Groups Geom. Dyn. 11 (2017), 121–138
DOI 10.4171/GGD/391

Groups, Geometry, and Dynamics

Isolated orderings on amalgamated free products

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Abstract. We show that an amalgamated free product $G *_A H$ admits a discrete isolated ordering, under some assumptions of G, H and A. This generalizes the author's previous construction of isolated orderings, and unlike known constructions of isolated orderings, can produce an isolated ordering with many non-trivial proper convex subgroups.

Mathematics Subject Classification (2010). Primary: 20F60; Secondary: 06F15.

Keywords. Orderable groups, isolated ordering, space of left orderings.

1. Introduction

A total ordering $<_G$ of a group G is a *left-ordering* if the relation $<_G$ is preserved by the left action on G itself, namely, $a <_G b$ implies $ga <_G gb$ for all $a, b, g \in G$. A group admitting a left-ordering is called *left-orderable*.

For $g \in G$, let U_g be the set of left-orderings $<_G$ of G that satisfy $1 <_G g$. The set of all left-orderings of G can be equipped with a topology so that $\{U_g\}_{g \in G}$ is an open sub-basis. We denote the resulting topological space by $\mathrm{LO}(G)$ and call it the *space of left-orderings of* G [16].

An *isolated ordering* is a left ordering which is an isolated point in $\mathrm{LO}(G)$. A left-ordering $<_G$ is isolated if and only if $<_G$ is determined by the sign of finitely many elements. That is, $<_G$ is isolated if and only if there exists a finite subset $\{g_1, \dots, g_n\}$ of G such that $\bigcap_{i=1}^n U_{g_i} = \{<_G\}$. We call such a finite subset a *characteristic positive set* of $<_G$. In particular, if the positive cone $P(<_G)$ of $<_G$, the sub semi-group of G consisting of $<_G$-positive elements, is finitely generated then $<_G$ is isolated.

Isolated orderings are quite interesting object in several points of view. First, if a group G has an isolated ordering whose positive cone is generated by finitely many elements $\{g_1, \dots, g_n\}$, then every non-trivial element $g \in G$ is written as either a positive or negative word over a finite alphabet $\{g_1, \dots, g_n\}$. This imposes a strong combinatorial feature on G. Moreover, isolated orderings can

[1] The author was partially supported by JSPS KAKENHI, Grant Number 25887030.

serve as a source of stimulating examples in non-commutative ring theory. In [7] we constructed a chain domain with exceptional ideals, whose existence was a question in the past [2, 5].

In a dynamics point of view, an isolated ordering can be seen as a "very rigid" action on the real line. Let us consider a countable group G and an isolated ordering with a characteristic positive set $\{g_1, \ldots, g_n\}$. Then up to conjugacy, its dynamical realization (see [12]), a faithful action of G on the real line $\mathbb{R}$ from the isolated ordering, is completely determined by finitely many conditions $0 < g_i(0)$ $(i = 1, \ldots, n)$.

An isolated ordering $<_G$ of G is *genuine* if $\mathrm{LO}(G)$ contains non-isolated points. This is equivalent to saying that $\mathrm{LO}(G)$ is not a finite set. Since the classification of groups with finitely many left-orderings (non-genuine isolated orderings) is known (see [10, Theorem 5.2.1]), we concentrate our attention to genuine isolated orderings. Several classes of groups do not have genuine isolated ordering. The non-existence of isolated orderings are observed for the free groups of rank > 1 [11], (in a different context), the free abelian groups of rank > 1 [16], and nilpotent groups [12]. More generally, it is shown that the free products of more than one groups [14], and virtually solvable groups [15] never admit a genuine isolated ordering.

Recent developments provide several examples of genuine isolated orderings, but our catalogues and knowledge are still limited and it is hard to predict when a left-orderable group admits an isolated ordering. At present, we have three ways of constructing (genuine) isolated orderings; Dehornoy-like orderings [8, 13], partially central cyclic amalgamation [9], and triangular presentations with certain special elements [4].

The aim of this paper is to extend a partially central cyclic amalgamated product construction of isolated orderings [9] in more general and abstract settings. Our argument brings a better understanding on how an isolated ordering arises when a group admits a graph of group decomposition.

To state the main theorem, we introduce the following two notions. Let A be a subgroup of a left-orderable group G. First we extend the notion of isolatedness in a relative setting.

Definition 1.1. Let $\mathsf{Res}\colon \mathrm{LO}(G) \to \mathrm{LO}(A)$ be the continuous map induced by the restriction of left orderings of G on A. We say that a left ordering $<_G$ of G is *relatively isolated* with respect to A if $<_G$ is an isolated point in the subspace $\mathsf{Res}^{-1}(\mathsf{Res}(<_G)) \subset \mathrm{LO}(G)$. Thus, $<_G$ is relatively isolated if and only if there exists a finite subset $\{g_1, \ldots, g_n\}$ of G such that $\mathsf{Res}^{-1}(\mathsf{Res}(<_G)) \cap \bigcap U_{g_i} = \{<_G\}$. We call such a finite set a *characteristic positive set* of $<_G$ relative to A.

The next property plays a crucial role in our construction of isolated orderings.

Definition 1.2. We say that a subgroup A is a *stepping* with respect to a left-ordering $<_G$ of G if for each $g \in G$ both the maximal and the minimal

$$\begin{cases} a(g) = \max_{<_G}\{a \in A \colon a \leq_G g\}, \\ a_+(g) = \min_{<_G}\{a \in A \colon g <_G a\}. \end{cases}$$

always exist.

For example, if A is an infinite cyclic subgroup generated by $a \in G$, then A is a stepping with respect to $<_G$ if and only if a is a cofinal element: for any $g \in G$, there exists $N \in \mathbb{Z}$ such that $a^{-N} <_G g <_G a^N$.

Using these notions our main theorem is stated as follows. Here is a situation we consider. Let A, G and H be left-orderable groups. We fix embeddings $i_G \colon A \hookrightarrow G$ and $i_H \colon A \hookrightarrow H$ so we always regard A as a common subgroup of G and H.

Theorem 1.3. *Let $<_G$ and $<_H$ be discrete orderings of G and H. Assume that $<_G$ and $<_H$ satisfy the following conditions.*

(a) *The restriction of $<_G$ and $<_H$ on A yields the same left ordering $<_A$ of A.*

(b) *A is a stepping with respect to both $<_G$ and $<_H$.*

(c) *$<_G$ is isolated and $<_H$ is relatively isolated with respect to A.*

Then the amalgamated free product $X = G *_A H$ admits isolated orderings $<_X^{(1)}$ and $<_X^{(2)}$ which have the following properties.

(1) *Both $<_X^{(1)}$ and $<_X^{(2)}$ extend the orderings $<_G$ and $<_H$: if $g <_G g'$ $(g, g' \in G)$ then $g <_X^{(i)} g'$, and if $h <_H h'$ $(h, h' \in H)$ then $h <_X^{(i)} h'$ $(i = 1, 2)$.*

(2) *If $\{g_1, \ldots, g_m\}$ is a characteristic positive set of $<_G$ and $\{h_1, \ldots, h_n\}$ is a characteristic positive set of $<_H$ relative to A, then*

$$\{g_1, \ldots, g_m, h_1, \ldots, h_n, h_{\mathsf{min}} a_{\mathsf{min}}^{-1} g_{\mathsf{min}}\}$$

is a characteristic positive set of $<_X^{(1)}$ and

$$\{g_1, \ldots, g_m, h_1, \ldots, h_n, g_{\mathsf{min}} a_{\mathsf{min}}^{-1} h_{\mathsf{min}}\}$$

is a characteristic positive set of $<_X^{(2)}$. Here a_{min}, g_{min} and h_{min} represent the minimal positive elements of the orderings $<_A$, $<_G$ and $<_H$, respectively. (Note that A is a stepping implies that $<_A$ is discrete, see Lemma 2.1).

(3) *$<_X^{(1)}$ is discrete with the minimal positive element $h_{\mathsf{min}} a_{\mathsf{min}}^{-1} g_{\mathsf{min}}$, and $<_X^{(2)}$ is discrete with the minimal positive element $g_{\mathsf{min}} a_{\mathsf{min}}^{-1} h_{\mathsf{min}}$.*

(4) *A is a stepping with respect to the orderings $<_X^{(1)}$ and $<_X^{(2)}$.*

The assumption (a) is an obvious requirement for X to have a left ordering extending both $<_G$ and $<_H$. The crucial assumptions are (b) and (c). It should be emphasized that the orderings $<_A$ and $<_H$ may not be isolated. We also note that, The property (4) allows us to iterate a similar construction, hence Theorem 1.3 produces huge examples of isolated orderings.

Remark 1.4. As for the existence of isolated orderings, Theorem 1.3 contains the main theorem of [9], but [9, Theorem 1.1] states much stronger results.

In [9], we treated the case that $A = \mathbb{Z}$ with additional assumptions that the isolated ordering $<_H$ is preserved by the right action of A, and that A is central in G. Under these assumptions, we proved that the positive cone of the resulting isolated ordering is *finitely generated*, and determined all convex subgroups. Moreover, one can algorithmically determine whether $x <_X x'$ or not.

On the other hand, for the isolated orderings $<_X^{(i)}$ in Theorem 1.3, we do not know whether its positive cone is finitely generated or not in general, and a computation of $<_X^{(i)}$ is more complicated. As for the computational issues, see Remark 2.13.

In light of the above remark, finding a generating set of the positive cone of $<_X^{(i)}$, and determining when it is finitely generated are quite interesting.

As for convex subgroups, in Proposition 2.14 we show that a convex subgroup of A with additional properties yields a convex subgroup of $(X, <_X)$. Thus, the resulting isolated ordering of X can admit many non-trivial convex subgroups. This also makes a sharp contrast in [9], where the obtained isolated ordering contains exactly one non-trivial proper convex subgroup. It should be emphasized that the Dubrovina-Dubrovin ordering of the braid groups [3, 6] are the only known examples of genuine isolated ordering with more than one proper non-trivial convex subgroup. In Example 2.15, starting from $\mathbb{Z}$ with standard ordering, the simplest isolated ordering, we construct many isolated orderings with more than one non-trivial convex subgroups.

2. Construction of isolated orderings

For a totally ordered set $(S, <_S)$ and $s, s' \in S$, we say that s' is the *successor* of s and we denote by $s \prec_S s'$, if s' is the minimal element in S that is strictly greater than s with respect to the ordering $<_S$.

A left ordering $<_G$ of a group G is *discrete* if there exists the successor g_{min} of the identity element. That is, $<_G$ admits the minimal $<_G$-positive element. By left-invariance, a discrete left ordering $<_G$ satisfy $gg_{\mathsf{min}}^{-1} \prec_G g \prec_G gg_{\mathsf{min}}$ for all $g \in G$.

Let us consider the situation in Theorem 1.3. Let G and H be groups admitting discrete left orderings $<_G$ and $<_H$, and A be a common subgroup of G and H, such that the restriction of $<_G$ and $<_H$ yield the same left ordering $<_A$.

The assumption that A is a stepping (assumption (b)) implies the following.

Lemma 2.1. *For a subgroup A of a left-orderable group G, if A is a stepping with respect to a left-ordering $<_G$, then the restriction of $<_G$ on A is discrete.*

Proof. From the definition of stepping,

$$a_{\mathsf{min}} = \min_{<_A}\{a \in A \colon 1 <_A a\} = \min_{<_G}\{a \in A \colon 1 <_G a\} = a_+(1)$$

exists. □

Thus $<_A$ is also discrete. We denote the minimal positive elements of $<_A$, $<_G$ and $<_H$ by a_{min}, g_{min} and h_{min}, respectively. We put $g_{\mathsf{M}} = a_{\mathsf{min}} g_{\mathsf{min}}^{-1}$ and $h_{\mathsf{M}} = a_{\mathsf{min}} h_{\mathsf{min}}^{-1}$, so $g_{\mathsf{M}} \prec_G a_{\mathsf{min}}$ and $h_M \prec_H a_{\mathsf{min}}$.

We start to construct an isolated ordering on a group $X = G *_A H$. We mainly explain the construction of the isolated ordering $<_X^{(1)}$, which we simply denote by $<_X$. Although the hypothesis on G and H are not symmetric, as we will discuss at the end of the proof of Theorem 1.3, the construction of $<_X^{(2)}$ is similar: the ordering $<_X^{(2)}$ is obtained by interchanging the role of G and H.

The amalgamated free product structure of X induces a filtration

$$\mathcal{F}_{-1}(X) \subset \mathcal{F}_{-0.5}(X) \subset \mathcal{F}_0(X) \subset \mathcal{F}_{0.5}(X) \subset \mathcal{F}_1(X) \subset \mathcal{F}_2(X) \subset \cdots \subset \mathcal{F}_i(X) \subset \cdots$$

defined by

$$\begin{cases} \mathcal{F}_{-1}(X) = \emptyset, \\ \mathcal{F}_{-0.5}(X) = A, \\ \mathcal{F}_0(X) = H, \\ \mathcal{F}_{0.5}(X) = G \cup H, \\ \mathcal{F}_{2i+1}(X) = G\mathcal{F}_{2i}, \\ \mathcal{F}_{2i}(X) = H\mathcal{F}_{2i-1}. \end{cases}$$

The non-integer parts of the filtrations are exceptional, and the filtration $\mathcal{F}_{0.5}(X)$ is the most important because it is the restriction on $\mathcal{F}_{0.5}(X)$ that eventually characterizes the isolated ordering $<_X$.

Starting from $<_G$ and $<_H$, we inductively construct a total ordering $<_i$ on $\mathcal{F}_i(X)$. To be able to extend $<_i$ to a left ordering of X, we need the following obvious property.

Definition 2.2. We say a total ordering $<_i$ on $\mathcal{F}_i(X)$ is *compatible* if for any $x \in X$ and $s, t \in \mathcal{F}_i(X)$, $xs <_i xt$ whenever $s <_i t$ and $xs, xt \in \mathcal{F}_i(X)$.

By definition, if $<_i$ is a restriction of a left ordering of X on $\mathcal{F}_i(X)$, then $<_i$ is compatible. Conversely, Bludov-Glass proved that a compatible ordering $<_i$ on $\mathcal{F}_i(X)$ can be extended to a compatible ordering $<_{i+1}$ of $\mathcal{F}_{i+1}(X)$ under some conditions [1]. This is a crucial ingredient of the proof of Bludov-Glass' theorem on necessary and sufficient conditions for an amalgamated free product to be left-orderable [1, Theorem A].

From the point of view of the topology of $\mathrm{LO}(G *_A H)$, it is suggestive to note that Bludov-Glass' extension of $<_i$ to $<_{i+1}$ is far from unique. This illustrates and explains the intuitively obvious fact that "most" left orderings of $G *_A H$ are not isolated. Our isolated ordering is constructed by specifying a situation in which Bludov-Glass' extension procedure must be unique.

As the first step of construction, we define an ordering $<_{\mathsf{base}}$ on $\mathcal{F}_{0.5}(X)$. Since we have assumed that A is a stepping with respect to both $<_G$ and $<_H$, we have the function

$$a \colon \mathcal{F}_{0.5}(X) \to A$$

defined by

$$a(x) = \begin{cases} \max_{<_G}\{a \in A : a \leq_G x\} & (x \in G), \\ \max_{<_H}\{a \in A : a \leq_H x\} & (x \in H). \end{cases} \tag{2.1}$$

Using the function a, we define the total ordering $<_{\mathsf{base}}$ as follows:

$$\begin{cases} g <_{\mathsf{base}} g' & \text{if } g, g' \in G \text{ and } g <_G g', \\ h <_{\mathsf{base}} h' & \text{if } h, h' \in H \text{ and } h <_H h', \\ h <_{\mathsf{base}} g & \text{if } h \in H - A, g \in G - A \text{ and } a(h) \leq_A a(g), \\ g <_{\mathsf{base}} h & \text{if } h \in H - A, g \in G - A \text{ and } a(g) <_A a(h). \end{cases} \tag{2.2}$$

The ordering $<_{\mathsf{base}}$ can be schematically understood by Figure 1.

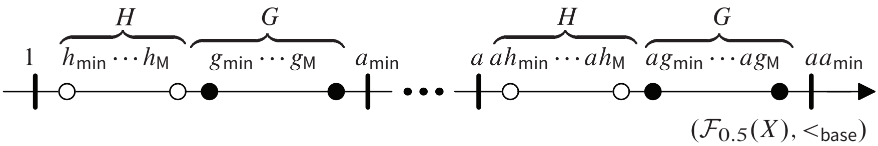

Figure 1. Ordering $<_{\mathsf{base}}$ on $\mathcal{F}_{0.5}(X)$.

Lemma 2.3. *The ordering $<_{\text{base}}$ is the unique compatible ordering of $\mathcal{F}_{0.5}(X)$ such that*

B1 the restriction of $<_{\text{base}}$ on G and H agrees with $<_G$ and $<_H$, respectively;

B2 $h_{\mathsf{M}} = a_{\min}h_{\min}^{-1} <_{\text{base}} g_{\min}$.

Proof. By definition, $<_{\text{base}}$ is a compatible ordering with B1 and B2. Assume that $<'$ is another compatible total ordering on $\mathcal{F}_{0.5}(X)$ with the same properties. To see the uniqueness, it is sufficient to show that for $g \in G - A$ and $h \in H - A$, $h <_{\text{base}} g$ implies $h <' g$.

By definition of $<_{\text{base}}$, $a(h) \leq_A a(g)$. If $a(h) <_A a(g)$, then $h <' a(h)a_{\min} \leq' a(g) <' g$ so $h <' g$. Assume that $a(h) = a(g)$ and put $a = a(g) = a(h)$. By B1, $1 <' a^{-1}h <' a_{\min}$ hence $1 <' a^{-1}h \leq' h_{\mathsf{M}} = a_{\min}h_{\min}^{-1}$. Similarly, $1 <' a^{-1}g$ so $g_{\min} \leq' a^{-1}g$. By B2,

$$a^{-1}h \leq' h_{\mathsf{M}} <' g_{\min} \leq' a^{-1}g,$$

hence $a^{-1}h <' a^{-1}g$. Since $<'$ is compatible, $h <' g$. □

Lemma 2.3, combined with our assumption (c) of Theorem 1.3, shows the following.

Proposition 2.4. *The compatible ordering $<_{\text{base}}$ is characterized by finitely many inequalities. Let $\{g_1, \dots, g_m\}$ be a characteristic positive set of $<_G$ and $\{h_1, \dots, h_n\}$ be a characteristic positive set of $<_H$ relative to A. Then $<_{\text{base}}$ is the unique compatible ordering on $\mathcal{F}_{0.5}(X)$ that satisfies the inequalities*

$$\begin{cases} 1 <_{\text{base}} g_i & (i = 1, \dots, m), \\ 1 <_{\text{base}} h_j & (j = 1, \dots, n), \\ a_{\min}h_{\min}^{-1} <_{\text{base}} g_{\min}. \end{cases} \tag{2.3}$$

Proof. The set of inequalities $\{1 <_{\text{base}} g_i\}$ uniquely determine the restriction of $<_{\text{base}}$ on G so in particular, determine the restriction of $<_{\text{base}}$ on A. Since $<_H$ is relatively isolated with respect to A, the additional inequalities $\{1 <_{\text{base}} h_i\}$ uniquely determine the restriction of $<_{\text{base}}$ on H. Therefore the family of inequalities (2.3) implies B1 and B2 in Lemma 2.3. □

The next step is to extend the ordering $<_{\text{base}}$ to a compatible ordering $<_1$ of $\mathcal{F}_1(X) = GH$. For $a \in A$, let

$$\begin{aligned} \Delta_a &= \{h \in H - A : a(h) = a\} \\ &= \{h \in H - A : a <_H h <_H aa_{\min}\} \\ &= \{h \in H - A : ah_{\min} \leq_H h \leq_H ah_{\mathsf{M}}\}. \end{aligned}$$

First we observe the following property which plays a crucial role in proving the uniqueness.

Lemma 2.5. *For $g, g' \in G$ and $h, h' \in H$, if $ga(h) = g'a(h')$ then $g\Delta_{a(h)} = g'\Delta_{a(h')}$.*

Proof. $ga(h) = g'a(h')$ implies that $g^{-1}g' = a(h)a(h')^{-1} \in A$. This shows $(g^{-1}g')\Delta_{a(h')} = a(h)a(h')^{-1}\Delta_{a(h')} = \Delta_{a(h)}$ hence $g\Delta_{a(h)} = g'\Delta_{a(h')}$. □

Proposition 2.6. *There exists a unique compatible total ordering $<_1$ on $\mathcal{F}_1(X)$ that extends $<_{\mathsf{base}}$.*

Proof. For each $a \in A$ and $g \in G - A$, we regard $g\Delta_a$ as a totally ordered set equipped with an ordering $<_1$ defined by $gh <_1 gh'$ ($h, h' \in \Delta_a$) if and only if $h <_H h'$.

First we check that this ordering $<_1$ is well-defined on each $g\Delta_a$. Assume that $g\Delta_a = g'\Delta_{a'}$ as a subset of $\mathcal{F}_1(X)$. Let $gh_0 = g'h'_0, gh_1 = g'h'_1$ be elements of $g\Delta_a = g'\Delta_{a'}$, where $h_0, h_1 \in \Delta_a$ and $h'_0, h'_1 \in \Delta_{a'}$. Note that $g\Delta_a = g'\Delta_{a'}$ implies that $g^{-1}g' \in A$. Therefore,

$$\begin{aligned} gh_0 <_1 gh_1 &\iff h_0 <_H h_1 \\ &\iff (g^{-1}g')h'_0 <_H (g^{-1}g')h'_1 \\ &\iff h'_0 <_H h'_1 \\ &\iff g'h'_0 <_1 g'h'_1. \end{aligned}$$

This shows that $<_1$ is a well-defined total ordering on $g\Delta_a$.

Since $\mathcal{F}_1(X) = \mathcal{F}_0(X) \cup \left(\bigcup g\Delta_a\right)$, we construct the desired ordering $<_1$ by inserting the ordered sets $g\Delta_a$ into $\mathcal{F}_0(X)$. We show that the way to inserting $g\Delta_a$ is unique.

Since $a <_{\mathsf{base}} h <_{\mathsf{base}} ag_{\mathsf{min}}$ for $h \in \Delta_a$, a compatible ordering $<_1$ must satisfy

$$ga <_1 gh <_1 gag_{\mathsf{min}} \quad (g \in G - A).$$

By definition of $<_{\mathsf{base}}$, $ga \prec_{\mathsf{base}} gag_{\mathsf{min}}$, that is, there are no elements of $\mathcal{F}_{0.5}(X)$ that lies between ga and gag_{min}. This says that to get a compatible ordering, we must insert the ordered set $g\Delta_a$ between ga and gag_{min}. Moreover, by Lemma 2.5, $ga(h) = g'a(h')$ implies $g\Delta_{a(h)} = g'\Delta_{a(h')}$. This means that the ordered set $g\Delta_a$ inserted between ga and gag_{min} must be unique.

Therefore there is the unique way of inserting $g\Delta_a$ into $\mathcal{F}_0(X)$ to get a compatible ordering on $\mathcal{F}_1(X)$. The process of inserting $g\Delta_a$ is schematically explained in Figure 2.

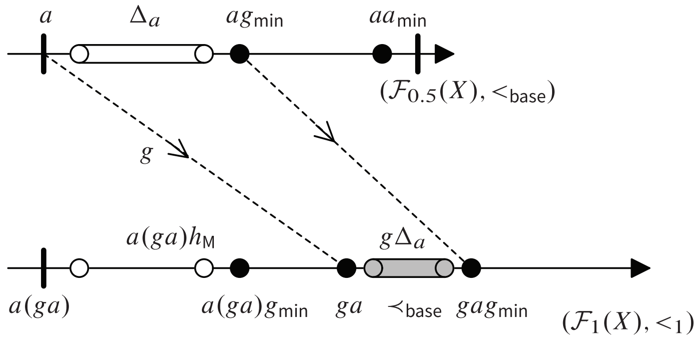

Figure 2. Ordering $<_1$: inserting $g\Delta_a$ between ga and gag_{min}.

The resulting ordering $<_1$ is written as follows. For $x = gh$ and $x' = g'h'$ $(g \in G, h \in H)$, we have

$$x <_1 x' \iff \begin{array}{lll} \text{either} & (1) & ga(h) <_{\mathsf{base}} g'a(h') \\ \text{or} & (2) & ga(h) = g'a(h') \text{ and } h <_{\mathsf{base}} (g^{-1}g')h'. \end{array} \tag{2.4}$$

Note that by the proof of Lemma 2.5, $ga(h) = g'a(h')$ implies $g^{-1}g' \in A$, hence $(g^{-1}g')h' \in \mathcal{F}_{0.5}(X)$. Hence the inequality $h <_{\mathsf{base}} (g^{-1}g')h'$ makes sense. □

In a similar manner, we extend the ordering $<_1$ of $\mathcal{F}_1(X)$ to a compatible ordering $<_2$ of $\mathcal{F}_2(X)$. We define the map $c_0\colon \mathcal{F}_1(X) - \mathcal{F}_0(X) \to \mathcal{F}_0(X)$ by

$$c_0(x) = \max_{<_1}\{y \in \mathcal{F}_0(X)\colon y <_1 x\},$$

and for $y \in \mathcal{F}_0(X)$, we put

$$\begin{aligned} \Delta_y &= \{x \in \mathcal{F}_1(X) - \mathcal{F}_0(X)\colon c_0(x) = y\} \\ &= \{x \in \mathcal{F}_1(X)\colon y <_1 x <_1 yh_{\mathsf{min}}\}. \end{aligned}$$

Note that Δ_y might be empty.

Lemma 2.7. *The map c_0 and the set Δ_y have the following properties.*

(1) *For $x = gh \in \mathcal{F}_1(X) - \mathcal{F}_0(X)$ $(g \in G - A, h \in H)$, $c_0(gh) = a(ga(h))h_{\mathsf{M}}$. Here $a\colon \mathcal{F}_{0.5}(X) \to A$ is the map defined by (2.1).*

(2) *For $x, x' \in \mathcal{F}_1(X) - \mathcal{F}_0(X)$ and $h, h' \in H$, if $hc_0(x) = h'c_0(x')$ then $h\Delta_{c_0(x)} = h'\Delta_{c_0(x')}$.*

Proof. Note that $a(ga(h)) <_1 ga(h) <_1 gh$. By definition of $<_1$ given in (2.4), there are no elements of $\mathcal{F}_0(X) = H$ between $ga(h)$ and gh. Moreover, for $g \in G$ $c_0(g) = a(g)h_{\mathsf{M}}$ (see Figure 2 again). This proves $c_0(gh) = c_0(ga(h)) = a(ga(h))h_{\mathsf{M}}$.

To see (2), write $x = gy$ and $x = g'y'$ ($g, g' \in G,\ y, y' \in \mathcal{F}_0$). Then by (1), $hc_0(x) = h'c_0(x')$ implies that $h^{-1}h' = c_0(x)c_0(x')^{-1} = a(ga(y))a(g'a(y'))^{-1} \in A$. This shows $(h^{-1}h')\Delta_{c_0(x')} = \Delta_{c_0(x)}$ hence $h\Delta_{c_0(x)} = h'\Delta_{c_0(x')}$. □

Proposition 2.8. *There exists a unique compatible total ordering $<_2$ on $\mathcal{F}_2(X)$ that extends $<_1$.*

Proof. For $h \in H$ and $y \in \mathcal{F}_1(X)$, we regard $h\Delta_y$ as a totally ordered set equipped with a total ordering $<_2$ defined by $hx <_2 hx'$ ($x, x' \in \Delta_y$) if and only if $x <_1 x'$. By the same argument as Proposition 2.6, this ordering is well-defined on each subset $h\Delta_y$.

$\mathcal{F}_2(X) = \mathcal{F}_1(X) \cup \left(\bigcup h\Delta_y\right)$ so we construct the desired ordering $<_2$ by inserting the ordered sets $h\Delta_y$ into $\mathcal{F}_1(X)$, as we have done in Proposition 2.6.

By the compatibility requirement, for $x \in \Delta_y$ and $h \in H$, a desired extension $<_2$ must satisfy

$$hy <_2 hx <_2 hyh_{\mathsf{min}}$$

so we need to insert $h\Delta_y$ between $hc_0(x)$ and $hc_0(x)h_{\mathsf{min}}$. By Lemma 2.7 (1), Δ_y is empty unless $y = ah_{\mathsf{M}}$ for some $a \in A$, and that if Δ_y is non-empty then $hy \prec_1 hyh_{\mathsf{min}}$ for $h \in H - A$. That is, there are no elements of $\mathcal{F}_1(X)$ between hy and hyh_{min}. Moreover, Lemma 2.7 (2) shows that an ordered set $h\Delta_y$ inserted between hy and hyh_{min} must be unique.

Thus, the process of inserting $h\Delta_y$ to $\mathcal{F}_1(X)$ is unique, and we get a well-defined compatible ordering $<_2$. Figure 3 gives schematic illustration of the inserting process.

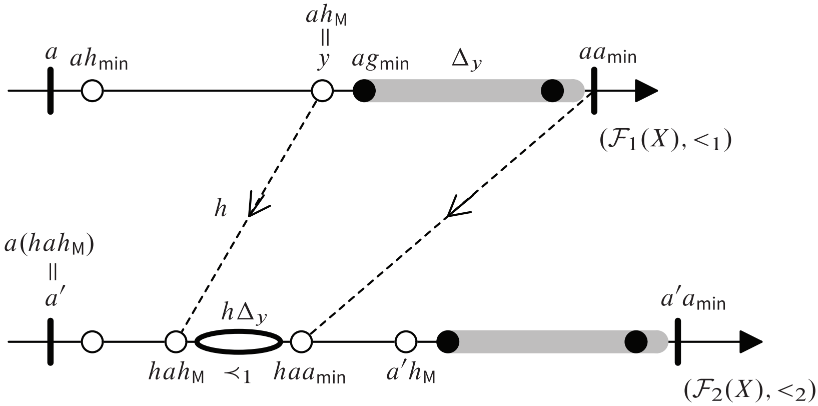

Figure 3. Ordering $<_2$: inserting $h\Delta_y$ between $hy = hah_{\mathsf{M}}$ and $hyh_{\mathsf{min}} = haa_{\mathsf{min}}$.

As a consequence, the ordering $<_2$ is given as follows. For $x = hy$ and $x' = h'y'$ $(h \in H, y \in \mathcal{F}_1(X))$, we have

$$x <_2 x' \iff \begin{array}{lll} \text{either} & (1) & hc_0(y) <_1 h'c_0(y') \\ \text{or} & (2) & hc_0(y) = h'c_0(y') \text{ and } y <_1 (h^{-1}h')y'. \end{array} \tag{2.5}$$

Note that $hc_0(y) = h'c_0(y')$ implies $h^{-1}h' \in A$ as we have seen in the proof of Lemma 2.7 (2), so the inequality $y <_1 (h^{-1}h')y' \in \mathcal{F}_1(X)$ makes sense. □

Now we inductively extend compatible orderings. Assume that we have defined a compatible ordering $<_i$ of $\mathcal{F}_{i+1}$. We define the map $c_{i-1}\colon \mathcal{F}_i(X) - \mathcal{F}_{i-1}(X) \to \mathcal{F}_{i-1}(X)$ by

$$c_{i-1}(x) = \max_{<_i}\{y \in \mathcal{F}_{i-1}(X) \mid y <_i x\}$$

and for $y \in \mathcal{F}_{i-1}(X)$, we put

$$\Delta_y = \{x \in \mathcal{F}_i(X) - \mathcal{F}_{i-1}(X) \mid c_{i-1}(x) = y\}.$$

Here we have assumed that c_{i-1} is well-defined, that is, the maximal exists.

We will say that $<_i$ satisfies the *ping pong property* if the ordering $<_i$ satisfies the following three properties.

P1. The maps c_{i-1} and c_{i-2} satisfy the equality

$$c_{i-1}(x) = \begin{cases} gc_{i-2}(y) & (x = gy, g \in G - A, y \in \mathcal{F}_{i-1}(X), \text{ if } i \text{ is odd}), \\ hc_{i-2}(y) & (x = hy, h \in H - A, y \in \mathcal{F}_{i-1}(X), \text{ if } i \text{ is even}). \end{cases}$$

Moreover, $c_{i-1}(x) \in \mathcal{F}_{i-2}(X) - \mathcal{F}_{i-3}(X)$.

P2. $c_{i-1}(x) \prec_{i-1} c_{i-1}(x)h_{\mathsf{min}}$.

P3. If $x \in \mathcal{F}_i(X) - \mathcal{F}_{i-2}(X)$, $x \prec_i xh_{\mathsf{min}}$.

The reason why we call these properties "ping pong" will be explained in Remark 2.12. Note that ping pong property P2 shows that

$$\Delta_y = \{x \in \mathcal{F}_i(X) - \mathcal{F}_{i-1}(X) \mid y <_i x <_i yh_{\mathsf{min}}\}. \tag{2.6}$$

Lemma 2.9. *The ordering $<_2$ satisfies the ping pong property.*

Proof. This is easily seen from the description (2.5) of $<_2$ (see Figure 3 again).

For $x = hy \in \mathcal{F}_2(X) - \mathcal{F}_1(X)$ $(h \in H - A, y \in \mathcal{F}_1(X) - \mathcal{F}_0(X))$, $hc_0(y) <_2 hy$. There are no elements of $\mathcal{F}_1(X) - \mathcal{F}_0(X)$ that lie between $hc_0(y)$ and hy so $c_1(x) = hc_0(y)$. In particular, $c_1(x) \in \mathcal{F}_0(X) = H$ hence by definition of $<_1$ given in (2.4) (see Figure 2 again), $c_1(x) \prec_1 c_1(x)h_{\mathsf{min}}$. Moreover, the description (2.5) of $<_2$ shows

$$\begin{cases} x \prec_2 xh_{\mathsf{min}} & \text{if } x \notin H, \\ x \prec_2 xh_{\mathsf{M}}^{-1}g_{\mathsf{min}} & \text{if } x \in H. \end{cases}$$

□

The ping pong property shows the counterparts of Lemma 2.5 and 2.7.

Lemma 2.10. *Assume that $<_i$ satisfies the ping pong property and let $x, x' \in \mathcal{F}_i(X) - \mathcal{F}_{i-1}(X)$.*

- *If i is odd, then $gc_{i-1}(x) = g'c_{i-1}(x')$ $(g, g' \in G)$ implies $g\Delta_{c_{i-1}(x)} = g'\Delta_{c_{i-1}(x')}$.*
- *If i is even, then $hc_{i-1}(x) = h'c_{i-1}(x')$ $(h, h' \in H)$ implies $h\Delta_{c_{i-1}(x)} = h'\Delta_{c_{i-1}(x')}$.*

Proof. We show the case i is odd. The case i is even is similar. Put $y = c_{i-1}(x)$ and $y' = c_{i-1}(x')$, respectively. We show $g'\Delta_{y'} \subset g\Delta_y$. The converse inclusion is proved similarly. By (2.6), $z' \in \Delta_{y'}$ if and only if $y' <_{i-1} z' <_{i-1} y'h_{\mathsf{min}}$. By compatibility,

$$y = g^{-1}g'y' <_{i-1} (g^{-1}g')z' <_{i-1} g^{-1}g'y'h_{\mathsf{min}} = yh_{\mathsf{min}}$$

so $(g^{-1}g')z' \in \Delta_y$. This proves $g'z' \in g\Delta_y$. □

The following proposition completes the construction of isolated ordering $<_X$.

Proposition 2.11. *If $<_i$ $(i > 1)$ is a compatible ordering with the ping pong property, then there exists a unique compatible ordering $<_{i+1}$ on $\mathcal{F}_{i+1}(X)$ that extends $<_i$. Moreover, this compatible ordering $<_{i+1}$ also satisfies the ping pong property.*

Proof. The construction of $<_{i+1}$ is almost the same as the construction of $<_2$. We treat the case i is even. The case i is odd is similar.

We regard each $g\Delta_y$ $(y \in \mathcal{F}_{i-1}(X), g \in G - A)$ as a totally ordered set, by equipping a total ordering $<_{i+1}$ defined by $gx <_{i+1} gx'$ $(x, x' \in \Delta_y)$ if and only if $x <_i x'$. By the same argument as Proposition 2.6, the ordering $<_{i+1}$ is well-defined on each $g\Delta_y$. The desired compatible ordering $<_{i+1}$ on $\mathcal{F}_{i+1}(X) = \mathcal{F}_i(X) \cup \left(\bigcup g\Delta_y\right)$ is obtained by inserting $g\Delta_y$ into $\mathcal{F}_i(X)$.

By the ping pong property P2, for $y \in \mathcal{F}_{i-1}(X)$ if Δ_y is non-empty, then $y \prec_{i-1} yh_{\mathsf{min}}$. Thus we need to insert $g\Delta_y$ between gy and gyh_{min}. By the ping pong property P3, $gy \prec_i gyh_{\mathsf{min}}$, so there are no elements of $\mathcal{F}_i(X)$ between gy and gyh_{min}. Moreover, Lemma 2.10 shows that there are exactly one ordered set of the form $g\Delta_y$ that should be inserted between gy and gyh_{min}. Therefore the process of insertions is unique, and the resulting ordering $<_{i+1}$ is given as follows. For $x = gy$ and $x' = g'y'$, $(g, g' \in G$ and $y, y' \in \mathcal{F}_i(X))$, we define

$$x <_{i+1} x' \iff \begin{array}{lll} \text{either} & (1) & gc_{i-1}(y) <_i g'c_{i-1}(y') \\ \text{or} & (2) & gc_{i-1}(y) = g'c_{i-1}(y') \text{ and } y <_i (g^{-1}g')y'. \end{array} \tag{2.7}$$

Next we show that $<_{i+1}$ also satisfies the ping pong property. We have inserted $x = gy \in \mathcal{F}_{i+1}(X) - \mathcal{F}_i(X)$ $(g \in G - A, y \in \mathcal{F}_i(X))$ between $gc_{i-1}(y)$ and $gc_{i-1}(y)h_{\mathsf{min}}$. By definition of $<_{i+1}$, there are no elements of $\mathcal{F}_i(X)$ that lie between x and $gc_{i-1}(x)$, hence $c_i(x) = gc_{i-1}(y)$. By P1 for $<_i$, $c_{i-1}(y) \in \mathcal{F}_{i-2}(X) - \mathcal{F}_{i-3}(X)$. Hence $c_i(x) = gc_{i-1}(y) \in \mathcal{F}_{i-1}(X) - \mathcal{F}_{i-2}(X)$ so $<_{i+1}$ satisfies P1. Moreover by P3 for $<_i$, $c_i(x) \in \mathcal{F}_{i-1}(X) - \mathcal{F}_{i-2}(X)$ implies that $c_i(x) \prec_i c_i(x)h_{\mathsf{min}}$ hence $<_{i+1}$ satisfies P2.

Finally we show that $<_{i+1}$ satisfies P3. Assume that $x \in \mathcal{F}_{i+1}(X) - \mathcal{F}_i(X)$, and put $x = gy$ $(g \in G - A,\ y \in \mathcal{F}_i(X) - \mathcal{F}_{i-1}(X))$. By P3 for $<_i$, we have $y \prec_i yh_{\mathsf{min}}$. Hence by definition of $<_{i+1}$ we have $x = gy \prec_{i+1} gyh_{\mathsf{min}} = xh_{\mathsf{min}}$.

If $x \in \mathcal{F}_i(X) - \mathcal{F}_{i-1}(X) \subset \mathcal{F}_i - \mathcal{F}_{i-2}(X)$, then by P3 for $<_i$ we have $x \prec_i xh_{\mathsf{min}}$. No elements of $\mathcal{F}_{i+1}(X) - \mathcal{F}_i(X)$ are inserted between x and xh_{min}, hence $x \prec_{i+1} xh_{\mathsf{min}}$. □

Proof of Theorem 1.3. For $x, x' \in X$, we define the isolated ordering $<_X = <_X^{(1)}$ by

$$x <_X x' \iff x <_N x'$$

where N is chosen to be sufficiently large so that $x, x' \in \mathcal{F}_N(X)$. Proposition 2.11 shows that $<_X$ is a well-defined left ordering of X. By Proposition 2.4, $<_X$ is isolated with characteristic positive set

$$\{g_1, \dots, g_m, h_1, \dots, h_n, h_{\mathsf{min}} a_{\mathsf{min}}^{-1} g_{\mathsf{min}}\},$$

if $\{g_1, \dots, g_m\}$ is a characteristic positive set of $<_G$ and $\{h_1, \dots, h_n\}$ is a characteristic positive set of $<_H$ relative to A.

It remains to show that A is a stepping with respect to $<_X$. To see this, for $x \in X$, define

$$a(x) = a \circ \cdots \circ c_{N-2} \circ c_N(x)$$

where N is taken so that $x \in \mathcal{F}_N(X)$. By definition of c_i, $a(x) = \max_{<_X}\{a \in A : a \leq_X x\}$.

A construction of isolated orderings $<_X^{(2)}$ is similar. Note that in the construction of $<_X^{(1)}$, we only used the assumption (b), and that we used the assumption (c), where the role of G and H are the not the same, only at Proposition 2.4. Hence by interchanging the role of G and H, we get another left ordering $<_X^{(2)}$ of X.

As in the case $<_X^{(1)}$, the ordering $<_X^{(2)}$ is uniquely determined by the restriction on $G \cup H$, which we denote by $<'_{\mathsf{base}}$. By the same argument as Proposition 2.4, the ordering $<'_{\mathsf{base}}$, hence $<_X^{(2)}$, is characterized by finitely many inequalities

$$\begin{cases} 1 <_{\mathsf{base}} g_i & (i = 1, \dots, m), \\ 1 <_{\mathsf{base}} h_j & (j = 1, \dots, n), \\ a_{\mathsf{min}} g_{\mathsf{min}}^{-1} <_{\mathsf{base}} h_{\mathsf{min}}. \end{cases}$$

□

Remark 2.12. Here we explain why we call the properties P1–P3 the ping pong property. This may help to understand the isolated ordering $<_X$ we constructed.

Let us divide $X - A$ into two disjoint subsets $\mathcal{E}$ and $\mathcal{O}$ as follows:

$$\begin{cases} \mathcal{E} = \bigcup_{a \in A} \{x \in X \colon a <_X x <_X a h_{\mathsf{M}}\}, \\ \mathcal{O} = \bigcup_{a \in A} \{x \in X \colon a g_{\mathsf{min}} <_X x <_X a a_{\mathsf{min}}\}. \end{cases}$$

By definition of $<_{\mathsf{base}}$, $\mathcal{F}_0(X) - A = H - A \subset \mathcal{E}$, and by definition of $<_1$, $\mathcal{F}_1(X) - \mathcal{F}_0(X) = GH - H \subset \mathcal{O}$. Now the ping pong property P1 says that

$$\begin{cases} g(\mathcal{F}_{2i}(X) - \mathcal{F}_{2i-1}(X)) \subset \mathcal{E} & (g \in G - A), \\ h(\mathcal{F}_{2i+1}(X) - \mathcal{F}_{2i-1}(X)) \subset \mathcal{O} & (h \in H - A). \end{cases}$$

Thus, we conclude

$$\begin{cases} \mathcal{E} = \{\text{even part}\} = \bigcup_i (\mathcal{F}_{2i}(X) - \mathcal{F}_{2i-1}(X)), \\ \mathcal{O} = \{\text{odd part}\} = \bigcup_i (\mathcal{F}_{2i+1}(X) - \mathcal{F}_{2i}(X)), \end{cases}$$

and for $g \in G - A$ and $h \in H - A$, we have

$$g(\mathcal{O}) \subset \mathcal{E}, \quad h(\mathcal{E}) \subset \mathcal{O}.$$

Therefore the subsets $\mathcal{O}$ and $\mathcal{E}$ provides the setting of a famous ping pong argument. The rest of the ping pong properties P2 and P3, as we have seen in the proof of Proposition 2.11, rather follows from P1. This explains why we call the properties P1–P3 the ping pong property.

Remark 2.13. Here we briefly explain the computability of the resulting isolated ordering $<_X$.

By (2.7), for $x \in \mathcal{F}_{i+1}(X) - \mathcal{F}_i(X)$, determining whether $1 <_X x$ (which is equivalent to $1 <_{i+1} x$) is reduced to the computation of $c_i(x)$ and the ordering $<_i$. By ping pong property P1, $c_i(x)$ is computed from the function c_{i-1}. Thus, eventually one can reduce to the computations of the base orderings $<_G$ and $<_H$ and the map $a \colon \mathcal{F}_{0.5}(X) \to A$. That is, *the ordering $<_X$ is algorithmically computable if and only if the orderings $<_G$, $<_H$ and the map $a : \mathcal{F}_{0.5}(X) \to A$ are algorithmically computable.*

The problem may occur when we want to compute the map a. Even if we have a nice algorithm to compute $<_G$ and $<_H$, this does not guarantee an algorithm to compute the map a, in general because it involves the maximum.

Finally we study convex subgroups. A subset C of a totally ordered set $(S, <_S)$ is *convex* if $c \leq_S s \leq_S c'$ $(c, c' \in C, s \in S)$ implies $s \in C$. For a subset T of $(S, <_S)$, the *convex hull* $\mathsf{Conv}_S(T)$ of T in S is the minimum convex subset that contains T. Namely,

$$\mathsf{Conv}_S(T) = \bigcap_{\{C \supset T:\text{convex}\}} C = \{s \in S \colon \exists t, t' \in T, t \leq_S s \leq_S t'\}.$$

Let $(G, <_G)$ be a left-ordered group and let A be a subgroup of G. We denote the restriction of $<_G$ on A by $<_A$. We say a convex subgroup B of $(A, <_A)$ is a $(G, <_G)$-*strongly convex* if its convex hull $\mathsf{Conv}_G(B)$ is a subgroup of G.

Proposition 2.14. *Let $<_X$ be an isolated ordering on $X = G *_A H$ as in Theorem 1.3. If a convex subgroup B of A is both $(G, <_G)$- and $(H, <_H)$-strongly convex, then B is $(X, <_X)$-strongly convex. In particular, if B and B' are different convex subgroups, then $\mathsf{Conv}_X(B)$ and $\mathsf{Conv}_X(B')$ yield different convex subgroups of $(X, <_X)$.*

Proof. The case $B = \{1\}$ is trivial so we assume that $B \neq \{1\}$. By induction on N, we prove that if $x \in \mathsf{Conv}_X(B) \cap \mathcal{F}_N(X)$ then $xx' \in \mathsf{Conv}_X(B)$ for any $x' \in \mathsf{Conv}_X(B)$.

First assume that $x \in \mathcal{F}_{0.5}(X) = G \cup H$. For $x' \in \mathsf{Conv}_X(B)$, take $b \in B$ so that $b^{-1} <_X x' <_X b$. Then $xb^{-1} <_X xx' <_X xb$. Since B is $(G, <_G)$- and $(H, <_H)$-strongly convex, $xb, xb^{-1} \in \mathsf{Conv}_G(B) \cup \mathsf{Conv}_H(B) \subset \mathsf{Conv}_X(B)$, hence $xx' \in \mathsf{Conv}_X(B)$.

To show general case, assume that $x \in \mathcal{F}_N(X) - \mathcal{F}_{N-1}(X)$ and put $x = gy$ $(g \in G - A, y \in \mathcal{F}_{N-1}(X))$. We consider the case N is odd, since the case N is even is similar.

By Theorem 1.3 (3), A is a stepping so

$$a(y) = \max_{<_X}\{a \in A \colon a <_X y\}$$

exists. On the other hand, $x \in \mathsf{Conv}_X(B)$ so there exists $b \in B \subset A$ such that $b^{-1} <_X x <_X b$. By definition of $a(y)$,

$$b^{-1} \leq_X ga(y) <_X x <_X b$$

hence $ga(y) \in \mathsf{Conv}_X(B)$. We have assumed that B is a non-trivial convex subgroup of A, so $a_{\min} \in B$. Since $1 <_X a(y)^{-1}y <_X a_{\min}$, $a(y)^{-1}y \in \mathsf{Conv}_X(B)$. By induction, $(a(y)^{-1}y)x' \in \mathsf{Conv}_X(B)$ if $x' \in \mathsf{Conv}_X(B)$. This shows that

$$xx' = (ga(y))(a(y)^{-1}y)x' \in \mathsf{Conv}_X(B)$$

as desired. □

We close the paper by giving new examples of isolated orderings obtained by Theorem 1.3.

Example 2.15. Let B_3 be the 3-strand braid group

$$B_3 = \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z} = \langle x, y \colon x^2 = y^3 \rangle = \langle \sigma_1, \sigma_2 \colon \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

By Theorem 1.3, B_3 admits an isolated ordering $<_{DD}$, which is known as the Duborvina-Dubrovin ordering [3, 6]. The Dubrovina-Dubrovin ordering is discrete with minimum positive element σ_2. For $p > 1$, let $A = A_p$ be the kernel of the mod p abelianization map $e \colon B_n \to \mathbb{Z}/p\mathbb{Z}$. Since for $x \in B_3$

$$\cdots \prec_{DD} x\sigma_2^{-1} \prec_{DD} x \prec_{DD} x\sigma_2 \prec_{DD} x\sigma_2^2 \prec_{DD} \cdots,$$

A is a stepping with respect to $<_{DD}$. The maximum and minimum functions are given by

$$a(x) = x\sigma_2^{-e(x)}, \quad a_+(x) = x\sigma_2^{p-e(x)} \quad (e(x) \in \{0, 1, \dots, p-1\}).$$

By Theorem 1.3,

$$\begin{aligned} X &= X_p \\ &= \langle \sigma_1, \sigma_2 \colon \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle *_{A_p} \langle s_1, s_2 \colon s_1s_2s_1 = s_2s_1s_2 \rangle \\ &= B_3 *_{A_p} B_3 \end{aligned}$$

admits an isolated ordering $<_X$. (One can apply the same construction for the Dubrovina-Dubrovin ordering on B_n for $n > 3$ to get more examples of groups admitting isolated orderings.)

The convex subgroup B of A generated by $\sigma_2^p = s_2^p$, which is the minimal positive element of $<_{DD} |_A$, is $(B_3, <_{DD})$-strongly convex. Hence by Proposition 2.14, $\mathsf{Conv}_X(B)$ is a non-trivial proper convex subgroup of $(X, <_X)$.

On the other hand, By Theorem 1.3(3), the $<_X$-minimum positive element is $s_2^{1-p}\sigma_2$ hence $(X, <_X)$ contains another non-trivial proper convex subgroup generated by $s_2^{1-p}\sigma_2$. Thus $(X, <_X)$ has at least two non-trivial proper convex subgroup.

Iterating this kind of construction, starting from $\mathbb{Z}$ we are able to construct isolated ordering with arbitrary many proper non-trivial convex subgroups.

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Received November 19, 2014

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