated to the generating set Σ_D of Γ.

Let us consider the map $f: (\mathcal{Z}, d_{x_0}) \to (\mathbb{Z}^n, \mathrm{euc})$ defined by sending each γ_i to the *i-th* vector of the standard basis of $\mathbb{R}^n$. We shall prove that f and f^{-1} are two Lipschitz maps, whose Lipschitz constants M and M' are bounded in terms of our geometric data n, D, Ω and σ; we will then extend f to a Lipschitz map $F: (X, d) \to (\mathbb{R}^n, \mathrm{euc})$. The purpose of the next lemmas is to estimate the constants M, M' by comparing with the dilatations of the following maps

$$f: (\mathcal{Z}, d_{x_0}) \longrightarrow (\mathcal{Z}, d_{\Sigma_D}|_{\mathcal{Z}}) \longrightarrow (\mathcal{Z}, d_{\widehat{\Sigma}_n}) \longrightarrow (\mathcal{Z}, d_{\Sigma_n}) \longrightarrow (\mathbb{Z}^n, \mathrm{euc}), \tag{3}$$

where

- $d_{\Sigma_D}|_{\mathcal{Z}}$ is the restriction of the word metric d_{Σ_D} to $\mathcal{Z}$;
- d_{Σ_n} is the word metric on $\mathcal{Z}$ relative to Σ_n;
- $d_{\widehat{\Sigma}_n}$ is the word metric on $\mathcal{Z}$ relative to the generating set $\widehat{\Sigma}_n$ of $\mathcal{Z}$ defined by

$$\widehat{\Sigma}_n = \{s_i\, \sigma\, s_j^{-1} \mid s_i \in S,\ \sigma \in \Sigma_D \text{ and } s_i\, \sigma\, s_j^{-1} \in \mathcal{Z}^*\}$$

(with $B_{(\mathcal{Z}, d_{x_0})}(r)$, $B_{(\mathcal{Z}, \Sigma_D)}(r)$, $B_{(\mathcal{Z}, \Sigma_n)}(r)$ and $B_{(\mathcal{Z}, \widehat{\Sigma}_n)}(r)$ the relative balls centered at e.)

Notice that $\Sigma_n \subset \widehat{\Sigma}_n$ (since $s_0 = e \in S$), but we might have $\Sigma_D \not\subset \widehat{\Sigma}_n$.

Moreover, remark that $(\mathcal{Z}, d_{\Sigma_n})$ is isometric to $\mathbb{Z}^n$ endowed with the canonical word metric $\|\ \|_1$, so we have

$$\frac{1}{\sqrt{n}} \cdot |\,\gamma\,|_{\Sigma_n} \leq \|f(\gamma)\|_{\mathrm{euc}} \leq |\,\gamma\,|_{\Sigma_n}, \tag{4}$$

$$\omega(\mathcal{Z}, d_{\Sigma_n}) = \frac{2^n}{n!}. \tag{5}$$

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[7] The elements with $d(\gamma x_0, x_0) \leq 2D$ suffice to generate Γ, cp. [11], p. 91; the constant $3D$ is chosen here to bound from below d_{x_0}/d_Σ.

Lemma 2.1 (cp. [11]). *The set Σ_D is a generating set for Γ such that*

$$\frac{\sigma}{2} \cdot |\,\gamma\,|_{\Sigma_D} \leq d(x_0, \gamma.x_0) \leq 3D \cdot |\,\gamma\,|_{\Sigma_D}.$$

Lemma 2.2. *For all $\gamma \in \mathcal{Z}$ we have*

$$|\gamma|_{\hat{\Sigma}_n} \leq |\gamma|_{\Sigma_D} \leq (2\,[\Gamma : \mathcal{Z}] + 1) \cdot |\gamma|_{\hat{\Sigma}_n}.$$

Proof. Let $\gamma = \gamma_1 \dots \gamma_\ell \in \mathcal{Z}$ with $\gamma_i \in \Sigma_D$. Assume that $\gamma_1 \dots \gamma_i\, \mathcal{Z} = s_{k_i}\, \mathcal{Z}$, then γ can be written as

$$\gamma = (s_{k_0}\gamma_1 \cdot s_{k_1}^{-1}) \cdot (s_{k_1}\, \gamma_2\, s_{k_2}^{-1}) \cdot \dots \cdot (s_{k_{\ell-2}}\, \gamma_{\ell-1}\, s_{k_{\ell-1}}^{-1}) \cdot (s_{k_{\ell-1}}\, \gamma_\ell s_{k_l})$$

with $s_{k_0} = s_{k_l} = e$, and any $s_{k_{i-1}}^{-1}\, \gamma_i s_{k_i}$ either is trivial or belongs to $\hat{\Sigma}_n$.

Therefore $|\,\gamma\,|_{\hat{\Sigma}_n} \leq |\,\gamma\,|_{\Sigma_D}$. For the second inequality, recall that any class $s_i\, \mathcal{Z}$ can be written as $s_i\, \mathcal{Z} = \gamma_1 \dots \gamma_k\, \mathcal{Z}$ with $\gamma_i \in \Sigma_D$ and $k \leq [\Gamma : \mathcal{Z}]$. So, every representative s_i, being Σ_D-minimal, satisfies $|\,s_i\,|_{\Sigma_D} \leq [\Gamma : \mathcal{Z}]$, which implies $|\,\gamma\,|_{\Sigma_D} \leq (2\,[\Gamma : \mathcal{Z}] + 1) \cdot |\,\gamma\,|_{\hat{\Sigma}_n}$. □

Lemma 2.3. *The subgroup $\mathcal{Z}$ satisfies*

(i) $$[\Gamma : \mathcal{Z}] \leq \frac{n!}{2^n}\Omega(3D)^n;$$

(ii) $$d_{x_0}(\gamma, \mathcal{Z}) \leq \frac{n!}{2^n}\Omega(3D)^{n+1}, \quad \textit{for any } \gamma \in \Gamma;$$

(iii) $$\operatorname{diam}(\mathcal{Z}\backslash X) \leq D + \Omega\frac{n!}{2^n}(3D)^{n+1}.$$

Proof. We consider the set $S = \{s_i\}_{i=0,\dots,d}$ of representatives of $\Gamma/\mathcal{Z}$ with minimal Σ_D-length. Let $M = \max_{s_i} |\,s_i\,|_{\Sigma_D}$. Then

$$\#B_{(\Gamma,\Sigma_D)}(R) \geq [\Gamma : \mathcal{Z}] \cdot \#B_{(\mathcal{Z},\Sigma_D)}(R - M) \geq [\Gamma : \mathcal{Z}] \cdot \#B_{(\mathcal{Z},d_{\Sigma_n})}(R - M).$$

Dividing by R^n and taking the limit for $R \to +\infty$ yields

$$[\Gamma : \mathcal{Z}] \leq \frac{\omega(\Gamma, d_{\Sigma_D})}{\omega(\mathcal{Z}, d_{\Sigma_n})}.$$

By Lemma 2.1 we have

$$\omega(\Gamma, d_{\Sigma_D}) \leq (3D)^n \omega(\Gamma, d) \leq (3D)^n \Omega,$$

while $\omega(\mathcal{Z}, d_{\Sigma_n}) = {}^{2^n}/_{n!}$ by (5); this proves (i).

To prove (ii), notice that the set $\{\gamma\mathcal{Z} \mid \gamma \in \Sigma_D\}$ generates $\Gamma/\mathcal{Z}$, and that every class $s_i\mathcal{Z}$ is the product of at most $[\Gamma : \mathcal{Z}]$ classes $\gamma_i\mathcal{Z}$ with $\gamma_i \in \Sigma_D$. Since any element of $\gamma \in \Gamma$ lies in some coset $s_i\mathcal{Z}$, the Σ_D-distance of γ from $\mathcal{Z}$ is at most $[\Gamma : \mathcal{Z}]$. Then, Lemma 2.1 yields $d_{x_0}(\gamma, \mathcal{Z}) \leq 3D \cdot [\Gamma : \mathcal{Z}]$.

Assertion (iii) then follows from (ii), as $\operatorname{diam}(\Gamma\backslash X) \leq D$. □

Lemma 2.4. *The generating set $\widehat{\Sigma}_n$ of $\mathcal{Z}$ satisfies*:

(i) $\omega(\mathcal{Z}, d_{\widehat{\Sigma}_n}) \leq (2^{n+3}n!)^n \cdot \Omega D^n \cdot (\Omega D^n + 1)$;

(ii)
$$\begin{aligned}|\hat{\gamma}|_{\Sigma_n} &\leq L(n, D, \Omega)\\ &= 2^{n^2+4n+3}(n!)^{n+1} \cdot \Omega D^n \cdot (\Omega D^n + 1)^n \quad \textit{for all } \hat{\gamma} \in \widehat{\Sigma}_n;\end{aligned}$$

(iii) $|\gamma|_{\Sigma_n} \leq L(n, D, \Omega) \cdot |\gamma|_{\widehat{\Sigma}_n}$ *for all* $\gamma \in \mathcal{Z}$.

Proof. By Lemmas 2.1 and 2.2 we have

$$|\gamma|_{\widehat{\Sigma}_n} \geq \frac{1}{3D(2\,[\Gamma : \mathcal{Z}] + 1)} d(\gamma x_0, x_0),$$

hence

$$\omega(\mathcal{Z}, d_{\widehat{\Sigma}_n}) \leq [3D(2\,[\Gamma : \mathcal{Z}] + 1)]^n \omega(\Gamma, d_{x_0}),$$

so (i) follows from Lemma 2.3.

To prove (ii), assume that $\hat{\gamma} \in \widehat{\Sigma}_n$ has Σ_n-length ℓ, so it can be written as a product $\hat{\gamma} = \gamma_{i_1} \ldots \gamma_{i_\ell}$, with every $\gamma_{i_k} \in \Sigma_n$. The sequence $(\gamma_{i_1}, \ldots, \gamma_{i_\ell})$ corresponds to a geodesic path c_0 in the Cayley graph $\mathcal{C}(\mathcal{Z}, \Sigma_n)$. Let c be the path in $\mathcal{C}(\mathcal{Z}, \Sigma_n)$ obtained by concatenation of all the paths $c_k = \hat{\gamma}^k.c_0$; notice that, since $(\mathcal{Z}, \Sigma_n)$ is isometric to $(\mathbb{Z}^n, |\ |_1)$, the path c is still geodesic. Consider now a new generating set: $\Sigma_n(\hat{\gamma}) = \Sigma_n \cup \{\hat{\gamma}\} \subset \widehat{\Sigma}_n$ and call for brevity $d_{\hat{\gamma}}$ the corresponding word metric. Chosen a radius $R = m\ell$, for $m > 0$, we consider the points $P_i = \hat{\gamma}^{2mi}$ on the geodesic c, and we remark that

$$\bigsqcup_{i=0}^{\lfloor \ell/2 \rfloor} B_{(\mathcal{Z}, d_{\Sigma_n})}(P_i, R - 2mi) \subset B_{(\mathcal{Z}, d_{\hat{\gamma}})}(e, R).$$

Actually, for $j \neq i \leq \ell/2$ the balls

$$B_{(\mathcal{Z}, d_{\Sigma_n})}(P_j, R - 2mj) \quad \text{and} \quad B_{(\mathcal{Z}, d_{\Sigma_n})}(P_i, R - 2mi)$$

are disjoint, since $d_{\Sigma_n}(P_i, P_j) \geq |\hat{\gamma}^{2m}|_{\Sigma_n} = 2m\ell = 2R$; moreover, these balls are all contained in $B_{(\mathcal{Z}, d_{\hat{\gamma}})}(e, R)$ as $d_{\hat{\gamma}}(e, P_i) = |\hat{\gamma}^{2mi}|_{\Sigma_n(\hat{\gamma})} \leq 2mi$.

Also, notice that, as $\omega(\mathcal{Z}, d_{\Sigma_n}) = 2^n/n!$, we have

$$\# B_{(\mathcal{Z}, \Sigma_n)}(P_j, R - 2mi) = \# B_{(\mathcal{Z}, \Sigma_n)}(m(\ell - 2i)) \geq \frac{2^{n-1}}{n!} m^n (\ell - 2i)^n$$

for $m \gg 0$. Thus

$$\# B_{(\mathcal{Z}, d_{\hat{\gamma}})}(R) \geq \sum_{i=0}^{\lfloor \ell/2 \rfloor} \frac{2^{n-1}}{n!} m^n (\ell - 2i)^n \geq \frac{2^{n-1}}{n!} m^n \ell^n \sum_{i=0}^{\lfloor \ell/3 \rfloor} \left(\frac{2}{3}\right)^n \geq \left(\frac{4}{3}\right)^n \frac{\ell R^n}{6 \cdot n!}$$

which shows that $\omega(\mathcal{Z}, d_{\hat{\gamma}}) \geq (\frac{4}{3})^n \frac{\ell}{6\cdot n!}$. On the other hand, we know by the above lemmas that

$$| \ |_{\Sigma_n(\hat{\gamma})} \geq | \ |_{\hat{\Sigma}_n} \geq \frac{2^{n-1}}{3D(n! \cdot \Omega(3D)^n + 2^n)} \cdot | \ |_{x_0},$$

so

$$\left(\frac{4}{3}\right)^n \frac{\ell}{6\cdot n!} \leq \omega(\mathcal{Z}, d_{\hat{\gamma}}) \leq (2^{n+3} \cdot n!)^n \cdot \Omega D^n (\Omega D^n + 1)^n,$$

which gives (ii). The third statement clearly follows from (ii). □

We deduce by the lemmas above that the map f defined in (3) is a bi-Lipschitz map, with Lipschitz constants given by

$$M(f) \leq M = M(n, \Omega, D, \sigma) = \frac{1}{\sigma} \cdot 2^{n^2+4n+4} \cdot (n!)^{n+1} \cdot \Omega D^n (\Omega D^n + 1)^n, \quad (6)$$

$$M(f^{-1}) \leq M' = M'(n, \Omega, D) = 8\left(\frac{3}{2}\right)^n \sqrt{n} \cdot n! \cdot \Omega D^{n+1}. \quad (7)$$

We will prove in the next section that we can get rid of the dependence on σ.

Now, we extend f to a M''-Lipschitz map $F : (X, d) \to (\mathbb{R}^n, \mathrm{euc})$, with $M'' = \sqrt{n}M$, by extending each coordinate function f_i of f as follows

$$F_i(x) = \inf_{\gamma \in \mathcal{Z}} (f_i(\gamma.x_0) + M(f) \cdot d(\gamma.x_0, x))$$

(notice that each F_i is M-Lipschitz, and then F is $\sqrt{n}M$-Lipschitz).

2.1. End of the proof of Theorem 1.1. We switch now to the additive notation for the abelian groups Γ and $\mathcal{Z}$, for easier comparison with $\mathbb{Z}^n$. Assume first that $\gamma \in \mathcal{Z}$, and let $\mathsf{c}: I = [0, \ell] \to X$ be a minimizing geodesic (i.e. $d(\mathsf{c}(t), \mathsf{c}(t')) = |t - t'|$) from x_0 to $2\gamma.x_0$. Then, we apply the following lemma due to D. Burago and G. Perelman to the path $\mathsf{c}_o = F \circ \mathsf{c}$, going from the origin o of $\mathbb{R}^n$ to $2f(\gamma).o$.

Lemma 2.5 (D. Burago and G. Perelman). *Let* $\mathsf{c}: I = [0, \ell] \to \mathbb{R}^n$ *be a Lipschitz path. There exists an open set* $A = \bigcup_{i=1}^m (a_i, b_i) \subset [0, \ell]$ *with* $m \leq n$ *and with Lebesgue measure* $\lambda(A) \leq 1/2\, \lambda(I) = \ell/2$ *such that*

$$\sum_{i=1}^m (\mathsf{c}_o(b_i) - \mathsf{c}_o(a_i)) = \frac{\mathsf{c}_o(\ell) - \mathsf{c}_o(0)}{2}.$$

This lemma provides a new path $\mathsf{c}_o/2: J = [0, \lambda(A)] \to \mathbb{R}^n$ going from the origin o to $f(\gamma).o = \sum_1^m (\mathsf{c}_o(b_i) - \mathsf{c}_o(a_i))$, defined concatenating the paths $\mathsf{c}_{o,i} = \mathsf{c}_o|_{[a_i, b_i]}$

$$\frac{\mathsf{c}_o}{2} = \mathsf{c}_{o,1} * \cdots * \mathsf{c}_{o,m} - \mathsf{c}_o(a_1)$$

(where $\alpha * \beta$ in $\mathbb{R}^n$ means that the path β is translated in order that its origin coincides with the endpoint of α). Consider now, for each $i = 1, \dots, m$, the orbit points $\alpha_i.x_0, \beta_i.x_0 \in \mathbb{Z}x_0$ closest respectively to $\mathsf{c}(a_i)$, $\mathsf{c}(b_i) \in X$, and let c_i be a minimizing geodesic from $\alpha_i.x_0$ to $\beta_i.x_0$. Then, let $\mathsf{c}'\colon [0, \ell'] \to X$ be the curve

$$\mathsf{c}' = \mathsf{c}'_1 * \dots * \mathsf{c}'_m - \alpha_1.x_0$$

that is, the concatenation of ($\mathbb{Z}$-translated of) the geodesics c_i such that the endpoint of $\mathsf{c}'_1 * \dots * \mathsf{c}'_{i-1}$ coincides with the origin of c'_i, and $\mathsf{c}'(0) = x_0$. Finally, let $\gamma'.x_0 = [\sum_1^m (\beta_i - \alpha_i)].x_0$ be the endpoint of c'.

Notice that, as the c'_i are geodesics,

$$\begin{aligned} d(x_0, \gamma'.x_0) &\leq \sum_{i=1}^{m} \ell(\mathsf{c}'_i) \\ &\leq 2n \operatorname{diam}(\mathbb{Z}\backslash X) + \sum_{i=1}^{m} d(\mathsf{c}(a_i), \mathsf{c}(b_i)) \\ &\leq 2n \cdot \operatorname{diam}(\mathbb{Z}\backslash X) + \frac{\ell}{2}. \end{aligned}$$

Moreover,

$$F(\gamma'.x_0) = \sum_{i=1}^{m} (F(\beta_i.x_0) - F(\alpha_i.x_0))$$

as $F = f$ on $\mathbb{Z}x_0$, so

$$\begin{aligned} &\|F(\gamma'.x_0) - F(\gamma.x_0)\|_{\text{euc}} \\ &\quad = \|\sum_i [F(\beta_i.x_0) - F(\alpha_i.x_0)] - \sum_i [\mathsf{c}_o(b_i) - \mathsf{c}_o(a_i)]\|_{\text{euc}} \\ &\quad \leq \sum_{i=1}^{m} \|F(\beta_i.x_0) - F(\mathsf{c}(b_i))\|_{\text{euc}} + \sum_{i=1}^{m} \|F(\alpha_i.x_0) - F(\mathsf{c}(a_i))\|_{\text{euc}} \\ &\quad \leq 2nM'' \cdot \operatorname{diam}(\mathbb{Z}\backslash X) \end{aligned}$$

and from this and the Lipschitz property of f^{-1} we deduce that

$$d(\gamma'.x_0, \gamma.x_0) = d(f^{-1}(F(\gamma'.x_0)), f^{-1}(F(\gamma.x_0))) \leq 2nM'M'' \cdot \operatorname{diam}(\mathbb{Z}\backslash X).$$

Then,

$$\begin{aligned} d(x_0, \gamma.x_0) &\leq d(x_0, \gamma'.x_0) + d(\gamma'.x_0, \gamma.x_0) \\ &\leq \frac{\ell}{2} + 2n(M'M'' + 1) \cdot \operatorname{diam}(\mathbb{Z}\backslash X) \end{aligned}$$

that is,

$$d(x_0, \gamma.x_0) \le \frac{1}{2} d(x_0, 2\gamma.x_0) + M'''$$

for a constant

$$M''' = M'''(n, D, \Omega, \sigma) = 2n(M'M'' + 1)\,\mathrm{diam}(\mathcal{Z}\backslash X)$$

which is given explicitly by (6), (7), and Lemma 2.3.

This implies the announced inequality $|d(x_0, \gamma.x_0) - \|\gamma\|_{\mathrm{st}}| \le M'''$ for all $\gamma \in \mathcal{Z}$. To get the inequality for all $\gamma \in \Gamma$, let $\gamma_0.x_0$ be a point of $\mathcal{Z}.x_0$ closest to $\gamma.x_0$; then

$$\begin{aligned} |d(x_0, \gamma.x_0) - \|\gamma\|_{\mathrm{st}}| &\le |d(x_0, \gamma_0.x_0) - \|\gamma_0\|_{\mathrm{st}}| + 2\, d(\gamma.x_0, \gamma_0.x_0) \\ &\le c(n, D, \Omega, \sigma) \end{aligned}$$

for

$$c(n, D, \Omega, \sigma) = 2\,\mathrm{diam}(\mathcal{Z}\backslash X)(nM'M'' + n + 1).$$

In the next section we show that the constant c actually does not depend on σ.

3. Stable systole and asymptotic volume

We prove here the two relations of (almost) inverse proportionality between $\omega(\Gamma, d)$ and $\mathrm{stsys}(\Gamma, d)$. First notice that, as $\|\cdot\|_{\mathrm{st}}$ is a true norm, the ball of radius $2D$ in $(\Gamma, \|\cdot\|_{\mathrm{st}})$ is compact; so, there exists $\gamma_1 \in \Gamma = \mathbb{Z}^n$ realizing the stable systole. Let $\sigma = \|\gamma_1\|_{\mathrm{st}} = \mathrm{stsys}(\Gamma, d)$ and $D = \mathrm{codiam}(\Gamma, d) = \mathrm{diam}(\Gamma\backslash X)$.

3.1. Proof of the abelian Margulis lemma, upper bound.

Let

$$\mathcal{D}_{\mathrm{st}} = \{p \mid \|p\|_{\mathrm{st}} < \|p - \gamma.o\|_{\mathrm{st}}\}$$

and

$$\widehat{\mathcal{D}}_{\mathrm{st}} = \{p \mid \|p\|_{\mathrm{st}} \le \|p - \gamma.o\|_{\mathrm{st}}\}$$

be respectively the *open* and *closed* Dirichlet domains of Γ acting on $\mathbb{R}^n$, centered at the origin, with respect to the stable norm, and let $M \ge \mathrm{diam}(\widehat{\mathcal{D}}_{\mathrm{st}})$. Notice that, in general, the closure of $\mathcal{D}_{\mathrm{st}}$ might be strictly included in $\widehat{\mathcal{D}}_{\mathrm{st}}$, and neither $\overline{\mathcal{D}}_{\mathrm{st}}$ nor $\widehat{\mathcal{D}}_{\mathrm{st}}$ a priori tile $\mathbb{R}^n$ under the action of Γ (think for instance of the Dirichlet domain of $2\mathbb{Z}\times\mathbb{Z}$ acting on $(\mathbb{R}^2, \|\ \|_\infty)$). So, let $\mathcal{F}$ be a closed fundamental domain such that $\mathcal{D}_{\mathrm{st}} \subset \mathcal{F} \subset \widehat{\mathcal{D}}_{\mathrm{st}}$; that is, $\bigcup_{\gamma\in\Gamma} \gamma.\mathcal{F} = \mathbb{R}^n$ and $\gamma.\mathring{\mathcal{F}} \cap \gamma'.\mathring{\mathcal{F}} = \varnothing$ for $\gamma \ne \gamma'$.

The open ball $B_{\mathrm{st}}(r) = \{p \mid \|p\|_{\mathrm{st}} < r\}$ is included in $\mathcal{D}_{\mathrm{st}}$ for $r = \frac{\sigma}{2}$, so

$$\begin{aligned} \frac{\#[B_{(X,d)}(x_0, R) \cap \Gamma]}{R^n} &\leq \frac{\#[B_{\mathrm{st}}(R) \cap \mathbb{Z}^n]}{R^n} \\ &\leq \frac{\mathrm{Vol}(B_{\mathrm{st}}(R+M))}{\mathrm{Vol}(\mathcal{F}) \cdot R^n} \\ &\leq \frac{\mathrm{Vol}(B_{\mathrm{st}}(R+M))}{\mathrm{Vol}(\mathcal{D}_{\mathrm{st}}) \cdot R^n} \\ &\leq \frac{\mathrm{Vol}(B_{\mathrm{st}}(R+M))}{\mathrm{Vol}(B_{\mathrm{st}}(\frac{\sigma}{2})) \cdot R^n} \\ &= \frac{2^n (R+M)^n}{\sigma^n \cdot R^n} \end{aligned} \tag{8}$$

and taking limits for $R \to \infty$ yields the announced inequality.

Clearly, this inequality is an equality for the standard lattice $\mathbb{Z}^n$ in $(\mathbb{R}^n, \| \, \|_\infty)$, but this is not the only case in which the equality is satisfied. Actually, assume that the equality $\omega(\Gamma, d) = \frac{2^n}{\sigma^n}$ holds: then, all the inequalities in (8) are equalities for $R \to \infty$, so $\mathrm{Vol}(B_{\mathrm{st}}(\frac{\sigma}{2})) = \mathrm{Vol}(\mathcal{D}_{\mathrm{st}}) = \mathrm{Vol}(\mathcal{F})$. Since $B_{\mathrm{st}}(\frac{\sigma}{2}) \subset \mathcal{D}_{\mathrm{st}} \subset \mathring{\mathcal{F}}$, we deduce that $B_{\mathrm{st}}(\frac{\sigma}{2}) = \mathcal{D}_{\mathrm{st}} = \mathring{\mathcal{F}}$. This implies that $\bar{\mathcal{D}}_{\mathrm{st}} \subset \mathcal{F}$ is a convex set (being the closure of a ball) which tiles $\mathbb{R}^n$ under the action of Γ. Actually, assume that there exists $p \in \mathbb{R}^n \setminus \Gamma.\bar{\mathcal{D}}_{\mathrm{st}}$. Then, $\mathbb{R}^n \setminus \bigcup_{\|\gamma\|_{\mathrm{st}} \leq \|p\| + 2M} \gamma.\bar{\mathcal{D}}_{\mathrm{st}}$ is a non-empty open set, containing a small ball $B_{\mathrm{st}}(p, \varepsilon)$ centered at p. As $\mathcal{F}$ tiles, there exists γ such that $\mathrm{Vol}(\mathcal{F} \cap \gamma B_{\mathrm{st}}(p, \varepsilon)) \neq 0$. This yields a contradiction, as $\bar{\mathcal{D}}_{\mathrm{st}} \subset \mathcal{F} \setminus \gamma B_{\mathrm{st}}(p, \varepsilon)$ but $\mathrm{Vol}(\bar{\mathcal{D}}_{\mathrm{st}}) = \mathrm{Vol}(\mathcal{F})$.

We show now that $\bar{\mathcal{D}}_{\mathrm{st}}$ is a polyhedron. For this, let us first show that the topological boundary $\partial \bar{\mathcal{D}}_{\mathrm{st}}$ is covered by a finite number of hyperplanes: actually, as the closed sets $\gamma \bar{\mathcal{D}}_{\mathrm{st}}$ tile, we have

$$\partial \mathcal{D}_{\mathrm{st}} = \bigcup_{0 < \|\gamma\| \leq 2M} (\partial \mathcal{D}_{\mathrm{st}} \cap \gamma.\partial \mathcal{D}_{\mathrm{st}}) = \bigcup_{0 < \|\gamma\| \leq 2M} (\bar{\mathcal{D}}_{\mathrm{st}} \cap \gamma.\bar{\mathcal{D}}_{\mathrm{st}})$$

and as $\bar{\mathcal{D}}_{\mathrm{st}} \cap \gamma.\bar{\mathcal{D}}_{\mathrm{st}}$ is a convex set with zero measure, it is contained in an affine hyperplane $H_\gamma = \{p \mid f_\gamma(p) = 1\}$, for some linear function f_γ; since $\bar{\mathcal{D}}_{\mathrm{st}}$ is convex, we may assume that $\bar{\mathcal{D}}_{\mathrm{st}} \subset H_\gamma^-$, where H_γ^- denotes the sub-level set $f_\gamma \leq 1$. Let Γ_0 be the subset of nontrivial elements $\gamma \in \Gamma$ such that $\bar{\mathcal{D}}_{\mathrm{st}} \cap \gamma.\bar{\mathcal{D}}_{\mathrm{st}} \neq \emptyset$. It then follows that $\bar{\mathcal{D}}_{\mathrm{st}} = \bigcap_{\gamma \in \Gamma_0} H_\gamma^-$. The inclusion $\bar{\mathcal{D}}_{\mathrm{st}} \subset \bigcap_{\gamma \in \Gamma_0} H_\gamma^-$ is clear. On the other hand, given $p \in \bigcap_{\gamma \in \Gamma_0} H_\gamma^-$, if $p \notin \bar{\mathcal{D}}_{\mathrm{st}}$ then the segment $\overline{op}$ intersects $\partial \mathcal{D}_{\mathrm{st}}$ at some point tp, for $0 < t < 1$, hence there exists some f_γ such that $f_\gamma(tp) = 1$; hence $f_\gamma(p) > 1$, a contradiction. This shows that $\bar{\mathcal{D}}_{\mathrm{st}} = \bar{B}_{\mathrm{st}}(\frac{\sigma}{2})$ is a convex polyhedron tiling $\mathbb{R}^n$ under the action of Γ, i.e. a Γ-parallelohedron (and $B_{\mathrm{st}}(1)$ as well).

Finally, as $\left|d - \| \ \|_{\mathrm{st}}\right| < c(n, D, \Omega)$ on $\Gamma.x_0$ by the QBD theorem, by identifying the orbit $\Gamma.x_0$ with $\mathbb{Z}^n$ we deduce a Γ-equivariant map

$$f : (X, d) \longrightarrow (R^n, \| \ \|_{\mathrm{st}})$$

which is a C-almost isometry for

$$\begin{aligned} C &= c(n, D, \Omega) + 2\,\mathrm{codiam}(\Gamma, d) + \mathrm{codiam}(\mathbb{Z}^n, \| \cdot \|_{\mathrm{st}}) \\ &\leq c(n, D, \Omega) + 2D + \sigma. \end{aligned}$$

3.2. Proof of the abelian Margulis lemma, lower bound. As γ_1 realizes the stable systole, for any $\varepsilon > 0$ there exists a K_ε such that

$$(1 - \varepsilon)\,|k|\,\sigma \leq |\gamma_1^k|_{x_0} \leq (1 + \varepsilon)\,|k|\,\sigma \text{ for all } |k| > K_\varepsilon.$$

Complete γ_1 to a set $\Sigma_n = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ of n linearly independent vectors, taking $\gamma_2, \dots, \gamma_n$ from the generating set $\Sigma'_D = \{\gamma \in \Gamma \mid d(x_0, \gamma.x_0) \leq 2D\}$, and let $\mathcal{Z} = \langle \Sigma_n \rangle$. Then, consider the norm $\| \ \|_{\sigma,2D}$ given by the weighted ℓ_1-norm on $\mathbb{R}^n$, relative to the basis Σ_n, with weights $\ell(\gamma_1) = \sigma$ and $\ell(\gamma_i) = 2D$ for $i \neq 1$. Finally, let $\mathcal{Z}_\varepsilon := \{\gamma_1^{k_1} \dots \gamma_n^{k_n} \mid |j_1| > K_\varepsilon\}$. Then, for all $\gamma \in \mathcal{Z}_\varepsilon$ we have

$$\begin{aligned} |\gamma|_{x_0} &\leq |\gamma_1^{k_1}|_{x_0} + \sum_{k=2}^{n} |\gamma_1^{k_i}|_{x_0} \\ &\leq (1 + \epsilon)|k_1| \cdot \sigma + 2D \sum_{k=2}^{n} |k_i| \\ &\leq (1 + \epsilon) \cdot \|\gamma\|_{\sigma,2D}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \omega(\Gamma, d_{x_0}) &\geq \omega(\mathcal{Z}, d_{x_0}|_{\mathcal{Z}}) \\ &\geq \omega(\mathcal{Z}_\varepsilon, d_{x_0}|_{\mathcal{Z}_\varepsilon}) \\ &\geq \frac{\omega(\mathcal{Z}_\varepsilon, \| \ \|_{\sigma,2D})}{(1 + \varepsilon)^n} \\ &= \frac{\omega(\mathcal{Z}_\varepsilon, d_{\Sigma_n})}{(1 + \varepsilon)^n\, \sigma (2D)^{n-1}} \end{aligned} \tag{9}$$

which gives the announced bound, as $\epsilon > 0$ is arbitrary and since

$$\omega(\mathcal{Z}_\varepsilon, d_{\Sigma_n}) = \omega(\mathcal{Z}, d_{\Sigma_n}) = \omega(\mathbb{Z}^n, \| \ \|_1) = \frac{2^n}{n!};$$

actually, the set $\mathcal{Z} \setminus \mathcal{Z}_\varepsilon$ has polynomial growth of order $n - 1$ and is negligible in the computation of the asymptotic volume, while $(\mathcal{Z}, d_{\Sigma_n})$ is isometric to $\mathbb{Z}^n$ endowed with the canonical word metric.

Notice that the equality holds for the action of the standard lattice $\mathbb{Z}^n$ on $(\mathbb{R}^n, \| \ \|_1)$. Furthermore, assume that, for Γ acting on (X, d), we have the equality

$$\omega(\Gamma, d_{x_0}) = \frac{2}{n! D^{n-1} \sigma}.$$

In particular, the first inequality in (9) is an equality, which implies $[\Gamma : \mathcal{Z}] = 1$. Moreover, we deduce that $\omega(\Gamma, \| \ \|_{\text{st}}) = \omega(\Gamma, d_{x_0}) = \omega(\Gamma, \| \ \|_{\sigma,2D})$. However, by construction, the stable and weighted norms satisfy $\| \ \|_{\sigma,2D} \geq \| \ \|_{\text{st}}$; then, being norms, we know that the equality of asymptotic volumes implies the equality of 1-balls, so $\| \ \|_{\sigma,2D} = \| \ \|_{\text{st}}$. Therefore $\| \ \|_{\text{st}}$ is affine equivalent to the ℓ_1 norm $\| \cdot \|_1$, via an affine map sending $\mathbb{Z}^n$ to the lattice $\Gamma_0 = \sigma \cdot \mathbb{Z} \times 2D \cdot \mathbb{Z}^{n-1}$ of $\mathbb{R}^n$. It follows by the QBD theorem that the action of Γ on (X, d) is equivalent, via an equivariant C-almost isometry $f : (X, d) \to (\mathbb{R}^n, \| \cdot \|_1)$, to the action of Γ_0 on $(\mathbb{R}^n, \| \cdot \|_1)$, for

$$\begin{aligned} C &= c(n, D, \Omega) + 2\,\text{codiam}(\Gamma, d) + \text{codiam}(\Gamma_0, \| \cdot \|_1) \\ &\leq c(n, D, \Omega) + (2n + 2)D. \end{aligned}$$

Remark 3.1. Let $\sigma = \text{sys}(\Gamma, d) \geq \sigma$ and $\omega(\Gamma, d) \leq \Omega$ as in Section §2.

(i) Using the lower bound given by the abelian Margulis lemma and the fact that $D \geq \sigma/2$, we find $\Omega D^n \geq \omega(\Gamma, d) D^n \geq 1/n!$. This estimate, together with (iii) of Lemma 2.3, plugged in the expressions (6) and (7) for M, M', and in the expressions for M'', M''' and $c = M''' + 2\,\text{diam}(\mathcal{Z} \backslash X)$ of §2.1, yields the following estimate for the constant c of the QBD theorem:

$$c(n, D, \Omega, \sigma) = c(n, D, \Omega) \leq 2^{n^2+6n+10} \cdot n^2 \cdot (n!)^{n+2} \cdot D(\Omega D^n + 1)^{n+4}$$

Notice that the quantity ΩD^n is scale invariant.

(ii) We also remark, for future reference, that the same computations show that the constant c that we find is $\gg nD$, namely $c(n, D, \Omega) \geq 2^{n^2+6n+8} n^2 (n!)^n D$.

Remark 3.2. As a consequence, we have the explicit bound

$$\left| \frac{|\gamma|_{x_0}}{\| \gamma \|_{\text{st}}} - 1 \right| \leq \frac{c(n, D, \Omega)}{\| \gamma \|_{\text{st}}}.$$

This should be compared with an asymptotics given by Gromov in [11], pp. 247–249:

$$\left| \frac{|\gamma|_{x_0}}{|\gamma|_{H_1}} - 1 \right| \leq \frac{c_{\bar{X}}}{|\gamma|_{H_1}^{n-1}} \tag{10}$$

for the mass of $\gamma \in H_1(\bar{X}, \mathbb{Z})$. Notice however that Gromov's bound is purely qualitative (no information can be deduced on the constant $c_{\bar{X}}$ from his argument) and that we always have $\| \gamma \|_{\text{st}} \leq |\gamma|_{H_1}$, by the characterization of the stable norm in real homology recalled in the introduction.

4. Examples

Here we show that the constant $c = c(n, D, \Omega)$ of Theorem 1.1 necessarily depends on each of the three parameters rank, diameter and asymptotic volume. We say that a sequence of actions of torsionless, discrete abelian groups Γ_k on (X_k, d_k) is *noncollapsing* if there exists $\sigma > 0$ such that $\mathrm{stsys}(\Gamma_k, d_k) > \sigma$ for all k.

Example 4.1 (collapsing actions with fixed rank and bounded co-diameter). Let $\mathbb{Z}$ act on $(X_k, d_k) = \mathcal{C}(\mathbb{Z}, S_k)$, the Cayley graph of $\mathbb{Z}$ with respect to the generating set $S_k = \{\pm 1, \pm k\}$, and let $\| \ \|_{\mathrm{st},k}$ be the associated stable norm. Then

(i) $\mathrm{codiam}(\mathbb{Z}, d_k) = 1$;

(ii) $\mathrm{sys}(\mathbb{Z}, d_k) = 1$, while $\mathrm{stsys}(\mathbb{Z}, d_k) \xrightarrow{k\to\infty} 0$, as

$$|1|_{\mathrm{st},k} = \lim_{m\to\infty} \frac{d_k(0, km)}{km} \leq \frac{1}{k};$$

(iii) $\omega(\mathbb{Z}, d_k) \to \infty$, as a consequence of Lemma 1.1;

(iv) $d_{2k}(0, k) = k$, while $\|k\|_{\mathrm{st},2k} \leq \frac{1}{2}$.

This example shows that, the rank and the co-diameter of (Γ_k, d_k) being fixed, without any assumption on the asymptotic volume (or the stable systole) the difference between the distance and the associated stable norm can be arbitrarily large.

It also shows that, whereas the collapse of the systole forces the asymptotic volume to diverge (by the abelian Margulis lemma), the converse is not true.

Notice that, with little effort, the example can be modified into a sequence of $\mathbb{Z}$-coverings of a compact Riemannian manifold with the same properties, in the following way. Start with the ϵ-tubular neighbourhood in $\mathbb{R}^3$ of a bouquet of two circles α, β with length 1, and consider its boundary $\bar{Y}$. Let (Y_k, d_k) the Riemannian covering of $\bar{Y}$ associated to the subgroup $N = \langle \alpha, \alpha^k \beta^{-1} \rangle$ of $H_1(\bar{Y}, \mathbb{Z})$: then, there exists a $(1 + \delta, \delta)$-quasi isometry between Y_k and the above graph X_k, with $\delta \approx \epsilon$, which is equivariant with respect to the actions of $\Gamma = H_1(\bar{Y}, \mathbb{Z})/N \cong \mathbb{Z}$. Therefore, Γ acts on Y_k with the same properties (up to multiplicative constants $1 + \delta$ in the above estimates (i), (ii), and (iv)).

Example 4.2 (noncollapsing actions with fixed rank and large co-diameter). Let $\mathbb{Z}$ act on $(X_k, d_k) = k \cdot \mathcal{C}(\mathbb{Z}, S_p)$, the Cayley graph of $\mathbb{Z}$ with respect to the generating set $S_p = \{\pm 1, \pm p\}$, with $p > 1$ fixed, and the graph metric dilated by a factor k. Let $\| \ \|_{\mathrm{st},k}$ be the associated stable norm. Then

(i) $\mathrm{diam}(X_k, d_k) = k$;

(ii) $\mathrm{stsys}(\mathbb{Z}, d_k) \geq \frac{k}{p}$, since

$$\|m\|_{\mathrm{st},k} = \lim_{h\to\infty} \frac{d_k(0, mhp)}{hp} = k \cdot \frac{m}{p} \geq \frac{k}{p};$$

(iii) $\frac{p}{k} \leq \omega(\mathbb{Z}, d_k) \leq \frac{2p}{k}$, as a consequence of Lemma 1.2;

(iv) $d_k(0,1) = k$, while $\|1\|_{\mathrm{st},k} = \frac{k}{p}$.

This example shows that, the rank and the asymptotic volume being bounded, without any assumption on the diameter the difference between the distance and the associated stable norm can be arbitrarily large.

Example 4.3 (noncollapsing actions with large rank and bounded co-diameter). Consider a round sphere (S^3, d) with north pole x_0, and remove an arbitrarily large number n of small, disjoint balls B_i, centered at m equatorial points, with boundary 2-spheres S_i (so that $d(x_0, S_i) \sim \pi/2$); then, take n copies T_i of a flat torus, each with a small ball B_i' removed and boundary spheres S_i', glue the (almost isometric) spheres S_i, S_i' through a cylinder of length ℓ, and smooth the metric to obtain a Riemannian manifold $\bar{X}_n$. We may assume that ℓ is much larger than the length σ of the shortest nontrivial 1-cycle in the flat torus T_i (which realizes the stable systole of $H_1(T_1, \mathbb{Z})$ acting on the universal covering of T_i), and that, nevertheless, $\mathrm{diam}(\bar{X}_n)$ stays bounded. The groups $\Gamma_n = H_1(\bar{X}_n, \mathbb{Z}) = \bigoplus_{i=1}^n H_1(T_i, \mathbb{Z}) \cong \mathbb{Z}^{3n}$ then act on the Riemannian homology coverings (X_n, d_n) of $\bar{X}_n$ without collapsing: actually, any class $\gamma_i \in H_1(T_i, \mathbb{Z})$ has length $\ell(\gamma_i)$ in X_n not smaller than its original length in T_i (the ball B_i' has been replaced by an almost flat cylinder), and any decomposable class $\gamma = \sum_i \gamma_i$ with $\gamma_i \in H_1(T_i, \mathbb{Z})$ has length greater than $\sum_i \ell(\gamma_i)$. Thus, $\mathrm{stsys}(\Gamma_n, d_n) \geq \sigma$ for all n. On the other hand, for every $\gamma = \sum_i \gamma_i$, with nontrivial components $\gamma_i \in H_1(T_i, \mathbb{Z})$ for all i, we have $d(x_0, \gamma x_0) \geq 2n\ell + \sum \ell(\gamma_i)$ (as the shortest geodesic loop representing γ must travel forth and back at least n cylinders), while $\|\gamma\|_{\mathrm{st}} \leq \sum_i \ell(\gamma_i)$; hence $d(x_0, \gamma x_0) - \|\gamma\|_{\mathrm{st}}$ diverges for $n \to \infty$. Notice that in these examples the asymptotic volume $\omega(\Gamma_n, d_n)$ stays bounded for $n \to \infty$, by Lemma 1.2, although the rank is arbitrarily large.

We conclude this section with an example showing that the bounded distance theorem may fail for abelian actions on metric spaces which are not length spaces. *Inner metric spaces,*[8] as defined by P. Pansu [14], are the closest spaces to length spaces: (X, d) is inner if, for every $\epsilon > 0$, there exist $\ell(\epsilon)$ such that for all $x, x' \in X$ there exists a sequence of points $x_0 = x, x_1, \ldots, x_{N+1} = x'$ with $d(x_i, x_{i+1}) \leq \ell(\epsilon)$ and $\sum_{i=1}^{N+1} d(x_{i-1}, x_i) \leq (1+\epsilon) d(x, x')$. The following is the simplest example of inner metric space where the bounded distance theorem does not hold.

Example 4.4 (noncollapsing $\mathbb{Z}$-actions on inner spaces with bounded co-diameter). Consider the group $\mathbb{Z}$ endowed with the left invariant metric induced by the norm $|||m||| = |m| + \sqrt{|m|}$. It is straightforward to check that $|||\ |||$ defines an inner

[8] Any finitely generated group Γ endowed with a word length, or with a geometric distance deduced from a cocompact action on a length space, is an inner metric space.

metric on $\mathbb{Z}$. Actually, given $\epsilon > 0$, choose an integer $\ell > 4/\epsilon^2$, and write any $m \in \mathbb{N}$ as $m = N\ell + r$, with $r < \ell$. If $N = 0$, there is nothing to prove; otherwise call $x_i = i\ell$ for $i \leq N$ and $x_{N+1} = m$, so

$$\begin{aligned}\frac{\sum_{i=1}^{N+1} |||x_i - x_{i-1}|||}{|||m|||} &\leq \frac{\sum_{i=1}^{N} |||\ell||| + |||r|||}{|||N\ell + r|||} \\ &= \frac{N(\ell + \sqrt{\ell}) + r + \sqrt{r}}{(N\ell + r) + \sqrt{N\ell + r}} \\ &\leq \frac{N\ell + r}{(N\ell + r) + \sqrt{N\ell + r}} + \frac{1}{\sqrt{\ell}} + \frac{\sqrt{r}}{N\ell} \\ &\leq 1 + \frac{2}{\sqrt{\ell}} \\ &< 1 + \epsilon.\end{aligned}$$

Then, $\mathbb{Z}$ acts by left translation on itself, and the stable norm associated to $|||\ |||$ coincides with the absolute value $|\ |$. Therefore, we have

$$\operatorname{codiam}(\mathbb{Z}, |||\ |||) = \frac{1}{2} + \sqrt{\frac{1}{2}}$$

and

$$\operatorname{stsys}(\mathbb{Z}, |||\ |||) = 1,$$

but

$$|||m||| - |||m|||_{\mathrm{st}} = \sqrt{|m|}$$

is not bounded.

5. On the number of connected components of optimal cycles

Let $\bar{X}$ be a Riemannian manifold, with torsion free homology covering (X, d). Let $x_0 \in X$ be fixed, let d_{x_0} be the induced distance on $\Gamma = H_1(\bar{X}, \mathbb{Z})$ acting on (X, d), and $\|\ \|_{\mathrm{st}}$ be the associated stable norm on $H_1(\bar{X}, \mathbb{R})$ as explained in §1. Let λ be the Busemann measure of the normed space $(H_1(\bar{X}, \mathbb{R}), \|\cdot\|_{\mathrm{st}})$, that is the Lebesgue measure assigning to its unit ball $B_{\mathrm{st}}(1)$ the volume of the unitary euclidean n-ball (which coincides with the n-dimensional Hausdorff measure).

Finally, let $\mathcal{F}$ be a closed fundamental domain included in the closed Dirichlet domain $\widehat{\mathcal{D}}_{\mathrm{st}}$ centered at the origin, for Γ acting on $(H_1(\bar{X}, \mathbb{R}), \|\cdot\|_{\mathrm{st}})$, as in §3.1.

As recalled in the introduction, an easy packing of fundamental domains shows that the asymptotic volume of the measure metric space $(H_1(\bar{X}, \mathbb{R}), \|\ \|_{\mathrm{st}}, \lambda)$ is

$$\omega_\lambda(\mathbb{R}^n, \|\ \|_{\mathrm{st}}) = \omega(\Gamma, \|\ \|_{\mathrm{st}}) \cdot \lambda(\mathcal{F}) \tag{11}$$

and, since $\|\ \|_{\mathrm{st}}$ is a norm, this also equals the volume $\lambda(B_{\mathrm{st}}(1))$ of the unit ball.

Then, as a consequence of the bounded distance theorem (even without any estimate of the constant c), one gets $\omega(\Gamma, d) = \omega(\Gamma, \| \ \|_{st}) = \lambda(B_{st}(1))/\lambda(\mathcal{F})$. Let us call $V = \lambda(\mathcal{F})$ and $V_1 = \lambda(B_{st}(1))$, so that $\omega(\Gamma, d) = V_1/V$.

Let us now fix some notations for balls and annuli and for the corresponding growth functions. We will write

$$B_{(\Gamma,d_{x_0})}(R) = \{\gamma \in \Gamma \mid |\gamma|_{x_0} < R\}$$

and

$$A_{(\Gamma,d_{x_0})}(r, R) = \{\gamma \in \Gamma \mid r \leq |\gamma|_{x_0} < R\},$$

and similarly $B_{(\Gamma,|\ |_{H_1})}$, $B_{(\Gamma,\mathrm{st})}$, and $A_{(\Gamma,|\ |_{H_1})}$, $A_{(\Gamma,\mathrm{st})}$ for balls and annuli in Γ with, respectively, the mass and the stable norm. We will write $v_\bullet(R)$, $v_\bullet(r, R)$ for the corresponding cardinalities. Finally, we will use B_{st} and A_{st} for ball and annuli in $(H_1(\bar{X}, \mathbb{R}), \| \ \|_{st})$, and write $v_{st}(R) = \lambda(B_{st}(R))$, $v_{st}(r, R) = \lambda(A_{st}(r, R))$.

A by-product of the QBD theorem is the following explicit estimate of the growth function of annuli in $H_1(\bar{X}, \mathbb{Z})$ with respect to the mass.

Proposition 5.1. *Assume $n = \mathrm{rank}\, H_1(\bar{X}, \mathbb{Z})$, $\mathrm{diam}(\bar{X}) \leq D$ and $\omega(X, d) \leq \Omega$. Let $c = c(n, D, \Omega)$ be as in the QBD theorem 1.1. If $\Delta > 4nD + c$ we have*

$$n\omega(\Gamma, d) \cdot \Delta(R - \Delta)^{n-1} \leq v_{(\Gamma,|\ |_{H_1})}(R - \Delta, R + \Delta) \leq 3n\omega(\Gamma, d) \cdot \Delta(R - 3\Delta)^{n-1}$$

for $R \geq 0$, and

$$A(k\Delta)^{n-1} \leq v_{(\Gamma,|\ |_{H_1})}(k\Delta, (k + 1)\Delta) \leq B(k\Delta)^{n-1}$$

for all $k \geq 1$, for constants $A = n \cdot \omega(\Gamma, d) \cdot \Delta$ and $B = 3^n \cdot A$.

Proof. By Theorem 1.1 we have $\| \gamma \|_{st} \leq | \gamma |_{H_1} \leq | \gamma |_{x_0} \leq \| \gamma \|_{st} + c$ and thus

$$A_{(\Gamma,\mathrm{st})}(r, R - c) \subseteq A_{(\Gamma,|\ |_{H_1})}(r, R) \subseteq A_{(\Gamma,\mathrm{st})}(r - c, R)$$

Notice that, if $D_{st} = \mathrm{diam}_{st}(\mathcal{F})$ is the diameter of $\mathcal{F}$ with respect to the stable norm, we have $D_{st} \leq 2nD$. Actually, choose n linearly independent vectors $v_1, \ldots, v_n \in \Sigma'_D$ from the generating set $\Sigma'_D = \{\gamma \in \Gamma \colon |\gamma|_{x_0} \leq 2D\}$: the n-parallelotope $\mathcal{P}$ determined by these vectors clearly contains $\mathcal{F}$, so $D_{st} \leq \mathrm{diam}_{st}(\mathcal{P}) \leq 2n\, D$. By an argument of packing and covering of the annuli $A_{st}(r, R)$ with copies of $\mathcal{F}$, we obtain

$$\frac{\lambda(A_{st}(r + 2n\, D, R - C - 2n\, D))}{V} \leq v_{(\Gamma,|\ |_{H_1})}(r, R) \leq \frac{\lambda(A_{st}(r - 2n\, D - C, R + 2n\, D))}{V}.$$

We estimate the left-hand inequality for $R \geq r + 4nD + c$ as

$$\lambda(A_{\rm st}(r + 2n\,D, R - C - 2n\,D)) = V_1 \cdot [(R - C - 2n\,D)^n - (r + 2n\,D)^n] \geq nV_1 \cdot (R - r - (4nD + c)) \cdot (r + 2nD)^{n-1}$$

and, similarly, the right-hand as

$$\lambda(A_{\rm st}(r - C - 2n\,D, R + 2n\,D)) \leq nV_1 \cdot (R - r + (4nD + c)) \cdot (R + 2nD)^{n-1}.$$

Choosing

$$\frac{R - r}{2} = \Delta \geq 4nD + c$$

we obtain

$$n\frac{V_1}{V} \cdot \Delta(R - \Delta)^{n-1} \leq v_{(\Gamma, |\ |_{H_1})}(R - \Delta, R + \Delta) \leq 3n\frac{V_1}{V} \cdot \Delta(R - 3\Delta)^{n-1}$$

which proves (i). The second statement follows from (i) taking $R = (k+1)\Delta$. □

Proof of Theorem 1.3. Let Δ be as in Proposition 5.1 above.

First, we consider the case where $N(\gamma) \leq v_{(\Gamma,H_1)}(2\Delta)$. As $\|\ \|_{\rm st} \leq |\ |_{H_1}$, we have

$$N(\gamma) \leq \frac{\lambda B_{\rm st}(2\Delta + D_{\rm st})}{V} \leq \frac{V_1}{V}(2\Delta + nD)^n = \omega(\Gamma, d)(9nD + c)^n$$

and using the explicit estimates for c given in Remark 3.1(i)&(ii) we get the announced bound $N(n, D, \Omega)$. Assume now that $N(\gamma) > v_{(\Gamma,H_1)}(2\Delta)$. Then, there exists $m = m(\gamma) \geq 1$ such that

$$\sum_{k=0}^{m} v_{(\Gamma,H_1)}(k\Delta, (k+1)\Delta) < N(\gamma) \leq \sum_{k=0}^{m+1} v_{(\Gamma,H_1)}(k\Delta, (k+1)\Delta).$$

Then, using the estimates of Proposition 5.1 we find

$$N(\gamma) \leq B \sum_{k=0}^{m+1} (k\Delta)^{n-1} \leq \frac{B\Delta^{n-1}}{n}(m+2)^n. \tag{12}$$

Now, observe that if c is an optimal cycle representing γ with $N(\gamma)$ connected components, then its components c_i are non-homologous to each other; thus, its total length is at least $\ell \geq \sum_{k=0}^{m} (k\Delta) \cdot v_{(\Gamma,H_1)}(k\Delta, (k+1)\Delta)$. Using the estimates of Proposition 5.1 we find

$$\ell = |\gamma|_{H_1} \geq A \sum_{k=0}^{m} (k\Delta)^n \geq \frac{A\Delta^n}{n+1} \cdot m^{n+1}. \tag{13}$$

Putting together the two estimates (12) and (13) above and we obtain

$$\begin{aligned} N(\gamma) &\leq \frac{B\,\Delta^{n-1}}{n}(m+2)^n \\ &\leq \frac{B\,\Delta^{n-1}}{n}\Big(\sqrt[n+1]{\frac{(n+1)\ell}{A\Delta^n}}+2\Big)^n \\ &\leq \frac{3^n\,\Delta^{n-1}B}{n\Delta^{\frac{n^2}{n+1}}}\cdot\Big(\frac{(n+1)\ell}{A}\Big)^{n/(n+1)} \end{aligned}$$

and as $A = n\omega(\Gamma,d)\Delta = 3^{-n}B$, this yields

$$N(\gamma) \leq 3^{2n}\cdot\sqrt[n+1]{\Big(1+\frac{1}{n}\Big)^n\cdot\omega(\Gamma,d)\ell^n}. \qquad \square$$

References

[1] G. Besson, G. Courtois, and S. Gallot, Un lemme de Margulis sans courbure et ses applications. *Prépublication de l'Institut Fourier* **595** (2003), 59 pp.

[2] D. Burago, Periodic metrics. In A. M. Vershik (ed.), *Representation theory and dynamical systems.* Advances in Soviet Mathematics, 9. American Mathematical Society, Providence, R.I., 1992, pp. 205–210. Zbl 0762.53023 MR 1166203

[3] D. Burago and S. Ivanov, Asymptotic volume of tori. *Geom. Funct. Anal.* **5** (1995), no. 5, 800–808. Zbl 0846.53043 MR 1354290

[4] D. Burago and S. Ivanov, On asymptotic volume of Finsler tori, minimal surfaces in normed spaces, and symplectic filling volume. *Ann. of Math.* (2) **156** (2002), no. 3, 891–914. Zbl 1040.53088 MR 1954238

[5] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry.* Graduate Studies in Mathematics, 33. American Mathematical Society, Providence, RI, 2001. Zbl 0981.51016 MR 1835418

[6] F. Cerocchi, Margulis Lemma, entropy and free products. Ann. Inst. Fourier (Grenoble) **64** (2014), no. 3, 1011–1030. Zbl 1311.53040 MR 3330162

[7] H. Federer, Real flat chains, cochains and variational problems. *Indiana Univ. Math. J.* **24** (1974/75), 351–407. Zbl 0289.49044 MR 0348598

[8] E. S. Fedorov, *Foundations of the theory of figures.* Acad. Sci. Peterburg, 1885.

[9] E. S. Fedorov, Reguläre Plan-und Raumteilung. *Abh. K. Bayer. Akad. Wiss* (II *Classe*) **20** (1899), 465–588.

[10] M. Gromov, *Asymptotic invariants of infinite groups.* Geometric group theory, Vol. 2 (Sussex, 1991). London Math. Soc. Lecture Note Ser., 182, Cambridge Univ. Press, Cambridge, 1993. Zbl 0841.20039 MR 1253544

[11] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces.* Progress in Mathematics, 152. Birkhäuser Boston, Inc., Boston, MA, 1999. Zbl 0953.53002 MR 2307192

[12] B. Grünbaum and G. C. Shephard, *Tilings and patterns.* W. H. Freeman and Company, New York, 1987. Zbl 0601.05001 MR 0857454

[13] B. Grünbaum and G. C. Shephard, Isohedral tilings of the plane by polygons. *Comment. Math. Helv.* **53** (1978), no. 4, 542–571. Zbl 0392.05020 MR 0511847

[14] P. Pansu, Croissances des boules et des géodésiques fermées dans les nilvariétés. *Ergodic Theory Dynam. Systems* **3** (1983), no. 3, 415–445. Zbl 0509.53040 MR 0741395

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