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A splitting theorem for spaces of Busemann non-positive curvature

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Abstract. In this paper we introduce a new tool for decomposing Busemann non-positively curved (BNPC) spaces as products, and use it to extend several important results previously known to hold in specific cases like CAT(0) spaces. These results include a product decomposition theorem, a de Rham decomposition theorem, and a splitting theorem for actions of product groups on certain BNPC spaces. We study the Clifford isometries of BNPC spaces and show that they always form Abelian groups, answering a question raised by Gelander, Karlsson, and Margulis. In the smooth case of BNPC Finsler manifolds, we show that the fundamental groups have the duality property and generalize a splitting theorem previously known in the Riemannian case.

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1. Introduction

In 1985, in a paper titled "A splitting theorem for spaces of nonpositive curvature" ([22]), Schroeder introduced a novel approach for studying Hadamard manifolds that was based on the work of Eberlein and Chen ([7], [5], and [8]), and proved a splitting theorem for non-positively curved manifolds of finite volume that generalized previous results of Gromoll and Wolf ([12]) and Lawson and Yao ([18]) for non-positively curved compact manifolds. While Schroeder's arguments were differential in essence, they have inspired a host of splitting theorems for actions on CAT(0) spaces (cf. Theorem II.6.21 in [3] and Theorem 9 in [19]). In this paper we adopt Schroeder's approach to study group actions on spaces satisfying a weaker notion of non-positive curvature, namely Busemann non-positive curvature (BNPC hereafter). We explore product decompositions and Clifford isometries of BNPC spaces and extend several results, previously known to hold in some specific cases like CAT(0) spaces, such as the product decomposition theorem (cf. Theorem II.2.14 in [3]), the de Rham decomposition theorem (cf. Theorem II.6.15 in [3]), and the splitting theorem (cf. Theorem 9 in [19]). These results are then combined in the smooth case to prove the following splitting theorem.

Theorem 1.1. *Let M be a complete reversible Finsler manifold of Busemann non-positive curvature and finite Busemann–Hausdorff volume. Suppose that the fundamental group Γ of M has a trivial center and that it splits as $\Gamma = \Gamma_1 \times \Gamma_2$, that M has a compact isometry group, and that the induced metric on the universal cover of M is uniformly convex. Then M is isometric to a product $M_1 \times M_2$ with $\pi_1(M_i) = \Gamma_i$.*

A reversible Finsler manifold is a smooth manifold equipped with a norm on the tangent space that varies smoothly (see precise definition below). The natural analogue of the sectional curvature in the context of Finsler manifolds is the flag curvature. However, as discussed for example in [9], non-positive flag curvature is in some ways not truly analogous to non-positive sectional curvature, and other notions of non-positive curvature may be more productive. In this work we consider the notion of Busemann for non-positive curvature, which has the advantage of being naturally comparable with the notion of CAT(0). Many of the techniques and proofs in this paper originate from the CAT(0) case.

A manifold (resp. geodesic metric space) is said to be BNPC if locally (resp. globally) the distance between any two constant speed geodesics is a convex function. A metric satisfying this property is said to be *convex*. BNPC spaces generalize CAT(0) spaces in a similar manner as Banach spaces generalize Hilbert spaces (cf. [11]). This notion of non-positive curvature was introduced by Busemann in [4] and it originates from the observation that a complete connected Riemannian manifold has non-positive sectional curvature if and only if its metric is locally convex. The notions of non-positive flag curvature and Busemann non-positive curvature coincide in the context of Riemannian manifolds, or more generally, in the context of *Berwald manifolds*, which are Finsler manifolds that are affinely equivalent to Riemannian manifolds (cf. [17] and [15]). However, in general these two notions are not equivalent (cf. [14]). Kristály and Roth conjectured in [16] that every Finsler manifold of Busemann non-positive curvature is necessarily a Berwald manifold. See Section 2 below for a further discussion and additional definitions.

If M has infinite volume or if Γ has a non-trivial center then the theorem fails already when M is Riemannian (cf. Theorem 1 and Corollary 1 in [22], and the discussion on Section 4.2 in [19]). The strategy used in [22], as well as in this paper, is to consider the action of Γ on the universal cover X of M and then show that X splits as $X = X_1 \times X_2$ and that Γ respects this splitting in the sense that Γ_i acts trivially on X_{3-i}. The main impediment is that the action of Γ on X might have fixed points at infinity. If M has finite volume then the action of Γ on X has the duality property (see Section 8), which implies that all the fixed points at infinity lie on the boundary of some flat factor of X. The fact that Γ has a trivial center ensures that X does not admit a non-trivial flat factor. Note that if M is Riemannian then the isometry group of M is compact (cf. [7]) and the induced metric on the universal cover is CAT(0) and hence uniformly convex (see Section 2). Thus Theorem 1.1 indeed generalizes Schroeder's splitting theorem (cf. Theorem 2 in [22]). The author is not familiar with any example of a complete reversible BNPC Finsler manifold of finite volume M such that $I(M)$ is not compact, or such that X is not UC. It is conceivable that Theorem 1.1 still holds without these two assumptions.

Like in [22], the splitting of the manifold in Theorem 1.1 is attained via a splitting of its universal cover, which by the Cartan–Hadamard theorem (Theorem II.4.1.(1) of [3]) is a BNPC space. However, unlike the CAT(0) case, there is no canonical metric to associate to the Cartesian product of two BNPC spaces. After providing the needed preliminaries in Section 2, and discussing the notion of parallel subsets in BNPC spaces in Section 3, we turn explore product decompositions of BNPC spaces in Section 4. We note some of the desirable properties of direct products of CAT(0) spaces and consider several approaches for defining product decompositions of BNPC spaces. On the basis of this discussion we suggest two types of BNPC product decompositions, symmetric and non-symmetric, according to the way the fibers in the ambient space intertwine. We prove that both types of decompositions have many of the desirable properties of direct products of CAT(0) spaces. In Section 5 we prove the product decomposition theorem (Theorem 5.3), which provides sufficient and necessary conditions for a cover of a BNPC space to induce a product decomposition.

Section 6 studies the *Clifford isometries* of BNPC spaces. A Clifford isometry is an isometry with a constant displacement function. We prove that a BNPC space admits a Clifford isometry if and only if it admits a BNPC decomposition with a flat factor. More generally, we prove an analogue of the de Rham decomposition theorem (compare with Theorem II.6.15 in [3]).

Theorem 1.2 (de Rham decomposition theorem). *Let X be a BNPC space. Then X admits a BNPC product decomposition $X = B \times Y$ where B is a strictly convex normed vector space and Y is a BNPC space with no non-trivial Clifford isometries. Every flat factor of X is contained in B. Every isometry of X respects this decomposition and every Clifford isometry of X acts trivially on Y and as a translation on B. If X is complete then B is a Banach space. If X is geodesically complete then the decomposition $X = B \times Y$ is symmetric.*

In particular, Theorem 1.2 answers affirmitvely a question raised in [11] (cf. Remark 2.5) on whether the set of Clifford isometries of a BNPC space forms a group.

Theorem 1.3. *The Clifford isometries of a BNPC space form an Abelian group.*

In Section 7 we prove a splitting theorem for group actions on BNPC spaces that generalizes previous splitting theorems on CAT(0) spaces (compare with Theorems II.6.21 and II.6.23 in [3] and Theorem 9 in [19]).

Theorem 1.4 (splitting theorem). *Let X be a complete BNPC space which is either locally compact or uniformly convex. Let $G = G_1 \times \ldots \times G_n$ be a group acting by isometries on X. If $d_G \to \infty$ then there exists a minimal closed, convex, G-invariant subspace $Z \subset X$ which has a G-equivariant symmetric BNPC decomposition $Z = Z_1 \times \ldots \times Z_n$ and each G_i acts trivially on Z_j when $j \neq i$.*

Here $d_G \to \infty$ means that the action is *non-weakly evanescent*, which is the analogue of having no fixed points at infinity for actions on spaces that are not locally compact. For the precise definition see Section 7 below. The proof of Theorem 1.4 given here is an adaptation of the proof of Theorem 9 in [19].

In Section 8 we turn to study Busmenan non-positive curvature in the context of smooth spaces, focusing on the *duality property*. We say that an action of a group G on a geodesic space X has the duality property if for every geodesic line $c\colon \mathbf{R} \to X$ there exists a sequence $g_n \in G$ such that $g_n c(0) \to c(\infty)$ and $g_n^{-1} c(0) \to c(-\infty)$. The duality property was first introduced by Chen and Eberlein in [5] as a replacement for cocompactness. The principal example of an action that satisfies this property is the action of the fundamental group of a Hadamard manifold of finite volume on the universal cover. The duality property played an essential role in the proof of the splitting theorem in [22] and it is also essential for the proof of Theorem 1.1. We prove the following theorem.

Theorem 1.5. *Let (M, F) be a complete BNPC Finsler manifold of finite volume. Then the action of $\pi_1(M)$ on the universal cover of M has the duality property.*

Note that it is unknown whether the duality property extends to more general classes of actions on CAT(0) spaces. In particular, it remains an open question whether a cocompact proper action on a geodesically complete CAT(0) space satisfies the duality property (cf. [1] and [2]).

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2. Preliminaries

The spaces considered in this work are *uniquely geodesic spaces*. This means that every two points x, y are connected by a unique *geodesic curve*, i.e., a unique isometry $c\colon [a,b] \to X$ such that $c(a) = x$ and $c(b) = y$. We will usually make no distinction between the map c and its image, which we denote by $[x, y]$, and will refer to either one as the *geodesic* (segment, curve, path, etc.) connecting x and y. Geodesic rays and lines are defined similarly. A geodesic space X is said to be (*uniquely*) *geodesically complete* if every geodesic segment in X can be extended (in a unique way) to a geodesic line. A *constant speed geodesic* is a curve $c\colon [0,1] \to X$ traveling at constant speed such that $c([0,1])$ is a geodesic.

A geodesic metric space (X, d) is said to be *Busemann non-positively curved* (BNPC) if d is a *convex metric*, i.e., if for every two constant speed geodesics c, c' we have

$$d\left(c\left(\frac{1}{2}\right), c'\left(\frac{1}{2}\right)\right) \le \frac{1}{2} d(c(0), c'(0)) + \frac{1}{2} d(c(1), c'(1)).$$

Standard examples of BNPC spaces include CAT(0) spaces and strictly convex normed vector spaces ("flat BNPC spaces" hereafter). Convex subsets and ceratin finite products of BNPC spaces are also BNPC (cf. Example 4.2 below). More generally, spaces of p-integrable maps to complete uniformly convex BNPC spaces are BNPC, and are generally not CAT(0) when $p \neq 2$. (cf. [11]). Another example of a BNPC space is an ellipse C in the plane endowed with the Hilbert metric (see Example 2.4 below). The resulting metric space is a BNPC Finsler manifold, which is CAT(0) if and only if C is a circle. We note that in the case of Finsler manifolds, Busemann's notion of non-positive curvature is one of several inequivalent notions that generalize sectional curvatue in differnt ways (cf. discussions in [15] and [9]).

Flat BNPC spaces and their linearly convex subsets (flat BNPC sets hereafter) play an important role in the theory of BNPC spaces, similar to the role played by flat CAT(0) spaces (i.e., inner product spaces) in the theory of CAT(0) spaces. Following are two useful facts on the relation between flat sets and affine maps. Recall that a function $f\colon X \to \mathbf{R}$ is said to be *affine* (resp. *convex*) if its composition with any geodesic curve in X is affine (resp. convex). More generally, a map $f\colon X \to Y$ between uniquely geodesic spaces is said to be an *affine* if the composition of f with any geodesic curve in X gives a constant speed geodesic curve in Y. If in addition f is a bijection then X and Y are said to be *affinely equivalent*. The following observation follows from Proposition 3.3 in [10].

Proposition 2.1. *Let X, Y be uniquely geodesic spaces and $f\colon X \to Y$ a continuous affine map. If $A \subset X$ is a flat BNPC set then so is $f(A)$.*

The following characterization of flat BNPC spaces was proved in [13].

Theorem 2.2. *Let X be a geodesic metric space. Then X is isometric to a flat BNPC set if and only if affine maps separate points in X, i.e., if for every $x \neq y \in X$ there exists an affine map $f: X \to \mathbf{R}$ such that $f(x) \neq f(y)$.*

It follows from the definition that BNPC spaces are uniquely geodesic spaces. The *midpoint* of $x, y \in X$ is the unique point $m \in [x, y]$ such that $d(m, x) = d(m, y)$ and we denote it by $\frac{x+y}{2}$. Convex metrics are in fact *strictly convex*, i.e., they satisfy $d(x, \frac{y_1+y_2}{2}) < \frac{d(x,y_1)+d(x,y_2)}{2}$ for every three non-aligned points x, y_1, y_2 (cf. Proposition 8.2.5 of [20]). Occasionally we will require a stronger notion of convexity, namely *uniform convexity*. X is said to be *weakly uniformly convex* (WUC) if all its *modulus of convexity functions* $\delta_{x,\varepsilon}: (0, \infty) \to \mathbf{R}$ ($x \in X$ and $\varepsilon > 0$) are strictly positive when $\delta_{x,\varepsilon}$ is defined by

$$\delta_{x,\varepsilon}(r) = \begin{cases} \infty & \text{if } M_{x,\varepsilon,r} = \emptyset, \\ r - \sup\left\{d(x, \frac{y_1+y_2}{2}): (y_1, y_2) \in M_{x,\varepsilon,r}\right\} & \text{else,} \end{cases}$$

where

$$M_{x,\varepsilon,r} = \{(y_1, y_2): \max\{d(x, y_1), d(x, y_2)\} \leq r \text{ and } d(y_1, y_2) \geq \varepsilon r\}.$$

We say that X is *uniformly convex* (UC) if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\delta_{x,\varepsilon}(r) \geq \delta(\varepsilon) r$ for every x and r. We note that in Theorems 1.1 and 1.4 the assumption that X is UC can be replaced with a slightly milder assumption, namely that every modulus of convexity function $\delta_{\varepsilon,x}$ satisfies $\liminf_{r\to\infty} \left(\frac{\delta_{x,\varepsilon}(r)}{r}\right) > 0$.

Recall that a metric space is said to be proper if every closed ball is compact. By the Hopf–Rinow theorem (cf. Theorem I.3.7 in [3]) complete and locally compact BNPC spaces are proper and thus WUC (cf. Corollary A.2 in [21]), but generally not UC (cf. Corollary A.12 in [21]).

Proposition 2.3. *Let X be a WUC BNPC space and suppose that $C \subset X$ is a non-empty complete convex subset of X then for every $x \in X$ there is a unique point $p(x) \in C$ such that $d(x, p(x)) = d(x, C) = \inf\{d(x, c): c \in C\}$. The projection $X \to C$ given by $x \mapsto p(x)$ is continuous and convex.*

The projection p is called the *closest point projection*. For a proof of the proposition see Corollary A.3 in [21]. We stress that unlike the CAT(0) case, closest point projections in BNPC spaces need not be Lipschitz (cf. Theorem A.9 in [21]). However, there are a few special cases where closest point projections are known to be Lipschitz, such as projections in 2-dimensional affine spaces (cf. Theorem A.7 in [21]) or projections to fibers of product decompositions (cf. Proposition 4.6 (a) below).

Let M denote a connected smooth manifold M. A *Finsler structure* on M is a continuous function $F: TM \to [0, \infty)$ which is smooth on $TM \setminus \{0\}$ and such that for all $p \in M$ the restriction F_p of F to T_pM is a Minkowski norm, i.e., F_p is positively homogeneous of degree one and with positive definite Hessian:

$$g(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(y + su + tv)]_{|s=t=0}.$$

A *Finsler manifold* is a smooth connected manifold M along with a Finsler structure F on M. If F_x is absolutely homogeneous (i.e., if F_x is a norm on T_xM) for every $x \in M$ then (M, F) is said to be *reversible*. The Finsler structure of reversible Finsler manifolds induces a length metric in the usual way. All Finsler structures considered in this paper are assumed to be reversible. A Finsler manifold (M, F) is siad to be of *Busemann non-positive curvature* if the induced length metric d_F of M is locally convex, or equivalently, if the universal cover of M is BNPC. The following is an example of a family of BNPC Finsler manifolds that are not CAT(0).

Example 2.4. Let C be a simple closed linearly convex subset in the plane. Given two distinct points p and q in C, the unique straight line connecting them intersects the boundary of C in two points, a and b, labeled so that $|ap| < |aq|$ where $| \, |$ denoted the Euclidean distance. We define the Hilbert distance between p, q by

$$d(p, q) = \frac{1}{2}\log\frac{|qa||bp|}{|pa|bq|}.$$

Then d defines a metric on C and in fact (C, d) is a reversible Finsler manifold of constant flag curvature = -1 (cf. [15]).However (C, d) is BNPC if and only the boundary of C is an ellipse and it is CAT(0) if and only if the boundary of C is a circle (cf. [14]).

3. Parallel sets in BNPC spaces

Let X be BNPC. We say that the segments $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ in X are *parallel* (denoted $I_1 \parallel I_2$) if

$$d(a_1, a_2) = d(b_1, b_2) = d\Big(\frac{a_1 + a_2}{2}, \frac{b_1 + b_2}{2}\Big).$$

If in addition

$$d(a_1, a_2) = d(I_1, I_2) = \inf\{d(x, y) \,|\, x \in I_1, y \in I_2\},$$

then we say that I_1 and I_2 are *opposite*. The following lemma is due to Busemann (cf. Theorem 3.14 in [4]).

Lemma 3.1 (Busemann's Lemma). *Let X be BNPC and let I_1, I_2 be two geodesic segments in X with parameterizations $c_i\colon [0,1] \to I_i$ such that $t \mapsto d(c_1(t), c_2(t))$ is affine. Then the set $\bigcup_{t\in[0,1]}[c_1(t), c_2(t)]$ is convex and isometric to a flat BNPC set.*

Busemann's lemma has several immediate implications. First note that parallel segments in BNPC spaces are either collinear or the opposite sides of a rectangle in some strictly convex normed plane. We therefore have the following corollary.

Corollary 3.2. *$[a,b] \parallel [c,d]$ if and only if $[a,c] \parallel [b,d]$.*

In particular, parallel segments in BNPC spaces are of equal length. Note that opposite segments cannot be collinear and thus opposite segments always span a flat BNPC rectangle. We emphasize however that Corollary 3.2 does not remain true if we replace "parallel" by "opposite," as illustrated in the following example.

Example 3.3. First note that a normed vector space V is BNPC if and only if the unit ball in V is strictly convex (cf. Remark II.1.18 in [3]), and that segments in flat BNPC spaces are parallel if and only if one segment is a translation of the other. Let $\| \ \|$ be a strictly convex norm on the plane with a unit ball B that satisfies $\|(0,1)\| = \|(1,0)\| = 1$ and $B \subset \{(x,y) \in \mathbf{R}^2 \colon |y| \leq 1\}$. Then the restrictions on $\| \ \|$ imply that $\|(t,-1),(s,1)\| \geq 2$ for every t, s and equality occurs if and only if $t = s$. Thus the segments $I_1 = [(-1,1),(1,1)]$ and $I_2 = [(-1,-1),(1,-1)]$ are opposite. On the other hand, a similiar argument implies that $I_3 = [(-1,1),(-1,-1)]$ and $I_4 = [(1,1),(1,-1)]$ are opposite if and only if $B \subset \{(x,y) \in \mathbf{R}^2 \colon |x| \leq 1\}$. Since the only additional restriction on B as a unit ball is that it is symmetric (i.e., $B = -B$), it is fairly straight at this point forward to produce examples of norms in which I_3 and I_4 are not opposite.

Note also that in general "being parallel" is not a transitive relation, not even in the CAT(0) case. For example, take two copies S_1 and S_2 of the Euclidean square $[0,1] \times [0,1]$ and paste them together by identifying $(x,y) \in S_1$ with $(x,y) \in S_2$ whenever $x \leq y$. Let X be the quotient space endowed with the induced length metric, then by Reshetnyak's theorem (Theorem II.11.1 of [3]) X is CAT(0). Let $p_i \colon S_i \to X$ denote the projection maps. Then on one hand, the two segments $p_1([(0,1),(1,1)])$ and $p_2([(0,1),(1,1)])$ have exactly one point in common and therefore these segments are not parallel in X. On the other hand, both these segments are parallel in X to $p_1([(0,0),(1,0)]) = p_2([(0,0),(1,0)])$. Thus "being parallel" is not a transitive relation.

The notion of "being parallel" can be extended to convex sets. We say that two convex subsets $A, B \subset X$ are *parallel* if they admit a surjective *parallel isometry*, i.e., a bijection $f \colon A \to B$ such that $[a_1, a_2]$ is parallel to $[f(a_1), f(a_2)]$ for every $a_1, a_2 \in A$. Note that Corollary 3.2 implies that $[a_1, f(a_1)] \parallel [a_2, f(a_2)]$ and thus that f is indeed an isometry. If f is the closest point projection to B then we say that A and B are *opposite*. Hereafter, the distance between two sets is defined by $d(A, B) = \inf\{d(x, y) \mid x \in A, y \in B\}$. In terms of distance, A and B are opposite if and only if $d(A, B)$ is attained at every point in $A \cup B$.

Next we list a few useful observations.

Proposition 3.4. *Let X be a BNPC space, $A, B \subset X$ two convex opposite subsets, and $C = \mathrm{conv}(A \cup B)$ their convex hull. Then,*

(a) *C is the disjoint union of convex sets A_α opposite to A*;

(b) *C admits closest point projections to the A_α that are 2-Lipschitz*;

(c) *C is affine if and only if A is affine*;

(d) *C is complete if and only if A is complete.*

The proof readily follows from the fact that C has a BNPC decomposition $C = A \times [0, d(A, B)]$. We will therefore postpone the proof of Proposition 3.4 to the end of Section 5, where we can use the properties of BNPC product decompositions.

4. Product decompositions of BNPC spaces

In this section we define product decompositions of BNPC spaces. We start by discussing what product decompositions of BNPC spaces should look like. Our first step is to characterize product decompositions of CAT(0) spaces.

Let (X, d) be a geodesic space and suppose X decomposes as $X = Y \times Z$ where Y and Z are not reduced to a point. The *Y-fibers* (resp. *Z-fibers*) of this decomposition are the subsets of the form $Y_z = Y \times \{z\}$ (resp. $Z_y = \{y\} \times Z$) and we assume that they are convex in X. We say that the Z-fibers are *transversal* to the Y-fibers, or that Z is *transversal* in X to Y, if $d((y, z), (y, z')) = d(Y_z, Y_{z'})$ for every $y \in Y$ and $z, z' \in Z$. Given some fiber Y_z (resp. Z_y) we denote by d_{Y_z} (resp. d_{Z_y}) the restriction of d to it. We say that X is the *direct product* of Y and Z, and denote $X = Y \oplus Z$, if there exist fibers Y_z and Z_y such that $d^2 = d_{Y_z}^2 + d_{Z_y}^2$. We consider the following characterizations of X:

(i) $X = Y \oplus Z$;

(ii) there exist some function $f\colon \mathbf{R}^2 \to \mathbf{R}$ and some fibers Y_z and Z_y such that $d = f(d_{Y_z}, d_{Z_y})$;

(iii) the Z-fibers in X are transversal to the Y-fibers.

Proposition 4.1. *If d is convex then* (i) $\implies$ (ii) $\implies$ (iii). *Furthermore, if* (iii) *is satisfied then d_{Y_z} and d_{Z_y} are independent on the choice of z and y and they induce convex metrics on Y and Z. If, in addition, (X, d) is* CAT(0) *then* (i), (ii), *and* (iii) *are equivalent.*

Proof. Fix $z_1 \neq z_2 \in Z$ and $y_1 \neq y_2 \in Y$. For $i = 1, 2$ let $c_i : [0, r_i] \to X$ denote the geodesic connecting y_1 and y_2 in Y_{z_i}. (i) $\implies$ (ii) is clear.

Assume (ii). Note that $d((y_1, z_1), (y_2, z_1)) = f(d((y_1, z), (y_2, z)), 0)$ is independent on z_1. We conclude that d_{Y_z} is independent on z. Similarly, d_{Z_y} is independent on y. Omitting the y and z from d_{Y_z} and d_{Z_y} respecively, we see that the Y-fibers are isometric to the space (Y, d_y) and the Z-fibers are isometric to the space (Z, d_Z). Let $c(t) : [0, r] \to Y$ denote the geodesic connecting y_1 and y_2 in Y then $c_1(t) = (c(t), z_1)$ and $c_2(t) = (c(t), z_2)$. For every fixed $0 < s < r$ define $g_s(t) = d(c_1(s), c_2(t)) = f(|t - s|, d_Z(z_1, z_2))$. Then $g_s(t)$ is strictly convex and symmetric around s and thus obtains it minima at $t = s$. It follows that $d(c_1(0), c_2(0)) < d(c_1(0), c_2(r)$. As y_1, y_2, z_1 and z_2 were chosen arbitrarily, it follows that Z is transversal to Y.

If Z is transversal to Y then $d((y, z_1), (y, z_2)) = d\left(Y_{z_1}, Y_{z_2}\right)$ is independent of y and thus d_{Z_y} is independent of y. Let $p : Y_{z_1} \to Y_{z_2}$ denote the closest point projection. The transversality of Z implies that $p((y', z_1)) = (y', z_2)$ for every $y' \in Y$. Set $m_i = c_i\left(\frac{r_i}{2}\right)$. Then by convexity of the metric, $d(m, m') \leq \frac{1}{2}(d(c_1(0), c_2(0)) + d(c_1(r_1), c_2(r_2)) = d(Y_{z_1}, Y_{z_2})$. We conclude that $[c_1(0), c_1(r_1)] \parallel [c_2(0), c_2(r_2)]$. By Corollary 3.2 it follows that $r_1 = r_2$. As y_1, y_2, z_1 and z_2 were taken arbitrary it follows that d_{Y_z} is independent of z. Being isometric to convex subsets of X, (Y, d_{Y_z}) and (Z, d_{Z_y}) are clearly BNPC.

Suppose X is CAT(0). Then by the Sandwich Lemma (cf. Exercise II.2.12.2 in [3]) the segments $[(y_1, z_1), (y_2, z_1)]$ and $[(y_1, z_2), (y_2, z_2)]$ span an Euclidean rectangle in X, isometric to the direct product of $[(y_1, z_1), (y_2, z_1)] \times [(y_1, z_2), (y_2, z_2)]$. It follows that $d^2((y_1, z_1), (y_2, z_2)) = d^2_{Y_z}(y_1, y_2) + d^2_{Z_y}(z_1, z_2)$. We conclude that if X is CAT(0) then (iii) $\implies$ (i). □

The following two examples demonstrate that (i), (ii), and (iii) are not equivalent in the context of BNPC spaces.

Example 4.2. The l_p-metric on the Cartesian product $X = Y \times Z$ is given by

$$d_p((y_1, z_1), (y_2, z_2)) = \|(d_Y(y_1, y_2), d_Z(z_1, z_2))\|_p$$

If Y and Z are BNPC and $1 < p < \infty$ then $(Y \times Z, l_p)$ is BNPC. Furthermore, the Y and Z fibers of $X = Y \times Z$ are convex subsets of X, pairwise opposite and transversal to one another. The restriction of d_p to every Y-fiber (resp. Z-fiber) coincides with d_Y (resp. d_Z). In the terms of Proposition 4.1, the l_p-metrics satisfy conditions (ii) and (iii) but X is isometric to the direct product of Y and Z only if $p = 2$. We conclude that the products of BNPC spaces are not rigid in the sense that the fibers can be composed together in various ways and there is no one common structure for all products, as in CAT(0) spaces.

Example 4.3. Let W be a flat Busemann space. Recall that given vectors $v, w \in W$ we say that v is *transversal* to w (denoted $v \dashv w$) if $\|v + tw\| \geq \|v\|$ for all $t \in \mathbf{R}$. Geometrically this means that the line $v + tw$ supports the ball $\{u \in W : \|u\| \leq \|v\|\}$. Equivalently, $v \dashv w$ if and only if $\mathbf{R}v$ is transversal to $\mathbf{R}w$ in the plane $\mathbf{R}v \times \mathbf{R}w$. Generally $\dashv$ is not a symmetric relation, i.e., $v \dashv w$ does not imply $w \dashv v$. Indeed, consider for example the case $W = (\mathbf{R}^2, \| \, \|)$ where $\| \, \|$ is given by $\|(x, y)\| = \sqrt{x^2 + 2y^2} + |y|$ and take $v = (1, 0)$ and $w = (1, 1)$. Observing that the unit ball of $\| \, \|$ is the intersection of the two Euclidean balls or radius $\sqrt{2}$ around $(0, 1)$ and $(0, -1)$ it is clear that v is transversal to w but not vise versa. Thus the decomposition $W = \mathbf{R}w \times \mathbf{R}v$ satisfies condition (iii) of Proposition 4.1 but not condition (ii), since otherwise applying Proposition 4.1 on the product $W = \mathbf{R}v \times \mathbf{R}w$ would imply that $\mathbf{R}w$ is transversal to $\mathbf{R}v$.

Proposition 4.1 provides three possible definitions for product decompositions while Examples 4.2 and 4.3 illustrate that they are not equivalent. The definition suggested at Proposition 4.1 (iii) is the most general as it implied by the other two. However, in Example 4.3 we saw that such definition would yield product decompositions which are not symmetric in the sense that $X = Y \times Z$ may be a decomposition while $X = Z \times Y$ might not. The reader may wonder whether working with the most general definition is really essential. The next example shows that if we want to generalize theorems such as the de Rham decomposition theorem (cf. Theorem II.6.15 in [3]) then we must endure non-symmetric decompositions.

Example 4.4. Let W, v, w and $\| \, \|$ be as in Example 4.3. Observe that v and $-v$ are the only unit vectors transversal to w. Thus there is no vector $u \in W$ such that $u \dashv w$ and $w \dashv u$. Define $V = \{sv + tw \,|\, s, t \in \mathbf{R} \ |s| \leq 1\}$ then V is a BNPC space which admits Clifford isometries. If there exists a de Rham decomposition of V it must be of the form $V = \mathbf{R}w \times U$ where U is some linear segment in W. The observation above shows a decomposition of this sort satisfies condition (iii) if and only if $U = [-1, 1]v$. Thus any definition for BNPC decomposition must allow decompositions like $V = \mathbf{R}w \times [-1, 1]v$ which are not symmetric.

Following the discussion above we are now ready to define product decompositions of BNPC spaces.

Definition 4.5. Let (X, d) be a BNPC space. We say that $X = Y \times Z$ is a *BNPC decomposition* if the Y-fibers and Z-fibers are convex subsets of X and if Z is transversal to Y. If in addition Y is transversal to Z then we say that the decomposition is *symmetric*.

Next we list several useful properties of BNPC product decompositions.

Proposition 4.6. *Suppose $X = Y \times Z$ be a BNPC decomposition where Y and Z are not reduced to a point.*

(a) *The factor maps $\pi_Y\colon X \to Y$ and $\pi_Z\colon X \to Z$ given by $\pi_Y(y,z) = y$ and $\pi_Z(y,z) = z$ are Lipschitz. In particular, X is complete if and only if Y and Z are complete.*

(b) *X is affinely equivalent to the direct product $Y \oplus Z$. Furthermore, if $c_Y(t)$ and $c_Z(t)$ are constant speed geodesics in Y and Z respectively, then $c(t) = (c_Y(t), c_Z(t))$ is a constant speed geodesic in X.*

(c) *If the BNPC decomposition is symmetric or if X is uniformly convex (UC) then X is quasi-isometric to $Y \oplus Z$.*

(d) *If $A \subset Y$ and $B \subset Z$ are flat subsets then so is $A \times B \subset X$. In particular, X is flat if and only if A and B are flat.*

(e) *Equality of slopes - Suppose c and c' are parallel segments in X then the projections of c and c' to each factor are also parallel and in particular have the same speed and length.*

(f) *If γ is an isometry of X that permutes the Y-fibers then γ preserves the decomposition of X and acts on each factor separately as an isometry.*

(g) *X is (uniquely) geodesically complete if and only if both Y and Z are (uniquely) geodesically complete. If X is UC then so are Y and Z.*

(h) *If $X = Y \times Z$ is a symmetric decomposition then the same fibers induce also a decomposition as $X = Z \times Y$. Furthermore, if Z has a BNPC decomposition $Z = Z_1 \times Z_2$ then X has a BNPC decomposition $X = (Y \times Z_1) \times Z_2$.*

Proof. (a) By definition Z is transversal to Y and thus, for any given $x = (y,z)$ and $x' = (y',z')$,

$$d_Z(p_Z(x), p_Z(x')) = d((y,z),(y,z')) = d(Y_z, Y_{z'}) \leq d(x,x')$$

and

$$d_Y(p_Y(x), p_Y(x')) = d((y,z),(y',z)) \leq d(x,x') + d(Y_z, Y_{z'}) \leq 2d(x,x').$$

The second part of the statement follows from the fact that the distance between any two Y-factors or two Z-factors is positive which implies that the Y-factors and Z-factors are closed subsets of X.

(b) Fix $y_1, y_2 \in Y$ and $z_1, z_2 \in Z$ and let $c_Y(t)$ (resp. $c_Z(t)$) denote the geodesic curve connecting y_1 and y_2 in Y (resp. z_1 and z_2 in Z). We will prove that the curve $c(t) = (c_Y(t), c_Z(t))$ is a constant speed geodesic in X. Indeed, by definition of BNPC decompositions, the segments $[(y_1,z_1),(y_2,z_1)]$ and $[(y_1,z_2),(y_2,z_2)]$ are opposite and thus, by Busemann's lemma, their convex hull is isometric to a flat rectangle R. Under this identification $c(t)$ travels on the diagonal between (y_1, z_1) and (y_2, z_2) at constant speed and thus it is a constant speed geodesic. We conclude that X is affinely equivalent to $Y \oplus Z$.

(c) By the triangle inequality, $d((y_1, z_1), (y_2, z_2)) \leq d(y_1, y_2) + d(z_2, z_2)$. We will prove that there exists $K > 0$ such that, for all $y_1, y_2 \in Y$ and $z_1, z_2 \in Z$,

$$K \cdot \|(d(y_1, y_2), d(z_1, z_2))\|_\infty \leq d((y_1, z_1), (y_2, z_2))$$

If the decomposition is symmetric then the equation is true with $K = 1$. Suppose then that X is UC with modulus of convexity function $\delta(\varepsilon)$ such that $\delta_{x,\varepsilon}(r) \geq \delta(\varepsilon)r$ for all x, ε, r. Let $\mathbb{U}$ denote the set of all strictly convex norms $\| \|$ of the plane satisfying

(i) $\|(0, 1)\| = \|(1, 0)\| = 1$ and

(ii) $\|\frac{x+y}{2}\| \leq 1-\delta(\varepsilon)$ for all $x, y \in \mathbf{R}^2$ such that $\|x\| = \|y\| = 1$ and $\|x-y\| \geq \varepsilon$.

Note that $\mathbb{U}$ is not empty. Indeed, suppose $z_1 \neq z_2 \in Z$ and let $c: [0, r] \to Y$ be any geodesic segment in Y. Then $[c(0), c(r)] \times \{z_1\}$ and $[c(0), c(r)] \times \{z_2\}$ are parallel segments. For every $0 \leq s \leq r$ let $c_s(t)$ denote the geodesic connecting $(c(s), z_1)$ and $(c(s), z_2)$. Then, by Busemann's Lemma 3.1, there exists some norm on the plane such that $d(c_s(t), c_{s'}(t') = \|(s, t), (s', t')\|$ for every s, t. This norm necessarily belongs to $\mathbb{U}$. Set $k = \sup\{\|v\|_\infty : v \in \mathbf{R}^2 \text{ and } \|v\| = 1 \text{ for some } \| \| \in \mathbb{U}\}$ then $k < \infty$ because the unit spheres of the normalized norms in $\mathbb{U}$ can not "stretch" too long from the origin without getting too "flat" and violating condition (ii). As $d((y_1, z_1), (y_2, z_2))$ equals the length of a diagonal of a rectangle in some normed plane with a norm from $\mathbb{U}$ it follows that the inequality above holds for $K = (1/k)$. We conclude that X is quasi-isometric to $Y \oplus Z$.

(d) Let $(a_1, b_1) \neq (a_2, b_2)$ be two distinct points in $A \times B$. Without loss of generality assume that $a_1 \neq a_2$ then by Theorem 2.2 there exists an affine function $f : A \to \mathbf{R}$ such that $f(a_1) \neq f(a_2)$. Extend f to $\hat{f} : A \times B \to \mathbf{R}$ by defining $\hat{f}(a, b) = f(a)$, then by the description of the geodesics in $A \times B \subset X$ given in (b) it follows that $\hat{f}$ is affine. Thus affine functions separate points in $A \times B$ and by Theorem 2.2 $A \times B$ is flat.

(e) The description of geodesics in X given in (b) implies that the projection maps π_Y and π_Z are affine maps. By Busemann's lemma (Lemma 3.1) the convex hull C of the segments c and c' is isometric to a flat rectangle, and in particular it is flat, and by Proposition 2.1 so are its projections $\pi_Y(C)$ and $\pi_Z(C)$. It follows by property (d) that the product $\pi_Y(C) \times \pi_Z(C)$ is affine. Thus we can reduce to the case where Y, Z and X are flat sets but in this setting c is a translation of c' by some vector v. Thus c_1 and c'_1 are parallel segments and in particular have the same length which implies that they have the same speed as well. Similar arguments apply to c_2 and c'_2.

The proofs of (f), (g), and (h) follow readily and are left to the reader as a simple exercise. □

5. The product decomposition theorem

The product decomposition theorem (cf. Theorem II.2.14 in [3]) states that a CAT(0) space admits an Euclidean factor if and only if it can be covered by pairwise parallel lines. The theorem relies on the following lemma (cf. Lemma II.2.15 of [3]).

Lemma 5.1. *Let X be a geodesic space and let $\{c_\alpha\}_{\alpha\in I}$ be a collection of pairwise parallel geodesic lines in X then it is possible to parameterize each c_α so that $d(c_\alpha(t), c_{\alpha'}(t)) = d(c_\alpha(\mathbf{R}), c_{\alpha'}(\mathbf{R}))$ for every $\alpha, \alpha' \in I$ and $t \in \mathbf{R}$.*

Lemma 5.1 motivates the following definition that characterizes those covers which are fibers of some product decomposition.

Definition 5.2. (a) Let C_1, C_2 be opposite sets. We say that $x_1 \in C_1$ and $x_2 \in C_2$ are *opposite* if $d(x_1, x_2) = d(C_1, C_2)$.

(b) Let $A = \{C_\alpha\}_{\alpha\in I}$ be a collection of convex pairwise opposite subsets. We say that A is *transitively opposite* if for any triplet $\{x_1, x_2, x_3\} \in X^3$ ($x_i \in C_{\alpha_i}$) if x_1 and x_2 are opposite and x_2 and x_3 are opposite then x_1 and x_3 are opposite.

(c) We say that A is a *foliation* of X if A is a transitively opposite collection of convex subsets of X and $X = \bigcup A$.

Lemma 5.1 can now be restated to say that every collection of pairwise parallel lines is transitively opposite. More generally, Lemma 5.1 implies that if A is any collection of geodesically complete subsets which are pairwise opposite then A is transitively opposite.

Theorem 5.3 (product decomposition theorem for BNPC spaces). *Let X be a BNPC space and suppose $A = \{Y_\alpha\}_{\alpha\in I}$ is a foliation of a convex subset $X_0 \subset X$ then there exists a unique BNPC decomposition $X_0 = Y \times Z$ of X_0 such that A coincides with the set of Y-fibers. If X, Y and Z are complete then so is X_0.*

Before getting to the proof of the theorem, we will first prove a useful lemma. Note that given a subspace $Y \in A$, every point X_0 belong to a unique Y_α and thus has a unique opposite point in Y. We can therefore define a "fiber map" $p_Y : X_0 \to Y$ that maps a point to its opposite point in Y. By definition, p_Y is the closest point projection to Y.

Lemma 5.4. *Let X, A, X_0 be as in Theorem 5.3. Suppose $Y \in A$, $y_0 \in Y$ and $Z = p_Y^{-1}(y_0)$. Then,*

(1) *X_0 is convex if and only if Z is convex*;

(2) *if X and Y are complete then the closure of X_0 in X is the union of the sets of a transitively opposite collection that contains A and is equal to A if and only if Z is complete.*

Proof. For every $y \in Y$ define $Z_y = p_Y^{-1}(y)$. For every $y \in Y$ and $\alpha \in I$ let $y^\alpha \in Z_y \cap Y_\alpha$ denote the opposite point of y in Y_α. Note that the $\{Z_y\}_{y\in Y}$ are pairwise isometric. Indeed, given $y_1, y_2 \in Y$, by definition $d(y_1^\alpha, y_1^{\alpha'}) = d(Y_\alpha, Y_{\alpha'}) = d(y_2^\alpha, y_2^{\alpha'})$ and thus the map $y_1^\alpha \mapsto y_2^\alpha$ is an isometry between Z_{y_1} and Z_{y_2}. We conclude that if Z is either convex or complete then so is Z_y for every $y \in Y$. Fix $y_1^\alpha, y_2^{\alpha'}$ in X_0. Since the segments $[y_1^\alpha, y_2^\alpha]$ and $[y_1^{\alpha'}, y_2^{\alpha'}]$ are opposite it follows by Busemann's lemma that $\frac{y_1^\alpha+y_2^{\alpha'}}{2}$ lies on the segment $\left[\frac{y_1^\alpha+y_2^\alpha}{2}, \frac{y_1^{\alpha'}+y_2^{\alpha'}}{2}\right]$ which is a subset of $Z_{\frac{y_1^\alpha+y_2^\alpha}{2}}$. Thus X_0 is convex if and only if Z is convex.

Suppose that both X and Y are complete and let (x_n) be a Cauchy sequence in X_0. For every n write $x_n = y_n^{\alpha_n}$ and $z_n = y_0^{\alpha_n}$. Note that (z_n) is a Cauchy sequence since $d(z_n, z_m) = d(Y_{\alpha_n}, Y_{\alpha_m}) \le d(x_n, x_m)$. By assumption X is complete and thus the sequence (z_n) converges to some element $\hat{y}_0$. We can repeat the process to obtain $\hat{y}$ for each $y \in Y$. Note that $d(\hat{y}, y) = \lim d(Y_{\alpha_n}, Y)$ is independent of y. Furthermore, for every $y, w \in Y$ we have $\widehat{\frac{y+w}{2}} = \frac{\hat{y}+\hat{w}}{2}$ by the convexity of the metric. It follows that $\widehat{Y} = \{\hat{y} : y \in Y\}$ is convex and by construction Y and $\hat{Y}$ are opposite. Since the construction of $\hat{y}$ depended only on the sets Z_y and not on y itself we can replace Y in the arguments above with any Y_α and reach the same $\widehat{Y}$ and the same conclusions. It follows that $A \cup \{\hat{Y}\}$ is a transitively opposite collection in $X_0 \cup \hat{Y}$. To complete the proof we will see that the limit of $(x_n)_{n\in \boldsymbol{N}}$ lies in $\hat{Y}$. Note that the sequence (y_n) is Cauchy since $d(y_n, y_m) \le d(x_n, x_m) + d\,(Y_{\alpha_n}, Y_{\alpha_m}) \le 2d(x_n, x_m)$. By assumption Y is complete and thus (y_n) converges to some $y \in Y$. We claim that $\hat{y}$ is the limit of (x_n). Indeed, since $d(x_n, \hat{y}) \le d(x_n, \hat{y}_n) + d(\hat{y}_n, \hat{y})$ and as $d(x_n, \hat{y}_n) = d(Y_{z_n}, \hat{Y})$ and $d(\hat{y}_n, \hat{y}) = d(y_n, y)$ it follows that both summands tend to zero. □

Proof of Theorem 5.3. Fix $Y \in A$, $y_0 \in Y$ and define $Z = p_Y^{-1}(y_0)$. By assumption X_0 is convex and by Lemm 5.4 (1) below so is Z. By construction Z is transversal to Y. Thus we indeed attained a BNPC decomposition $X_0 = Y \times Z$. Suppose $X_0 = Y \times Z'$ is a BNPC decomposition and let Z_0' be the Z'-fiber that contain y_0. By definition Z_0' is transversal to Y and thus Z_0' intersects each Y_α in the unique point in Y_α that is opposite to y_0. We conclude that $Z_0' = Z$ and thus the BNPC decomposition $X_0 = Y \times Z$ is unique. The last claim of the theorem follows from Lemma 5.4 (2). □

Proof of Theorem 3.4. Let $p : A \to B$ denote the parallel isometry, which by definition is also the closest point projection, and set $r = d(A, B)$. For any $s \in [0, r]$ and $a \in A$ let a_s denote the unique point on $[a, f(a)]$ of distance s from a and define $A_s = \{a_s\}_{a\in A}$. By Busemann's lemma, given any $a, a' \in A$, the convex hull $conv\,([a, a'] \cup [f(a), f(a')])$ is isometric to a flat rectangle. It follows that $[a, f(a')]$ lies in $\bigcup A_s$ and also that for any given s, $[a_s, a_s']$ is a subset of A_s. If $d(a_s, a_{s'}') < d(a_s, a_{s'})$ for some a, a', s and s' then $d(a_0, a_r') < (a_0, a_r)$, which contradicts the fact that f is closest point projection. We conclude that the A_s

form a foliation of C and by Theorem 5.3 it follows that C admits a decomposition $C = A \times [0, r]$. The proof now follows from Proposition 4.6. $\square$

6. Clifford isometries and the de Rham decomposition theorem

The goal of this section is to prove de Rham decomposition theorem (Theorem 1.2).

Recall that a *Clifford isometry* is an isometry γ with a constant displacement function d_γ where $d_\gamma(x) = d(x, \gamma(x))$. Equivalently, a Clifford isometry is an isometry that attains its minimal translation $|\gamma|$ at every point where $|\gamma| = \inf\{d_\gamma(x) \mid x \in X\}$. When dealing with CAT(0) spaces the Clifford isometries coincide with the translations of the maximal Euclidean factor and in particular form an Abelian group. Theorem 1.2 provides an analogous result for Clifford isometries of BNPC spaces. Before getting to the proof, we need to make a few observations regarding Clifford isometries and BNPC decompositions.

Proposition 6.1. *Let X denote a BNPC space then for any map $\gamma\colon X \to X$ the following are equivalent*:

(a) *γ is a Clifford isometry*;

(b) *$[x, y] \parallel [\gamma(x), \gamma(y)]$ for every $x, y \in X$*;

(c) *$[x, \gamma(x)] \parallel [y, \gamma(y)]$ for every $x, y \in X$*;

(d) *γ is an isometry and the axes of γ are pairwise parallel and cover X*;

(e) *X admits a BNPC decomposition $X = \mathbf{R} \times Y$ and γ respects this splitting and acts trivially on Y and as a translation on $\mathbf{R}$.*

Proof. (a) $\Longrightarrow$ (b) follows by definition and (b) $\Longleftrightarrow$ (c) follows from Corollary 3.2. Conversely, assume both (b) and (c). Then for every $x, y \in X$ (c) implies that $d(x, y) = d(\gamma(x), \gamma(y))$ and (b) implies that $d_\gamma(x) = d_\gamma(y)$. We conclude that (a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (c).

Assume (a) then by Proposition 11.4.2 in [20] every element of X lies in some axis of γ. As $d(x, y) = d(\gamma^n x, \gamma^n y)$ for every $n \in \mathbb{Z}$ it follows that the axes of γ are pairwise parallel. Thus (a) $\Longrightarrow$ (d). Conversely, assume (d). Then we can parameterize the axes of γ so that $\gamma(c(t)) = c(t + |\gamma|))$ for any axis c and every t. As the axes are parallel it follows that (d) $\Longrightarrow$ (c) and consequently (a) $\Longleftrightarrow$ (d).

Assume (d) then by Lemma 5.1 the axes of γ form a foliation of X and by the product decomposition theorem (Theorem 5.3) X admits a decomposition $X = \mathbf{R} \times Y$ where the $\mathbf{R}$-fibers are the axes of γ. Thus (d) $\Longrightarrow$ (e). Conversely, assume (e). Then the $\mathbf{R}$-fibers of this decomposition are by definition pairwise parallel axes of γ that cover X. We conclude that (e) $\Longleftrightarrow$ (d). $\square$

Lemma 6.2. *Let $X = X_1 \times X_2$ be a BNPC decomposition. Let γ_i be a Clifford isometry of X_i. Then $\gamma = (\gamma_1, \gamma_2)$ is a Clifford isometry of $X_1 \times X_2$.*

Recall that $[a, b] \parallel [c, d]$ means that $d(a, c) = d(b, d) = d\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$.

Proof. By Proposition 6.1 it will suffice to show that

$$[x, y] \parallel [\gamma x, \gamma y] \quad \text{for all } x, y \in X. \tag{1}$$

Fixing $x = (x_1, x_2),\ y = (y_1, y_2) \in X_1 \times X_2$ (1) comes down to

$$[(x_1, x_2), (y_1, y_2)] \parallel [(\gamma_1 x_1, \gamma_2 x_2), (\gamma_1 y_1, \gamma_2 y_2)]. \tag{2}$$

As γ_1 is a Clifford isometry of X_1 it follows by Proposition 6.1 that

$$[(x_1, x_2), (y_1, y_2)] \parallel [(\gamma_1 x_1, x_2), (\gamma_1 y_1, y_2)]. \tag{3}$$

since

$$d(x_1, \gamma_1 x_1) = d(y_1, \gamma_1 y_1) = d\left(\frac{x_1 + y_1}{2}, \frac{\gamma_1 x_1 + \gamma_1 y_1}{2}\right) = |\gamma_1|.$$

Here we used the fact that

$$\frac{\gamma_1 x_1 + \gamma_1 y_1}{2} = \gamma_1\left(\frac{x_1 + y_1}{2}\right).$$

Similar arguments imply that

$$[(\gamma_1 x_1, x_2)(\gamma_1 y_1, y_2)] \parallel [(\gamma_1 x_1, \gamma_2 x_2), (\gamma_1 y_1, \gamma_2 y_2)] \tag{4}$$

Things would have been simple if "being parallel" was a transitive relation and thus (3) and (4) would just imply (2). Unfortunately "being parallel" is not necessarily a transitive relation, not even in the case of geodesic segments in CAT(0) spaces. However, Busemann's lemma implies that the convex hull C_1 of $[x_1, y_1] \cup [\gamma_1 x_1, \gamma_1 y_1]$ in X_1 is flat and so is the convex hull C_2 of $[x_2, y_2] \cup [\gamma_2 x_2, \gamma_2 y_2]$ in X_2. By (d) of Proposition 4.6 the product $C = C_1 \times C_2$ in $X_1 \times X_2$ is also flat. As C contains the three segments in (3) and (4) and as $\parallel$ is a transitive relation on segments in flat spaces, we conclude that (2) follows from (3) and (4) and the lemma is proven. □

Lemma 6.3. *Let $X = X_1 \times X_2$ be a BNPC decomposition then every Clifford isometry of X respects this decomposition and acts on the X_i as a Clifford isometry.*

Proof. Fix a Clifford isometry η. If η acts trivially on Y or Z then there is nothing to prove. Otherwise let $\{c_\alpha\}_{\alpha \in I}$ denote the axes of η. For every α there exist lines $c^i_\alpha(t)\colon \mathbf{R} \to X$ $(i = 1, 2)$ and constants r_α and k_α, such that $c_\alpha(t) = (c^1_\alpha(r_\alpha t), c^2_\alpha(k_\alpha t))$. By the equality of slopes property (Proposition 4.6 (e)) r_α and k_α are independent of α and we can replace them by some constants r and k respectively. The same property implies that the c^i_α are pairwise parallel lines which cover X_i. By definition $\eta(c_\alpha(t)) = c_\alpha(t + |\eta|)$, which implies that $\eta(c^1_\alpha(rt), c^2_\alpha(kt)) = (c^1_\alpha(rt + r|\eta|), c^2_\alpha(kt + k|\eta|))$. Thus η respects the decomposition $X = X_1 \times X_2$ and acts on X_1 (resp. X_2) by translation along parallel lines with a fixed displacement $r|\eta|$ (resp. $k|\eta|$). The lemma now follows from Proposition 6.1. □

We are now ready to prove that the Clifford isometries of X form an Abelian group.

Proof of Theorem 1.3. Let γ, η be Clifford isometries of some BNPC space X. Following Proposition 6.1 let $X = \mathbf{R} \times Y$ be the product decomposition induced by the axes of γ. By Lemma 6.3, η respects this splitting and acts on Y as a Clifford isometry and on $\mathbf{R}$ as a translation. It follows that γ and η commute and that $\eta \circ \gamma$ acts as a Clifford isometry on each factor. Lemma 6.3 now implies that $\eta \circ \gamma$ is a Clifford isometry of X. □

Proof of Theorem 1.2. By Theorem 1.3 the Clifford isometries of X form an Abelian group which we denote by $\mathrm{CL}(X)$. $\mathrm{CL}(X)$ can be naturally endowed with a structure of a real vector space by setting $r \cdot \gamma$ to be the Clifford isometry translating along the axes of γ by $r|\gamma|$. We can also endow $\mathrm{CL}(X)$ with a strictly convex norm by defining $\|\gamma\| = |\gamma|$. We denote the resulting space by B. The proof that with these definitions B is indeed a strictly convex normed vector space is very similar to the proof given in the CAT(0) case and it is left to the reader (cf. Theorem II.6.15 in [3]).

Observe that every orbit of $\mathrm{CL}(X)$ in X is naturally isometric to B through the map $\psi \mapsto \psi(x)$ and in particular that all the orbits are geodesically complete. Since every two orbits are of finite Hausdorff distance from one another it follows that the orbits of $\mathrm{CL}(X)$ form a foliation of X by copies of B. By the product decomposition theorem (Theorem 5.3) X has a BNPC decomposition $X = B \times Y$, which we call the *de Rham decomposition* of X. By Lemma 6.2 every Clifford isometry ψ of Y extends to a Clifford isometry (Id, ψ) of X. As each axis of (Id, ψ) lies in some fiber of B it follows that ψ must the identity, i.e., that Y has no non-trivial Clifford isometries. Next we prove that the de Rham decomposition is unique. Suppose $X = B' \times Y'$ is another BNPC decomposition such that B' is affine and Y' has no non-trivial Clifford isometries. By Theorem 5.3, it will suffice to show that B-fibers and the B'-fibers coincide, or equivalently, that $\mathrm{CL}(X)$ coincides with the translations of B'. On one hand, if $\gamma\colon X \to X$ acts on B' as a translation and on Y' as the identity then by Lemma 6.2 $\gamma \in \mathrm{CL}(X)$. Conversely, if $\gamma \in \mathrm{CL}(X)$ then by Lemma 6.3 γ respects the decomposition

$X = B' \times Y'$ and acts on B' and Y' by Clifford isometries. By assumption Y' does not admit non-trivial Clifford isometries and thus γ is a translation of B'. We conclude that the de Rham decomposition is unique. If X is complete then so are the orbits of $\mathrm{CL}(X)$ and thus so is B. If X is geodesically complete then so is Y and by Proposition 4.6 (g) it follows that the Y-fibers form a transitively opposite collection. The symmetry of the decomposition $X = B \times Y$ follows from the next lemma. ☐

Lemma 6.4. *Let $X = V \times Y$ be a BNPC decomposition where V is flat. If the Y-fibers are pairwise opposite and* $\mathrm{CL}(Y)$ *is trivial then the decomposition is symmetric.*

Proof. For every $v \in V$ let Y_v denote the Y-fiber $\{v\} \times Y$ and let $p_v \colon Y_0 \to Y_v$ denote the closest point projection. For every $y \in Y_0$ and a unit vector $v \in V$ define a curve $c_{y,v}(t) = p_{tv}(y)$. Then $\{c_{y,v}(\mathbf{R})\}$ is a collection of parallel geodesic lines that cover X. Indeed, fix $s \in \mathbf{R}$ and let $c\colon [0,r] \to X$ denote the geodesic path connecting $(0, y)$ and $c_{y,v}(s)$. By Proposition 4.6 (b) $c(\frac{r}{2})$ must belong to $Y_{\frac{s}{2}v}$ and since $c_{y,v}(\frac{s}{2})$ is by definition the closest point in $Y_{\frac{s}{2}v}$ to $(0, y)$ it follows that $c(\frac{r}{2}) = c_{y,v}(\frac{s}{2})$. By continuity $c(\alpha r) = c_{y,v}(\alpha s)$ for every $0 \le \alpha \le 1$ and as s was arbitrary it follows that $c_{y,v}(\mathbf{R})$ is a geodesic line. The fact that the Y-fibers are pairwise opposite implies that the projections p_v are parallel isometries and thus that the lines $c_{y,v}(\mathbf{R})$ are pairwise opposite. It follows that $\{c_{y,v}\}$ are the axes of some Clifford isometry γ of X. Since Y does not admit non-trivial Clifford isometries it follows that the axes of γ lie in V which by construction implies that V is transversal to Y_0. As $Iso(X)$ acts transitively on V it follows that V is transversal to every Y-fiber, i.e., that $X = V \times Y$ is a symmetric BNPC decomposition. ☐

7. The splitting theorem

In this section we prove the splitting theorem (Theorem 1.4). The proof given here is an adaptation of the proof given in [19] for the CAT(0) case (cf. Theorem 9).

Suppose G is some group acting by isometries on a BNPC space X. A non-empty subset $C \subset X$ is said to be a *G-minimal subset* if

$$C = \overline{\mathrm{conv}(Gx)} \quad \text{for every } x \in C,$$

i.e., if C is a minimal non-empty closed convex G-invariant subset of X. If X is a complete BNPC space that is UC or locally compact and $d_G \to \infty$ then X admits G-minimal subsets (cf. Lemma 2.10 in [11]). Recall that $d_G \to \infty$ means that the action of G is *non-weakly evanescent*, i.e., that there exists a finite subset $Q \subset G$ such that $d_Q(x_n) \to \infty$ whenever $x_n \to \infty$. Here $d_Q(x) = \sup_{q \in Q} d(qx, x)$

is the displacement function with respect to Q, and $x_n \to \infty$ means that x_n is a sequence of points in X that eventually leave every ball in X. If X is proper then the action of G is non-weakly evanescent if and only if it does not fix points at the boundary of X (see [19] for more details).

Proof of Theorem 1.4. By the preceding paragraph and Proposition 4.6 (h) we can reduce to the case where X is G-minimal and $n = 2$.

Step i. X admits a G_1-minimal subset. Fix $x_0 \in X$ and define

$$C = \overline{\operatorname{conv}(G_1 \cdot x_0)}.$$

For any $g_2 \in G_2$ and $x = g_1' x_0 \in G_1 \cdot x_0$ we have $d_{g_2}(x) = d(g_2 g_1' x_0, g_1' x_0) = d(g_2 x_0, x_0) = d_{g_2}(x_0)$. By the convexity and continuity of d_{g_2} it follows that $d_{g_2}(x) \le d_{g_2}(x_o)$ for every $x \in C$. If C is bounded then the action of G_1 on C is trivally non-weakly evanescent. Otherwise, let x_n be a sequence in C such that $x_n \to \infty$ then there is some $g = g_1 g_2$ in G so that $d_g(x_n)$ is unbounded. It follows by the triangle inequality and the fact that d_{g_2} is bounded on (x_n) that d_{g_1} must be unbounded on (x_n). We conclude that the action of G_1 on C is non-weakly evanescent and thus C admits a G_1-minimal subset.

Let Σ be the collection of all G_1-minimal subsets of X and define $Z = \bigcup \Sigma$.

Step ii. Σ is a foliation of Z. First note that the elements of Σ are transitively opposite. Indeed, suppose $Z_1, Z_2 \in \Sigma$ and $z, z' \in Z_1$ are such that $d(z, Z_2) < d(z', Z_2)$ then $\{x \in Z_1 \mid d(x, Z_2) \le d(z, Z_2)\}$ is a closed convex G_1-invariant proper subset of Z_1, contradicting the fact that Z_1 is G_1-minimal. Next let Z_1, Z_2, Z_3 be G_1-minimal and let p_i denote the projection to Z_i. Define $\gamma = p_1 \circ p_3 \circ p_{2|Z_1}$ then d_γ is G_1-equivariant and the same argument as above shows that it must be constant, i.e., γ is a Clifford isometry. If γ is non-trivial then it has an axis l_1 in Z_1 on which it acts by translations. But l_1, $p_2(l_1)$ and $p_3 \circ p_2(l_1)$ are three parallel lines and by Lemma 5.1 they form a transitively opposite collection meaning that the restriction of γ to l_1 is trivial, a contradiction. Thus γ is the identity and we conclude that Σ is transitively opposite. It remains to show that Z is convex in X. It will suffice to show that if $Z_1, Z_2 \in \Sigma$ and $z_i \in Z_i$ then $\frac{z_1+z_2}{2} \in Z$. By Busemann's Lemma we have

$$\left\{\frac{w_1 + w_2}{2} \colon w_i \in Z_i\right\} = \left\{\frac{w_1 + w_2}{2} \colon w_i \in Z_i \text{ and } d(w_1, w_2) = d(Z_1, Z_2)\right\}.$$

The left-hand set contains $\frac{z_1+z_2}{2}$ and the right-hand set is G_1-minimal and thus a subset of Z.

Step iii: Z has a $(G_1 \times G_2)$-equivariant BNPC decomposition. Fix a G_1-minimal subset X_1 and $o \in X_1$. Let p denote the projection to X_1, restricted to Z, and set $X_2 = p^{-1}(o)$. Then by the product decomposition theorem (Theorem 5.3) Z has a BNPC decomposition $Z = X_1 \times X_2$ where the X_1-fibers are the G_1-minimal subsets. Since G_1 and G_2 commute it follows that Z is G-invariant. We claim that G_1 acts trivially on X_2 and G_2 acts trivially on X_1. The first assertion is obvious since the X_1-fibers are G_1-invariant. We will see that the X_2-fibers are G_2-invariant as well. For every $h \in G_2$ define $h^*\colon X_1 \to X_1$ by $h^*(x) = p(h \cdot x)$. Then h^* is a G_1-equivariant isometry of X_1 and as X_1 is G_1-minimal it follows that h^* is a Clifford isometry. Fix $g_2 \in G_2$. If g_2^* does not act trivially on X_1 then it admits an axis c in X_1. Every $g_1 \in G_1$ commutes with the action of g_2^* and thus takes c to an opposite line. In particular, d_{g_1} is bounded on c. Similarly, as Clifford isometries commute (Theorem 1.3), it follows that g_2^* commutes with h^* for every $h \in G_2$. Thus and thus h^* takes c to an opposite line. Since the restriction of p to $h \cdot X_1$ is a parallel isometry it follows that h takes c to an opposite line and thus d_h is bounded on c. The triangle inequality now implies that d_g is bounded on c for every $g \in G$, contradicting the fact that the action of G on X is non-weakly evanescent. We conclude that the action of g_2^* must be trivial, i.e., that the X_2-fibers are G_2-invariant.

Step iv: Z=X. We saw that Z is a G-invariant convex subset of X. By the minimality of the G-action it follows that $X = \bar{Z}$. By Lemma 5.4, X has a foliation $\bar{\Sigma}$ that contains Σ. Recall from the proof of Lemma 5.4 that each set $C \in \bar{\Sigma} \setminus \Sigma$ is obtained from a sequence of sets $C_n \in \Sigma$ in the sense that every point $c \in C$ is a limit of a Cauchy sequence of pairwise opposite points $c_n \in C_n$. Thus the elements of $\bar{\Sigma}$ are G_1-invariant and being opposite to X_1 they are G_1-minimal. We conclude that $\Sigma = \bar{\Sigma}$ and that X has a foliation by G_1-minimal sets, i.e., $X = Z = X_1 \times X_2$.

Step v: $X = X_1 \times X_2$ is a symmetric BNPC decomposition. As X is G-minimal and the action of G on X is $G_1 \times G_2$-equivariant it follows that the X_2-fibers are exactly the G_2-minimal sets. Thus if we interchange G_1 and G_2 in the previous steps then we will obtain a BNPC decomposition $X = X_2 \times X_1$ with the same fibers. We conclude that the decomposition $X = X_1 \times X_2$ is symmetric. This completes the proof. □

8. The duality property

All along this section let (M, F) denote a complete reversible Finsler manifold of Busemann NPC and finite volume. Let X denote the universal cover of M endowed with the metric d induced by d_F. By the Cartan–Hadamard theorem

(X, d) is a proper geodesically complete BNPC metric space. Set $\Gamma = \pi_1(M)$ and note that Γ acts on X by isometries freely and properly discontinuously. The aim of this section is to prove that the action of Γ satisfies the duality property which we now define.

Definition 8.1. Let G be a group acting by isometries on a geodesically complete BNPC space Y. We say that (the action of) G has the *duality property* if for every geodesic line $c\colon \mathbf{R} \to Y$ there exists a sequence $g_n \in G$ such that $g_n c(0) \to c(\infty)$ and $g_n^{-1} c(0) \to c(-\infty)$.

We start by expressing the duality property in terms of the geodesics of M. The following lemma is due to Eberlein.

Lemma 8.2. *Γ has the duality property if and only if for every line $c(t)$ in X there exist $s_m \in \mathbf{R}$, $\gamma_m \in \Gamma$ and lines $c_m(t)$ in X such that $s_m \to \infty$, $c_m \to c$ and $\gamma_m c_m(t + s_m) \to c(t)$.*

Proof. Suppose Γ has the duality property. Then given a line $c(t)$ there exist $\gamma_n \in \Gamma$ such that $\gamma_n(x) \to c(\infty)$ and $\gamma_n^{-1}(x) \to c(-\infty)$ for every fixed $x \in X$. For every $m, n \in \mathbf{N}$ define geodesic segments $c_{n,m}\colon [-m, t_{n,m}] \to [c(-m), \gamma_n(c(m))]$. For every fixed m choose a minimal $N(m) \in \mathbf{N}$ such that $N(m) > N(m-1)$ and such that for every $n > N(m)$,

(a) $t_{n,m} > 2m$,

(b) $d(c(m), c_{n,m}(m)) < \frac{1}{m}$,

(c) $d(c(-m), \gamma_n^{-1} \circ c_{n,m}(t_{n,m} - 2m))$.

Such $N(m)$ exists because for every fixed m, $\gamma_n^{\pm 1}(c(\pm m)) \to c(\pm\infty)$. For every m let c_m be the line extending $c_{N(m)+1,m}$. Set $s_m = t_{n,m} - m$ and note that $s_m > m$. Then, by construction, $s_m \to \infty$, $c_m \to c$ and $\gamma_m c_m(t + s_m) \to c(t)$.
Conversely, suppose that for any given line c there exist γ_m, s_m and c_m like in the statement of the lemma. Set $c'_m(t) = \gamma_m \circ c_m(t + s_m)$ then on one hand, by the assumptions, $c'_m \to c$, which implies that $\gamma_m(c_m(0)) = c'_m(-s_m) \to c(-\infty)$. On the other hand, $c_m \to c$ implies that $\gamma_m^{-1} c'_m(0) = c_m(+s_m) \to c(\infty)$. As $c'_m(0) \to c(0)$ we conclude that $\gamma_m^{-1} c(0) \to c(\infty)$. As c was arbitrary we conclude that Γ has the duality property. □

Corollary 8.3. *Γ has the duality property if and only if for every geodesic line c in M there exist lines c_m in M and numbers $s_m \to \infty$ such that $c_m \to c$ and $c_m(t + s_m) \to c(t)$.*

Let SM denote the unit bundle ($F \equiv 1$) of M. There is a one-to-one correspondence between SM and the geodesic lines in M where $v \in SM$ corresponds to the unique geodesic $c_v\colon \mathbf{R} \to M$ such that $\dot{c_v}(0) = v$. The *geodesic flow* f_t

on SM is defined by $f_t(v) = \dot{c_v}(t) \in S_{c_v(t)}M$. The unit bundle admits a measure, called the *Liouville measure*, which is invariant under the geodesic flow. When M has finite volume then the Liouville measure is a finite measure with full support (cf. [24], [6], and [9]). We now have everything we need for proving Theorem 1.5.

Proof of Theorem 1.5. By the discussion above SM has a finite measure invariant under the geodesic flow. Suppose c is a line in M and denote $v = \dot{c}(0)$. Then by Poincaré recurrence theorem there exists a sequence $s_m \to \infty$ and $v_m \in SM$ such that $v_m \to v$ and $f_{s_m}(v) \to v$. Let c_m denote the lines corresponding to c_{v_n} respectively then by definition $c_m \to c$ and $c_m(t + s_m) \to c$. By Corollary 8.3 we conclude that Γ has the duality property. □

9. A splitting theorem for Finsler manifolds of finite volume

We now turn to the proof of Theorem 1.1. The heart of the proof lies in the following proposition which shows that all the Γ-fixed points at the boundary of the universal cover lie in some flat factor (compare with Theorem 4.2 in [5]).

Proposition 9.1. *Let M and Γ be as in Theorem* 1.1 *and let X be the universal cover of M, endowed with the induced length metric. Then X has a symmetric BNPC decomposition $X = V \times Y$ such that*

(a) *V is a linear subspace of the de Rham factor of X*;

(b) *Γ respects the decomposition and acts on V by translations*;

(c) *the induced action of Γ on Y is without fixed point at infinity*;

(d) *the centralizer of Γ in $G = Iso(X)$ consists precisely of the Clifford isometries of V.*

Suppose ζ, ξ are points at the (visual) boundary of X, then we say that they are *visually opposite* if there is a geodesic line $c\colon \mathbf{R} \to X$ such that $c(\infty) = \xi$ and $c(-\infty) = \zeta$. If ζ has a unique visually opposite point then we denote it by $-\zeta$.

Lemma 9.2. *Let ξ be a fixed point in the boundary of X then ξ has a unique visually opposite point.*

Proof. Fix a visually opposite point ζ of ξ and suppose that ζ has another visually opposite point ξ'. Suppose $c\colon \mathbf{R} \to X$ is a geodesic line in X such that $c(-\infty) = \zeta$ and $c(\infty) = \xi$. By the duality property there exists a sequence γ_n such that $\gamma_n x \to \xi'$ and $\gamma_n^{-1} x \to \zeta$ for any $x \in X$. We will prove that $\gamma_n(c(0)) \to \xi$. The proof is quite technical so it may be helpful to start with an outline of the argument. Intuitively speaking, we will show that the “angles” between the segments $[c(0), \gamma_n(c(0)]$ and the ray $[c(0), \xi)$ tend to zero. Formally, we will show that for any fixed $\varepsilon > 0$ and sufficiently large n, if $r_n = d(c(0), \gamma_n(c(0)))$, then $d(c(r_n), \gamma_n(c(0))) < \varepsilon r_n$. To ease the notation, for every $t \in \mathbf{R}$ and $n \in \boldsymbol{N}$ set $x_t = c(t)$ and $r_n = d(x_0, \gamma_n^{-1}(x_0))$, and let $f_{t,n}\colon [0, d(x_t, \gamma^{-1}(x_0))] \to X$ denote the parameterizations of the segments $[\gamma_n^{-1}(x_0), x_t]$. With these notations we will show that $d(\gamma_n^{-1}(x_{r_n})), x_0) < \varepsilon r_n$ for all sufficiently large n (see Figure 1). Note that if n is fixed and $t \to \infty$ then $f_{t,n}(r_n) \to \gamma_n^{-1}(x_{r_n})$. Thus, it will suffice to show that $d(f_{t,n}(r_n), x_0) < \varepsilon r_n$ for all sufficiently large n.

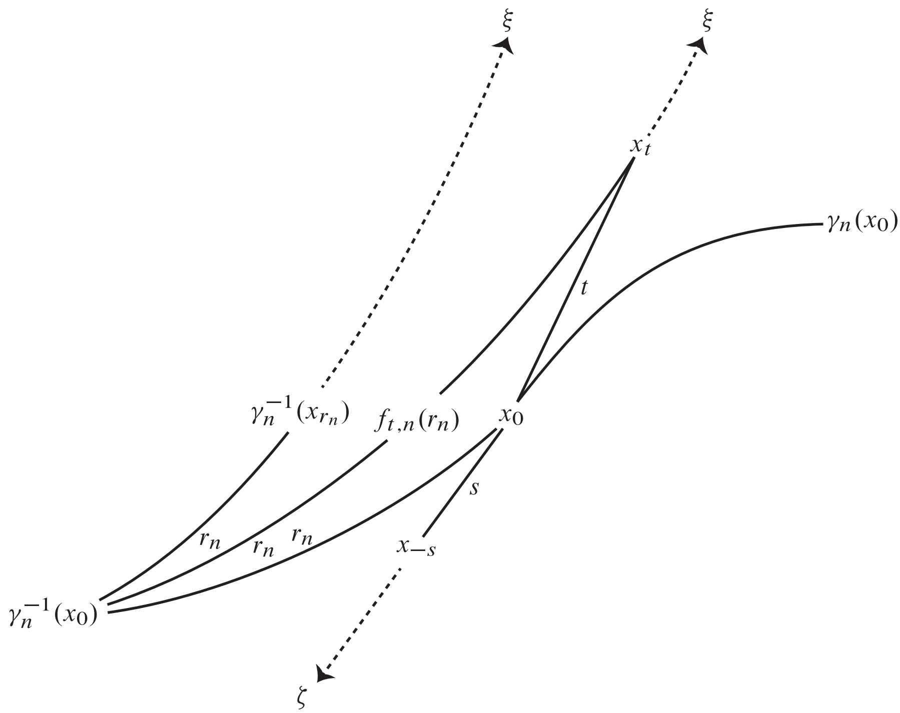

Figure 1

Fix $\varepsilon > 0$ and $s > 0$ and set $\varepsilon_n = d(x_{-s}, f_{0,n}(r_n - s))$. The fact that $\gamma_n^{-1}(x_0) \to \zeta$ implies that $\varepsilon_n \to 0$ and that $r_n \to \infty$. Since the map $u \mapsto d(x_t, f_{0,n}(u))$ is convex and as $d(x_t, f_{0,n}(r_n - s)) \geq t + s - \varepsilon_n$ it follows that $d(\gamma_n^{-1}(x_0), x_t) \geq t + r_n \frac{t+s-\varepsilon_n-t}{s} = t + r_n - \frac{\varepsilon_n r_n}{s}$. By assumption X is UC and so there exists $\delta > 0$ such that $\frac{\delta_{x,\varepsilon}(r)}{r} > \delta$ for every x and sufficiently large r. Suppose that $d(f_{t,n}(r_n), x_0) \geq \varepsilon r_n$. Then $d\big(\gamma_n^{-1}(x_0), \frac{x_0+f_{t,n}(r_n)}{2}\big) \leq r_n - \delta \cdot r_n$. Since $d(x_t, x_0) = t$ and $d(x_t, f_{t,n}(r_n)) = d(x_t, \gamma^{-1}(x_0)) - r_n \leq t$, it follows by the convexity of d that $d\big(x_t, \frac{x_0+f_{t,n}(r_n)}{2}\big) \leq t$. We conclude that, for all sufficiently large n,

$$t + r_n - \frac{\varepsilon_n r_n}{s} \leq d(\gamma_n^{-1}(x_0), x_t) \leq t + r_n - \delta \cdot r_n$$

and hence that $\delta < \frac{\varepsilon_n}{s}$, a contradiction since $\varepsilon_n \to 0$. Thus $d(f_{t,n}(r_n), x_0) < \varepsilon r_n$ for all sufficiently large n, and as discussed above, this implies that $\gamma^n(x_n) \to \xi$ and $\xi' = \xi$. □

Proof of Theorem 9.1. Let $X = B \times Z$ denote the de Rham decomposition of X and let F denote the set of points at the boundary of X fixed by Γ. We start by showing that every point of F lies on the boundary of B. Suppose g is a non-trivial isometry of X which centralizes Γ. Since Γ has the duality property and X is geodesically complete Γ acts minimally and thus g is a Clifford isometry. As g and Γ commute it follows that Γ permutes the axes of g and globally fixes their end-points. We conclude that the centralizer of Γ in $Iso(X)$ is isometric to a linear subspace V of B whose boundary is a subset of F. Conversely, suppose ξ is a fixed boundary point of Γ then by the previous lemma all the lines in X having one end at ξ are parallel. By the product decomposition theorem (Theorem 5.3) it follows that X admits a BNPC decomposition with a $\mathbf{R}$-factor such that $\partial\mathbf{R} = \{\pm\xi\}$. As ξ is a fixed point of Γ it follows that every $\gamma \in \Gamma$ permutes the $\mathbf{R}$-fibers and thus preserves the splitting and acts on the $\mathbf{R}$-factor by translation. Thus every translation of the $\mathbf{R}$-factor centralizes Γ and we conclude that F coincides with the boundary of V. For every $x \in B$ let V_x denote the minimal affine subspace of B that contains x and whose boundary is F and let A denote the set of all such subspaces. Note that the V_x are closed, convex and geodesically complete subsets of B parallel to V. By the remark It follows by Lemma 5.1 that A is transitively opposite, and thus a foliation of B. By the product decomposition theorem, B admits a BNPC decomposition $B = V \times V$. Set $Y = W \times Z$. The action of Γ on B permutes the V_x and thus Γ respects the splitting $X = V \times Y$ and acts on V by translations and on Y without fixed points at infinity. The induced actions on W and Y still have the duality property. It remains to see that $X = V \times Y$ is a symmetric BNPC decomposition. By Lemma 6.4, if the decomposition is not symmetric then W admits a non-trivial Clifford isometry that commutes with the action of Γ. This contradicts the fact that the centralizer of Γ coincides with the translations of V. □

Proof of Theorem 1.1. Let $X = V \times Y$ be the decomposition attained in Proposition 9.1. By (c) of that proposition the induced action of Γ on Y is without fixed points at infinity. We can now invoke the splitting theorem (Theorem 1.4) on the action of Γ on Y and obtain a Γ-equivariant symmetric BNPC decomposition $Y = Y_1 \times Y_2$ where Γ_i acts trivially on Y_{3-i}. The desired decomposition of M will now follow if we will show that $\dim(V) = 0$ or equivalently that Γ has no globally fixed points at the boundary of X. By (d) of Proposition 9.1 the Clifford isometries of V coincide with the centralizer $Z(\Gamma)$ of Γ. Thus $Z(\Gamma)$ is connected and since Γ is discrete it follows that $Z(\Gamma) = N_0(\Gamma)$, the identity component of the normalizer of Γ. $I(M) = N(\Gamma)/\Gamma$ is a Lie group (cf. [23]) and by assumption it is compact and it follows that $I_0(M) = N_0(\Gamma)/\Gamma = Z(\Gamma)/\Gamma$ is a k-torus where $k = \dim(V)$. Since every circle group $I_0(M)$ lifts to a group in $Z(\Gamma)$ it follows that k is no greater then the rank of the center of Γ and we conclude that $\dim(V) = 0$. □

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