

Representation zeta functions of self-similar branched groups

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Abstract. We compute the numbers of irreducible linear representations of self-similar branched groups, by expressing these numbers as the coefficients r_n of a Dirichlet series $\sum r_n n^{-s}$.

We show that this Dirichlet series has a positive abscissa of convergence and satisfies a functional equation thanks to which it can be analytically continued (through root singularities) to the right half-plane.

We compute the abscissa of convergence and the functional equation for some prominent examples of branched groups, such as the Grigorchuk and Gupta–Sidki groups.

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1. Introduction

Let G be a group, and let \hat{G} denote its set of equivalence classes of irreducible, finite-dimensional complex linear representations; assume that there are finitely

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many such representations in each degree (G is then called *rigid*). The *representation zeta function* of G is the Dirichlet series with integer coefficients

$$\zeta_G(s) = \sum_{\rho \in \widehat{G}} (\deg \rho)^{-s} = \sum_{n \geq 1} r_n n^{-s},$$

with r_n denoting the degree- n representations in \widehat{G} , that is, irreducible representations of G in $\mathrm{GL}_n(\mathbb{C})$. If the numbers r_n grow polynomially, then analytic properties of ζ_G yield asymptotic information on r_n and conversely. For example, let $\sigma_0(G)$ denote the abscissa of convergence of ζ_G ; then, assuming $\sum r_n = \infty$,

$$\sigma_0(G) = \limsup_{n \rightarrow \infty} \frac{\log \sum_{j=1}^n r_j}{\log n}, \quad (1.1)$$

so the partial sums $\sum_{j=1}^n r_j$ grow approximately as n^{σ_0} . More precisely, the Landau-Phragmén theorem implies that $\zeta_G(s)$ has a singularity at σ_0 , and if a limiting behaviour $\zeta_G(s) = (s - \sigma_0)^e g(s) + h(s)$ is known with g, h holomorphic in $\{\Re(s) \geq \sigma_0\}$ and $e \in \mathbb{R} \setminus \mathbb{N}$, then

$$\sum_{j=1}^n r_j \approx \frac{g(\sigma_0)}{\sigma_0 \Gamma(-e)} n^{\sigma_0} (\log n)^{-e-1},$$

see [26, Theorem 15, p. 243] and [8].

Note that it is easy to deduce the number of linear representations of given degree out of the number of irreducible ones, and vice versa; indeed every linear representation decomposes into a direct sum of irreducibles whose multiplicities are uniquely determined. Letting R_n denote the number of representations of degree n , we have the Euler-product formula

$$\sum_{n \geq 0} R_n t^n = \prod_{n \geq 1} \left(\frac{1}{1 - t^n} \right)^{r_n}.$$

1.1. Self-similar branched groups. Representation zeta functions have been extensively investigated for linear groups (see §1.2 for a quick summary); in this article, we focus on *self-similar branched groups*. They are certain kinds of groups G equipped with an injective homomorphism $\psi: G \rightarrow G^d \rtimes \mathfrak{S}_d$, and possessing a finite-index subgroup K such that $\psi(K)$ contains K^d ; see Definitions 2.1, 2.2, and 5.1 for the exact definitions.

Thus in particular G and G^d have isomorphic finite-index subgroups. The integer $d > 1$ is called the *degree* of the branched group, and one says that G is branched *over* K . Iterating the map ψ on its components, one obtains for every branched group an action by permutation on the set X^* of words over an alphabet X of cardinality d .

Self-similar branched groups constitute a well-studied class of groups, containing such prominent examples as Grigorchuk's torsion group of intermediate word growth [12] and Gupta–Sidki's examples [13]. Their topological closures in $\text{Aut}(X^*)$ may be thought of as analogues of algebraic groups, defined by equations over infinitely many variables indexed by X^* , see [25].

On the one hand, branched groups have many finite quotients coming from the action of the group on X^n for all $n \in \mathbb{N}$; so have many finite, and in particular linear, representations. On the other hand, they contain abelian subgroups of arbitrarily large rank and large normalizer, so they are quite different from linear groups.

In this article, I show that the zeta function of a self-similar branched group admits quite remarkable properties:

Theorem A. *Let G be a self-similar group of degree $d > 1$, branched over its subgroup K . Then G is rigid if and only if $K/[K, K]$ is finite. In that case, its representation zeta function ζ_G*

- (1) *has a positive, finite abscissa of convergence σ_0 , so that the coefficients r_n grow polynomially;*
- (2) *is a linear combination of the solutions $\zeta_i(s)$ of a system of functional equations of the form*

$$\begin{aligned} &F_i(\zeta_1(s), \zeta_1(2s), \dots, \zeta_1(ds), \\ &\quad \zeta_2(s), \zeta_2(2s), \dots, \zeta_2(ds), \\ &\quad \vdots \\ &\quad \zeta_N(s), \zeta_N(2s), \dots, \zeta_N(ds)) = \zeta_i(s), \quad i = 1, \dots, N, \end{aligned} \tag{1.2}$$

for some $N, P \in \mathbb{N}$ and some Dirichlet polynomials

$$F_1, \dots, F_N \in \mathbb{Q}[z_{1,1}, \dots, z_{N,d}, 2^{-s}, \dots, P^{-s}];$$

furthermore, if $z_{j,k}$ have degree k , then the polynomials F_i are homogeneous of degree d ;

- (3) *can be continued to a bounded, multivalued analytic function on the half-plane $\Re(s) > 0$, with only root singularities;*
- (4) *has a Puiseux series expansion at σ_0 of the form*

$$\zeta_G(s) = \sum_{n=0}^{\infty} a_n (s - \sigma_0)^{n/e},$$

for some integer $e \leq d$.

The functional equation can be determined algorithmically out of the description of G as a self-similar branched group, and has been implemented in GAP code; it is part of the author's package FR freely available on Internet. This code was used to compute the various examples in §2.

Note that in general the functional equation (1.2) is not sufficient to determine ζ_G . However, under a judicious choice of finite data extracted from G , it determines ζ_G and permits a very efficient calculation of its coefficients.

It is easy to generalize Theorem A to more general character series. Let us say that an element $g \in G$ of a self-similar group is *finite-state* if there exists a finite subset $W \subseteq G$, containing g , such that $\psi(W) \subseteq W^d \times \mathfrak{S}_d$. In words, the element g is defined by a finite set of recursive rules via the map ψ . For any $g \in G$, write its Lambda series

$$\Lambda(g, s) = \sum_{\rho \in \hat{G}} \text{tr } \rho(g) \text{ deg}(\rho)^{-s}.$$

In particular, $\Lambda(1, s) = \zeta_G(s - 1)$. The proof of Theorem A actually gives:

Theorem A'. *Let G be a self-similar branched group of degree $d > 1$, and let $g \in G$ be finite-state. Then all properties of ζ_G claimed in Theorem A also hold for $\Lambda(g, s)$.*

In particular, the variables in the functional equation shall be of the form $\Lambda_i(w, s)$ for all $w \in W$ and $i = 1, \dots, N$, and the coefficients in the functional equation will belong to the field generated by the character values of G . I omit details.

Theorem A extends the main results of [4], in which the group G was assumed to be isomorphic to $G \wr_X Q$. Here and below the *wreath product* $G \wr_X Q$ of the group G with the group Q , along the Q -set X , is by definition $G^X \rtimes Q$, and we write $G \wr Q$ if $X = Q$ with its regular Q -action. I will make liberal use of results from [4].

1.2. Historical background. If the group G is a topological or algebraic group, then it is natural to restrict to continuous, respectively rational representations. Since these behave usually much better, part of the art is to relate the representation zeta function of a topological (e.g. Lie) group to that of its lattices.

It seems that the first occurrence of representation zeta functions is in [28], which relates $\zeta_G(2g - 2)$ to the moduli space of flat connections of G -principal bundles over Σ_g , for G a compact, simple, simply connected Lie group and Σ_g an orientable surface of genus $g \geq 2$. However, ζ_G was already implicitly considered earlier; for example, it follows from Weyl's theory that, if ℓ be G 's rank and κ be the number of positive roots of G 's Lie algebra over \mathbb{C} , then there exists a polynomial P of degree κ in ℓ variables such that

$$\zeta_G(s) = \sum_{n_1 \geq 0, \dots, n_\ell \geq 0} P(n_1, \dots, n_\ell)^{-s}.$$

It follows that the abscissa of convergence of ζ_G is ℓ/κ , and that ζ_G extends to a meromorphic function on the whole plane; see [18, Theorem 5.1].

Larsen and Lubotzky consider in [18] arithmetic lattices in semisimple algebraic groups \mathbf{G} , and show, under the “congruence subgroup property”, that these lattices $\Gamma = \mathbf{G}(\mathcal{O})$ are products of local factors $\mathbf{G}(\mathcal{O}_v)$ and archimedean factors $\mathbf{G}(\mathbb{C})$; consequently, the representation zeta function ζ_Γ is the product of the respective zeta functions; for example,

$$\zeta_{\mathrm{SL}_3(\mathbb{Z})}(s) = \zeta_{\mathrm{SL}_3(\mathbb{C})}(s) \prod_{p \text{ prime}} \zeta_{\mathrm{SL}_3(\mathbb{Z}_p)}(s).$$

A careful study of the abscissæ of convergence of the $\zeta_{\mathbf{G}(\mathcal{O}_v)}$ as a function of v allowed Avni to prove, in [2], that ζ_Γ has a rational abscissa of convergence; though its precise value is still mysterious.

The local factors $G = \mathbf{G}(\mathcal{O}_v)$ are compact p -adic analytic groups, and Jaikin-Zapirain shows in [16] that the representation zeta function of such a group may be written as

$$\zeta_G(s) = \sum_{i=1}^k n_i^{-s} f_i(p^{-s})$$

for natural numbers n_1, \dots, n_k and rational functions $f_1, \dots, f_k \in \mathbb{Q}(p^{-s})$.

1.3. “Quoi de neuf, docteur?” Here is a quick summary of the main differences between this article and [4].

Firstly, Isaacs’ notion of “character triples” is fundamental to the calculations done here. I found it necessary to express character triples slightly differently, by making explicit a marking with a given finite group. This makes also more transparent the extent to which character triples are convenient computational tools to study and manipulate cohomological information. Thus while character triples are triples (χ, N, G) with $\chi \in \widehat{N}$ and $N \triangleleft G$, I prefer to fix a group B , and call B -character triple a pair (χ, f) with $\chi \in \widehat{\ker f}$ and f a homomorphism to B . One recovers the classical notion by taking for f the natural map $G \mapsto G/N$.

Secondly, I associate a *branch structure* to a branched group G . This is a data structure made of a finite group B , a subgroup B_+ of $B \wr_X Q$, and a surjective map $B_+ \twoheadrightarrow B$. It seems to capture in an efficient manner the important properties of a branched group. The group G itself is not determined by the branch structure, but one may construct out of the branch structure a profinite group $G(B)$ with a canonical map $G \rightarrow G(B)$.

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2. Illustrations

I describe here some examples of self-similar branched groups, and some information on their representation zeta functions. Let us start by the precise definition of self-similar groups that we will use. The definition of branched groups will appear in §5.

Definition 2.1. A *self-similar group* is a group G endowed with an injective homomorphism $\psi: G \rightarrow G \wr_X Q$, for a permutation group Q acting on a finite set X . The map ψ is called a *self-similarity structure*, and the integer $\#X$ is called its *degree*. Usually, the self-similarity structure is implicit, and one simply denotes by G the self-similar group.

The notation $\langle\langle g_1, \dots, g_d \rangle\rangle q$ refers to the element of $G \wr_X Q$ with $(g_1, \dots, g_d) \in G^X$ and $q \in Q$. So as to avoid degenerate cases, we make the following restriction.

Definition 2.2. An *effective self-similar group* is a self-similar group whose branch structure satisfies the following conditions:

- (1) the degree $\#X$ is at least 2;
- (2) the action of Q on X is transitive;
- (3) the projection $\psi(G) \rightarrow Q$ is surjective, and for each $x \in X$, the projection $\psi(G) \cap G^X \rightarrow G$ on coordinate x is surjective.

The second condition could, in fact, be relaxed to the requirement that Q act without fixed points on X . The third condition may be ensured by replacing Q by the image of ψ and/or replacing G by the projection of $\psi(G) \cap G^X$ to a coordinate (possibly after post-composing the self-similarity structure by an automorphism of $G \wr_X Q$). All self-similar groups in this text are assumed to be effective.

The map ψ can be applied diagonally to all entries in G^X , yielding a map $G^X \rightarrow (G \wr_X Q)^X$, and therefore a map $G \wr_X Q \rightarrow (G \wr_X Q) \wr_X Q = G \wr_{X \times X} (Q \wr_X Q)$; more generally, we write $\wr_X^n Q$ for the iterate $Q \wr_X \cdots \wr_X Q$, and get maps $G \wr_{X^n} (\wr_X^n Q) \rightarrow G \wr_{X^{n+1}} (\wr_X^{n+1} Q)$ which we all denote by ψ . We may compose these maps, and write ψ^n for the iterate $\psi^n: G \rightarrow G \wr_{X^n} (\wr_X^n Q)$.

By projecting to the permutation part, we then have homomorphisms $G \rightarrow \mathfrak{S}_{X^n}$ for all $n \in \mathbb{N}$ and, assembling these homomorphisms together, we get a permutational action of G on $X^* = \bigsqcup_{n \geq 0} X^n$; one may identify X^* with the vertex set of a rooted $\#X$ -regular tree, by connecting $v_1 \dots v_n$ to $v_1 \dots v_n v_{n+1}$ for all $v_i \in X$. In this manner, G acts by graph isometries. This action need not be faithful; if it is, then G is called a *faithful self-similar group*. In first three examples below, this action is faithful; while in the fourth it is not.

Lemma 2.3. *Let G be an effective self-similar group. Then its action on X^n is transitive for all $n \in \mathbb{N}$. In particular, G is infinite.*

Proof. We proceed by induction, the case $n = 1$ being given by the second condition. Then, assuming that the action of G is transitive on X^n , it follows from the third condition that the action of $\psi(G) \cap G^X$ on xX^n is transitive for all $x \in X$, so the orbits of $\psi(G) \cap G^X$ are precisely $\{xX^n\}_{x \in X}$. Now since $\psi(G)$ maps onto Q which is transitive, these orbits form a single G -orbit on X^{n+1} . Infiniteness of G follows from the first assertion. \square

The examples of groups that we consider below will be described by the following data: a finite group Q , a finite Q -set X , a finitely presented group F , and a homomorphism $\tilde{\psi}: F \rightarrow F \wr_X Q$. Define normal subgroups of F by $R_0 = 1$ and $R_{n+1} = \tilde{\psi}^{-1}(R_n^X)$ for all $n \geq 0$. The *injective quotient* of F is by definition the self-similar group $G := F / \bigcup_{n \geq 0} R_n$. The homomorphism $\tilde{\psi}$ descends to an injective map $\psi: G \hookrightarrow G \wr_X Q$.

2.1. The Alëshin and Grigorchuk groups. The Grigorchuk group is obtained as follows. The cyclic group of order 2 is written C_2 . Set

$$F = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd \rangle = C_2 * (C_2 \times C_2),$$

and define $\tilde{\psi}: F \rightarrow F \wr C_2$ by

$$\tilde{\psi}(a) = \langle\langle 1, 1 \rangle\rangle(1, 2), \quad \tilde{\psi}(b) = \langle\langle a, c \rangle\rangle, \quad \tilde{\psi}(c) = \langle\langle a, d \rangle\rangle, \quad \tilde{\psi}(d) = \langle\langle 1, b \rangle\rangle.$$

Let G be the injective quotient of F . It acts faithfully on $\{1, 2\}^*$. A related group (see below) was first considered by Alëshin in [1], providing a ‘‘tangible’’ example of an infinite, finitely generated, residually finite, torsion group (the first examples of groups with these properties are due to Golod [11]). Grigorchuk proved in [12] that G ’s word growth is strictly between polynomial and exponential. See [14, Chapter VIII] for an elementary introduction to G . For its structure as a branched group, see §5.1.

Since G is a 2-group, all its irreducible representations are 2^n -dimensional for some n ; therefore $\zeta_G(s) = f(2^{-s})$ for a power series $f \in \mathbb{N}[[2^{-s}]]$. Let us write $q = 2^{-s}$ for brevity; then the first values are

$$\begin{aligned} f(q) = & 8 + 10q + 29q^2 + 100q^3 + 413q^4 + 1990q^5 + 9787q^6 + 50810q^7 \\ & + 278797q^8 + 1593796q^9 + 9572828q^{10} + 60125360q^{11} \\ & + 396548538q^{12} + 2732836832q^{13} + 19674348692q^{14} \\ & + 147148989714q^{15} + \dots \end{aligned}$$

and, for illustration, there are $5554240222 \dots 8648974784 \approx 5.5 \cdot 10^{93}$ irreducible representations of degree 2^{100} . This calculation took 4 minutes on a 2010 laptop using GAP and the author’s package FR. The functional equation involves 62 variables $\zeta_1, \dots, \zeta_{62}$.

The abscissa of convergence of ζ_G is computed as described in §7.3, and is $\sigma_0(G) \approx 3.293330470$.

Here is a brief description of Alëshin's group \tilde{G} and its relation to G . The Alëshin group can be viewed as a group acting on $\{1, 2\}^*$, generated by two elements A, B . The recursions defining the generator's actions are

$$\psi^2(A) = \langle\langle\langle a, c \rangle\rangle, \langle\langle 1, d \rangle\rangle\rangle, \quad \psi^2(B) = \langle\langle\langle 1, 1 \rangle\rangle, \langle\langle 1, 1 \rangle\rangle(1, 2)\rangle(1, 2).$$

Lemma 2.4. *The groups G and \tilde{G} have a common finite-index subgroup.*

Proof. Consider the normal closure \tilde{G}_0 of A in \tilde{G} . Clearly \tilde{G}_0 has index 4 in \tilde{G} , and the generators of \tilde{G}_0 are involutions. The derived subgroup \tilde{G}'_0 therefore has finite index in \tilde{G} . Now $\psi^2(\tilde{G}'_0)$ contains

$$\psi^2([A, A^B]) = [\langle\langle\langle a, c \rangle\rangle, \langle\langle 1, d \rangle\rangle\rangle, \langle\langle\langle d, 1 \rangle\rangle, \langle\langle a, c \rangle\rangle\rangle] = \langle\langle\langle [a, d], 1 \rangle\rangle, \langle\langle 1, 1 \rangle\rangle\rangle,$$

so it contains $\langle\langle\langle L, L \rangle\rangle, \langle\langle L, L \rangle\rangle\rangle$ for the subgroup $L = \langle [a, d] \rangle^G$ of G . A direct computation shows that L has index 32 in G . Therefore, L and K have a common finite-index subgroup, so every two of the following subgroups of the automorphism group of the binary tree \tilde{G} , $\psi^{-2}\langle\langle\langle L, L \rangle\rangle, \langle\langle L, L \rangle\rangle\rangle$, $\psi^{-2}\langle\langle\langle K, K \rangle\rangle, \langle\langle K, K \rangle\rangle\rangle$, K and G have a common finite-index subgroup. \square

It was already shown in [17, p. 229] that G is a section of \tilde{G} ; they poetically describe the extraction of G from \tilde{G} as “tearing off Adam's rib.”

Corollary 2.5. *The representation zeta functions of G and \tilde{G} have the same abscissa of convergence.*

Proof. By Lemma 2.4, the groups G and \tilde{G} are commensurable. For two Dirichlet series $\eta(s) = \sum a_n n^{-s}$ and $\theta(s) = \sum b_n n^{-s}$, let us write $\eta \leq \theta$ to mean $\sum_{j \leq n} a_j \leq \sum_{j \leq n} b_j$ for all $n \in \mathbb{N}$. It follows from [19, Lemma 2.2] that if G, H are groups and H is a finite-index subgroup of G , then

$$\zeta_H(s) \leq [G : H]^{1+s} \zeta_G(s) \text{ and } \zeta_G(s) \leq [G : H] \zeta_H(s), \quad (2.1)$$

so ζ_H and ζ_G have the same domain of convergence. \square

2.2. The Gupta–Sidki group. The Gupta–Sidki groups are obtained as follows. The cyclic group of order p is written C_p . For each prime $p \geq 3$, set

$$F_p = \langle a, t \mid a^p, t^p \rangle = C_p * C_p,$$

and define $\tilde{\psi}: F_p \rightarrow F_p \wr C_p$ by

$$\tilde{\psi}(a) = \langle\langle 1, \dots, 1 \rangle\rangle(1, \dots, p), \quad \tilde{\psi}(t) = \langle\langle a, a^{-1}, 1, \dots, 1, t \rangle\rangle.$$

Let G_p be the injective quotient of F_p . It acts faithfully on $\{1, \dots, p\}^*$.

These groups G_p are shown in [13] to be infinite, finitely-generated torsion p -groups. For their structure as branched groups, see §5.2. The study of their representations was initiated by Passman and Temple [21]; their main result, in the present paper's language, is $\sigma_0(G_p) \geq p - 2$.

We restrict our consideration to the case $p = 3$. Since G_3 is a 3-group, all its irreducible representations are 3^n -dimensional for some n ; therefore $\zeta_{G_3}(s) = f(3^{-s})$ for a power series $f \in \mathbb{Z}[[3^{-s}]]$. Writing $q = 3^{-s}$, the first values are

$$\begin{aligned} f(q) = & 9 + 26q + 402q^2 + 6876q^3 + 178160q^4 + 7527942q^5 + 461931336q^6 \\ & + 31704156696q^7 + 2421457788330q^8 \\ & + 197775615899520q^9 + 16915932297409064q^{10} + \dots \end{aligned}$$

and there are 1386068855...8306590020 $\approx 1.3 \cdot 10^9$ representations of degree 3^{50} . This calculation took 6 seconds on a 2010 laptop using GAP and the author's package FR. The functional equation involves 8 variables. It may be written in the slightly simplified form as

$$\begin{aligned} \zeta_{G_3}(s) = & \frac{1}{9}q^2\zeta_1(s) + q\zeta_2(s) + q\zeta_3(s) + 2q\zeta_4(s) + (9 + 2q)\zeta_6(s), \\ \zeta_1(s) = & \frac{1}{9}q^2\zeta_1(s)^3 + \frac{1}{3}q^2\zeta_1(s)^2\zeta_2(s) + \frac{1}{3}q^2\zeta_1(s)^2\zeta_3(s) + \frac{2}{3}q^2\zeta_1(s)^2\zeta_4(s) \\ & + q^2\zeta_1(s)^2\zeta_6(s) + \frac{1}{3}q^2\zeta_1(s)\zeta_2(s)^2 + \frac{2}{3}q^2\zeta_1(s)\zeta_2(s)\zeta_3(s) \\ & + \frac{4}{3}q^2\zeta_1(s)\zeta_2(s)\zeta_4(s) + 2q^2\zeta_1(s)\zeta_2(s)\zeta_6(s) + \frac{1}{3}q^2\zeta_1(s)\zeta_3(s)^2 \\ & + \frac{4}{3}q^2\zeta_1(s)\zeta_3(s)\zeta_4(s) + 2q^2\zeta_1(s)\zeta_3(s)\zeta_6(s) + q^2\zeta_1(s)\zeta_4(s)^2 \\ & + 2q^2\zeta_1(s)\zeta_4(s)\zeta_6(s) + q\zeta_2(s)^3 + \frac{1}{3}q^2\zeta_2(s)^2\zeta_3(s) + \frac{2}{3}q^2\zeta_2(s)^2\zeta_4(s) \\ & + 9q\zeta_2(s)^2\zeta_6(s) + \frac{4}{3}q^2\zeta_2(s)\zeta_3(s)\zeta_4(s) + \frac{2}{3}q^2\zeta_2(s)\zeta_4(s)^2 \\ & + 18q\zeta_2(s)\zeta_6(s)^2 + \frac{1}{9}q^2\zeta_3(s)^3 + \frac{2}{3}q^2\zeta_3(s)^3\zeta_4(s) \\ & + \left(3q + \frac{2}{3}q^2\right)\zeta_3(s)\zeta_4(s)^2 + 18q\zeta_3(s)\zeta_4(s)\zeta_6(s) + 18q\zeta_3(s)\zeta_6(s)^2 \\ & + \frac{2}{9}q^2\zeta_4(s)^3 + 6q\zeta_4(s)^2\zeta_6(s) + 36q\zeta_4(s)\zeta_6(s)^2 + 72\zeta_6(s)^3 \\ & - q^2\zeta_1(3s) - 9q\zeta_2(3s) - q^2\zeta_3(3s) - 2q^2\zeta_4(3s) - 18\zeta_6(3s) \\ = & 54 + \mathcal{O}(q), \end{aligned}$$

$$\begin{aligned}
\zeta_2(s) &= \frac{1}{3}q^2\zeta_1(s)\zeta_4(s)^2 + 2q^2\zeta_1(s)\zeta_4(s)\zeta_6(s) + 3q^2\zeta_1(s)\zeta_6(s)^2 \\
&\quad + \frac{1}{3}q^2\zeta_2(s)\zeta_3(s)^2 + 2q^2\zeta_2(s)\zeta_3(s)\zeta_6(s) + \frac{2}{3}q^2\zeta_2(s)\zeta_4(s)^2 \\
&\quad + 4q^2\zeta_2(s)\zeta_4(s)\zeta_6(s) + 9q\zeta_2(s)\zeta_6(s)^2 + q^2\zeta_3(s)^2\zeta_6(s) \\
&\quad + \frac{1}{3}q^2\zeta_3(s)\zeta_4(s)^2 + 2q^2\zeta_3(s)\zeta_4(s)\zeta_6(s) + 9q\zeta_3(s)\zeta_6(s)^2 \\
&\quad + \frac{2}{3}q^2\zeta_4(s)^3 + (3q + 3q^2)\zeta_4(s)^2\zeta_6(s) + 18q\zeta_4(s)\zeta_6(s)^2 \\
&\quad + (9 + 18q)\zeta_6(s)^3 - 3\zeta_6(3s) \\
&= 6 + \mathcal{O}(q),
\end{aligned}$$

$$\zeta_3(s) = q^2\zeta_1(3s) + 3q\zeta_2(3s) + q^2\zeta_3(3s) + 2q^2\zeta_4(3s) + 6\zeta_6(3s) = 6 + \mathcal{O}(q),$$

$$\zeta_4(s) = 3q\zeta_2(3s) + 6\zeta_6(3s) = 6 + \mathcal{O}(q),$$

$$\zeta_6(s) = \zeta_6(3s) = 1.$$

The abscissa of convergence of ζ_{G_3} is computed as described in §7.3, and is $\sigma_0(G_3) \approx 4.250099133$. In view of the Passman-Temple result mentioned above, it would be interesting to examine the dependency of $\sigma_0(G_p)$ on p .

2.3. Wreath products. There exist sundry residually-finite, finitely generated groups that are isomorphic to their wreath product with a non-trivial finite group; here is such an example. Set

$$F = A_5 * A_5,$$

with A_5 the alternating group on five letters, and distinguish both copies of A_5 by writing ‘ \bar{a} ’ for permutations in the second copy. Set $X = \{1, \dots, 5\}$, and define $\tilde{\psi}: F \rightarrow F \wr_X A_5$ by

$$\tilde{\psi}(a) = \langle\langle 1, \dots, 1 \rangle\rangle a, \quad \tilde{\psi}(\bar{a}) = \langle\langle \bar{a}, a, 1, 1, 1 \rangle\rangle.$$

Let W be the injective quotient of F ; it acts faithfully on X^* .

This example was considered, among others, in [4, Example 4]; it is a branched group, and more precisely ψ is an isomorphism. The representation zeta function of W starts as

$$\begin{aligned}
\zeta_W(s) &= 1 + 2 \cdot 3^{-s} + 4^{-s} + 5^{-s} + 6 \cdot 15^{-s} + 3 \cdot 20^{-s} + 3 \cdot 25^{-s} + 2 \cdot 45^{-s} \\
&\quad + 60^{-s} + 19 \cdot 75^{-s} + 4 \cdot 90^{-s} + 9 \cdot 100^{-s} + \dots,
\end{aligned}$$

and has abscissa of convergence

$$\sigma_0(W) \approx 1.17834859575464,$$

computed as described in §7.3.

2.4. Non-faithful self-similar groups. The group W acts on the tree X^* , and therefore on its boundary X^∞ . Consider the ray $\rho = 1^\infty$ in it, and its orbit \mathcal{O} in X^∞ . Consider then the permutational wreath product $G := C_2 \wr_{\mathcal{O}} W$. This group is also self-similar; to see that, consider now

$$F = \langle A_5, \bar{A}_5, s \mid s^2, [s, \bar{a}] \text{ for all } \bar{a} \in \bar{A}_5 \rangle,$$

extend $\tilde{\psi}$ by

$$\tilde{\psi}(s) = \langle\langle s, 1, 1, 1, 1 \rangle\rangle,$$

and let G be the injective quotient of F . Remark that s acts trivially on X^* , so that G does not act faithfully on X^* . The group G is also branched, see §5.3. The zeta function of G starts as

$$\begin{aligned} \zeta_G(s) &= 2 + 4 \cdot 3^{-s} + 2 \cdot 4^{-s} + 8 \cdot 5^{-s} + 4 \cdot 10^{-s} + 26 \cdot 15^{-s} + 14 \cdot 20^{-s} \\ &\quad + 48 \cdot 25^{-s} + 8 \cdot 45^{-s} + 24 \cdot 50^{-s} + 28 \cdot 60^{-s} + 172 \cdot 75^{-s} + 12 \cdot 80^{-s} \\ &\quad + 24 \cdot 90^{-s} + 132 \cdot 100^{-s} + \dots, \end{aligned}$$

and has abscissa of convergence

$$\sigma_0(G) \approx 1.64046292658488,$$

as follows from §7.3.

3. Representations of extensions

I recall Clifford's construction of representations of an extension. First, a *linear representation* of a group G is a homomorphism $\rho: G \rightarrow \mathrm{GL}_n(\mathbb{C})$. Two linear representations $\rho, \rho': G \rightarrow \mathrm{GL}_n(\mathbb{C})$ are *equivalent*, written \sim , if there exists $T \in \mathrm{GL}_n(\mathbb{C})$ such that $\rho(g)T = T\rho'(g)$ for all $g \in G$.

A *projective representation* of a group G is a homomorphism

$$\rho: G \longrightarrow \mathrm{PGL}_n(\mathbb{C}) := \mathrm{GL}_n(\mathbb{C})/\mathbb{C}^\times.$$

Two projective representations ρ, ρ' are *equivalent* if there exists $T \in \mathrm{PGL}_n(\mathbb{C})$ such that $\rho(g)T = T\rho'(g)$ for all $g \in G$.

Let ρ be a linear or projective representation, to $\mathrm{GL}_n(\mathbb{C})$ or $\mathrm{PGL}_n(\mathbb{C})$. Its *degree* $\deg(\rho)$ is n . The *contragredient* representation ρ^\vee is defined by $\rho^\vee(g) = \rho(g^{-1})^*$, the matrix adjoint. For linear representations ρ, σ of degree m, n respectively, the *tensor product* $\rho \otimes \sigma$ is the linear representation $g \mapsto \rho(g) \otimes \sigma(g)$ into $\mathrm{GL}_{mn}(\mathbb{C})$; and if ρ, σ are both projective representations, their tensor product is a projective representation into $\mathrm{PGL}_{mn}(\mathbb{C})$.

Let $\rho: G \rightarrow \mathrm{PGL}_n(\mathbb{C})$ be a projective representation. Choose a lift

$$\tilde{\rho}: G \longrightarrow \mathrm{GL}_n(\mathbb{C}).$$

Define then

$$\tilde{c}_{\tilde{\rho}}: G \times G \longrightarrow \mathbb{C}^\times$$

by

$$\tilde{c}_{\tilde{\rho}}(g, h) = \tilde{\rho}(g)\tilde{\rho}(h)/\tilde{\rho}(gh).$$

A quick calculation shows that $\tilde{c}_{\tilde{\rho}}$ satisfies the 2-cocycle identity

$$\tilde{c}_{\tilde{\rho}}(g, h) / \tilde{c}_{\tilde{\rho}}(g, hk) \times \tilde{c}_{\tilde{\rho}}(gh, k) / \tilde{c}_{\tilde{\rho}}(h, k) = 1,$$

and therefore defines a cohomology class c_ρ in $H^2(G, \mathbb{C}^\times)$, which depends on ρ only, and not on the choice of lift $\tilde{\rho}$.

3.1. Exact sequences. Let now

$$1 \longrightarrow N \longrightarrow G \xrightarrow{f} Q \longrightarrow 1$$

be an exact sequence. If ρ be a representation (linear or projective) of N , its *inertia* is the group $G_\rho = \{g \in G \mid {}^g\rho \sim \rho\}$ consisting of those $g \in G$ such that the conjugate representation ${}^g\rho: n \mapsto \rho(n^g)$ is equivalent to ρ . The representation ρ is said to be *inert in H* whenever $H \leq G_\rho$.

Assume now that ρ is an irreducible, degree- n linear representation of N . Then ρ extends to a unique projective representation $\bar{\rho}$ of G_ρ , as follows. Fix a right transversal X of N in G_ρ . For each $x \in X$, choose $T_x \in \mathrm{GL}_n(\mathbb{C})$ such that $T_x\rho(h^x) = \rho(h)T_x$ for all $h \in N$; this T_x is unique up to scalars, by Schur's Lemma. For $g = hx \in G_\rho$, set $\tilde{\rho}(g) = \rho(h)T_x$, and let $\bar{\rho}(g)$ be $\tilde{\rho}(g)$'s image in $\mathrm{PGL}_n(\mathbb{C})$. Then, since the T_x are uniquely determined, $\bar{\rho}$ is a projective representation. Furthermore, the 2-cocycle $\tilde{c}_{\tilde{\rho}}$ vanishes on $N \times N$, so defines a cohomology class $c_\rho \in H^2(G_\rho/N, \mathbb{C}^\times)$.

Let χ be an irreducible projective representation of G_ρ/N with cohomology class c_ρ^{-1} ; then $\bar{\rho} \otimes (\chi \circ f)$ is a projective representation of G_ρ with trivial cohomology class. Say χ is of degree m , and let $\tilde{\chi}$ be a lift $G_\rho/N \rightarrow \mathrm{GL}_m(\mathbb{C})$ of χ ; then $\tilde{\rho} \otimes (\tilde{\chi} \circ f)$ is a lift of $\bar{\rho} \otimes (\chi \circ f)$, so its 2-cocycle is a coboundary, namely the 2-cocycle $(\delta b)(g, h) = b(g)b(h)/b(gh)$ associated with a function $b: G_\rho/N \rightarrow \mathbb{C}^\times$. Furthermore, b is unique up to multiplication by a homomorphism $\mu \in H^1(G_\rho/N, \mathbb{C}^\times)$. Then $g \mapsto \tilde{\rho}(g) \otimes \tilde{\chi}(f(g))/b(g)$ is a linear representation of G_ρ , which we denote by $\sigma'_{\rho, \chi}$.

We call such $\sigma'_{\rho, \chi}$ *extensions* of ρ ; they are irreducible representations whose restriction to N is a direct sum of copies of ρ . Finally, let $\sigma_{\rho, \chi, \mu}$ be the induced representation of $\sigma'_{\rho, \chi} \otimes \mu$ up to G .

Theorem 3.1 (Clifford [7]). *With the notation above, $\sigma_{\rho,\chi,\mu}$ is an irreducible representation of G , and every irreducible representation of G is equivalent to some $\sigma_{\rho,\chi,\mu}$.*

The multiplicity of $\sigma_{\rho,\chi,\mu}$ in that list behaves as follows: for a group Q and a class $c \in H^2(Q, \mathbb{C}^\times)$, denote by \widehat{Q}^c the set of equivalence classes of projective representations of Q with cocycle c ; then the correspondence $(\rho, \chi, \mu) \mapsto \sigma_{\rho,\chi,\mu}$ is a map

$$\sigma: \bigsqcup_{\rho \in \widehat{N}} (\widehat{G_\rho/N}^{c_\rho^{-1}} \times H^1(G_\rho/N, \mathbb{C}^\times)) \longrightarrow \widehat{G}$$

which is surjective, and such that every $\sigma_{\rho,\chi,\mu}$ has $\#H^1(G_\rho/N, \mathbb{C}^\times) \cdot [G : G_\rho]$ preimages. \square

We will need to understand how the inertia subgroup changes under extension. I state the following property as a general lemma:

Lemma 3.2. *Let G be a group with normal subgroup N . Let ρ be a representation of N . Consider a subgroup H with $N \leq H \leq G_\rho$. Let σ be an extension of ρ to H . Then $G_\sigma \leq G_\rho$.*

Proof. Since σ is an extension of ρ and ρ is inert in H , the restriction of σ to N is a direct sum of $[H : N]$ copies of ρ . Consider $g \in G_\sigma$, and write T_g as a $[H : N] \times [H : N]$ block matrix. Then $(T_g)_{ij}\rho(n^g) = \rho(n)(T_g)_{ij}$ for all $i, j \in \{1, \dots, [H : N]\}$; and since T_g is invertible, the $(T_g)_{ij}$ span $M_n(\mathbb{C})$ so some linear combination U_g of them is invertible; then $U_g\rho(n^g) = \rho(n)U_g$ so $g \in G_\rho$. \square

4. Representation triples

I recall Isaacs' notion of *character triple*, with a slightly different notation. See also [16, §5] for a more modern formulation.

Definition 4.1. Let B be a finite group. A *B-representation triple* is a pair $\Theta = (\rho, f)$, with $f: G \rightarrow B$ a homomorphism with kernel N and $\rho \in \widehat{N}$ a representation that is inert in G .

(The reader may wonder why they are called triples and not pairs. Isaacs' original definition involves triples (χ, N, G) with χ an N -character that is inert in G . We explicitly add a marking by a group B to the data, and remove B, G and N from the notation.)

We introduce the following terminology: for a B -representation triple $\Theta = (\rho, f)$, its *source* is $\text{src}(\Theta) := \text{src}(f) := G$; its *image* is $\text{im}(\Theta) := \text{im}(f) := f(G) \leq B$; its *representation* is $\rho(\Theta) := \rho$; its *marking* is $f(\Theta) := f$. If $\Theta = (\rho, f)$, we also define $\Theta^\vee = (\rho^\vee, f)$ the triple with same marking but contragredient representation.

A morphism between two B -representation triples (ρ, f) and (ρ', f') is a map $\sigma: \text{src}(f) \rightarrow \text{src}(f')$ such that $f' = f \circ \sigma$ and $\rho \sim \rho' \circ \sigma$. There is also a weaker notion than isomorphism of B -representation triples, that of *equivalence*, which we describe now.

For G a group with normal subgroup N and $\rho \in \widehat{N}$, let $\mathcal{R}(G|\rho)$ denote the monoid of representations of G whose restriction to N is a multiple of ρ . It is an abelian monoid, freely generated by the irreducible representations of G that restrict to a multiple of ρ , and admits a scalar product $\langle | \rangle$ making the irreducible representations an orthonormal basis.

Definition 4.2 (essentially [15, Definition 11.23]). Two B -representation triples (ρ, f) and (ρ', f') are *equivalent* if $\text{im}(f) = \text{im}(f')$ and for every $H \leq \text{im}(f)$ there exists an isometry

$$\sigma_H: \mathcal{R}(f^{-1}(H)|\rho) \longrightarrow \mathcal{R}((f')^{-1}(H)|\rho')$$

such that, for every $N \leq H \leq \text{im}(f)$ and every $\chi \in \mathcal{R}(f^{-1}(H)|\rho)$, we have

$$\sigma_{f^{-1}(N)}(\chi_{f^{-1}(N)}) = (\sigma_{f^{-1}(H)}(\chi))_{(f')^{-1}(N)},$$

$$\sigma_{f^{-1}(H)}(\chi \otimes (\beta \circ f)) = \sigma_{f^{-1}(H)}(\chi) \otimes (\beta \circ f') \quad \text{for all } \beta \in \widehat{\text{im}(f)}.$$

Schur considered projective representations in [23] and [24]. In modern language, he showed that $H_2(G, \mathbb{Z})$ is finite for every finite group G , and that there exists at least one extension

$$1 \longrightarrow H_2(G, \mathbb{Z}) \longrightarrow \widetilde{G} \xrightarrow{f} G \longrightarrow 1$$

such that $H_2(G, \mathbb{Z})$ is contained in $[\widetilde{G}, \widetilde{G}]$; this implies in particular that the lift of any generating set of G is a generating set of \widetilde{G} . One calls \widetilde{G} a *Schur cover* of G , and the epimorphism f a *Schur covering map*.¹

Theorem 4.3 (Isaacs [15, Theorem 11.28]). *Every B -representation triple is equivalent to a B -representation triple (χ, f) with $f: \widetilde{H} \rightarrow H \leq B$ a Schur covering map, and $\chi \in \widehat{H_2(H, \mathbb{Z})} = H^2(H, \mathbb{C}^\times)$.*

In particular, there are finitely many equivalence classes of B -representation triples. A B -representation triple $\Theta = (\rho, f)$ is a convenient way of keeping track of a group $\text{im}(f)$ and a cohomology class in $H^2(\text{im}(f), \mathbb{C}^\times)$.

¹ \widetilde{G} is sometimes called a *stem cover*.

The two procedures at the heart of Clifford's description from §3 – extension and induction – can be rephrased in terms of representation triples.

Consider a B -representation triple $\Theta = (\rho, f)$, and a homomorphism $g: B \rightarrow C$. Let L denote the kernel of $g \circ f$; we have $\ker(f) = N \leq L \leq G = \text{src}(f)$. Let $\{\rho_1, \dots, \rho_n\}$ denote those irreducible representations of L that restrict on N to a multiple of ρ . For $i = 1, \dots, n$, let G_i denote the inertia of ρ_i in G . The g -extensions of Θ are the C -representation triples

$$\Theta_1 = (\rho_1, (g \circ f)|_{G_1}), \quad \dots, \quad \Theta_n = (\rho_n, (g \circ f)|_{G_n}).$$

Lemma 4.4. *The equivalence classes of the C -triples $(\Theta_i)_{1 \leq i \leq n}$ depend only on the equivalence class of Θ .*

Proof. Follows immediately from Definition 4.2 and Lemma 3.2. \square

Note that extension of triples covers both extension and induction; the induction is performed from $\ker(g) \cap \text{im}(f)$ to $\ker(g)$, or, equivalently, from $\text{im}(f)$ to $\text{im}(f) \ker(g)$, and in fact does not modify the triple at all. This is seen as follows. Consider a B -representation triple $\Theta = (\rho, f)$ with $\rho \in \widehat{N}$ and $f: G \rightarrow B$. Let H, M be groups with $N \triangleleft G \leq H, N \leq M \triangleleft H$ and $M \cap G = N$ and $MG = H$ and $H_\rho = G$. Then $G/N \cong H/M$; define $h: H \rightarrow B$ by $h(xy) = f(y)$ for $x \in M, y \in G$; this is well-defined because $M \cap G = N = \ker(f)$. Note $\ker(h) = M$. Induce ρ to M , and let Θ' be the B -representation (ρ^M, h) .

Lemma 4.5 (see [16, Corollary 5.3]). *The triples Θ and Θ' are equivalent.*

Proof. Follows immediately from Definition 4.2. The map

$$\sigma: \mathcal{R}(G|\rho) \longrightarrow \mathcal{R}(H|\rho^M)$$

is simply given by induction to H , namely $\chi \mapsto \chi^H$. \square

We may deduce from Theorem 3.1 a formula expressing the representation zeta function of a group in terms of representations of a normal subgroup. Consider an exact sequence

$$1 \longrightarrow N \longrightarrow G \xrightarrow{f} B \longrightarrow 1$$

For a B -representation triple Θ , define the Dirichlet series

$$\zeta_{G, \Theta}(s) = \sum_{\substack{\rho \in \widehat{N} \\ (\rho, f) \sim \Theta}} (\deg \rho)^{-s}.$$

Proposition 4.6. *With the notation above,*

$$\zeta_G(s) = \sum_{\Theta \in \{B\text{-representation triples}\}} \zeta_{G,\Theta}(s) \zeta_{\Theta^\vee}(s) [B : \text{im}(\Theta)]^{-1-s}.$$

Proof. Consider an irreducible representation ρ of N with character triple Θ ; such representations are counted by $\zeta_{G,\Theta}(s)$. According to Theorem 3.1, a representation of G is obtained by tensoring ρ with a representation χ of opposite cocycle, so as to obtain a linear representation of ρ 's inertia subgroup; such χ are counted by $\zeta_{\Theta^\vee}(s)$. This representation is then induced to a representation of G ; induction increases the degree by $[B : \text{im}(\Theta)]$, and yields $[B : \text{im}(\Theta)]$ copies of the same representation of G . \square

5. Branched groups

We turn now to the notion of *self-similar branched group*, presenting it in a slightly more general and algebraic manner than is usual; see [20] or [3] for classical references.

Let G be a self-similar group with self-similarity structure $\psi: G \rightarrow G \wr_X Q$.

Definition 5.1. The self-similar group G is *branched* if there exists a finite-index subgroup $K \leq G$ such that $\psi(K) \geq K^X$. One says then that G is branched *over* K .

The subgroup K may be assumed to be normal; and in fact there exists a maximal such K , because if K_0, K_1 both satisfy $\psi(K_i) \geq K_i^X$ then $\langle K_0, K_1 \rangle$ also satisfies that property.

For purposes of computation, it is useful to introduce a finite structure capturing important features of branched groups.

Definition 5.2. A *branch structure* is a pair (B, ϕ) such that

- (1) B is a finite group;
- (2) ϕ is an epimorphism from a subgroup B_+ of $B \wr_X Q$ onto B .

Let G be a self-similar group. A *branch structure for* G is a branch structure (B, ϕ) such that

- (1) there exists an epimorphism $f: G \twoheadrightarrow B$;
- (2) for a map f as in (1), denoting f_1 the natural map $f \wr 1: G \wr_X Q \rightarrow B \wr_X Q$, we have $B_+ = f_1 \psi(G)$ and $f = \phi f_1 \psi$:

$$\begin{array}{ccc} G & \xrightarrow{\psi} & \psi(G) \subseteq G \wr_X Q \\ f \downarrow & & \downarrow f_1 \\ B & \xleftarrow{\phi} & B_+ \subseteq B \wr_X Q. \end{array}$$

Lemma 5.3. *A self-similar group is branched if and only if it has a branch structure.*

Proof. Assume first that G is branched over its normal subgroup K . Define $B = G/K$ with natural map $f: G \rightarrow B$. Define then f_1 as in Definition 5.2, and set $B_+ = f_1\psi(G)$. Define finally $\phi: B_+ \twoheadrightarrow B$ by $\phi(f_1(\psi(g))) = f(g)$. This map is well-defined because $K^X \leq \psi(K)$.

Conversely, if (B, ϕ) is a branch structure for G then let K denote the kernel of a map $f: G \rightarrow B$ as in Definition 5.2, and note that G is branched over K . \square

Note that, just as there exists a maximal subgroup K in Definition 5.1, there exists a minimal branch structure (B, ϕ) .

The branch structure captures all the information we will need of G , so that we may forget G altogether when we have its branch structure. In fact, let (B, ϕ) be a branch structure for G . Define then a sequence of groups G_n , with maps $\phi_n: G_n \rightarrow G_{n-1}$, as follows: $G_{-1} = B$, $G_0 = B_+$, $\phi_0 = \phi$, and $G_{n+1} = \{\langle\langle g_x \rangle\rangle q \in G_n \lambda_X Q \mid \langle\langle \phi_n(g_x) \rangle\rangle q \in G_n\}$, with $\phi_{n+1}(\langle\langle g_x \rangle\rangle q) = \langle\langle \phi_n(g_x) \rangle\rangle q$. Finally form the inverse limit

$$G(B) = \varprojlim (G_n, \phi_n).$$

Lemma 5.4. *If (B, ϕ) be a branch structure, then the group $G(B)$ is a profinite self-similar branched group, and B is a branch structure for $G(B)$.*

If furthermore (B, ϕ) be a branch structure for G , then there exists a canonical map $\iota: G \rightarrow G(B)$ interlacing the self-similarity structures of G and $G(B)$, and ι is injective if G is faithful.

Proof. It is clear that $G(B)$ is profinite, being defined as a limit of finite groups.

An element of $G(B)$ is a sequence $h = (\cdots \twoheadrightarrow h_n \twoheadrightarrow h_{n-1} \twoheadrightarrow \cdots)$, with $h_n \in G_n$, namely $h_n = \langle\langle g_{n,x} \rangle\rangle q_n$, with $q_n = q$ for all $n \geq 0$. Define $\psi(h) = \langle\langle (\cdots \twoheadrightarrow g_{n,x} \twoheadrightarrow g_{n-1,x})_x \rangle\rangle q$. This shows that $G(B)$ is self-similar.

We next show that B is a branch structure for $G(B)$. Projection on the last group G_{-1} defines a homomorphism $G(B) \rightarrow B$, and $\psi(G(B))$ projects to $G_0 \subseteq B \lambda_X Q$.

Suppose finally that B is a branch structure for the self-similar branched group G with self-similarity structure $\psi: G \rightarrow G \lambda_X Q$. Define inductively maps $\iota_n: G \rightarrow G_n$ by $\iota_{-1} = f$ and $\iota_n(g) = \langle\langle \iota_{n-1}(g_x) \rangle\rangle q$ if $\psi(g) = \langle\langle g_x \rangle\rangle q$, for all $n \geq 0$. Then $\phi_n \circ \iota_n = \iota_{n-1}$ for all $n \geq 0$, so the maps ι_n assemble into a map $\iota: G \rightarrow \varprojlim G_n$.

If G is faithful, then $\bigcap_{n \geq 0} \psi^{-n}(K^{X^n}) = 1$, so $\bigcap_{n \geq 0} \ker(\iota_n) = 1$ and ι is injective. \square

Note then that $G(B)$ defines a topology on G , which is intermediate between the congruence topology (in which neighbourhoods of the identity are stabilizers of large subtrees) and the profinite completion (in which every finite-index subgroup is a neighbourhood). This topology is Hausdorff precisely when G is faithful. See [5] for details on these topologies.

Proposition 5.5. *Let G be a self-similar branched group over K , and consider a linear representation $\rho: G \rightarrow \mathrm{GL}_n(\mathbb{C})$ of G . Then, for all $\ell \in \mathbb{N}$ large enough depending only on n , the kernel of ρ contains $\psi^{-\ell}([K, K]^{X^\ell})$.*

Proof. Assume $[K, K] \neq 1$, otherwise there is nothing to show. The image of $\psi^{-\ell}(K^{X^\ell})$ in $\mathrm{GL}_n(\mathbb{C})$ has bounded rank, so that there exists a constant b , depending only on n , with the following property: for all ℓ there exists a subset $\Omega \subseteq X^\ell$, with $\#(X^\ell \setminus \Omega) \leq b$, such that $\ker \rho \cap \psi^{-\ell}(K^{X^\ell})$ maps onto $\psi^{-\ell}(K^\Omega)$. In particular, for $\ell \gg 0$ one has $\Omega \neq \emptyset$, say $\omega \in \Omega$; then $[\ker \rho, \psi^{-\ell}(1 \times \cdots \times K \times \cdots \times 1)] = \psi^{-\ell}(1 \times \cdots \times [K, K] \times \cdots \times 1)$, with the non-trivial entry each times in position ω . Since the action of G on X^n is transitive, we get $\psi^{-\ell}([K, K]^{X^\ell}) \leq \ker \rho$. \square

Corollary 5.6. *Let G be a self-similar group, branched over K . Then G is rigid if and only if $K/[K, K]$ is finite.*

Proof. If $K/[K, K]$ is infinite, then it has infinitely many irreducible 1-dimensional representations, so $G/[K, K]$ has infinitely many representations of degree at most $[G : K]$.

Conversely, assume $K/[K, K]$ is finite, and consider $n \in \mathbb{N}$. By Proposition 5.5, there exists $\ell \in \mathbb{N}$ such that all n -dimensional representations of G factor through $G/\psi^{-\ell}([K, K]^{X^\ell})$, which is finite; so there are finitely many n -dimensional representations. \square

Remark 5.7. In case $[K, K]$ contains $\psi^{-\ell}(K^{X^\ell})$ for some $\ell \in \mathbb{N}$, then the sharper statement holds that every linear representation has kernel containing $\psi^{-\ell}(K^{X^\ell})$ for some $\ell \in \mathbb{N}$.

Remark 5.8. If the self-similar group G is branched over K , then it is also branched over $[K, K]$, so that there exists a branch structure with $B = G/[K, K]$ and with the additional property that every linear representation $\rho: G \rightarrow \mathrm{GL}_n(\mathbb{C})$ factors through G_ℓ for some ℓ large enough.

Therefore, the representation zeta function of G coincides with the zeta function counting all continuous representations of the profinite group $G(B)$.

We now turn to the examples introduced in §2, and describe their branch structures.

5.1. The Grigorchuk group. The maximal branching subgroup of the Grigorchuk group (see §2.1) is well-known; we recall it briefly.

In the Grigorchuk group G , consider the subgroup $K = \langle [a, b] \rangle^G$. A direct computation shows that K has index 16 in G , using the relations $a^2 = b^2 = c^2 = d^2 = bcd = (ad)^4 = 1$. The computation $\psi(\langle [a, b], d \rangle) = \langle \langle 1, [a, b] \rangle \rangle$ shows that $\psi(K)$ contains $K \times K$.

In the corresponding branch structure, one has $B = C_2 \times D_8$.

Another direct computation shows that $[K, K]$ contains $\psi^{-3}(K^{2^3})$, so that, by Remark 5.7, the representations of G and $G(B)$ are in bijection.

5.2. The Gupta–Sidki groups. The maximal branching subgroups of the Gupta–Sidki groups (see §2.2) are well-known; we recall them briefly.

In the Gupta–Sidki group G_p , consider the subgroup $K = [G_p, G_p]$. A direct computation shows that K has index p^2 in G_p . If $p \geq 5$, then the computation $\psi(\langle [t, t^a] \rangle) = \langle \langle [a, t], 1, \dots, 1 \rangle \rangle$ shows that $\psi(K)$ contains K^p . For $p = 3$, the computation is slightly different: $\psi(\langle [t t^a, t^a t^{a^2}] \rangle) = \langle \langle [t^{-1}, a^{-1}], 1, 1 \rangle \rangle$.

In the corresponding branch structure, one has $B = C_p \times C_p$.

Another direct computation shows that $[K, K]$ contains $\psi^{-2}(K^{p^2})$, so that, by Remark 5.7, the representations of G and $G(B)$ are in bijection.

5.3. Non-faithful actions. If G is a self-similar branched group, but is not faithful, it may still be possible to construct a branch structure for it. Consider the example of §2.4: it is a group of the form $G = H \wr_{\mathcal{O}} W$, for an abelian group H , a self-similar branched group W and an orbit \mathcal{O} of W on the boundary of the tree X^* .

Let (B, ϕ) be a branch structure for W , with $B \wr_X Q \supseteq B_+ \xrightarrow{\phi} B$. Set $B' = H \times B$ and $B'_+ = H^X \times B_+ \subseteq B' \wr_X Q$, and define $\phi': B'_+ \rightarrow B'$ by

$$\phi'((h_x)_{x \in X}, b) = \left(\prod_{x \in X} h_x, \phi(b) \right). \quad (5.1)$$

Then (B', ϕ') is a branch structure for G .

6. Proof of Theorem A

The criterion “ G is rigid if and only if $K/[K, K]$ is finite” is Corollary 5.6.

6.1. Abscissa of convergence. The next statement of the Theorem asserts that the abscissa of convergence of ζ_G is finite and positive. The proof follows very closely that in [4], so I only describe its main steps.

Proposition 6.1 (see [4, Proposition 13]). *The abscissa of convergence of ζ_G is positive.*

Proof. We let r_n denote the number of irreducible degree- n representations of K . As a first step, there are infinitely many irreducible representations of K , so that, for every $B \in \mathbb{N}$, there exists n such that $\sum_{j \leq n} r_j \geq B$.

For every integer ℓ , there are then at least B^{d^ℓ} representations of K^{X^ℓ} of degree at most n^{d^ℓ} .

Induce and extend these representations to G , and apply (2.1): the index of $\psi^{-\ell}(K^{X^\ell})$ in G is $[K : \psi^{-1}(K^X)]^{(d^\ell-1)/(d-1)} [G : K] \leq k^{d^\ell}$ for some constant k , so there are at least $(B/k)^{d^\ell}$ irreducible representations of G of degree at most $(nk)^{d^\ell}$. Choosing any $B > k$ gives the desired inequality $\sigma_0 \geq \log(B/k)/\log(nk)$. \square

Proposition 6.2 (see [4, Proposition 12]). *The abscissa of convergence of ζ_G is finite.*

Proof. Since the proof follows closely [4, Proposition 12], let me only sketch the proof. Furthermore, the finiteness of the abscissa of convergence also implicitly follows from the functional equation.

Let r_n denote the number of irreducible, n -dimensional complex representations of K . We claim that there exist constants $A \in \mathbb{N}$ and $t > 1$ such that

$$r_n \leq \bar{r}_n := A(n/\sigma_0(n))^t, \quad (6.1)$$

with $\sigma_0(n)$ denoting the number of divisors of n .

Up to replacing X by X^ℓ for some $\ell \in \mathbb{N}$, we may assume that K acts non-trivially on X . Indeed G acts transitively on X^ℓ , so since K has finite index in G it acts with boundedly many orbits.

From Proposition 5.5, all representations of K may be seen as representations of $K/\psi^{-\ell}([K, K]^{X^\ell})$ for some $\ell \in \mathbb{N}$. Let us denote by $r_{n,\ell}$ the number of those representations of K that factor through $K/\psi^{-\ell}([K, K]^{X^\ell})$. We have $r_n = \sup_\ell r_{n,\ell}$, and $r_{n,0} = 0$ for all $n \geq 2$ while $r_{n,0} = [K : [K, K]]$. We prove by induction on ℓ that (6.1) holds for all $\ell \in \mathbb{N}$.

To compute $r_{n,\ell+1}$ in terms of $r_{m,\ell}$ for all $m|n$, we apply Theorem 3.1. We tensor d representations of K to obtain a representation of K^X , extend it to its inertia subgroup $I \leq K$, and induce it to a representation of K . Therefore

$$r_{n,\ell+1} \leq \sum_{\substack{\psi^{-1}(K^X) \leq I \leq K \\ n = n_1 \cdots n_d e [K:I]}} r_{n_1} \cdots r_{n_d} N_e,$$

with N_e denoting the number of e -dimensional projective representations of $I/\psi^{-1}(K^X)$. We consider only $n \geq 3$. The summands with $e[K : I] \geq 2$ are

easily controlled by a bound of the form $\bar{r}_n/2$, if t is large enough (independently of ℓ). Consider then summands with $K = I$ and $e = 1$. The d -tuple of representations of K we are inducing must then be constant on K -orbits, and since these orbits are non-trivial, there are repetitions in the K -tuple, diminishing the number of factors $r_{n_1} \cdots r_{n_d}$; so this term may again be bounded by $\bar{r}_n/2$, if t is large enough (independently of ℓ). \square

6.2. Functional equation. We fix a branched group G , a branch structure (B, ϕ) , and an epimorphism $f: G \rightarrow B$. Let $K = \ker(f)$ denote the branching subgroup. Up to replacing K by $[K, K]$ if needed, we assume by Proposition 5.5 that every representation of G factors through $G/\psi^{-\ell}(K^{X^\ell})$ for some $\ell \in \mathbb{N}$.

Let \mathcal{T} denote a complete set of equivalence class representatives of B -representation triples. Recall that \mathcal{T} is finite, being the disjoint union of the second cohomology groups of all subgroups of B . For Θ a B -representation triple, we denote by $[\Theta]$ its representative in \mathcal{T} .

Without loss of generality, we assume that whenever $\Theta, \Theta' \in \mathcal{T}$ are representation triples such that $\text{im}(\Theta)$ and $\text{im}(\Theta')$ are conjugate in B , say by $b \in B$, then $\text{src}(\Theta) = \text{src}(\Theta')$ and ("conjugation by b ") $\circ f(\Theta) = f(\Theta')$.

In order to compute the zeta function $\zeta_G(s)$, we introduce Dirichlet series

$$\zeta_{G, \Theta}(s) = \sum_{\substack{\rho \in \hat{K} \\ [(\rho, f)] = \Theta}} (\deg \rho)^{-s}.$$

Then, by Proposition 4.6, these series can then be assembled into ζ_G as follows:

$$\zeta_G(s) = \sum_{\Theta=(\rho, f) \in \mathcal{T}} \zeta_{G, \Theta}(s) \zeta_{\Theta^\vee}(s) [B : \text{im}(\Theta)]^{-1-s}. \quad (6.2)$$

In fact, the functional equation we derive will have the following form, equivalent to (1.2), for polynomials F_Θ to be defined in (6.3):

Equation (6.2), and $\zeta_{G, \Theta}(s) = F_\Theta(\{\zeta_{G, \Theta'}(s), \dots, \zeta_{G, \Theta'}(ds)\}_{\Theta' \in \mathcal{T}})$ for all $\Theta \in \mathcal{T}$.

For greater clarity, we consider

$$G_+ = \psi(G) = \{\langle\langle g_x \rangle\rangle q \in G \wr_X Q \mid \langle\langle f(g_x) \rangle\rangle q \in B_+\},$$

and produce a functional equation relating the zeta functions of G and G_+ . Since G and G_+ are isomorphic (via ψ), we will be done.

In a different language, we know from Remark 5.8 that the zeta functions of G and of the profinite group $G(B)$ coincide, and $G(B) = \varprojlim G_n$. The zeta function of G is the coefficient-wise limit of the zeta functions of the finite groups G_n , and the functional equation (6.3) may also be interpreted as a functional equation

between the zeta functions of G_n and G_{n+1} , with G_n taking the role of G and G_{n+1} taking the role of G_+ . Starting from $G_{-1} = B$, we obtain by iteration and taking a limit the zeta function of $G(B)$.

For brevity of notation, we consider the free module Ω with base \mathcal{T} over the ring of Dirichlet series, and its element

$$\zeta_{G,\mathcal{T}} := \sum_{\Theta \in \mathcal{T}} \zeta_{G,\Theta} \cdot \Theta.$$

An equation in Ω is a convenient way of writing $\#\mathcal{T}$ equations among zeta functions.

Theorem 3.1 asserts that all representations of K may be obtained by running through all choices of ρ_x , extending $\bigotimes_{x \in X} \rho_x$ to its inertia in $K \leq G_+$, tensoring by a projective representation, and inducing to K . We show that the equivalence class of the obtained representation triple depends only on the equivalence classes of the representation triples (ρ_x, f) and the datum of which ρ_x are equivalent:

Proposition 6.3. *Let $(\rho_x)_{x \in X}$ be a collection of irreducible representations of K , with associated representation triples $\Theta_x := [(\rho_x, f_x)] \in \mathcal{T}$. Write*

$$\zeta(s) = \sum_{\sigma \in \widehat{K} \text{ extending } \bigotimes_x \rho_x} \dim(\sigma)^{-s} [(\sigma, f)] \in \Omega.$$

Then $\zeta(s) \prod_{x \in X} \dim(\rho_x)^s$ depends only on the Θ_x and on the relation

$$\{(x, y): \rho_x \sim \rho_y\} \subseteq X^2.$$

Proof. The inertia of $\bigotimes_x \rho_x$ in $G \wr_X Q$ has the form $(\prod_x G_{\rho_x})^{\tilde{c}} \rtimes P$, for some $\tilde{c} \in G^X$ and the subgroup $P \leq Q$ consisting of all $q \in Q$ such that $\rho_x \sim \rho_{qx}$ for all $x \in X$. It is also the preimage by $f_1: G \wr_X Q \rightarrow B \wr_X Q$ of $H = (\prod_x \text{im}(\Theta_x) \rtimes P)^c$ for some $c \in B^X$, and is therefore determined by the character triples Θ_x and the relation $\{(x, y): \rho_x \sim \rho_y\}$.

Define then $I = \prod_x \text{src}(\Theta_x) \rtimes P$, and $f_+: I \rightarrow B \wr_X Q$ by $f_+(\langle\langle g_x \rangle\rangle p) = (\langle\langle f_x(g_x) \rangle\rangle p)^c$. On $N := \ker(f_+) = \prod_x \ker(f_x)$, define the representation $\rho_+ = \bigotimes_x \rho_x$. Then (ρ_+, f_+) is a $(B \wr_X Q)$ -representation triple.

Write $I_+ = f_+^{-1}(B_+)$, and denote still by f_+ the restriction of f_+ to I_+ . We obtain a B_+ -representation triple (ρ_+, f_+) . Set $N_+ = f_+^{-1}(\ker \phi)$. Let σ run over all the extensions of ρ_+ to N_+ , and note that σ 's inertia still lies in I_+ , by Lemma 3.2.

Note that the representation ρ_+ was extended from N to N_+ ; this extension degree is therefore expressible as $\dim(\sigma) / \dim(\rho_+)$.

Consider then the induced representation triple $(\sigma, \phi \circ f_+)$. The induction degree is $[\ker \phi : \ker \phi \cap \text{im}(f_+)]$.

This recipe is based on Theorem 3.1, and follows Proposition 4.6 producing all representations of K out of representations of its normal subgroup $\psi^{-1}(K^X)$. The equivalence class of the representation triple $(\sigma, \phi \circ f_+)$ depends only on the classes Θ_x and on the choice of subgroup H , which in turn was dictated by the relation $\{(x, y) : \rho_x \sim \rho_y\}$. Furthermore, the extension and induction degrees are determined by character triples as required. \square

We are now ready to construct the functional equation expressing $\zeta_{G_+, \mathcal{T}}$ in terms of $\zeta_{G, \mathcal{T}}$. We follow Proposition 6.3 in writing $\zeta_{G_+, \mathcal{T}}$ as a sum, over all d -tuples of character triples $(\Theta_x)_{x \in X}$, of all representations of $K \leq G_+$ whose restriction to K^X is a multiple of $\bigotimes_x \rho_x$ for representations ρ_x of K with $[(\rho_x, f)] = \Theta_x$ for all $x \in X$.

Once a family $(\Theta_x)_{x \in X} \in \mathcal{T}^X$ of B -representation triples has been fixed, we sum over all possible inertias of the corresponding tensor product of representations. Since the inertia contains K^X , it suffices to consider its image in $B \wr_X Q$. We are therefore led to enumerate all subgroups $H \leq B \wr_X Q$ satisfying the following two properties: $H \cap B^X = \prod_x \text{im}(\Theta_x)$; and, denoting by $P \leq Q$ the image of H in Q , the family $(\Theta_x)_{x \in X}$ is constant on P -orbits. The first condition implies that abstractly $H \cong \prod_x \text{im}(\Theta_x) \rtimes P$, and in fact $H = (\prod_x \text{im}(\Theta_x) \rtimes P)^c$ for some $c \in B^X$.

We then consider all representations induced and extended from all irreducible representations $\bigotimes_x \rho_x$ of K^X such that $[(\rho_x, f)] = \Theta_x$ and $\rho_x \sim \rho_y$ if and only if $x \in Py$. For a P -orbit Y , we write $\Theta_Y := \Theta_y$ for any $y \in Y$.

To conclude the enumeration, observe that the subgroups H as above form a lattice, under *reverse* inclusion, so that the lattice's maximal element is $\prod_x \text{im}(\Theta_x)$. Let μ denote the lattice's Möbius function [22]; so $\sum_{H \geq H' \geq H''} \mu(H, H') = \delta_{H, H''}$.

It is convenient to replace the condition “if and only if $x \in Py$ ” by “if $x \in Py$ ”, and apply inclusion-exclusion on the lattice of subgroups H . Indeed, then, the contribution to $\zeta_{G_+, \mathcal{T}}(s)$ is $\zeta(s) \prod_x \dim \rho(\Theta_x)^s \prod_{P\text{-orbits } Y} \zeta_{G, \Theta_Y}(s)$ with $\zeta(s)$ as in Proposition 6.3.

We have arrived at the following formula expressing $\zeta_{G_+, \mathcal{T}}$ in terms of $\zeta_{G, \mathcal{T}}$; recall the notation $\Theta_+ = (\rho_+, f_+)$ from the proof of Proposition 6.3:

$$\begin{aligned} \zeta_{G_+, \mathcal{T}}(s) &= \sum_{(\Theta_x) \in \mathcal{T}^X} \sum_{\prod_x \text{im}(\Theta_x) \leq H \leq B \wr_X Q} \sum_{\substack{\sigma \text{ induced from } (\rho_+, f_+) \\ (\rho_+, f_+) \text{ extending } \bigotimes_x \rho(\Theta_+)}} \\ &\quad [\ker \phi : \ker \phi \cap \text{im}(f_+)]^{-1-s} \left(\frac{\dim(\sigma)}{\dim(\rho_+)} \right)^{-s} \\ &\quad \sum_{\prod_x \text{im}(\Theta_x) \leq H' \leq H} \mu(H, H') \prod_{Y \text{ orbit of } H' \text{ on } X} \zeta_{G, \Theta_Y}(\#Ys) \cdot [(\sigma, \phi \circ f_+)]. \end{aligned} \tag{6.3}$$

This equation takes place in the module Ω ; by writing it in the basis \mathcal{T} , one obtains equations $\zeta_{G_+, \Theta} = F_{\Theta}(\{\zeta_{G, \Theta'} : \Theta' \in \mathcal{T}\})$ for polynomials F_{Θ} with coefficients in $\mathbb{Q}(2^{-s}, \dots, P^{-s})$ for $P = \#B_+$, as required.

6.3. Singularities. We recall some arguments from [4]. Let as usual σ_0 denote the abscissa of convergence of ζ_G ; it is the maximum of the abscissæ of convergence of $\zeta_{G, \Theta}$ for all $\Theta \in \mathcal{T}$, since all $\zeta_{G, \Theta}$ are positive-coefficient power series counting subsets of the representations counted by ζ_G , and combining to ζ_G by (6.2). Let $\mathcal{H}(k)$ denote the ring of holomorphic functions in $\{\Re(s) > 2^{-k}\sigma_0\}$. Observe then that $\zeta_{G, \Theta}$ converges in $\mathcal{H}(0)$ for all $\Theta \in \mathcal{T}$, and that

$$\mathcal{H}(0) \subset \mathcal{H}(1) \subset \dots \subset \bigcup_{k \geq 0} \mathcal{T}(k) = \{f: \{\Re(s) > 0\} \longrightarrow \mathbb{C}\}.$$

Treating all variables $\zeta_{G, \Theta}(ks)$ with $k \geq 2$ as coefficients, the functional equation (6.3) may be viewed as a polynomial equation system in unknowns $\zeta_{G, \Theta}(s)$ and coefficients in $\mathcal{H}(1)$. As such, it defines the $\zeta_{G, \Theta}(s)$ as algebraic functions, in a finite extension of $\mathcal{H}(1)$. More generally, let $\overline{\mathcal{H}(k)}$ denote an algebraic closure of $\mathcal{H}(k)$; then, for every $k \geq 1$, the functional equation (6.3) may be viewed as a polynomial equation system in unknowns $\zeta_{G, \Theta}(s)$ and coefficients in $\overline{\mathcal{H}(k)}$, hence describing $\zeta_{G, \Theta}(s) \in \overline{\mathcal{H}(k)}$.

It remains to check that the leading coefficients in the functional equation never vanish. To see that, consider a monomial $S = \zeta_{G, \Theta_1}(s) \dots \zeta_{G, \Theta_d}(s)$ in a term of (6.3). It is associated with representations that extend/induce from $\rho_1 \otimes \dots \otimes \rho_d$ whose inertia is precisely $\prod_x G_{\rho_x}$, namely for which the group P as above is trivial. There is therefore no inclusion-exclusion, and the coefficient of S in the functional equation is the Dirichlet polynomial counting representations of $(\prod_x G_{\rho_x} \cap G_+)/K^X$ with given cocycle; in particular, this coefficient is holomorphic in $\{\Re(s) > 0\}$, and bounded away from 0.

We have therefore shown that all the singularities in $\{\Re(s) > 0\}$ of $\zeta_{G, \Theta}$ are algebraic; since an algebraic closure of the ring of holomorphic functions may be taken as the ring of convergent Puiseux series (see e.g. [9, Corollary 13.15]), we have power series expansions in $s^{1/e}$ about all $s \in \mathbb{C}$ with $\Re(s) > 0$, and in particular in σ_0 . The root order e at σ_0 is at least 2, because σ_0 is a singularity of ζ_G , and it bounded by the degree d of the functional equation.

6.4. Layered groups. In the special case that $G \cong G \wr_X Q$, we recover Theorem 3 from [4] as follows: $B = 1$, and there is a single representation triple. The subgroups H are then in bijection with subgroups of Q . Theorems 1 and 3 in [4] were in fact written in terms of the lattice of partitions of X ; however, if two subgroups Q, Q' induce the same orbit partition on X , then these subgroups contribute many times to (6.3), but that multiplicity is compensated by the Möbius function. Since the cohomology classes in question are all trivial, the summation on all σ may in fact be written via the zeta function of Q .

7. Implementation details

The proof given in §6 is constructive enough that it can be implemented easily in a computer algebra system such as GAP [10]. The code is freely available, and is part of my package FR designed to manipulate self-similar groups. Some changes to the method given in §6 made the computation more efficient.

7.1. Representation triples. Representation triples are objects consisting of a linear representation and a homomorphism. Cohomology classes in $H^2(G, \mathbb{C}^\times)$ are represented as 2-cocycles, namely, as lists of cyclotomic numbers indexed by $G \times G$.

A function computes the cocycle of a representation triple.

Another function converts a representation triple to an equivalent one in which the marking is a Schur covering map.

More precisely, this function finds, given a representation triple Θ and a list \mathcal{T} of representation triples, the one from the list that is equivalent to Θ .

A function computes all the B -representation triples up to equivalence. This is done by enumerating subgroups of B ; computing their Schur cover; and for each subgroup enumerating the characters of the kernel of its Schur covering map.

A function computes all projective representations of a group with given cocycle; the group and cocycle are respectively given to the function as image and representation of a representation triple.

A function, given a projective representation ρ of G that is equivalent to a linear one and an epimorphism $f: G \rightarrow B$ such that the restriction of ρ to $\ker(f)$ is linear, computes all linear representations of G that are equivalent to ρ . These are in bijection with $H^1(B, \mathbb{C}^\times)$.

Finally, a function computes, given a linear representation ρ of H and a group $G \geq H$, all irreducible representations of G that extend ρ .

7.2. Constructing the functional equation. The parameters stated in Theorem A are $N = \#\mathcal{T}$ and $P = \#B$. In particular, the partial zeta functions $\zeta_i(s)$ are really $\zeta_{G, \Theta}(s)$, and the homogeneous polynomials F_i are really F_Θ .

It is too costly to enumerate all subgroups H as in §6. Rather, given the triples $(\Theta_x)_{x \in X}$, we first compute all admissible partitions of X , namely those $\mathcal{P} = (Y_1, \dots, Y_k)$ such that if x, y are in the same part then $\Theta_x = \Theta_y$. We denote by $Q_{\mathcal{P}}$ the stabilizer of \mathcal{P} in Q . We then define subsets \mathcal{C}_x of B , for every $x \in X$, as follows. For each part Y_i , we choose a representative x_i ; we let \mathcal{C}_{x_i} be a right transversal of the normalizer of $\text{im}(\Theta_{x_i})$ in B . For the other $x \in Y_i$, we let \mathcal{C}_x be a right transversal of $\text{im}(\Theta_x)$ in B .

The corresponding subgroup H of B_+ is $(\prod_x \text{im}(\Theta_x) \rtimes Q_{\mathcal{P}})^c$ for an arbitrary choice of $c \in \prod_x \mathcal{C}_x$. We do not construct H explicitly, but rather let \mathcal{I} , the “possible inertias”, be the list, for all choices of a partition \mathcal{P} and $c \in \prod_x \mathcal{C}_x$, of the

homomorphism f from $I = \prod_x \text{src}(\Theta_x) \rtimes Q_{\mathcal{P}}$ to $B \lambda_X Q$ given by $(\prod_x f(\Theta_x)) \times \text{id}$ followed by conjugation by c .

We then construct a $\mathcal{I} \times \mathcal{I}$ -matrix ι , with $\iota(f, f') = 1$ if $\text{im}(f) \leq \text{im}(f')$ and $\iota(f, f') = 0$ otherwise. The Möbius function of \mathcal{I} is just the matrix inverse of ι .

Now, for every $f \in \mathcal{I}$, we compute the extensions σ of $\prod_x \rho(\Theta_x)$ to $f^{-1}(\ker \phi)$; and keep track of the extension degree e and the induction degree i , as well as the representative of $\Theta' = (\sigma, \phi \circ f)$ in \mathcal{T} . Summing over all $f' \in \mathcal{I}$ the expression $\mu(f, f') e^{-s} i^{-1-s}$, we have just computed a term of $F_{\Theta'}$. We repeat this for all tuples $(\Theta_x)_{x \in X} \in \mathcal{T}^X$.

7.3. Using the functional equation. To compute the coefficient of n^{-s} in ζ_G , it is sufficient to work with Dirichlet series truncated at degree n . One starts with the Dirichlet series $\zeta_{B, \mathcal{T}}$, which can easily be computed because B is a finite group, and iterates the functional equation to obtain a fixed point. The iteration converges because the polynomials $F_i - z_{i,1}$ are homogeneous of degree at least two. This is how high-degree coefficients were computed.

On the other hand, to continue ζ_G analytically, one starts by computing a large number of terms of $\zeta_{G, \mathcal{T}}$ as above, up to, say, degree $n = 10^{10}$, obtaining a Dirichlet polynomial. For $s \in \mathbb{C}$ with sufficiently large real part, $\zeta_G(s)$ is well approximated by the Dirichlet polynomials of $\zeta_{G, \mathcal{T}}$ and (6.2). For smaller values of s , one goes through the functional equation (6.3), and replaces $\zeta_{G, \Theta}(ks)$, whenever $k \geq 2$, by its value using the Dirichlet polynomial. What remains is a sequence of $\#\mathcal{T}$ polynomials with complex coefficients and in variables $\{\zeta_{G, \Theta}(s)\}_{\Theta \in \mathcal{T}}$. Such a system can be solved numerically, e.g. using PHC [27] or the more recent Bertini [6]. The system usually has more than one solution, and one picks the relevant one; in particular, for real s , one picks (following analytic continuation) the solution in $\mathbb{C}^{\mathcal{T}}$ that is closest to the one computed for a neighbouring s .

Finally, to obtain the abscissa of convergence, one restricts oneself to real s ; and finds, by repeated subdivision, the minimal s such that the solutions returned by numerically solving for $\zeta_{G, \Theta}(s)$ remain all real. By the Landau-Phragmén theorem mentioned in the Introduction, the abscissa of convergence is a number σ_0 such that all $\zeta_{G, \Theta}(k\sigma_0)$ may be accurately computed using the Dirichlet polynomial truncation, while the polynomial system derived from the functional equation has a multiple root at σ_0 .

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