

Decision problems, complexity, traces, and representations

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Abstract. In this article, we study connections between representation theory and efficient solutions to the conjugacy problem on finitely generated groups. The main focus is on the conjugacy problem in conjugacy separable groups, where we measure efficiency in terms of the size of the quotients required to distinguish a distinct pair of conjugacy classes.

Mathematics Subject Classification (2010). 20E26, 20E05, 20C99.

Keywords. Residually finite, conjugacy separable, character variety, trace equivalence, fully residually free.

1. Introduction

Given an infinite, finitely presented group Γ , two basic decision problems posed by Dehn [21] in 1911 are the word and conjugacy problems. In 1927, in solving the word problem for free groups, Schreier [61] proved that free groups are residual finite. That seems to be the first connection between decision problems and residual properties. In 1940, Mal'cev [47] proved that finitely presented, residually finite groups have a solution to the word problem, and noted a similar connection between the conjugacy problem and conjugacy separability in [48].

Once an algorithm to solve the word or conjugacy problem is given, one can study the efficiency of the algorithm. For free groups, it is straightforward to see that both problems have algorithms that terminate in linear steps as a function of word length via cyclic reduction. Bou-Rabee [8] introduced a function $F_\Gamma(n)$ that quantified the efficiency of the solution to the word problem on Γ given by residual finiteness. We say a group Γ is *residually finite* if for each $\gamma \in \Gamma$ with $\gamma \neq 1$, there exists a homomorphism $\varphi: \Gamma \rightarrow Q$ with $|Q| < \infty$ and $\varphi(\gamma) \neq 1$. The function introduced by Bou-Rabee measures the efficiency by the size of the groups Q needed over all the elements of length at most n in the verification of residual finiteness. Several papers have addressed the growth rate of this function for various classes of groups; [8], [9], [10], [11], [13], [14], [15], [17], [19], [35], [36], [37], [38], [51], [52], [59], and [67]. By work of Mal'cev [47], a finitely generated

linear group Γ is residually finite. In [15], using an effective proof of [47], it was shown that $F_\Gamma(n) \leq n^d$ where d depends only on a linear realization of Γ .

We say that Γ is *conjugacy separable* if for any non-conjugate $\gamma, \eta \in \Gamma$, there exists a homomorphism $\varphi: \Gamma \rightarrow Q$ with $\varphi(\gamma), \varphi(\eta)$ not conjugate in Q and $|Q| < \infty$. One of the goals of this article is to extend some of the above results with conjugacy separability in place of residual finiteness. Unfortunately, issues arise immediately. Stebe [65] proved that the linear groups $\mathrm{SL}(n, \mathbf{Z})$ are not conjugacy separable for $n > 2$. More generally, the groups of integer points of a semi-simple \mathbf{Q} -algebraic group with the congruence subgroup property are never conjugacy separable; see [54, Chapter 8]. However, free and surface groups [65, 46] (see also [50] and [69]), virtual polycyclic groups [23, 58], fundamental groups of compact, orientable 3-manifolds [31], and right-angled Artin groups [66] are conjugacy separable; see also [20] for more examples.

A faithful linear representation reduces the verification of the non-triviality of an element to showing some matrix coefficient is non-zero. We want a similar solution to the conjugacy problem through representation theory and must replace the coefficients of the matrix by conjugacy invariants. We use traces and the following properties to effectively distinguish conjugacy classes.

- (A) There exists an integer n and $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(n, \mathbf{C}))$ such that $\mathrm{Tr}(\rho(\gamma)) \neq \mathrm{Tr}(\rho(\eta))$ for any non-conjugate $\gamma, \eta \in \Gamma$.
- (B) For each $\gamma \in \Gamma$, there exists $\rho_\gamma \in \mathrm{Hom}(\Gamma, \mathrm{SL}(n_\gamma, \mathbf{C}))$ such that $\mathrm{Tr}(\rho_\gamma(\gamma)) \neq \mathrm{Tr}(\rho_\gamma(\eta))$ for every non-conjugate $\eta \in \Gamma$.
- (C) For any finite set $S = \{\gamma_i\}_{i=1}^s$ of conjugacy classes in Γ , there exists $\rho_S \in \mathrm{Hom}(\Gamma, \mathrm{SL}(n_S, \mathbf{C}))$ such that $\mathrm{Tr}(\rho_S(\gamma_i)) \neq \mathrm{Tr}(\rho_S(\gamma_j))$ for $\gamma_i, \gamma_j \in S$ and $i \neq j$.
- (D) For each non-conjugate $\gamma, \eta \in \Gamma$, there exists $\rho_{\gamma, \eta} \in \mathrm{Hom}(\Gamma, \mathrm{SL}(n_{\gamma, \eta}, \mathbf{C}))$ such that $\mathrm{Tr}(\rho_{\gamma, \eta}(\gamma)) \neq \mathrm{Tr}(\rho_{\gamma, \eta}(\eta))$.

We have (A) \implies (B) \implies (D) \iff (C). All of these implications are immediate from the definitions except for the equivalence of (C) and (D), which is elementary. We thank Greg Kuperberg for pointing that out to us. We say one of the above (B), (C), or (D) is *uniformly satisfied* if $n_\gamma, n_{\gamma, \eta}$, or n_S is bounded over all choices of $\gamma, \{\gamma, \eta\}$, or S . That is, the dimension of the representations do not depend on $\gamma, \{\gamma, \eta\}$, or S . In those cases, we say Γ *uniformly* has (B), (C), or (D). Note, it is less clear if uniform (C) and uniform (D) are equivalent.

Remark 1. Since SL_n and consequently $\mathrm{Hom}(\Gamma, \mathrm{SL}_n)$ are \mathbf{Z} -schemes, the above properties (A)–(D) can be restated with \mathbf{C} replaced by any algebraically closed field \mathbf{F} . When we are not working over \mathbf{C} we will refer to these properties as (A')–(D'). For example, with respect to a fixed algebraically closed field \mathbf{F} , (A') states there exists an integer n , and $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(n, \mathbf{F}))$ such that $\mathrm{Tr}(\rho(\gamma)) \neq \mathrm{Tr}(\rho(\eta))$ for any non-conjugate $\gamma, \eta \in \Gamma$. Properties (B')–(D') are similarly written.

Theorem 1.1. *If Γ uniformly has (C), then Γ has (A). In fact, if Γ uniformly has (D) for some n_0 and $\text{Hom}(\Gamma, \text{SL}(n_0, \mathbf{C}))$ is irreducible, then Γ has (A).*

Throughout, by a surface group, we mean the fundamental group of a closed, orientable surface of genus $g \geq 2$. We have the following corollary:

Corollary 1.2. *If Γ is either a finitely generated free group or a surface group, then Γ uniformly has (D) if and only if Γ has (A). Moreover, for any connected algebraic subgroup $\mathbf{G} < \text{SL}(n, \mathbf{C})$, the following are equivalent for a free group F_r of rank r :*

- (a) *for each $\rho \in \text{Hom}(F_r, \mathbf{G})$, there exist non-conjugate $\gamma, \eta \in F_r$ with $\text{Tr}(\rho(\gamma)) = \text{Tr}(\rho(\eta))$;*
- (b) *there exist non-conjugate $\gamma, \eta \in \Gamma$ such that $\text{Tr}(\rho(\gamma)) = \text{Tr}(\rho(\eta))$ for each $\rho \in \text{Hom}(F_r, \mathbf{G})$.*

We also record the following result which first appeared in Bass and Lubotzky [2, Proposition 3.1] where they also prove the converse.

Proposition 1.3 (Bass and Lubotzky). *If Γ satisfies (D), then Γ is conjugacy separable.*

Similar to the function $F_\Gamma(n)$ associated to the word problem using residual finiteness, we define a function $\text{Conj}_\Gamma(n)$ for the conjugacy problem using conjugacy separability (see §2 for the definition).

Theorem 1.4. *If Γ has (A), then $\text{Conj}_\Gamma(n) \leq n^d$ for some $d \in \mathbf{N}$. Moreover, for some $n_0 \in \mathbf{N}$, the finite quotients used in proving conjugacy separability of Γ are subgroups of the finite groups $\text{SL}(n_0, \mathbf{F}_p)$ where \mathbf{F}_p denotes a field of prime order p .*

We define a relative version of the function $\text{Conj}_\Gamma(n)$ by fixing a conjugacy class $[\gamma]$ in Γ and denote this function by $\text{Conj}_{\Gamma, \gamma}(n)$. The analog of Theorem 1.4 holds with (B) and $\text{Conj}_{\Gamma, \gamma}(n)$ in place of (A) and $\text{Conj}_\Gamma(n)$.

Theorem 1.5. *If Γ has (B), then for each $\gamma \in \Gamma$, there exists $d_\gamma \in \mathbf{N}$ such that $\text{Conj}_{\Gamma, \gamma}(n) \leq n^{d_\gamma}$.*

Property (A). We now address the likelihood a group satisfies (A) or (B). We begin with (A). The obvious test case to begin investigating with regard to property (A) is finitely generated free groups. For $n = 2$, Horowitz [32] proved that there exist non-conjugate $\gamma, \eta \in F_2$ such that for any representation $\rho: F_2 \rightarrow \text{SL}(2, \mathbf{C})$, we have $\text{Tr}(\rho(\gamma)) = \text{Tr}(\rho(\eta))$. We say such elements are SL_2 -trace equivalent. It seems to have been, for some time now, a folklore question

as to whether or not there exists SL_n -trace equivalent elements in F_2 for $n > 2$. In Section 4, we discuss whether or not the elements constructed by Horowitz can be SL_n -trace equivalent, and see that if they are, an unexpected trace relation must hold. Ginzburg and Rudnick [25] investigated when a given element has a SL_2 -trace companion and gave a conjectural condition on the element to ensure that it does not have such a companion. Anderson [1] provided a broader context for the construction of Horowitz and a conjectural picture for what such pairs of SL_2 -trace equivalent elements should look like. Additionally, Leininger [43] and Kapovich, Levitt, Schupp, and Shpilrain [34] gave a more geometric/topological take (see also [25], [40], [41], [42]). Of course, we have trivially that any SL_3 -trace equivalent pair is also an SL_2 -trace equivalent pair. The failure of Anderson's general construction to produce SL_3 -trace equivalent pairs would provide some evidence that free groups have (A).

The most compelling evidence against free groups having (A) is Theorem 1.4. By [67] and [8], the function $F_{F_r}(n)$ satisfies $n(\log \log(n))^{9/2}/(\log(n))^2 \leq F_{F_r}(n) \leq n^3$. We believe that the growth rate of $\text{Conj}_{F_r}(n)$ should be much greater since conjugacy separability requires separating a fixed element γ from an infinite set while residual finiteness requires only separating γ from the trivial element. It is this reason why many linear groups are not conjugacy separable. However, if F_r has (A), then by Theorem 1.4, we would have, for some fixed d , the asymptotic inequalities $\text{Conj}_{F_r}(n) \leq n^d$. In particular, $\text{Conj}_{F_r}(n) \leq (F_{F_r}(n))^{3d}$. For any finitely generated abelian group, these two functions are the same, and the best setting to hope for a power relationship like $\text{Conj}_\Gamma(n) \leq (F_\Gamma(n))^d$ is the class of torsion free, finitely generated nilpotent groups where conjugacy classes are relatively small. However, by [8] and [53], a torsion free, finitely generated nilpotent group satisfies such a power relationship if and only if the group is virtually abelian.

Property (B). Following a construction of Wehrfritz [68] for free groups, we can prove that finitely generated free groups and surface groups have (B).

Theorem 1.6. *If Γ is a finitely generated free group or surface group, then Γ has (B).*

From Theorem 1.5 and Theorem 1.6, we obtain:

Corollary 1.7. *If Γ is a finitely generated free group or surface group and $\gamma \in \Gamma$, then there exists $d_\gamma \in \mathbf{N}$ such that $\text{Conj}_{\Gamma, \gamma}(n) \leq n^{d_\gamma}$. Moreover, one can take $d_\gamma \approx \|\gamma\|^2$ and thus $\text{Conj}_\Gamma(n) \leq n^{n^2}$.*

The degree d_γ in Corollary 1.7 is directly related to the smallest index of a finite index subgroup Γ of F_r where $\gamma \in \Gamma$ is primitive. In the case of surface groups, it is directly related to the smallest degree of a cover where the curve associated to γ has a lift to a simple closed curve. Patel [51] and Gupta and Kapovich [28] have given upper bounds in the case of surface groups and free groups, respectively, on the order of order $\|\gamma\|$. Gaster [24], improving on work of Gupta and Kapovich [28], has shown that there exist γ that require a cover of degree on the order of $\|\gamma\|$. We conjecture that there is no polynomial upper bound for $\text{Conj}_{F_r}(n)$, and coupled with Theorem 1.4, that would imply that free groups do not have (A).

Conjecture 1. *Finitely generated free groups do not have (A).*

Finally, we prove a result that shows that for fully residually free groups, one can recover the profinite topology via the topology generated by the Zariski topologies for faithful representations into $\text{SL}(n, \mathbf{C})$ as we vary over all $n \in \mathbf{N}$. Recall that Γ is *fully residually free* if for each finite subset $S \subset \Gamma$ of non-trivial elements, there exists $r_S \in \mathbf{N}$ and a homomorphism $\psi_S: \Gamma \rightarrow F_{r_S}$ such that the restriction of ψ_S to S is injective. Examples of fully residually free groups are free groups and surface groups.

Theorem 1.8. *Let Γ be a fully residually free group, Δ a finite index, normal subgroup of Γ , and $p \in \mathbf{N}$ a prime. Then there exists an integral domain $R \subset \mathbf{C}$, an ideal $\mathfrak{m} \subset R$ with $R/\mathfrak{m} = \mathbf{F}_p$, and a faithful homomorphism $\rho: \Gamma \rightarrow \text{SL}(n_\Delta, R)$ such that $\Delta = \ker(r_\mathfrak{m} \circ \rho)$ where $r_\mathfrak{m}: \text{SL}(n_\Delta, R) \rightarrow \text{SL}(n_\Delta, \mathbf{F}_p)$ is the reduction modulo \mathfrak{m} homomorphism and $n_\Delta = 2[\Gamma : \Delta]$.*

The ring R can be taken to be finitely generated over \mathbf{Z} (see Remark 2), and when Γ is a free group, we can take $R = \mathbf{Z}$ (see Remark 3). When Γ is an arithmetic lattice in a \mathbf{Q} -algebraic group \mathbf{G} , the congruence subgroup property asserts that every finite index subgroup $\Lambda < \Gamma$ contains $\ker(r_m)$ for some integer $m \in \mathbf{N}$. Every non-abelian free group F_r can be realized as a finite index subgroup of $\text{SL}(2, \mathbf{Z})$ and it is well-known that $\text{SL}(2, \mathbf{Z})$ does not have the congruence subgroup property. The above result provides a weaker property than the congruence subgroup property when Γ is a limit group. As we mentioned above, lattices in semi-simple Lie groups with the congruence subgroup property are not conjugacy separable and so do not have (D). These groups are super-rigid and the Zariski topology associated to the standard representation, which is the congruence topology, is too coarse for separating conjugacy classes. That free groups and surface groups are conjugacy separable is due to their much richer representation theory. We believe conjugacy separability requires linear representations of arbitrarily large dimension or finite quotients with arbitrarily large representation dimension. If Conjecture 1 is false, then free groups would be conjugacy

separable via the Zariski topology associated to a fixed finite dimensional representation. In fact, for sufficiently large n and a generic (in the Baire Category sense) $\rho \in \text{Hom}(F_r, \text{SL}(n, \mathbf{C}))$, every conjugacy class in F_r would be closed in the Zariski topology associated to ρ .

Acknowledgements. The authors would like to thank Nigel Boston, Khalid Bou-Rabee, Frank Calegari, Ted Chinburg, Kelly Delp, Nathan Dunfield, Patrick Eberlin, Carolyn Gordon, Adrian Ioana, Mike Jablonski, Ilya Kapovich, Greg Kuperberg, Chris Leininger, Darren Long, Alex Lubotzky, Dave Morris, Alan Reid, Mark Sapir, Juan Souto, and Pete Storm for stimulating conversations on the topics of this article. Special thanks is given to Dunfield who many years ago posed the question to the third author on the possibility of generalizing Horowitz's construction. The first author was partially supported by NSF grant 1309376 and Simons grant 245642. The third author was partially supported by NSF grant 1105710. Lastly, we want to thank an anonymous referee for very good remarks that helped improve the paper.

2. Preliminaries

2.1. Quantitative separability functions. Given $f, g: \mathbf{N} \rightarrow \mathbf{N}$, we say $f \leq g$, if $f(n) \leq Cg(Cn)$ for some constant $C \in \mathbf{N}$ and for all $n \in \mathbf{N}$. If $f \leq g$ and $g \leq f$, we write $f \approx g$. Throughout, Γ will denote an infinite, finitely generated group unless stated otherwise. Given a finite generating set X of Γ and $\gamma \in \Gamma$, we denote the word length of γ with respect to X by $\|\gamma\|_X$ (or simply $\|\gamma\|$) and n -ball with respect to the associated word metric by $B_{\Gamma, X}(n)$. Given Γ , we define $D_{\Gamma}: \Gamma - \{1\} \rightarrow \mathbf{N} \cup \{\infty\}$ by

$$D_{\Gamma}(\gamma) = \min\{[\Gamma : \Delta]: \gamma \notin \Delta, \Delta \triangleleft \Gamma\}$$

and $F_{\Gamma, X}(n)$ by

$$F_{\Gamma, X}(n) = \max_{\gamma \in B_{\Gamma, X}(n) - \{1\}} D_{\Gamma}(\gamma).$$

For any two finite generating sets X, Y , we have $F_{\Gamma, X} \approx F_{\Gamma, Y}$ (see [8, Lemma 1.1]). Consequently, we suppress the dependence on X in our notation. For a finitely generated group Γ and $\gamma \in \Gamma$, we denote the Γ -conjugacy class of γ by $[\gamma]$ and denote the set of Γ -conjugacy classes by C_{Γ} . For $[\gamma] \in C_{\Gamma}$, we define $\|[\gamma]\|_X = \min\{\|\gamma'\|_X : \gamma' \in [\gamma]\}$, and $\text{CD}_{\Gamma}: C_{\Gamma} \times C_{\Gamma} \rightarrow \mathbf{N} \cup \{\infty\}$ by

$$\text{CD}_{\Gamma}([\gamma], [\eta]) = \min\{|Q|: \varphi: \Gamma \longrightarrow Q, [\varphi(\gamma)]_Q \neq [\varphi(\eta)]_Q\}.$$

By definition, for $\gamma, \eta \in \Gamma$ with $[\gamma] \neq [\eta]$, we have

$$\text{CD}_{\Gamma}([\gamma], [\eta]) \geq \max\{D_{\Gamma}(\gamma^{-1}\eta'): \eta' \in [\eta]\}.$$

We define

$$B_X(C_\Gamma, n) = \{[\gamma]: \|\gamma\|_X \leq n\}$$

and

$$\text{Conj}_{\Gamma, X}: \mathbf{N} \longrightarrow \mathbf{N} \cup \{\infty\}$$

via

$$\text{Conj}_{\Gamma, X}(n) = \max_{[\gamma], [\eta] \in B_X(C_\Gamma, n), [\gamma] \neq [\eta]} \text{CD}_\Gamma([\gamma], [\eta]).$$

For $[\gamma] \in C_\Gamma$, we define

$$\text{CD}_{\Gamma, \gamma}: C_\Gamma - \{[\gamma]\} \longrightarrow \mathbf{N} \cup \{\infty\}$$

to be

$$\text{CD}_{\Gamma, \gamma}([\eta]) = \text{CD}_\Gamma([\gamma], [\eta])$$

and

$$\text{Conj}_{\Gamma, \gamma, X}(n) = \max_{[\eta] \in B_X(C_\Gamma, n), [\eta] \neq [\gamma]} \text{CD}_{\Gamma, \gamma}([\eta]).$$

For any two finite generating sets X, Y of Γ , we have $\text{Conj}_{\Gamma, X}(n) \approx \text{Conj}_{\Gamma, Y}(n)$ and $\text{Conj}_{\Gamma, \gamma, X}(n) \approx \text{Conj}_{\Gamma, \gamma, Y}(n)$. The proof is similar to the proof of the comparable statement for the function $F_\Gamma(n)$; see [8, Lemma 1.1]. As a result, we suppress the dependence on the generating set X in our notation.

2.2. Representation varieties. We refer the reader to [18, §5], [26, §2], and [56, Chapter V] for the material in this subsection. If G is a Lie (resp. algebraic) group and $\Gamma = F_r$, then $\text{Hom}(F_r, G) = G^r$ is an analytic (resp. algebraic) variety. More generally, when Γ is finitely generated, $\text{Hom}(\Gamma, G)$ will be an analytic (resp. algebraic) subvariety of $\text{Hom}(F_r, G)$ for some r ; see [18, §5]. For each $\gamma \in \Gamma$, we have an analytic (resp. algebraic) function $\text{Hom}(\Gamma, G) \rightarrow G$ given by $\rho \mapsto \rho(\gamma)$. If $G < \text{GL}(n, \mathbf{C})$, the function $\text{Tr}_\gamma: \text{Hom}(\Gamma, G) \rightarrow \mathbf{C}$ given by $\text{Tr}_\gamma(\rho) = \text{Tr}(\rho(\gamma))$ is analytic (resp. algebraic). When G is a K -algebraic group with K a characteristic zero field, $\text{Hom}(\Gamma, G)$ is a K -algebraic set (not necessarily irreducible or connected), and so has finitely many irreducible (and connected) components. In particular, for $G = \text{SL}(n, \mathbf{C})$, the space $\text{Hom}(\Gamma, \text{SL}(n, \mathbf{C}))$ is a complex algebraic variety with finitely many irreducible components. For a connected, reductive algebraic group \mathbf{G} , the \mathbf{G} -character variety $\mathfrak{X}(\Gamma, \mathbf{G})$ is the GIT quotient of $\text{Hom}(\Gamma, \mathbf{G})$ by the \mathbf{G} -conjugation action, and for $\Gamma = F_r$, we set $\mathfrak{X}_r(\mathbf{G}) = \mathfrak{X}(F_r, \mathbf{G})$. Though we do not require it here, we include the following result on algebraic points of character varieties that we could not find explicitly in the literature (it is implicit in [56, Proposition 6.6]).

Theorem 2.1. *If \mathbf{G} is a connected, reductive, affine algebraic group, then $\mathrm{Hom}(\Gamma, \mathbf{G}(\overline{\mathbf{Q}}))$ is classically dense in $\mathrm{Hom}(\Gamma, \mathbf{G}(\mathbf{C}))$, and $\mathfrak{X}(\Gamma, \mathbf{G}(\overline{\mathbf{Q}}))$ is classically dense in $\mathfrak{X}(\Gamma, \mathbf{G}(\mathbf{C}))$.*

Proof. First note that for any d -dimensional affine variety V defined over \mathbf{Q} the Noether normalization map $V \rightarrow \mathbf{A}^d$ is surjective and defines a finite cover off its branch locus. Since the branch locus is nowhere dense, the $\overline{\mathbf{Q}}$ -points are both Zariski and classically dense in the \mathbf{C} -points of V . According to [7, p. 220], $\mathbf{G}(K)$ is Zariski dense in \mathbf{G} for any infinite subfield $K \subset \mathbf{C}$. Since \mathbf{G} is defined over \mathbf{Q} and the relations in Γ are defined over \mathbf{Z} , $\mathrm{Hom}(\Gamma, \mathbf{G})$ is an affine variety defined over \mathbf{Q} and

$$\mathbf{C}[\mathrm{Hom}(\Gamma, \mathbf{G})] = \mathbf{Q}[\mathrm{Hom}(\Gamma, \mathbf{G})] \otimes_{\mathbf{Q}} \mathbf{C}. \quad (1)$$

Hence, $\mathrm{Hom}(\Gamma, \mathbf{G}(\overline{\mathbf{Q}}))$ is both Zariski and classically dense in $\mathrm{Hom}(\Gamma, \mathbf{G}(\mathbf{C}))$. Let f_1, \dots, f_N be a set of generators for $\mathbf{C}[\mathrm{Hom}(\Gamma, \mathbf{G})]^{\mathbf{G}}$, and define

$$f: \mathrm{Hom}(\Gamma, \mathbf{G}) \longrightarrow \mathbf{C}^N$$

by

$$f(g_1, \dots, g_r) = (f_1(g_1, \dots, g_r), \dots, f_N(g_1, \dots, g_r)).$$

Since $\mathfrak{X}(\Gamma, \mathbf{G}) = \mathrm{Spec}(\mathbf{C}[\mathrm{Hom}(\Gamma, \mathbf{G})]^{\mathbf{G}})$, we have $\mathfrak{X}(\Gamma, \mathbf{G}) = f(\mathrm{Hom}(\Gamma, \mathbf{G}))$; see [62] for example. As $\mathbf{C}[\mathrm{Hom}(\Gamma, \mathbf{G})]^{\mathbf{G}} \subset \mathbf{C}[\mathrm{Hom}(\Gamma, \mathbf{G})]$, Equation (1) implies that f_1, \dots, f_N may be chosen to have \mathbf{Q} -coefficients. Thus,

$$f(\mathrm{Hom}(\Gamma, \mathbf{G}(\overline{\mathbf{Q}}))) \subset \mathfrak{X}(\Gamma, \mathbf{G}(\overline{\mathbf{Q}})).$$

As f is a continuous surjective function, we conclude that $f(\mathrm{Hom}(\Gamma, \mathbf{G}(\overline{\mathbf{Q}})))$ is classically dense in $\mathfrak{X}(\Gamma, \mathbf{G}(\mathbf{C}))$. Hence, $\mathfrak{X}(\Gamma, \mathbf{G}(\overline{\mathbf{Q}}))$ is classically dense in $\mathfrak{X}(\Gamma, \mathbf{G}(\mathbf{C}))$ as it contains $f(\mathrm{Hom}(\Gamma, \mathbf{G}(\overline{\mathbf{Q}})))$. \square

Corollary 2.2. *If $\mathbf{G} = \mathrm{SL}(n, \mathbf{C})$, the integral points are infinite in $\mathfrak{X}_r(\mathbf{G})$.*

Proof. For $\mathbf{G} = \mathrm{SL}(n, \mathbf{C})$, the group schemes and invariant rings in the above proof are defined over $\mathbf{Z}[1/n]$. So, the result follows from the above proof noting that $\mathrm{Hom}(F_r, \mathbf{G}) \cong \mathbf{G}^r$. \square

From the work of Long and Reid [44], one can infer that Corollary 2.2 is false for $\mathrm{SL}(2, \mathbf{C})$ when Γ is a surface group.

3. Property (C): proof of Theorem 1.1 and Proposition 1.3

3.1. Proof of Theorem 1.1 and Corollary 1.2. We now prove that either uniform (C), or uniform (D) with the irreducibility of $\mathrm{Hom}(\Gamma, \mathrm{SL}(n, \mathbf{C}))$ imply property (A).

Proof of Theorem 1.1. We assume first that Γ uniformly has (C). We enumerate the conjugacy classes of Γ by $\{[\gamma_1], [\gamma_2], \dots\}$ and for each $j \in \mathbf{N}$, set $S_j = \{[\gamma_i]\}_{i=1}^j$. By assumption, there exists $n \in \mathbf{N}$ and for each r , we have a representation $\rho_r: \Gamma \rightarrow \mathrm{SL}(n, \mathbf{C})$ such that $\mathrm{Tr}(\rho(\gamma_i)) \neq \mathrm{Tr}(\rho(\gamma_j))$ for all $i \neq j \leq r$. As $\mathrm{Hom}(\Gamma, \mathrm{SL}(n, \mathbf{C}))$ has only finitely many irreducible components, there exists a component that contains infinitely many of the representations ρ_r , say $V_0 \subset \mathrm{Hom}(\Gamma, \mathrm{SL}(n, \mathbf{C}))$. By selection, the trace functions Tr_γ restricted to V_0 are distinct algebraic functions for each conjugacy class $[\gamma]$. In particular, $\mathrm{Tr}_{\gamma_i} - \mathrm{Tr}_{\gamma_j} \neq 0$ is a non-constant algebraic function on V_0 for each pair $i \neq j$. In particular, the sets

$$Z_{i,j} = \{\rho \in V_0: \mathrm{Tr}_{\gamma_i}(\rho) - \mathrm{Tr}_{\gamma_j}(\rho) = 0\}$$

are proper algebraic subvarieties of V_0 . By the Baire Category Theorem, $V = V_0 - \bigcup_{i,j} Z_{i,j}$ is dense and so non-empty. By construction, any $\rho \in V$ has the property that $\mathrm{Tr}(\rho(\gamma)) = \mathrm{Tr}(\rho(\eta))$ if and only if γ, η are conjugate in Γ . In particular, Γ has property (A).

In the case we uniformly have (D) and $\mathrm{Hom}(\Gamma, \mathrm{SL}(n, \mathbf{C}))$ is irreducible, we know that by assumption that for each pair of conjugacy classes $\gamma, \eta \in \Gamma$, we have a representation $\rho: \Gamma \rightarrow \mathrm{SL}(n, \mathbf{C})$ with $\mathrm{Tr}(\rho(\gamma)) \neq \mathrm{Tr}(\rho(\eta))$. Since $\mathrm{Hom}(\Gamma, \mathrm{SL}(n, \mathbf{C}))$ is irreducible, we can proceed as before with $V_0 = \mathrm{Hom}(\Gamma, \mathrm{SL}(n, \mathbf{C}))$. □

Before we prove Corollary 1.2, we note that in the special case of the genus 1 surface, the fundamental group \mathbf{Z}^2 has (A). Take any two algebraically independent numbers $\alpha, \beta \in \mathbf{R}$. Fixing a \mathbf{Z} -basis v, w , we have the representation

$$\rho: \mathbf{Z}^2 \longrightarrow \mathrm{GL}(2, \mathbf{R})$$

given by

$$\rho(av + bw) = \begin{pmatrix} \alpha^a & 0 \\ 0 & \beta^b \end{pmatrix}.$$

By selection of α, β , distinct elements in \mathbf{Z}^2 will have distinct traces. The groups \mathbf{Z}^n also have (A) for any $n \in \mathbf{N}$.

Proof of Corollary 1.2. The first part of Corollary 1.2 follows immediately from the irreducibility of $\mathrm{Hom}(F_r, G) = G^r$ for any connected algebraic group over \mathbf{C} in the case of free groups. For a closed, orientable surface Σ_g of genus $g \geq 2$, $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SL}(n, \mathbf{C}))$ is irreducible by [57] and [4, Lemma 2.5] (the same holds for $g = 1$; see [22, Proposition 5.16]). For the second part, we must prove that the following two statements are equivalent:

- (a) there exists non-conjugate $\gamma, \eta \in F_r$ that have $\text{Tr}(\rho(\gamma)) = \text{Tr}(\rho(\eta))$ for every $\rho \in \text{Hom}(F_r, G)$;
- (b) for each $\rho \in \text{Hom}(F_r, G)$, there exist non-conjugate $\gamma, \eta \in F_r$ such that $\text{Tr}(\rho(\gamma)) = \text{Tr}(\rho(\eta))$.

It is clear that (a) implies (b). To prove that (b) implies (a), we assume that (b) holds but not (a) and derive a contradiction. Since (a) does not hold, then for each non-conjugate pair $\gamma, \eta \in F_r$, the function $\text{Tr}_\gamma - \text{Tr}_\eta$ on $\text{Hom}(F_r, G)$ is a non-constant algebraic function. Since $\text{Hom}(F_r, G)$ is irreducible,

$$V_{\gamma, \eta} = \{\rho \in \text{Hom}(F_r, G) : \text{Tr}_\gamma(\rho) - \text{Tr}_\eta(\rho) = 0\}$$

is nowhere dense. Taking $V = \bigcup_{[\gamma] \neq [\eta]} V_{\gamma, \eta}$, by the Baire Category Theorem, V is nowhere dense. Let $\rho \in \text{Hom}(F_r, G) - V$ and note that by construction, no two non-conjugate elements have the same trace under ρ . That contradicts our assumption that (b) holds for every $\rho \in \text{Hom}(F_r, G)$. \square

3.2. Proof of Proposition 1.3. The proof of Proposition 1.3 is similar to Mal'cev's proof of residual finiteness for linear groups. As we will use some of the setup later, we give a proof here. A proof can also be found in [2].

Proof of Proposition 1.3. Given non-conjugate $\gamma, \eta \in \Gamma$, we must find a homomorphism $\varphi: \Gamma \rightarrow Q$ where Q is a finite group such that $\varphi(\gamma), \varphi(\eta)$ are not conjugate in Q . By assumption, Γ has Property (D) and so there exists $\rho \in \text{Hom}(\Gamma, \text{SL}(n, \mathbf{C}))$ such that $\text{Tr}(\rho(\gamma)) \neq \text{Tr}(\rho(\eta))$. Since Γ is finitely generated, the field L generated over \mathbf{Q} by the coefficients of the elements $\rho(\lambda)$ as we vary over all $\lambda \in \Gamma$ has the form $L = K(x_1, \dots, x_r)$, where K/\mathbf{Q} is a finite extension and x_1, \dots, x_r are indeterminants. It follows that $\rho(\Gamma) < \text{SL}(n, R)$, where $R = S[1/\beta_1, \dots, 1/\beta_r]$, $S = \mathcal{O}_K[x_1, \dots, x_r]$, and \mathcal{O}_K is the ring of K -integers. We see then that $\text{Tr}(\rho(\lambda)) \in R$ for each $\lambda \in \Gamma$. We know that $\text{Tr}(\rho(\gamma)) - \text{Tr}(\rho(\eta)) = F(x_1, \dots, x_{r'}) \in R$ is a non-zero polynomial in the variables $x_1, \dots, x_{r'}$ with coefficients in S . Since F is non-zero, we can find $\alpha_1, \dots, \alpha_{r'} \in S$ such that $\alpha = F(\alpha_1, \dots, \alpha_{r'}) \neq 0$ with $\alpha \in S$. As there are only finitely many prime ideals \mathfrak{p} in S such that $\alpha \equiv 0 \pmod{\mathfrak{p}}$, we select a prime \mathfrak{p} for which $\alpha \not\equiv 0 \pmod{\mathfrak{p}}$. For such a prime, the ring homomorphisms $R \rightarrow S \rightarrow S/\mathfrak{p} \cong \mathbf{F}_q$ induce homomorphisms $\Gamma \rightarrow \text{SL}(n, R) \rightarrow \text{SL}(n, S) \rightarrow \text{SL}(n, \mathbf{F}_q)$. Set $\varphi: \Gamma \rightarrow \text{SL}(n, \mathbf{F}_q)$ to be the resulting map. By construction $\text{Tr}(\varphi(\gamma)) \neq \text{Tr}(\varphi(\eta))$ and so $\varphi(\gamma), \varphi(\eta)$ are not conjugate in $\text{SL}(n, \mathbf{F}_q)$. \square

3.3. Ultraproducts. For a fixed $n \in \mathbf{N}$, we say that a group Γ is *n-trace distinguished* if for each non-conjugate pair $\gamma, \eta \in \Gamma$, there exists a finite field \mathbf{F}_q and a homomorphism $\varphi: \Gamma \rightarrow \text{SL}(n, \mathbf{F}_q)$ such that $\text{Tr}(\varphi(\gamma)) \neq \text{Tr}(\varphi(\eta))$. We say Γ is *fully n-trace distinguished* if for any finite set $S = \{\gamma_j\}_{j=1}^s \subset \Gamma$ of pairwise non-conjugate elements, there exists a finite field \mathbf{F}_q and a homomorphism $\varphi: \Gamma \rightarrow \text{SL}(n, \mathbf{F}_q)$ such that $\text{Tr}(\varphi(\gamma_i)) \neq \text{Tr}(\varphi(\gamma_j))$ for all $1 \leq i < j \leq s$.

Theorem 3.1. *If Γ is finitely generated and fully n -trace distinguished for some $n \in \mathbf{N}$, then Γ has (A).*

In the proof of Theorem 3.1, we employ ultraproducts. We refer the reader to [29] for an introduction to these methods.

Proof. To begin, we enumerate the conjugacy classes of Γ by $\{[\gamma_1], [\gamma_2], \dots\}$ and for each $j \in \mathbf{N}$, set $S_j = \{[\gamma_i]\}_{i=1}^j$. By assumption, for each $j \in \mathbf{N}$, there exists a finite field \mathbf{F}_{q_j} and a homomorphism $\varphi_j: \Gamma \rightarrow \mathrm{SL}(n, \mathbf{F}_{q_j})$ such that $\mathrm{Tr}(\varphi_j(\gamma_i)) \neq \mathrm{Tr}(\varphi_j(\gamma_{i'}))$ for all $1 \leq i < i' \leq j$. Picking a non-principal ultrafilter ω on \mathbf{N} , the ultraproduct $\prod_{\omega} \mathbf{F}_{q_j} = \mathbf{K}_{\omega}$ is a field and we have an induced homomorphism $\prod_{\omega} \varphi_j = \Phi_{\omega}$, where $\Phi_{\omega}: \Gamma \rightarrow \mathrm{SL}(n, \mathbf{K}_{\omega})$. By selection of the homomorphisms φ_j , it follows that $\mathrm{Tr}(\Phi_{\omega}(\gamma_i)) \neq \mathrm{Tr}(\Phi_{\omega}(\gamma_{i'}))$ for all $i \neq i'$. Hence, Γ has (A). \square

The field \mathbf{K}_{ω} may have positive characteristic and so in the definition of (A), we must allow for algebraically closed fields of positive characteristic (see Remark 1). Using the methods from the proof of Proposition 1.3, it is straightforward to see that if Γ has (A'), then Γ is fully n -trace distinguished.

We can also consider a relative version of n -trace distinguished. For $\gamma \in \Gamma$ and $n \in \mathbf{N}$, we say γ is *n -trace distinguished in Γ* if for each non-conjugate $\eta \in \Gamma$, there exists a finite field \mathbf{F}_q and a homomorphism $\varphi: \Gamma \rightarrow \mathrm{SL}(n, \mathbf{F}_q)$ such that $\mathrm{Tr}(\varphi(\gamma)) \neq \mathrm{Tr}(\varphi(\eta))$. We say γ is *fully n -trace distinguished in Γ* if for any finite set $S = \{\gamma_j\}_{j=1}^s \subset \Gamma$, none of which is conjugate to γ , there exists a finite field \mathbf{F}_q and a homomorphism $\varphi: \Gamma \rightarrow \mathrm{SL}(n, \mathbf{F}_q)$ such that $\mathrm{Tr}(\varphi(\gamma)) \neq \mathrm{Tr}(\varphi(\gamma_j))$ for all $1 \leq j \leq s$.

Theorem 3.2. *If Γ is finitely generated and for each $\gamma \in \Gamma$, there exists $n_{\gamma} \in \mathbf{N}$ such that γ is fully n_{γ} -trace distinguished, then Γ has (B).*

Proof. To begin, we enumerate the conjugacy classes of Γ by $\{[\gamma_1] = [\gamma], [\gamma_2], [\gamma_3], \dots\}$ and for each $j \in \mathbf{N}$, set $S_j = \{[\gamma_i]\}_{i=2}^j$. By assumption, for each $j \geq 2$, there exists a finite field \mathbf{F}_{q_j} and a homomorphism $\varphi_j: \Gamma \rightarrow \mathrm{SL}(n, \mathbf{F}_{q_j})$ such that $\mathrm{Tr}(\varphi_j(\gamma_i)) \neq \mathrm{Tr}(\varphi_j(\gamma))$ for all $2 \leq i \leq j$. Picking a non-principal ultrafilter ω on \mathbf{N} , the ultraproduct $\prod_{\omega} \mathbf{F}_{q_j} = \mathbf{K}_{\omega}$ is a field and we have an induced homomorphism $\prod_{\omega} \varphi_j = \Phi_{\omega}$, where $\Phi_{\omega}: \Gamma \rightarrow \mathrm{SL}(n, \mathbf{K}_{\omega})$. By selection of the homomorphisms φ_j , it follows that $\mathrm{Tr}(\Phi_{\omega}(\gamma_i)) \neq \mathrm{Tr}(\Phi_{\omega}(\gamma))$ for all $i \geq 2$. Hence, Γ has (B). \square

As before, allowing for algebraically closed fields of positive characteristic in our definition of (B), the converse holds assuming (B').

3.4. Proof of Theorem 1.4 and Theorem 1.5.

We now prove Theorem 1.4.

Proof of Theorem 1.4. We assume that Γ has (A) for some integer $m \in \mathbf{N}$, and so there exists $\rho \in \text{Hom}(\Gamma, \text{SL}(m, \mathbf{C}))$ such that $\text{Tr}(\rho(\gamma)) \neq \text{Tr}(\rho(\eta))$ for any non-conjugate $\gamma, \eta \in \Gamma$. For simplicity, we assume that $\rho(\Gamma) < \text{SL}(m, \overline{\mathbf{Q}})$, as the alternative $\rho(\Gamma) < \text{SL}(m, K[x_1, \dots, x_r])$, where K/\mathbf{Q} is a finite extension, is handled similarly (see [15]). We must prove that for any non-conjugate pair $\gamma, \eta \in \Gamma$ with $\|\gamma\|, \|\eta\| \leq n$, that $\text{CD}_\Gamma(\gamma, \eta) \leq Cn^{m^2-1}$ for a constant C that is independent of γ, η . To begin, we can find finite extensions K/\mathbf{Q} and S/\mathcal{O}_K such that $\rho(\Gamma) < \text{SL}(m, S)$. With this setup, we know for any non-conjugate γ, η that $\text{Tr}(\rho(\gamma)) - \text{Tr}(\rho(\eta)) \in S$ and also is non-zero. We need an ideal \mathfrak{a} of S such that $\text{Tr}(\rho(\gamma)) - \text{Tr}(\rho(\eta)) \not\equiv 0 \pmod{\mathfrak{a}}$ and with $|S/\mathfrak{a}|$ small. We achieve this goal using the methods of [8] (or [15]). First, we control the size of the coefficients of $\rho(\gamma), \rho(\eta)$ as a function of word length. To that end, it follows (see [8] or [15]) that there exists constants α and C_0 depending only on the generators of Γ such that

$$\max\{ |(\rho(\gamma))_{i,j}| : i, j \in \{1, \dots, m\} \} \leq \alpha^{C_0 \|\gamma\|}.$$

In particular, given non-conjugate $\gamma, \eta \in \Gamma$ with $\|\gamma\|, \|\eta\| \leq n$, we see that

$$|\text{Tr}(\rho(\gamma)) - \text{Tr}(\rho(\eta))| \leq |\text{Tr}(\rho(\gamma))| + |\text{Tr}(\rho(\eta))| \leq 2m\alpha^{C_0 n}.$$

By [8, Theorem 2.4], we can find a prime ideal \mathfrak{p} with

$$|S/\mathfrak{p}| \leq C_1 \log(C_1 2m\alpha^{C_0 n}) \leq C_1 C_0 n \log(C_1 2m\alpha)$$

such that $\text{Tr}(\rho(\gamma)) \not\equiv \text{Tr}(\rho(\eta)) \pmod{\mathfrak{p}}$. The constant C_1 depends only on the ring S . Let $r_{\mathfrak{p}}: \text{SL}(n, S) \rightarrow \text{SL}(n, S/\mathfrak{p})$ be the reduction modulo \mathfrak{p} homomorphism and set $\rho_{\mathfrak{p}}: \Gamma \rightarrow \text{SL}(n, S/\mathfrak{p})$ by $\rho_{\mathfrak{p}} = r_{\mathfrak{p}} \circ \rho$. By selection of \mathfrak{p} , we see that $\rho_{\mathfrak{p}}(\gamma), \rho_{\mathfrak{p}}(\eta)$ have distinct traces and hence have non-conjugate images. We also have

$$|\rho_{\mathfrak{p}}(\Gamma)| \leq |\text{SL}(n, S/\mathfrak{p})| \leq |S/\mathfrak{p}|^{m^2-1} \leq (C_1 n \log(C_1 2m\alpha))^{m^2-1} = Cn^{m^2-1}$$

where C is the constant $(C_1 C_0 \log(C_1 2m\alpha))^{m^2-1}$. In particular, $\text{CD}_\Gamma([\gamma], [\eta]) \leq Cn^{m^2-1}$ for some constant C depending only on Γ and ρ . As this holds for all $[\gamma], [\eta] \in B(C_\Gamma, n)$, we see that $\text{Conj}_\Gamma(n) \leq n^{m^2-1}$. The assertion that one only needs subgroups of $\text{SL}(n_0, \mathbf{F}_p)$ in proving conjugacy separability for Γ follows from the Čebotarev Density Theorem. \square

Proof of Theorem 1.5. We proceed similarly to the proof of Theorem 1.4. By assumption, we have $\rho \in \text{Hom}(\Gamma, \text{SL}(n_\gamma, \mathbf{C}))$ such that $\text{Tr}(\rho(\gamma)) \neq \text{Tr}(\rho(\eta))$ for any $\eta \in \Gamma$ that is not conjugate to γ . Using $\text{Tr}(\rho(\gamma)) - \text{Tr}(\rho(\eta))$, we can employ the same methods used in the proof of Theorem 1.4 to find the desired homomorphism to a finite group where γ, η have non-conjugate images. \square

4. Horowitz’s construction

In this section we show that the cyclically reduced words constructed in Example 8.2 in [32] that do have the same trace over $SL(2, \mathbf{C})$ are not likely to have the same trace over $SL(n, \mathbf{C})$ for $n > 2$. Since $SL(n - 1, \mathbf{C})$ embeds into $SL(n, \mathbf{C})$ it suffices to show that this failure occurs for $n = 3$.

4.1. Reduction to free groups. The following lemma reduces the search for trace equivalent pairs in non-elementary hyperbolic groups to finding them in F_r

Lemma 4.1. *Let $n, r \geq 2$ be integers. If there exists a non-conjugate pair $w_1, w_2 \in F_r$ such that w_1, w_2 are SL_n -trace equivalent, then for any non-elementary hyperbolic group Δ , there exists non-conjugate $\delta_1, \delta_2 \in \Delta$ that are SL_n -trace equivalent.*

Proof. By I. Kapovich [33, Theorem C], Δ has a malnormal subgroup Δ_0 that is isomorphic to F_r . Fixing any isomorphism $\psi: F_r \rightarrow \Delta_0$, we set $\delta_j = \psi(w_j)$. For any representation $\rho: \Delta \rightarrow SL(n, \mathbf{C})$, it follows that $\text{Tr}(\rho(\delta_1)) = \text{Tr}(\rho(\delta_2))$. As δ_1, δ_2 are non-conjugate in Δ_0 and Δ_0 is malnormal in Δ , we see that δ_1, δ_2 are non-conjugate in Δ . □

Since free groups are hyperbolic, it follows that for any integers $r, s \geq 2$, F_r has a non-conjugate SL_n -trace equivalent pair if and only if F_s has a non-conjugate SL_n -trace equivalent pair. In particular, we need only consider the existence of trace equivalent pairs in F_2 . In fact, for any finitely generated group Γ with a malnormal free subgroup, we see that $\text{Conj}_{F_2}(n) \leq \text{Conj}_\Gamma(n)$. Moreover, $\text{Conj}_{F_r}(n) \approx \text{Conj}_{F_s}(n)$ for any integers $r, s \geq 2$. We also note that Lemma 4.1 implies that if F_2 does not have (A), then no non-elementary hyperbolic group can have (A). Indeed, no finitely generated group with a malnormal free subgroup can have (A).

4.2. Horowitz’s construction. Let $F_2 = \langle a, b \rangle$. Horowitz’s words are defined recursively by $w_0 = a$ and

$$w_m(\epsilon_1, \dots, \epsilon_m) := w_{m-1}^{-\epsilon_m} b^{2m} w_{m-1}^{\epsilon_m} b^{2m-1} w_{m-1}^{-\epsilon_m} b^{2m} w_{m-1}^{\epsilon_m}$$

for $\epsilon_i = \pm 1$. Horowitz shows that for $(\epsilon_1, \dots, \epsilon_m) \neq (\epsilon_1^*, \dots, \epsilon_m^*)$, the corresponding words will not be cyclically equivalent for any $m > 0$ and they are all SL_2 -trace equivalent. Hence, there are arbitrarily large collections of SL_2 -trace equivalent non-conjugate words. For $w_1(1) = a^{-1}b^2aba^{-1}b^2a$ and $w_1(-1) = ab^2a^{-1}bab^2a^{-1}$, one can find a representation $\rho = (A, B) \in SL(3, \mathbf{C})^2$ where $\text{Tr}(\rho(w_1(1))) - \text{Tr}(\rho(w_1(-1))) \neq 0$. In particular, this pair is not SL_3 -trace equivalent. Below, we further elaborate on why it is unlikely that the above SL_2 -trace pairs are also SL_3 -trace pairs. First, we review in more detail why these

pairs are SL_2 -trace equivalent. The first step in showing SL_2 -trace equivalence is a proof that

$$\begin{aligned} & \mathrm{Tr}(w_m(\epsilon_1, \dots, \epsilon_{j-1}, +1, \epsilon_{j+1}, \dots, \epsilon_m)) \\ &= \mathrm{Tr}(w_m(\epsilon_1, \dots, \epsilon_{j-1}, -1, \epsilon_{j+1}, \dots, \epsilon_m)) \end{aligned}$$

for $1 \leq j \leq m$ for $\mathrm{SL}(2, \mathbf{C})$. By the recursive definition of w_m , Horowitz shows that

$$w_m(\epsilon_1, \dots, \epsilon_{j-1}, +1, \epsilon_{j+1}, \dots, \epsilon_m) = W(u^{-1}bu, b)$$

whereas

$$w_m(\epsilon_1, \dots, \epsilon_{j-1}, -1, \epsilon_{j+1}, \dots, \epsilon_m) = W(ubu^{-1}, b)$$

where $u = w_{j-1}(\epsilon_1, \dots, \epsilon_{j-1})$ and W is a word in two letters. What works for $\mathrm{SL}(2, \mathbf{C})$ is that there exists a polynomial P_W in three variable so that $\mathrm{Tr}(W(u, v)) = P_W(\mathrm{Tr}(u), \mathrm{Tr}(v), \mathrm{Tr}(uv))$. In the case above, these three traces are identical when evaluated at (ubu^{-1}, b) and $(u^{-1}bu, b)$ respectively since the trace is invariant under cyclic permutations, and hence their polynomials are equal too. One can argue inductively to establish the general result.

However, this first step fails for $\mathrm{SL}(3, \mathbf{C})$. The comparable statement is that there exists a polynomial P_W in nine variables (see [39]) such that

$$\begin{aligned} \mathrm{Tr}(W(u, v)) = P_W(\mathrm{Tr}(u), \mathrm{Tr}(u^{-1}), \mathrm{Tr}(v), \mathrm{Tr}(v^{-1}), \mathrm{Tr}(uv), \mathrm{Tr}(u^{-1}v^{-1}), \\ \mathrm{Tr}(uv^{-1}), \mathrm{Tr}(u^{-1}v), \mathrm{Tr}(uvu^{-1}v^{-1})). \end{aligned}$$

Upon checking, one finds that the first 6 variables are equal. However, the seventh variables become $\mathrm{Tr}(ubu^{-1}b^{-1})$ and $\mathrm{Tr}(u^{-1}bub^{-1}) = \mathrm{Tr}(bub^{-1}u^{-1}) = \mathrm{Tr}((ubu^{-1}b^{-1})^{-1})$. These traces of words are generically not equal (see [39]); in fact they are equal if and only if the $\mathrm{SL}(3, \mathbf{C})$ representations are transpose fixed. Likewise the eighth variables will differ as well. In fact, we would have an expression of the form $P_W(a_1, \dots, a_6, a_7, a_8, \mathrm{Tr}(w)) = P_W(a_1, \dots, a_6, a_8, a_7, \mathrm{Tr}(w^{-1}))$ since the 7th and 8th variables are in fact permuted (switching the roles of u and v) and the first 6 are identical (just by cyclic permutation), and the 9th is cyclically equivalent to the trace of its inverse. The 9th word is $ubu^{-1}bub^{-1}u^{-1}b^{-1}$. Note also that there is a polynomial P in the 8 algebraically independent variables so that $\mathrm{Tr}(w^{-1}) = P - \mathrm{Tr}(w)$. If we had equality we would have a non-trivial relation (symmetric in two variables), which is unlikely for a fixed w .

4.3. Candidate words. By Lemma 6.8 in [32] any SL_2 -trace equivalent pair in F_2 must have the same number of each generator represented in the word, up to plus or minus exponents. Thus, the same result holds for words that are SL_n -trace equivalent for any n . It is easy to see that pairs of the form (w, w^{-1}) are SL_2 -trace equivalent. However, by [6] the word map is dominant for non-trivial words, and so (w, w^{-1}) are never SL_n -trace equivalent for $n \geq 3$ since $\mathrm{Tr}(A) \neq \mathrm{Tr}(A^{-1})$ for

a generic (in the Baire sense) $A \in \mathrm{SL}(3, \mathbf{C})$. Along the same lines, we have the following lemma.

Lemma 4.2. *Let $r(w)$ be the reverse of the word w , and assume $r(w)$ is not conjugate to w . Then $r(w)$ and w are always SL_n -trace equivalent if and only if $n = 2$.*

Proof. Since $\mathrm{Tr}(w) = \mathrm{Tr}(w^{-1})$ for $n = 2$, we obtain

$$\mathrm{Tr}(w(a, b)) = \mathrm{Tr}(w(a, b)^{-1}) = \mathrm{Tr}(r(w(a^{-1}, b^{-1}))).$$

Therefore, $\mathrm{Tr}(r(w(a, b))) = \mathrm{Tr}(w(a^{-1}, b^{-1}))$. By the Fricke–Vogt Theorem (see for instance [27]), $\mathrm{Hom}(F_2, \mathrm{SL}(2, \mathbf{C})) // \mathrm{SL}(2, \mathbf{C}) \cong \mathbf{C}^3$ parametrized by $(\mathrm{Tr}(a), \mathrm{Tr}(b), \mathrm{Tr}(ab))$. Thus, there exists a unique polynomial $P \in \mathbf{C}[x, y, z]$ such that $\mathrm{Tr}(w(a, b)) = P(\mathrm{Tr}(a), \mathrm{Tr}(b), \mathrm{Tr}(ab))$. We conclude

$$\begin{aligned} \mathrm{Tr}(r(w(a, b))) &= P(\mathrm{Tr}(a^{-1}), \mathrm{Tr}(b^{-1}), \mathrm{Tr}(a^{-1}b^{-1})) \\ &= P(\mathrm{Tr}(a), \mathrm{Tr}(b), \mathrm{Tr}(ab)) = \mathrm{Tr}(w(a, b)). \end{aligned}$$

Conversely, $\mathrm{Hom}(F_2, \mathrm{SL}(3, \mathbf{C})) // \mathrm{SL}(3, \mathbf{C})$ is a branched double cover of \mathbf{C}^8 (see [39]). The branch locus is exactly determined by

$$\mathrm{Tr}(aba^{-1}b^{-1}) = \mathrm{Tr}(b^{-1}a^{-1}ba);$$

showing that for $r = 2$ the pairs $(w, r(w))$ are not generally SL_n -trace equivalent for $n \geq 3$. \square

We expect that non-conjugate reverse pairs are never SL_3 -trace equivalent. A more provocative conjecture is the following; in the statement, positive words have only non-negative powers of the generators:

Conjecture 2. *Let $n \geq 2$. There exists SL_n -trace equivalent pairs (u, v) if and only if there exists positive pairs (u', v') that are SL_n -trace equivalent.*

Before giving a heuristic proof for the above conjecture, we mention two related conjectures. Ginzburg and Rudnick [25, Conjecture 1.1] have a conjectural condition to ensure a word does not have an SL_2 -trace companion (aside from its inverse); in their terminology, such a word has stable multiplicity one. Anderson [1, Conjecture 4.1] gave conjectural picture for all SL_2 -trace companions.

We now give a heuristic for the validity of the conjecture. As the reverse implication is obvious, we discuss only the direct implication. For $n = 2$, Lemma 4.2 establishes the statement. For $n > 2$ we describe an algorithm (that depends on n) that takes a non-conjugate SL_n -trace equivalent pair and produces a pair, that we expect that is positive, SL_n -trace equivalent, and not conjugate.

We have implemented the algorithm for $n = 2$ and it does produce a positive pair (u', v') that is SL_2 -trace equivalent but u' is conjugate to v' ; we expect this to be a problem only with $n = 2$.

In what follows, let $\rho(a) = \mathbf{A}$ be a n by n matrix. Recall the Cayley–Hamilton formula gives

$$\mathbf{0} = \sum_{k=0}^n (-1)^{n-k} C_k^n(\mathbf{A}) \mathbf{A}^k,$$

where the coefficients $C_k^n(\mathbf{A})$ arise from the characteristic equation $\det(t\mathbf{I} - \mathbf{A}) = \sum_{k=0}^n (-1)^{n-k} C_k^n(\mathbf{A}) t^k$. We know that $C_n^n(\mathbf{A}) = 1$, $C_{n-1}^n(\mathbf{A}) = \mathrm{Tr}(\mathbf{A})$ and $C_0^n(\mathbf{A}) = \det(\mathbf{A})$. By Newton's trace formulas each $C_k^n(\mathbf{A})$ is a polynomial in the traces of non-negative powers of the matrix \mathbf{A} . Since $\det(\mathbf{A}) = 1$, we can multiply the Cayley–Hamilton formula by a word $\mathbf{UA}^{-1} := \rho(ua^{-1})$ on the left and another word $\mathbf{V} := \rho(v)$ on the right. This results in

$$\mathbf{UA}^{n-1}\mathbf{V} + \sum_{k=1}^{n-1} (-1)^{n-k} C_k^n(\mathbf{A}) \mathbf{UA}^{k-1}\mathbf{V} = (-1)^{n+1} \mathbf{UA}^{-1}\mathbf{V}.$$

Thus, by taking the trace of both sides, we have:

$$\mathrm{Tr}(\mathbf{UA}^{-1}\mathbf{V}) = (-1)^{n+1} \mathrm{Tr}(\mathbf{UA}^{n-1}\mathbf{V}) + \sum_{k=1}^{n-1} (-1)^{k-1} C_k^n(\mathbf{A}) \mathrm{Tr}(\mathbf{UA}^{k-1}\mathbf{V}).$$

That shows that given any word w with negative exponents, one can iteratively apply the preceding formula in the coordinate ring $\mathbf{C}[\mathrm{Hom}(F_2, \mathrm{SL}(n, \mathbf{C})) // \mathrm{SL}(n, \mathbf{C})]$, which is generated by traces of words by results of Procesi [55], to obtain an expression for $\mathrm{Tr}(w)$ as a polynomial in traces of positive words.

Now, suppose (u, v) is SL_n -trace equivalent but are not conjugate. After cyclically reducing u and v , given results of Horowitz ([32]), we can assume that u and v have the same word length and the same (signed) multiplicity of each letter. Applying the preceding algorithm to $\mathrm{Tr}(u)$, $\mathrm{Tr}(v)$ results in polynomial expressions P_u, P_v in terms of traces of only positive words. By inspection of the replacement formula defining the algorithm, one sees that there will be a monic trace term with a longest word. That is $P_u = \mathrm{Tr}(u') + L$, and likewise $P_v = \mathrm{Tr}(v') + L'$ where both L, L' contain terms of products of traces of shorter positive words. We expect that $\mathrm{Tr}(u') = \mathrm{Tr}(v')$ since $\mathrm{Tr}(u) = \mathrm{Tr}(v)$ to begin with. Also, given that $n \geq 3$, we expect that u' is not conjugate to v' given that u is not conjugate to v .

It is not presently clear to us how to complete the above argument, that is, to prove that the last two lines are valid. We thank Greg Kuperberg for conversations about the validity of the above sketch.

We now indicate our interest in this conjecture. For the free group $F_2 = F_2(a, b)$, the smallest positive exponent SL_2 -trace equivalent pair is $\{babbaa, abaabb\}$. To find examples of $SL(3, \mathbf{C})$ words, if the conjecture is true, we need only check words with the same number of letters in each word having only positive exponents. Moreover, since by restricting, the trace equivalence must also hold for $SL(2, \mathbf{C})$, we need only check words of the above type that work for $SL(2, \mathbf{C})$. We expect that non-conjugate reverse pairs will never be SL_3 -trace equivalent, and so we further wish to only consider positive non-conjugate pairs that are not reverse but are SL_2 -trace equivalent; the first examples occurs at length 12 with one explicit pair being $\{aababbaabbab, aababbabaabb\}$. We end this section with two questions about such words.

- (1) What is a classification of these words, or generating families?
- (2) What is the growth rate as a function the length of these words?

As we expect SL_n -trace equivalent words exist, our guess is that the above words are rather plentiful. However, by computer search, there are no SL_3 -trace equivalent pairs of length up to 20.

5. Efficient solutions to the conjugacy problem

In this section, we provide two different approaches to solving the conjugacy problem in free groups using finite quotients, neither of which are originally due to us.

5.1. Lower central and derived series. Recall, the lower central and derived series are defined inductively by $\Gamma_0 = \Gamma$, $\Gamma_j = [\Gamma, \Gamma_{j-1}]$, and $\Gamma^j = [\Gamma^{j-1}, \Gamma^{j-1}]$. We set $N_j(\Gamma) = \Gamma/\Gamma_j$ and $S_j(\Gamma) = \Gamma/\Gamma^j$. By [46, p. 27, Prop. 4.9], we know that $\gamma, \eta \in F_r$ are conjugate in F_r if and only if they have conjugate image in $S_j(F_r)$ (or $N_j(F_r)$) for all j . Since the groups $S_j(F_r)$ and $N_j(F_r)$ are conjugacy separable for all j (see [5], [23], and [58]), we see that F_r is conjugacy separable. In order to implement these methods effectively, we must first estimate $j_{\gamma, \eta}$ as a function of the word length of γ, η where $j_{\gamma, \eta}$ is the smallest $j \in \mathbf{N}$ such that γ, η have non-conjugate images in $N_j(F_r)$ (or $S_j(F_r)$). Second, we must effectively solve the conjugacy problem in torsion free nilpotent or polycyclic groups. Malestein and Putman [49] address the first problem. Pengitore [53] addresses the second problem. As our current goal is deciding whether or not the function $\text{Conj}_{F_r}(n)$ has a polynomial bound, we note that it is already known that the above method cannot work. Specifically, neither the lower central or derived series provides a polynomial upper bound for the function $F_{F_r}(n)$; see [8] and [9].

5.2. Proof of Theorem 1.6. In this subsection, we prove Theorem 1.6. The construction of the representation needed to verify Theorem 1.6 in the case of free groups follows Wehrfritz [68]. The surface group case is similar.

We now produce the representation for the case of free groups. To begin, given a conjugacy class $[\gamma]$ in F_r , we first pass to a finite index subgroup Γ where $\Gamma = \langle \gamma \rangle * \Delta$. That such can be done follows from work of Hall [30]. Since γ is part of a free basis, it follows that there exists a representation $\rho_0: \Gamma \rightarrow \mathrm{SL}(2, \mathbf{R})$ where γ has a unique, non-zero trace up to conjugation and inverses. As $\mathrm{SL}(2, \mathbf{R}) < \mathrm{SL}(3, \mathbf{C})$ by the standard inclusion into the upper two by two block, we see that there exists a representation $\rho_1: \Gamma \rightarrow \mathrm{SL}(3, \mathbf{C})$ such that $\rho_1(\gamma)$ has a unique, non-zero trace up to conjugation and inverses. Since $\mathrm{Tr}(\rho(\gamma^{-1})) \neq \mathrm{Tr}(\rho(\gamma))$ for a generic $\mathrm{SL}(3, \mathbf{C})$ representation ρ (in the Baire Category sense), we can further assume that $\rho(\gamma)$ has a unique, non-zero trace up to conjugation. For any $\eta \in \Gamma$ that is not conjugate to γ in Γ , we know that $\mathrm{Tr}(\rho(\gamma)) - \mathrm{Tr}(\rho(\eta))$ is a non-constant function of ρ . Consequently, by the Baire Category Theorem, we can assume that $k_1 \mathrm{Tr}(\rho_1(\gamma)) \neq k_2 \mathrm{Tr}(\rho_1(\eta))$ for any pair of integers $1 \leq k_1, k_2 \leq m = [F_r : \Gamma]$. For any such $\rho_1 \in \mathrm{Hom}(\Gamma, \mathrm{SL}(3, \mathbf{C}))$, the induced representation $\rho = \mathrm{Ind}_{\Gamma}^{F_r}(\rho_1)$ is the needed representation to verify Theorem 1.6 in the free group case.

Proof of Theorem 1.6: Free Case. If $\eta \in F_r$ is not conjugate in F_r into Γ , then $\mathrm{Tr}(\rho(\eta)) = 0$ by the Frobenius formula for traces of induced representations. If $\eta \in F_r$ is conjugate in F_r to some $\eta' \in \Gamma$, then $\mathrm{Tr}(\rho(\eta)) = k_{\eta} \mathrm{Tr}(\rho_1(\eta'))$ for some integer $1 \leq k_{\eta} \leq m$. As $\mathrm{Tr}(\rho(\gamma)) = k_{\gamma} \mathrm{Tr}(\rho_1(\gamma))$ for some $1 \leq k_{\gamma} \leq m$, it follows from our selection of ρ_1 that $\rho(\gamma)$ has a unique trace up to conjugation, as needed for Theorem 1.6. \square

We now produce the representation for the case of surface groups. To begin, given a conjugacy class $[\gamma]$ in $\pi_1(\Sigma_g)$, we first pass to a cover where a lift of the curve associated to $[\gamma]$ is simple. That such can be done follows from work of Scott [63]. We fix a finite index subgroup of $\pi_1(\Sigma_g)$ associated to this finite cover which we denote by Γ . Since the curve associated to $[\gamma]$ has a simple lift, it follows that there exists a representation $\rho_0: \Gamma \rightarrow \mathrm{SL}(2, \mathbf{C})$ where γ has a unique, non-zero trace up to conjugation and inverses. The remainder of the construction of ρ is identical to the free case of the proof of Theorem 1.6. Note that to ensure $\mathrm{Tr}(\rho(\gamma)) \neq \mathrm{Tr}(\rho(\gamma^{-1}))$ for a generic $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}(3, \mathbf{C}))$ (in the Baire sense), we can use [12] in place of [6].

In either the free or surface case, we can use the methods from [15] to establish Corollary 1.7. In particular, the degree of the polynomial in Corollary 1.7 depends only on m and the coefficient ring of the representation, both of which are constant for a fixed γ . By Patel [51] and Gupta and Kapovich [28], we have $m \leq \|\gamma\|$, and so when $\|\gamma\|, \|\eta\| \leq n$, we see that $\mathrm{CD}_{\Gamma}(\gamma, \eta) \leq C n^{Cn^2}$.

6. Proof of Theorem 1.8

Given a fully residually free group Γ with a finite index, normal subgroup Δ and a prime $p \in \mathbf{Z}$, we will construct a faithful homomorphism $\rho: \Gamma \rightarrow \mathrm{SL}(n_\Delta, R_\omega)$ such that $\Delta = \ker(r_{\mathfrak{m}_\omega} \circ \rho)$ where $n_\Delta = 2[\Gamma : \Delta]$, R_ω is a local domain, $\mathfrak{m}_\omega < R_\omega$ is the unique maximal ideal with residue field $R_\omega/\mathfrak{m}_\omega = \mathbf{F}_p$, the field of p elements, and $r_{\mathfrak{m}_\omega}: \mathrm{SL}(n_\Delta, R_\omega) \rightarrow \mathrm{SL}(n_\Delta, \mathbf{F}_p)$ is the reduction modulo \mathfrak{m}_ω homomorphism. We enumerate the non-trivial elements of Δ via $\{\delta_1, \delta_2, \dots\}$. Since subgroups of fully residually free groups are fully residually free, for each $t \in \mathbf{N}$, there exists a homomorphism $\psi_t: \Delta \rightarrow F_{r_t}$ such that ψ_t is injective when restricted to the finite subset $\{\delta_1, \dots, \delta_t\}$. Recall that the ring of p -adic integers \mathbf{Z}_p is a local integral domain with a unique maximal ideal \mathfrak{m}_p . Via the ping pong lemma, the homomorphism $\psi_{(p)}: F_2 \rightarrow \mathrm{SL}(2, \mathbf{Z}) < \mathrm{SL}(2, \mathbf{Z}_p)$ induced by sending a free basis a, b of F_2 to the matrices

$$a, b \mapsto \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$$

is an isomorphism. By the Nielsen–Schreier theorem, we have a faithful homomorphism $F_{r_t} \rightarrow F_2$ for each $r_t \in \mathbf{N}$ and fix one such homomorphism for each $r_t \in \mathbf{N}$. Respectfully, we define $\rho_{p,t} := \psi_{(p)} \circ \psi_t$ and note $\rho_{p,t}(\Delta) < \ker r_{\mathfrak{m}_p}$.

We restrict $\psi_{(p)}$ to the image of $F_{r_t} < F_2$ and for notational simplicity denote the resulting homomorphism by ψ . Taking a non-principal ultrafilter ω on \mathbf{N} , the ultraproduct $R_\omega = \prod_\omega \mathbf{Z}_p$ is a local integral domain with unique maximal ideal $\mathfrak{m}_\omega = \prod_\omega \mathfrak{m}_p$ (see [60, Chapter 1] for instance). The associated residue field $R_\omega/\mathfrak{m}_\omega$ is given by $\prod_\omega \mathbf{Z}_p/\mathfrak{m}_p$. Since the latter is an ultraproduct of \mathbf{F}_p , it follows that $R_\omega/\mathfrak{m}_\omega$ is isomorphic to \mathbf{F}_p (see [29, p. 184] for instance). The ultraproduct ρ_ω of the representations $\rho_t = \psi \circ \psi_t$ yields a representation $\rho_\omega: \Delta \rightarrow \mathrm{SL}(2, R_\omega)$. By selection of ψ_t and ψ , ρ_ω is faithful with $\rho_\omega(\Delta) < \ker r_{\mathfrak{m}_\omega}$. Setting $\rho = \mathrm{Ind}_\Delta^\Gamma(\rho_\omega)$, we obtain a faithful representation $\rho: \Gamma \rightarrow \mathrm{SL}(2d, R_\omega)$ where $d = [\Gamma : \Delta]$. By construction of ρ_ω , the definition of Ind, and the normality of Δ in Γ , we see that $\Delta = \ker(r_{\mathfrak{m}_\omega} \circ \rho)$. \square

Remark 2. The ring R_ω embeds into $\prod_\omega \mathbf{Q}_p$ which is a field of characteristic zero. Since fully residually free groups are finitely presentable ([64, 4.4]), the ring R generated over \mathbf{Z} by the coefficients of the matrix entries of $\rho(\Gamma)$ is finitely generated. Setting $\mathfrak{m} = R \cap \mathfrak{m}_\omega$, we obtain a maximal ideal in R with residue field $R/\mathfrak{m} = \mathbf{F}_p$ such that $\rho(\Gamma) < \mathrm{SL}(2[\Gamma : \Delta], R)$ and $\Delta = \ker(r_{\mathfrak{m}} \circ \rho)$. Moreover, we have an embedding of R into \mathbf{C} ; the field of fractions of R embeds into \mathbf{C} via the axiom of choice.

Remark 3. If Γ is a free group, we can take $\rho = \text{Ind}_{\Delta}^{\Gamma}(\rho_0)$ where ρ_0 is the representation given by $\Delta \rightarrow F_2 \rightarrow \text{SL}(2, \mathbf{Z})$. The first homomorphism $\Delta \rightarrow F_2$ is given by the Nielsen–Schreier theorem and the second homomorphism $F_2 \rightarrow \text{SL}(2, \mathbf{Z})$ is given by $\psi_{(p)}$. In total, we obtain a faithful representation $\rho: \Gamma \rightarrow \text{SL}(2[\Gamma : \Delta], \mathbf{Z})$ such that $\Delta = \ker(r_p \circ \rho)$.

Theorem 1.8 can also be proven by using work of Barlev–Gelder [3], which followed the work of Breuillard, Gelder, Souto, and Storm [18]. Barlev–Gelder [3, Theorem 1.2] proved that if G is a compact topological group with a non-abelian free subgroup, then G contains an isomorphic copy of every non-abelian limit group. Since \mathbf{Z}_p is a compact topological ring, $\text{SL}(2, \mathbf{Z}_p)$ is a compact topological group. Moreover, the finite index subgroup $\ker r_{m_p} < \text{SL}(2, \mathbf{Z}_p)$ is a compact topological group with $F_2 < \ker r_{m_p}$ from above. Hence by [3, Theorem 1.2], $\ker r_{m_p}$ contains an isomorphic copy of every non-abelian limit group. Given a non-abelian limit group Γ with a finite index, normal subgroup Δ , we can apply this observation to obtain a faithful representation $\rho_0: \Delta \rightarrow \ker r_{m_p} < \text{SL}(2, \mathbf{Z}_p)$. It follows then that $\rho = \text{Ind}_{\Delta}^{\Gamma}(\rho_0)$ is a faithful representation into $\text{SL}(2[\Gamma : \Delta], \mathbf{Z}_p)$ with $\Delta = \ker(r_{m_p} \circ \rho)$.

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Received July 21, 2014

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