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**Groups, Geometry, and Dynamics**


# Minimal exponential growth rates of metabelian Baumslag–Solitar groups and lamplighter groups

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**Abstract.** We prove that for any prime $p \geq 3$ the minimal exponential growth rate of the Baumslag–Solitar group $\mathrm{BS}(1,p)$ and the lamplighter group $\mathcal{L}_p = (\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}$ are equal. We also show that for $p = 2$ this claim is not true and the growth rate of $\mathrm{BS}(1,2)$ is equal to the positive root of $x^3 - x^2 - 2$, whilst the one of the lamplighter group $\mathcal{L}_2$ is equal to the golden ratio $(1+\sqrt{5})/2$. The latter value also serves to show that the lower bound of A.Mann from [9] for the growth rates of non-semidirect HNN extensions is optimal.

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## 1. Introduction

Let $G$ be a finitely generated group. For any finite generating set $S$ of $G$ we can consider the *exponential growth rate* of $G$ with respect to $S$ which is defined as follows. Any element $g \in G$ can be written as a finite product of elements in $S \cup S^{-1}$ and we define the length $\ell_{G,S}(g)$ of $g$ as the minimum number of elements in such a product. The growth function $F_{G,S}(n)$ is the number of elements $g \in G$ for which $\ell_{G,S}(g) \leqslant n$. Finally the *exponential growth rate* of $G$ with respect to $S$ is the limit

$$\omega(G,S) = \lim_{n\to\infty} (F_{G,S}(n))^{\frac{1}{n}} \geq 1.$$

Note that this limit always exists by submultiplicativity of the growth function (see [7, VI.C.56]).

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The exponential growth rate $\omega(G, S)$ clearly depends on the choice of the generating set $S$ and one obtains a group invariant by considering the infimum over all finite generating sets:

$$\Omega(G) = \inf_{|S|<\infty} \{\omega(G, S)\}. \tag{1.1}$$

It is now natural to ask if there exists a generating set $S$ for which the equality $\Omega(G) = \omega(G, S)$ is realized. For the free group $\mathbb{F}_n$ of rank $n$, Gromov remarked in [5, Example 5.13] that $\Omega(\mathbb{F}_n)$ is exactly $2n-1$ and is realized on any free generating set (with $n$ elements). Except for this example, very few exact values for $\Omega(G)$ have been computed. Known cases include free products $\mathbb{Z}_2 * \mathbb{Z}_{p^k}$ [15] (the cases $p^k = 3, 4$ were proven earlier in [9]), the free product $\mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and the Coxeter group $\mathrm{PGL}(2, \mathbb{Z})$ [2] and a few more examples in the references [2, 9, 15]. But the question of de la Harpe and Grigorchuk whether $\Omega(\pi_1(\Sigma_g))$ is realized on the canonical generators of the fundamental group of a closed surface $\Sigma_g$ with $g \geq 2$ is still open (see [6, p. 55]). While in many cases, the value $\omega(G, S)$ can be computed for some particular generating set $S$, it is usually much harder to find a generating set $S$ such that $\Omega(G) = \omega(G, S)$ and sometimes even impossible due to the existence of groups for which the infimum in (1.1) is not attained (see [11, 16]).

We consider two classes of metabelian groups: Baumslag–Solitar groups $\mathrm{BS}(1, n)$ and lamplighter groups $\mathcal{L}_n = (\mathbb{Z}/n\mathbb{Z}) \wr \mathbb{Z}$. The growth functions of the Baumslag–Solitar groups

$$\mathrm{BS}(1, n) = \langle a, t \mid tat^{-1} = a^n \rangle \tag{1.2}$$

with respect to the canonical generating set $S = \{a, t\}$ were computed by Collins, Edjvet and Gill in [4]. The restricted wreath products $\mathcal{L}_n = (\mathbb{Z}/n\mathbb{Z}) \wr \mathbb{Z}$ can be presented as

$$\mathcal{L}_n = \langle a, t \mid a^n = 1,\ \ [t^k a t^{-k}, a] = 1\ (k = 1, 2, \ldots) \rangle. \tag{1.3}$$

To compute the growth function of $\mathcal{L}_n$ with respect to the set $\{a, t\}$ one can use formulas given by Parry in [10]. Even though the formulas for the growth functions of $\mathrm{BS}(1, n)$ and $\mathcal{L}_n$ were obtained by completely different methods and by use of different properties of the groups, we find that remarkably for all odd $n = 2k + 1$

$$\omega(\mathrm{BS}(1, n), \{a, t\}) = \omega(\mathcal{L}_n, \{a, t\}) = \omega_k, \tag{1.4}$$

where $\omega_k$ is the unique positive root of

$$T_k(x) = x^{k+1} - x^k - 2x^{k-1} - \cdots - 2x - 2,$$

for $k \geq 1$. This is easily deduced from [4] and [10] in Lemma 8. Interestingly, this equality never holds for even $n$. We will see the case $n = 2$ in more details.

Some inference for the equality (1.4) can be seen in the actions of the groups $\mathrm{BS}(1,n)$ and $\mathcal{L}_n$ on their corresponding Bass–Serre trees. There is indeed a very strong similarity between these actions, which we exploit to prove the main result of the paper:

**Theorem 1.** *Let $p$ be a prime. The minimal growth rate of the Baumslag–Solitar group $\mathrm{BS}(1,p)$ and lamplighter groups $\mathcal{L}_p$ are realized on the canonical generators $\{a,t\}$:*

$$\Omega(\mathcal{L}_p) = \Omega(\mathrm{BS}(1,p)) = \omega_k, \quad \text{for } p = 2k+1,$$

$$\Omega(\mathcal{L}_2) = \frac{1+\sqrt{5}}{2} < \Omega(\mathrm{BS}(1,2)) = \beta,$$

*where $\beta \sim 1.69572$ is the unique positive root of $z^3 - z^2 - 2$.*

The exact computation $\Omega(\mathcal{L}_2) = (1+\sqrt{5})/2$ gives a positive answer to the question of Mann [9] whether the lower bound $\Omega(G) \geq (1+\sqrt{5})/2$ can be realized on a non-semidirect HNN extension. (The fact that $\mathcal{L}_2$ is indeed a non-semidirect HNN extension will be shown in Section 2). Note that it follows from Theorem 1 that this lower bound could never be realized on any of the Baumslag–Solitar groups $\Omega(\mathrm{BS}(1,n))$ also for arbitrary integers $n \geq 2$.

The lower bounds for the growth rates in Theorem 1 are obtained by looking at the actions on the corresponding Bass–Serre trees, finding free submonoids using a local variant of the classical ping-pong lemma (Lemma 6 here) and computing their growth with Lemma 7. Interestingly, all the minimal growth rates are in fact realized as the growth rate of some free submonoid. The Bass–Serre trees of $\mathcal{L}_p$ and $\mathrm{BS}(1,p)$ are both $(p+1)$-regular trees, but the corresponding actions are of course different. Nevertheless, when $p$ is odd, the same method applies to give the lower bound of Theorem 1, which we abstract in the following theorem:

**Theorem 2.** *Let $G = H*_\theta$ be an HNN extension relative to an isomorphism $\theta\colon A \to B$ with $A = H$ and $B$ a normal subgroup of prime index $p$ in $H$. Then*

$$\Omega(G) \geq \frac{1+\sqrt{5}}{2}, \quad \text{for } p = 2,$$

$$\Omega(G) \geq \omega_k, \quad \text{for } p = 2k+1.$$

Together with the equalities (1.4) proven in Lemma 8 this immediately implies Theorem 1, except in the case of $\mathrm{BS}(1,2)$. For this last group, a finer analysis of its action on its Bass–Serre tree will be needed.

The question of Mann mentioned above was prompted by his proof of the lower bound $\Omega(G) \geq (1 + \sqrt{5})/2$ for any non-semidirect HNN extension $G$ (see [9]), using the cute algebraic observation that a hyperbolic element and a nontrivial conjugate of it generate a free monoid with growth rate equal to the golden ratio. Our proof for the case $p = 2$ of Theorem 2 also holds for any non-semidirect HNN extension and gives an alternative geometric proof to Mann's inequality.

Finally, as an application of Theorem 1, we can compute the minimal growth rate of the wreath product $\mathbb{Z} \wr \mathbb{Z}$. Indeed, as was already noted by Shukhov in [12], one can deduce from [4] that

$$\lim_{n \to \infty} \omega(\mathrm{BS}(1, n), \{a, t\}) = 1 + \sqrt{2}. \tag{1.5}$$

Since the wreath product $\mathbb{Z} \wr \mathbb{Z}$ can be viewed as an extension of the groups $\mathcal{L}_p$, combining Theorem 1 and Parry's computations for $\mathbb{Z} \wr \mathbb{Z}$, we obtain

**Corollary 3.** *The minimal growth rate of the restricted wreath product*

$$\mathbb{Z} \wr \mathbb{Z} = \langle a, t \mid [t^k a t^{-k}, a] = 1 \ (k = 1, 2, \ldots) \rangle$$

*is realized on the set* $\{a, t\}$ *and*

$$\Omega(\mathbb{Z} \wr \mathbb{Z}) = \omega(\mathbb{Z} \wr \mathbb{Z}, \{a, t\}) = 1 + \sqrt{2}.$$

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## 2. Bass–Serre tree for an HNN extension

Let $G = H*_\theta$ be the HNN extension of $H$ relative to the isomorphism $\theta: A \to B$ between the two subgroups $A, B$ of $H$. Following [9] we call $H*_\theta$ a *non-semidirect* HNN extension if at least one of the subgroups $A$ or $B$ is a proper subgroup in $H$. If $H = \langle S_H \mid R_H \rangle$ is a presentation of $H$, then $G$ admits the presentation

$$G = \langle S_H, t \mid R_H, tat^{-1} = \theta(a) \text{ for all } a \in A \rangle.$$

There is a natural surjection $\varphi: G \to \mathbb{Z}$ defined by sending the generators $S_H$ to 0 and $t$ to 1.

The vertices of the associated Bass–Serre tree $T$ of $G$ are the right cosets of $G$ by $H$ and the edges are the right cosets of $G$ by $B$,

$$T^0 = G/H, \quad T^1 = G/B.$$

The edge $gB \in T^1$ has vertices $gH$ and $gtH$. This is a tree of valency $[H : A] + [H : B]$. The group $G$ acts on $T$ by left multiplication.

Since the natural surjection $\varphi: G \to \mathbb{Z}$ is trivial on $H$, it induces a map $\bar{\varphi}: T^0 \to \mathbb{Z}$ which sends vertices $v, w$ of an edge of $T^1$ to images satisfying $|\bar{\varphi}(v) - \bar{\varphi}(w)| = 1$. This allows us to define an orientation on the edges by giving an edge from $v$ to $w$ with $\bar{\varphi}(w) - \bar{\varphi}(v) = 1$ the positive orientation. This allows us to distinguish between two types of neighbors to any vertex $v$: the $[H : A]$ vertices $w$ such that $\bar{\varphi}(w) = \bar{\varphi}(v) - 1$ which we call the *direct ascendants* of $v$, and the $[H : B]$ vertices $w$ such that $\bar{\varphi}(w) = \bar{\varphi}(v) + 1$, which we call the *direct descendants* of $v$. We further call a vertex $z$ an *ascendant*, respectively a *descendant*, of $v$ if there is a sequence $v = w_0, w_1, \dots, w_\ell = z$ such that $w_i$ is a direct ascendant, resp. direct descendant, of $w_{i-1}$ for $1 \le i \le \ell$. In our examples, $[H : A] = 1$, which means that there is only one direct ascendant to any vertex. We will also use the terminology that a vertex $v$ is *above*, respectively *below*, a vertex $w$ if $v$ is an ascendant, resp. descendant, of $w$.

Since the action of $G$ on $T$ preserves the orientation on the edges defined above, it is immediate that $G$ acts on $T$ without inversions. Thus there are two types of elements: elliptic and hyperbolic. Elliptic elements $g \in G$ have a fixed point on $T$ and are thus conjugated to $H$. Hyperbolic elements $g \in G$ have no fixed point and possess a unique invariant geodesic $L_g$, called the axis of $g$, on which $g$ acts by translation. Note that any element $g \in G$ which is not in the kernel of $\varphi: G \to \mathbb{Z}$ necessarily is hyperbolic, so in particular, any generating set of $G$ contains a hyperbolic element. Such hyperbolic elements will be called positive, respectively negative according to their image acting as a positive or negative translation on $\mathbb{Z}$.

Let us look at the first of our two main examples: the Baumslag–Solitar group $\mathrm{BS}(1,n)$. The Baumslag–Solitar group $\mathrm{BS}(1,n)$ is an HNN extension for $H = A = \mathbb{Z}$, $B = n\mathbb{Z}$ and $\varphi: \mathbb{Z} \to n\mathbb{Z}$ given by multiplication by $n$,

$$\mathrm{BS}(1,n) = \langle a, t \mid tat^{-1} = a^n \rangle.$$

Its Bass–Serre tree is depicted in Figure 2.1.

First we note that the standard presentation for a restricted wreath product $G \wr \mathbb{Z}$ provides an HNN extension, but the subgroups $A, B$ are both equal to $G$, so the corresponding Bass–Serre tree is a line, and the corresponding action of $G$ on a line is not useful for our goals. Still, it is possible to find yet another HNN decomposition. It was shown in [3, Theorem 2.5] that a finitely generated group $G$ is a non-semidirect HNN extension, once there exists a homomorphism $G \to \mathbb{Z}$ with infinitely generated kernel. Even earlier in [14], it has been pointed out that for any wreath product $G \wr \mathbb{Z}$ there exists an HNN extension presentation with indices $|G|$ and 1 so that the corresponding Bass–Serre tree is a regular tree of valency $|G| + 1$. For completeness, we include a proof of this fact for $\mathcal{L}_n = (\mathbb{Z}/n\mathbb{Z}) \wr \mathbb{Z}$.

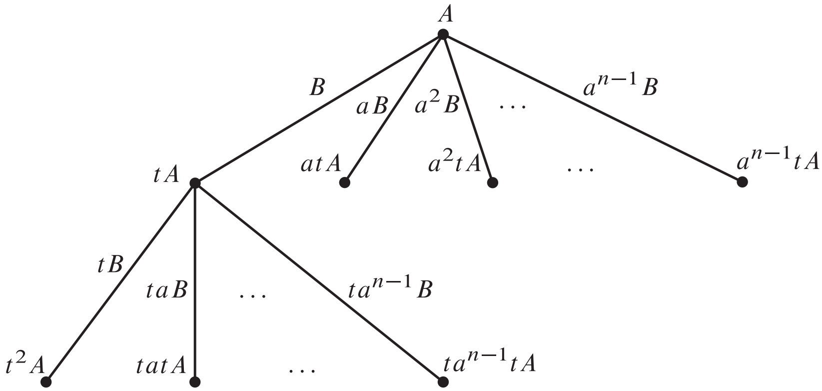


Figure 2.1. Bass–Serre tree of $\mathrm{BS}(1, n)$

**Lemma 4.** *The lamplighter group* $\mathcal{L}_n = (\mathbb{Z}/n\mathbb{Z}) \wr \mathbb{Z}$ *can be decomposed as an HNN extension* $D*_\theta$ *with indices of the subgroups* $[D : A] = 1, [D : B] = n$.

*Proof.* We will show that $\mathcal{L}_n$ is a non-semidirect HNN extension of an abelian group with countable generating set. Consider the infinite direct sum $D = \bigoplus_{\mathbb{N}_0} (\mathbb{Z}/p\mathbb{Z})$ canonically generated by the set of elements $\{a_0, a_1, a_2, \ldots\}$. Obviously,

$$D = \langle a_0, a_1, a_2, \cdots \mid a_i^n = 1,\ [a_i, a_j] = 1,\ i, j \in \mathbb{N}_0 \rangle. \tag{2.1}$$

Take the HNN extension $D*_f$ given by the subgroups $A = D$ and $B = \langle a_1, a_2, \ldots \rangle$ and the isomorphism $f(a_i) = a_{i+1}$. Then the group $D*_f$ can be presented as

$$D*_f = \langle t, a_0, a_1, a_2 \cdots \mid a_i^n = 1,\ [a_i, a_j] = 1,\ t a_i t^{-1} = a_{i+1},\ i, j \in \mathbb{N}_0 \rangle. \tag{2.2}$$

The relations $a_{i+1} = t a_i t^{-1}$ imply that

$$a_i = t^i a_0 t^{-i} \quad \text{for } i \geq 1. \tag{2.3}$$

The relations $a_i^n = 1$ with $i \geq 1$ are redundant in (2.2) because they follow from the relation $a_0^n = 1$ and the relations (2.3). Moreover, using the equalities (2.3) we can exclude the generators $a_i$ with $i \geq 1$ from the presentation (2.2) to obtain

$$D*_f = \langle t, a_0 \mid a_0^n = 1,\ [t^i a_0 t^{-i}, t^j a_0 t^{-j}] = 1,\ i, j \in \mathbb{N}_0 \rangle. \tag{2.4}$$

Since each relation $[t^i a_0 t^{-i}, t^j a_0 t^{-j}] = 1$ follows from $[a_0, t^{j-i} a_0 t^{i-j}] = 1$ and $[a_0, t^k a_0 t^{-k}]$ follows from $[a_0, t^{-k} a_0 t^k]$, we can reduce the presentation (2.4) to

$$D *_f = \langle t, a_0 \mid a_0^n = 1,\ [a_0, t^k a_0 t^{-k}] = 1,\ k \in \mathbb{N} \rangle,$$

which is the presentation of the lamplighter group $\mathcal{L}_n$. □

**Lemma 5.** *Let $G$ be an HNN extension such that $A = H$ and $B$ is a normal subgroup of $H$ of odd prime index $p = 2k+1$. Let $g \in G$ be an elliptic element. For any vertex $v$ of the Bass–Serre tree $T$ either $g(v) = v$ or the $p = 2k+1$ vertices*

$$g^{-k}(v), \dots, g^{-1}(v), v, g(v), \dots, g^k(v)$$

*are distinct.*

*Proof.* Let $a \in A = H$ be any element not in the kernel of the natural surjection $A \to A/B \cong \mathbb{Z}_p$. Then $A = \sqcup_{j=-k}^{k} a^j B$. In the Bass–Serre tree of $G$, the $p$ direct descendants of the vertex $A$ are the vertices $a^{-k}tA, \dots, tA, \dots, a^k tA$ and are joined to $A$ through the edges $a^{-k}B, \dots, B, \dots, a^k B$ respectively. Observe that since $B$ is normal in $A$, any element $b \in B$ acts trivially on the direct descendants of the vertex $A$. Furthermore, $a$ and any of its powers $a^j$ where $p$ does not divide $j$ obviously acts cyclically on the first descendants of $A$.

By conjugation, we can suppose that our elliptic element is in fact $h = a^j b \in H = A$, with $b \in B$ and $-k \leq j \leq k$. If $j = 0$ then $h$ acts trivially on the direct descendants of $A$, while if $j \neq 0$ then $h$ acts as a cyclic permutation of order $p$. This implies the lemma. □

The following lemma is an immediate application of the classical ping-pong lemma for semigroups [7, Proposition VII.2] taking as ping-pong sets, the descendants of $x_i v$, for every $i$:

**Lemma 6** (ping pong lemma). *Let $x_1, x_2, \dots, x_r \in \mathrm{BS}(1,p)$ act as positive hyperbolic automorphisms on the corresponding Bass–Serre tree $T$. Suppose that there exists a vertex $v \in T^0$ such that $\{x_1 v, x_2 v, \dots, x_r v\}$ are descendant leaves of a tree rooted at $v$. Then the set $\{x_1, \dots, x_r\}$ freely generates a free monoid.*

## 3. Growth rates computations and estimates

We collect in this section some explicit computations and estimates on growth rates. Lemma 7, which is proved in [2, Lemma 6], will be used extensively in the proofs of Theorems 1 and 2 in combination with our ping pong lemma 6. The exact growth rates of some Baumslag–Solitar groups and lamplighters groups are computed in Lemma 8 and the last Lemma 10 allows us to compare some particular roots.

**Lemma 7.** *Let $G$ be a group generated by a finite set $S$. Suppose that there exists a set $\{x_1, \dots, x_k\} \subset G$ generating a free monoid inside $G$. Set $\ell_i = \ell_{G,S}(x_i)$, for $i = 1, \dots, k$, and $m = \max\{\ell_1, \dots, \ell_k\}$. Then $\omega(G, S)$ is greater or equal to the unique positive root of the polynomial*

$$Q(z) = z^m - \sum_{i=1}^{k} z^{m-\ell_i}. \tag{3.1}$$

As mentioned in the introduction we can easily compute the growth rate of the lamplighters and Baumslag–Solitar group with respect to the canonical generators from the growth functions found by Parry [10] and Collins, Edjvet and Gill [4] respectively. Recall that for any integer $k \geq 1$ we consider the polynomial

$$T_k(x) = x^{k+1} - x^k - 2x^{k-1} - \cdots - 2x - 2.$$

Due to Descartes rule of signs, $T_k$ has a single positive root, which we denote by $\omega_k$.

**Lemma 8.** (a) *The growth rate $\omega(\mathcal{L}_2, \{a, t\})$ is equal to $\varphi = \frac{1+\sqrt{5}}{2}$.*

(b) *For any $k \geq 1$ we have that*

$$\omega(\mathrm{BS}(1, 2k+1), \{a, t\}) = \omega(\mathcal{L}_{2k+1}, \{a, t\}) = \omega_k,$$

(c) *The growth rate $\omega(\mathrm{BS}(1, 2), \{a, t\})$ is equal to the positive root of $x^3 - x^2 - 2$.*

The equality $\omega(\mathcal{L}_2, \{a, t\}) = \varphi$ was also mentioned in [8, p.1997] by Lyons-Pemantle-Peres, and follows from the observation that there is a subtree in the Cayley graph of $\mathcal{L}_2$ which is a Fibonacci tree.

*Proof.* (a) For the wreath product $G \wr \mathbb{Z}$ one can compute the exact growth series using the following formula of Parry from [10, Corollary 3.3]. Let $\Sigma_{G,S}(x) = \sum_{m=0}^{\infty} f_{G,S}(m) x^m$ be the growth series of the group $G$ with respect to the finite generating set $S$. Then the growth series of $G \wr \mathbb{Z}$ with respect to the set $S \cup \{t\}$ can be obtained as

$$\Sigma_{G \wr \mathbb{Z}, S \cup \{t\}}(x) = \frac{\Sigma_{G,S}(x)(1 - x^2)^2 (1 + x\Sigma_{G,S}(x))}{(1 - x^2 \Sigma_{G,S}(x))^2 (1 - x\Sigma_{G,S}(x))}. \tag{3.2}$$

We use this formula to compute the growth series for $\mathcal{L}_2$.

$$\begin{aligned} \Sigma_{\mathcal{L}_2, \{a,t\}}(x) &= \frac{(1 + x)(1 - x^2)^2 (1 + x(1 + x))}{(1 - x^2(1 + x))^2 (1 - x(1 + x))} \\ &= \frac{(1 + x)(1 - x^2)^2 (1 + x + x^2)}{(1 - x^2 - x^3)^2 (1 - x - x^2)}. \end{aligned}$$

The factors in the numerator have roots on the unit circle, whilst the factors of the denominator give two roots inside the unit circle, whose reciprocals are the golden ratio $\varphi = (1 + \sqrt{5})/2$ and $\psi \approx 1.325$ (which is the so-called "plastic number"). Since $\varphi > \psi$, we get $\omega(\mathcal{L}_2, \{a, t\}) = \varphi$.

(b) Another elegant formula by Parry (see [10, Theorem 4.1]) allows to compute the growth rate of the wreath product $G \wr \mathbb{Z}$. If $S$ is a finite generating set for the group $G$ then $\omega(G \wr \mathbb{Z}, S \cup \{t\}) = 1/\kappa$, where $\kappa$ is the smallest positive zero of the function $1 - x\Sigma_{G,S}(x)$. Taking $\Sigma_{\mathbb{Z}/(2k+1)\mathbb{Z},\{a\}}(x) = 1 + 2x + 2x^2 + \cdots + 2x^{k-1}$ we get that $\omega(\mathcal{L}_{2k+1}, \{a, t\}) = 1/\kappa_k$, where $\kappa_k$ is the smallest positive root of the polynomial $R_k(x) = 1 - x - 2x^2 - \cdots - 2x^{k+1}$. The polynomials $R_k$ and $T_k$ are reciprocal, so indeed we get that $\omega(\mathcal{L}_{2k+1}, \{a, t\}) = 1/\omega_k$.

To prove that $\omega(\mathrm{BS}(1, 2k + 1), \{a, t\}) = \omega_k$ we use the following explicit formula from [4], which gives a power series $\Sigma_n(x) = \sum_{m=0}^{\infty} f(m)x^m$ for the growth function $f(m) = f_{\mathrm{BS}(1,n),\{a,t\}}(m)$. For the case $n = 2k + 1$ they obtain

$$\Sigma_n(x) = \frac{(1 + x^2 - 2x^{k+2})(1 + x - 2x^{k+2})(1 + x)^2(1 - x)^3}{(1 - x - x^2 - x^3 + 2x^{k+3})^2(1 - 2x - x^2 + 2x^{k+2})}. \tag{3.3}$$

Then the growth rate $\omega(\mathrm{BS}(1, 2k + 1), \{a, t\})$ is equal to $1/\alpha$, where $\alpha$ is the smallest positive pole of the function $\Sigma_n(x)$. Since $1 < \omega(\mathrm{BS}(1, 2k + 1), \{a, t\})$, we obtain $\alpha \in (0, 1)$. We will first prove that $\alpha = \gamma_2$, where $\gamma_2$ is the smallest positive root of the second factor

$$Q_2(x) = 1 - 2x - x^2 + 2x^{k+2}$$

of the denominator in (3.3). Let $\gamma_1$ be the smallest positive root of the first factor $Q_1(x) = 1 - x - x^2 - x^3 + 2x^{k+3}$. Note that $Q_1(0) = Q_2(0) = 1$ and $Q_1(1) = Q_2(1) = 0$, so the numbers $\gamma_1, \gamma_2$ are well defined and $0 < \gamma_1, \gamma_2 \leq 1$.

Since the difference function

$$Q_1(x) - Q_2(x) = x - x^3 + 2x^{k+2} - 2x^{k+3} = x(1 - x^2) + 2x^{k+1}(1 - x)$$

is non-negative on $[0, 1]$, we obtain that $\gamma_1 \geq \gamma_2$.

To show that $\alpha = \gamma_2$ we are left to prove that $\gamma_2$ is not a root of the numerator. Since $Q_2(1/2) = 1/2^{k+1} - 1/4 \leq 0$, we obtain that $\gamma_2 \in (0, 1/2)$. The factors $(1 + x)^2$ and $(1 - x)^3$ do not have roots on the interval $I = (0, 1/2)$, and we will check that $P_1(x) = 1 + x^2 - 2x^{k+2}$ and $P_2(x) = 1 + x - 2x^{k+2}$ have no common roots with $Q_2(x)$ on $I$. This is true, since otherwise either $Q_2(x) + P_1(x) = 2 - 2x$ or $Q_2(x) + P_2(x) = (2 + x)(1 - x)$ would have a root on $(0, 1/2)$, which is false.

We can factorize $Q_2(x)$ as $(1 - x)Z(x)$ with

$$Z(x) = 1 - x - 2x^2 - \cdots - 2x^{k+1}.$$

Since the polynomial $Z(x)$ is reciprocal to the polynomial $T(x)$ from the statement, the part (b) of the lemma is proved.

(c) Here we use another formula from [4] that is

$$\Sigma_2(x) = \frac{(1-x)^2(1+x)^2 H(x)}{(1-x-2x^3)(1-x^2-2x^5)^2},$$

where

$$H(x) = 1 + 3x + 8x^2 + 12x^3 + 16x^4 + 20x^5 + 22x^6 + 16x^7 + 14x^8 + 12x^9 + 4x^{10}.$$

We follow the same strategy as in the part (b), and first make sure that the positive root of the polynomial $Q_1(x) = 1 - x - 2x^3$ is smaller than the one of $Q_2(x) = 1 - x^2 - 2x^5$, because $Q_2(x) - Q_1(x) = x(1-x) + 2x^3(1-x^2) > 0$ on $(0, 1)$. Then, making tedious computations or using a computer, one gets that $\mathrm{GCD}(H(x), Q_1(x)) = 1$, so the smallest pole of $\Sigma_2(x)$ indeed comes from $Q_1(x)$. Again, $Q_1(x)$ is reciprocal to $x^3 - x^2 - 2$, and the part (c) is also proved. □

Now we can show that the classic lamplighter $\mathcal{L}_2$ gives the answer to Mann's question about growth of non-semidirect HNN extensions (see [9, Problem 1]), proving a part of the Theorem 1. Indeed, as $\mathcal{L}_2$ is a non-semidirect HNN extension due to Lemma 4, we may apply the Theorem 1 from [9] to get the lower bound $\Omega(\mathcal{L}_2) \geqslant \varphi$ and finally conclude that $\Omega(\mathcal{L}_2) = \varphi$.

**Remark 9.** The constant $\psi$ is quite notable. It is the smallest Pisot number and is sometimes called the "plastic number". It is shown in [2] that $\psi = \Omega(\mathrm{GL}(2, \mathbb{Z})) = \Omega(\mathrm{PGL}(2, \mathbb{Z}))$. It may be interesting to find a natural (maybe geometric) reason for the group $\mathcal{L}_2$ to have $\psi$ as "the second growth rate."

The next lemma will allow us to compare $\omega_k$ with the growth rate of some free monoid in the proof of Theorem 2.

**Lemma 10.** *Let $k \geq 1$ be an integer and $\delta_k$ be the unique positive root of the polynomial $D_k(x) = x^{2k+1} - 2x^{2k} - 2x^{2k-2} - \cdots - 2x^2 - 2$. Then*

$$\frac{1+\sqrt{5}}{2} \leq \omega_k \leq \delta_k < 1 + \sqrt{2}.$$

*Proof.* The inequality $(1+\sqrt{5})/2 \leq \omega_k$ may be proven directly, but actually we already know that $\omega(\mathrm{BS}(1, 2k+1), \{a, t\}) = \omega_k$ and $\Omega(\mathrm{BS}(1, 2k+1)) \geq (1+\sqrt{5})/2$ as proved by Mann.

Since $T_k(1), D_k(1) < 0$ and $T_k(+\infty) = D_k(+\infty) = +\infty$ we get $\delta_k, \omega_k > 1$. Consider the polynomials $D(x) = (x^2-1)D_k(x)$ and

$$T(x) = (x^2-1)T_k = (x+1)(x-1)T(x).$$

After a simple calculation we get

$$D(x) = x^{2k+3} - 2x^{2k+2} - x^{2k+1} + 2,$$

$$T(x) = x^{k+3} - x^{k+2} - 3x^{k+1} - x^k + 2x + 2.$$

As $(x^2-1) > 0$ on $(1, +\infty)$ and $D(1+\sqrt{2}) = 2 > 0$, we get that $\delta_k \in (1, 1+\sqrt{2})$.

Since $T(1) = D(1) = 0$ and $T(1+\varepsilon), D(1+\varepsilon) > 0$ for small $\varepsilon$, in order to show the inequality $\omega_k \leq \delta_k$ it suffices to show that $T(x) \geq D(x)$ on the interval $(1, 1+\sqrt{2})$.

Consider the difference function

$$\begin{aligned} D(x) - T(x) &= x^{2k+3} - 2x^{2k+2} - x^{2k+1} - x^{k+3} + x^{k+2} + 3x^{k+1} + x^k - 2x \\ &= (x^k - 1)(x^{k+1} - 1)(x^2 - 2x - 1) - (x^2 - 1). \end{aligned}$$

Since the polynomials $x^k - 1$ and $x^{k+1} - 1$ are positive on $(1, +\infty)$ and $x^2 - 2x - 1$ is negative on $(1, 1+\sqrt{2})$, we indeed have that $D(x) - T(x) < 0$ on $(1, 1+\sqrt{2})$, which proves the lemma. □

## 4. Proofs of Theorems 1 and 2

*Proof of theorem* 2. Let $G = H*_\theta$ be an HNN extension relative to an isomorphism $\theta\colon A \to B$ with $A = H$ and $B$ a normal subgroup of prime index $p$ in $H$. Let $S$ be any generating set for $G$. We need to show that $\omega(G, S) \geq (1+\sqrt{5})/2$ for $p = 2$ and $\omega(G, S) \geq \omega_k$ for $p = 2k+1$.

As explained above (see Section 2), the natural surjection $\varphi\colon G \to \mathbb{Z}$ ensures the existence of a hyperbolic element in $S$. Furthermore, upon replacing $x$ by $x^{-1}$ we can suppose that $x$ is a positive element. Since the action of $G$ is transitive on its $(p+1)$-regular Bass–Serre tree, there exists an element in $S$ not preserving the axis $L_x$ of $x$. We distinguish two cases according to this element being elliptic or hyperbolic.

Case 1 (elliptic). There exists an elliptic element $z \in S$ such that $z(L_x) \neq L_x$.

For $p = 2$, we consider the set

$$M = \{x,\ zx\},$$

while for odd primes $p = 2k+1$,

$$M = \{x,\ zx, z^2x, \ldots, z^kx,\ z^{-1}x, z^{-2}x, \ldots, z^{-k}x\}.$$

In either cases, we will show that $M$ freely generates a free monoid.

Since any vertex has only one direct ascendant, if a vertex is in the fixed point set of $z$, then all its ascendants are. For the same reason, any two ascending rays meet, so there exists a vertex of the axis of $x$ which is fixed by $z$. Let $v$ be the lowest vertex on $L_x \cap \mathrm{Fix}(z)$. Then $x(v)$ is a descendant of $v$, which is not in the set $\mathrm{Fix}(z)$, hence the vertices

$$x(v),\ zx(v), \quad \text{for } p = 2,$$

and by Lemma 5, the vertices

$$x(v),\ zx(v), \dots, z^k x(v),\ z^{-1}x(v), \dots, z^{-k}x(v), \quad \text{for odd } p = 2k+1,$$

are all distinct leaves of a tree rooted at $v$, so $M$ freely generates a free monoid due to the ping pong Lemma 6. Lemma 7 now implies that $\omega(G, S)$ is greater or equal to the unique positive root of

$$z^2 - z - 1, \quad \text{for } p = 2,$$

which is precisely the golden ratio $(1+\sqrt{5})/2$, while for $p = 2k+1$, it is greater or equal to the unique positive root of

$$T_k(z) = z^{k+1} - z^k - 2z^{k-1} - \cdots - 2z - 2,$$

which is $\omega_k$ by definition.

Case 2 (hyperbolic). There exists a hyperbolic element $y \in S$ such that $y(L_x) \neq L_x$. Upon replacing $y$ by its inverse, we can suppose that $y$ is positive hyperbolic. Since $y$ preserves its axis $L_y$, this implies that the axes $L_x$ and $L_y$ are different. This already implies that

$$\omega(\mathrm{BS}(1,p), S) \geq 2$$

(see [1, Lemma] or Lemma 7 with $\ell_1 = \ell_2 = 1$). Since for $p = 2, 3$ we have

$$\omega(\mathrm{BS}(1,2), \{a,t\}) < \omega(\mathrm{BS}(1,3), \{a,t\}) = 2,$$

we can suppose that $p \geq 5$, and again $p = 2k+1$.

We consider four subcases, according to the situations when

A. $\ell(x) = \ell(y)$,

B. $2\ell(y) < \ell(x)$,

C. $\ell(x) = 2\ell(y)$, and

D. $\ell(y) < \ell(x) < 2\ell(y)$.

Case 2a: $\ell(x) = \ell(y)$. Note that the element $yx^{-1}$ is elliptic and $yx^{-1}(L_x) \neq L_x$. We can apply the claim of Case 1 to $x$ and $z = yx^{-1}$ to conclude that the set

$$\{x, y, yx^{-1}y, \dots, (yx^{-1})^{k-1}y, xy^{-1}x, \dots, (xy^{-1})^k x\}$$

is a basis of a free monoid. Then Lemma 7 shows that

$$\omega(\mathrm{BS}(1, 2k+1), S) \geq \delta_k,$$

where $\delta_k$ is the single positive root of the polynomial

$$D_k(x) = x^{2k+1} - 2\sum_{m=0}^{k} x^{2m}.$$

Finally, Lemma 10 gives the desired inequality

$$\omega(\mathrm{BS}(1, 2k+1)) \geq \delta_k \geq \omega_k.$$

We can now suppose that $\ell(y) < \ell(x)$ and distinguish three further subcases.

Case 2b: $2\ell(y) < \ell(x)$. We will show that the infinite family

$$\{y^{-2}x, y^{-1}x, x, yx, y^2x, \dots, y^s x, \dots,$$
$$yx^{-1}yx, y^2x^{-1}yx, \dots, y^s x^{-1}xy, \dots\}$$

which is maybe better described as

$$\{y^s x \mid s \geq -2\} \cup \{y^s x^{-1} yx \mid s \geq 1\}$$

freely generates a free monoid. Then, taking as free generators only the $2k+1$ elements

$$x, yx, y^2x, \dots, y^k x, y^{-1}x, y^{-2}x, yx^{-1}yx, y^2x^{-1}yx, \dots, y^{k-2}x^{-1}xy$$

we get that $\omega(G, S)$ is by Lemma 7 greater or equal to the unique positive root of

$$T_k(z) = z^{k+1} - z^k - 2z^{k-1} - \dots - 2z - 2,$$

which is $\omega_k$ by definition.

To prove that the above infinite family freely generates a monoid, let $v_0$ be the lowest vertex on $L_x \cap L_y$ and let $v_x \in L_x$ and $v_y \in L_y$ be the corresponding direct descendants of $v_0$. We aim at applying the ping pong Lemma 6 to the vertex $w = x^{-1}(v_x)$, see Figure 4.1.

First notice that since $v_x \notin L_y$, the translates $y^s x(w) = y^s(v_x)$ are all distinct, branching from $L_y$ at $y^s(v_0)$. Furthermore, for $-2 \leq s$, the highest such translate is $y^{-2}x(w) = y^{-2}(v_x)$ which is strictly below $y^{-2}(v_0)$ by construction. Now $w = x^{-1}(v_x)$ is equal or above $y^{-2}(v_0)$ since $2\ell(y) < \ell(x)$. This already implies that the infinite subfamily $\{y^s x \mid -2 \leq s\}$ freely generates a free monoid.

Second consider the vertex $y(v_x)$. It is branching from $L_x$ at $v$ and the first vertex from $L_x \cap L_y$ to $y(v_x)$ is $v_y$. It follows that $x^{-1}y(v_x)$ does not belong to $L_x$ either and is branching at $x^{-1}(v)$ from $L_x$ and hence also from $L_y$. It follows that all the translates $y^s x^{-1} y x(w) = y^s x^{-1} y(v_x)$ belong to different branches of $L_y$, branching at $y^s x^{-1}(v_0)$. Since $\ell(y) \geq 1$, for $1 \leq s$ the branch points are below or equal to $w = x^{-1}(v_x)$.

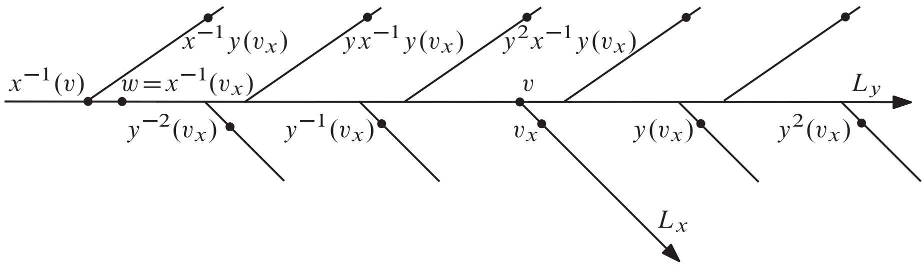


Figure 4.1. Case $2\ell(y) < \ell(x)$.

If $\ell(x)$ is not a multiple of $\ell(y)$ the two families of branching points are different and we are done. If $\ell(x) = m\ell(y)$ for some $m > 2$ we need to check that $y^{n+m}x^{-1}(v_y) \neq y^n v_x$ and it is enough to check it for $n = 0$. Consider the elliptic element $y^m x^{-1}$. It fixes $v_0$, sends $v_x$ to $v_y$ and $v_y$ to $y^m x^{-1}(v_x)$ which cannot be equal to $v_x$ otherwise the action on the direct descendants of $v_0$ of the elliptic element $y^m x^{-1}$ would not be transitive, contradicting Lemma 5.

Case 2c: $\ell(x) = 2\ell(y)$. It is enough to show that the set

$$\{x, y, xy^{-1}x, xy^{-2}x, xy^{-1}xy^{-1}x, y^2x^{-1}y, xyx^{-1}y\}$$

is a basis of a free monoid. Then, using Lemma 7 we get that $\omega(\mathrm{BS}(1,k))$ is at least $\gamma$, where $\gamma$ is the root of the polynomial $F(x) = x^5 - 2x^4 - x^2 - 3x - 1$. Since $F(x) = (x^2 - 2x - 1)(x^3 + x + 1)$, we get that $\gamma = 1 + \sqrt{2}$, and again Lemma 10 gives the desired inequality $\omega(G, S) \geq \omega_k$.

Let as above $v$ be the lowest vertex on $L_x \cap L_y$. We aim at applying the ping pong Lemma 6 to the vertex $v$. Let $v_x \in L_x$ and $v_y \in L_y$ be the corresponding direct descendants of $v_0$.

The elliptic transformation $b = y^2x^{-1}$ fixes $v$ and takes $v_x$ to $v_y$. Thus its action on the direct descendants of $v$ is nontrivial and hence transitive. Since we assume $p \geq 4$, it follows by Lemma 5 that the image $v_+ = y^2x^{-1}(v_y)$ of $v_y$ and the preimage $v_- := xy^{-2}(v_x)$ of $v_x$ give four distinct direct descendants of $v_0$ as depicted in Figure 4.2.

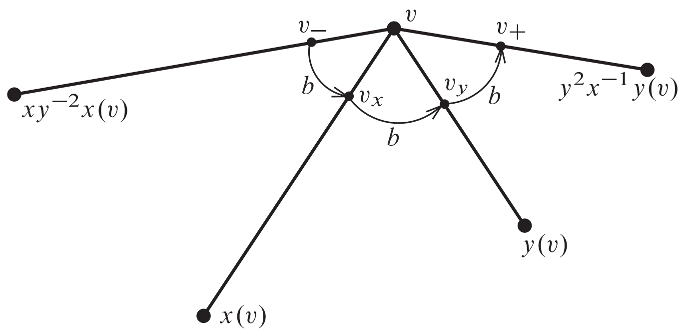


Figure 4.2. Case $\ell(x) = 2\ell(y)$: the action of the elliptic element $b = y^2x^{-1}$.

Observe that $y^2x^{-1}y(v)$ is on the branch through $v$ and $v_+$, while $xy^{-2}x(v)$ is on the branch through $v_0$ and $v_-$. Thus the four elements $xv$, $yv$, $xy^{-2}x(v)$ and $y^2x^{-1}y(v)$ have distinct geodesics to $v$.

We now forget about $xy^{-2}x(v)$ and look at the image of the tree rooted at $v$ of the three remaining elements through the hyperbolic transformation $xy^{-1}$. The root $v$ is mapped on the segment from $v$ to $x(v)$. The vertex $y(v)$ is mapped to $x(v)$, and the two remaining leaves are sent to vertices branching from $L_x$ at $xy^{-1}(v)$.

Iterating this procedure but only on $xy^{-1}(v)$, $x(v)$ and $xy^{-1}x(v)$ shows that $xy^{-1}xy^{-1}x(v)$ is branching from the segment between $xy^{-1}(v)$ and $xy^{-1}x(v)$. We have thus proven that the seven vertices are leaves of a tree rooted at $v$, as illustrated in Figure 4.3, which finishes the proof of this case.

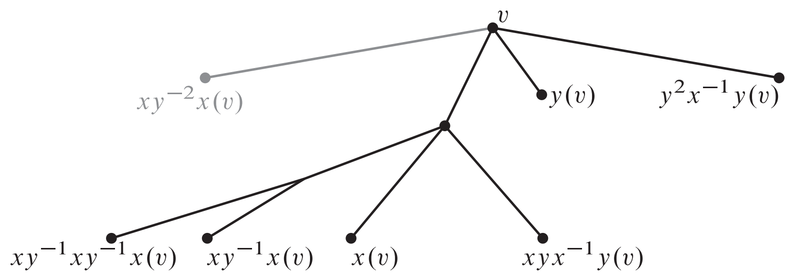


Figure 4.3. Case $\ell(x) = 2\ell(y)$: The subtree to which we apply the ping pong Lemma 6.

CASE 2D: $\ell(y) < \ell(x) < 2\ell(y)$. We will show that the set

$$\{x, y, xy^{-1}x, xy^{-2}x, yx^{-1}y\}$$

is a basis of a free monoid. Since the corresponding polynomial $x^4 - 2x^3 - 2x - 1 = x(x^2+1)(x^2-2x-1)$ has only one positive root $1+\sqrt{2}$, this will prove this case.

Set $a = \ell(x)$ and $b = \ell(y)$. The proof decomposes in the two cases $b < a \leq (3/2)b$ and $(3/2)b \leq a < 2b$ with an additional small argument needed in the equality case.

In case $b < a \leq (3/2)b$ we aim at applying the ping pong Lemma 6 to the vertex $w = xy^{-2}(v)$. (See Figure 4.4.) This vertex is on the intersection of the axes $L_x \cap L_y$ at distance $2b - a$ above $v$. Of the five images of $w$, only $x(w)$ is on the axis $L_x$, at distance $a$ below $w$ and hence $2(a-b)$ below $v$. The four other images are not in $L_x$ and we will determine their projection on $L_x$.

The image $y(w)$ is on the axis $L_y$ at distance $b$ below $w$ and hence at distance $a - b$ from its projection $v \in L_x$. Since the axis of the hyperbolic transformation $xy^{-2}$ contains $L_x \cap L_y$ and at least the vertex $v_y \in L_y$, the segment $[v, x(w)]$, which intersects $L_{xy^{-2}}$ only at $v$ is mapped by $xy^{-2}$ to the segment $[w, xy^{-2}x(w)]$ which intersect $L_{xy^{-2}}$ and hence $L_x$ only in $w$. Similarly, the axis of $xy^{-1}$ contains $L_x \cap L_y$ and at least the vertex $v_x \in L_x$, so that the hyperbolic transformation $xy^{-1}$ takes the segment $[v, x(v)]$ to the segment $[xy^{-1}(v), xy^{-1}x(v)]$ which intersects $L_{xy^{-1}}$ and hence $L_x$ precisely in $xy^{-1}(v)$ which is at distance $a - b$ from both $v$ and $x(v)$. Finally, the axis of $yx^{-1}$ contains $L_x \cap L_y$ and at least the vertex $v_y \in L_y$, so that applying $yx^{-1}$ to the segment $[v, y(w)]$ we obtain the segment $[yx^{-1}(v), yx^{-1}y(w)]$ which intersects $L_x$ in $yx^{-1}(v)$ which is at distance $a - b$ above $v$ and hence at distance $3b - 2a \geq 0$ below $w$. If the inequality is strict, the claim immediately follows from the ping pong Lemma 6. If $3b - 2a = 0$, we will see below how to show that the segments $[yx^{-1}(v), yx^{-1}y(w)]$ and $[w, xy^{-2}x(w)]$ only intersect at $w = yx^{-1}(v)$.

If $(3/2)b \leq a < 2b$ the argument is completely analogous, except that the vertex $yx^{-1}(v)$ is above or equal to $w = xy^{-2}(v)$. Thus we want to replace $w$ by $w' := yx^{-1}(v)$ and apply the ping pong Lemma 6 to this vertex $w'$. (See Figure 4.5.) This vertex is on the intersection of the axes $L_x \cap L_y$ at distance $a - b$ above $v$. Of the five images of $w'$, only $x(w')$ is on the axis $L_x$, at distance $a$ below $w$ and hence $b$ below $v$. The four other images are not in $L_x$ and we will determine their projection on $L_x$.

The image $y(w')$ is on the axis $L_y$ at distance $b$ below $w$ and hence at distance $2b - a$ from its projection $v \in L_x$. For the three other image points, the proof is identical to the above case, replacing $w$ by $w'$.

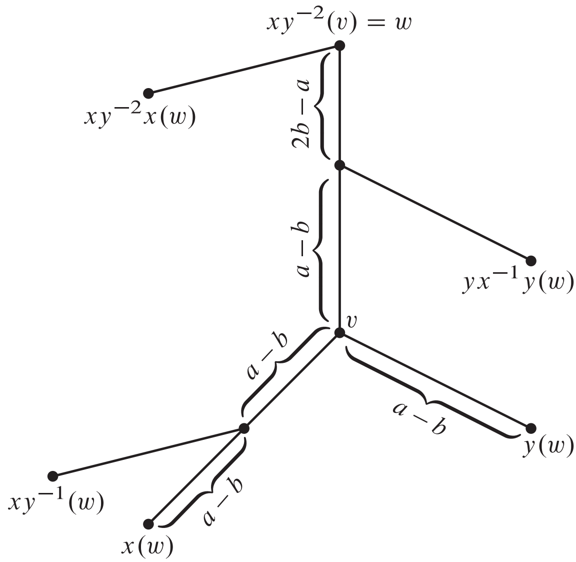


Figure 4.4. Case $b < a < 3/2b$.

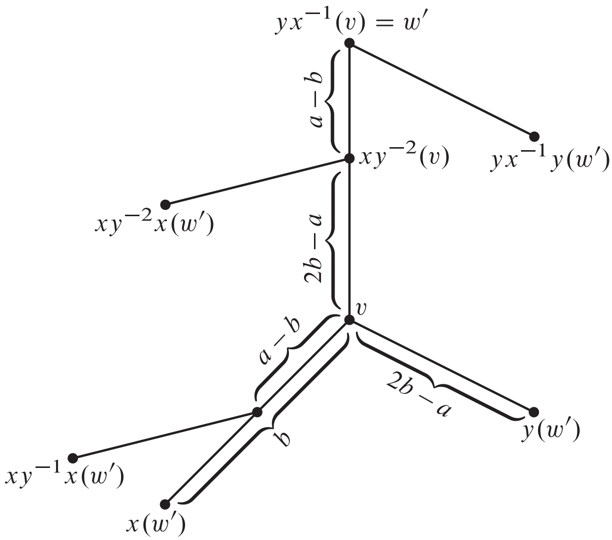


Figure 4.5. Case $3/2b < a < 2b$.

In the equality case the two vertices $w = w'$ agree. Let $v_1$, respectively $v_2$ be the first vertex after $w$ on the geodesic to $xy^{-2}(w)$, respectively $yx^{-1}y(w)$. We need to show that $v_1 \neq v_2$. Let $v_a$ be the direct descendant of $w$ on the geodesic to $v$. The ordered pair $(v_1, v_a)$ is mapped to $(v_x, v_y)$ by $y^2x^{-1}$, which are further mapped to $(v_a, v_2)$ by $yx^{-1}$. Thus the elliptic element $yx^{-1}y^2x^{-1}$ sends the ordered pair $(v_1, v_a)$ to $(v_a, v_2)$ and since $p \geq 3$ and elliptic elements act either trivially or transitively on direct descendants of a fixed point by Lemma 5 it follows that $v_1 \neq v_2$, which finishes the proof of this case and of the theorem. □

*Proof of Theorem* 1. In view of Lemma 8, Theorem 1 follows immediately from Theorem 2 except in the case of $\mathrm{BS}(1,2)$ where we need a better understanding of its action on the Bass–Serre tree to obtain the accurate lower bound of

$$\omega(\mathrm{BS}(1,2), \{a, t\}) = \beta,$$

where $\beta$ is the unique real root of $x^3 - x^2 - 2$.

Let $S$ be a generating set for $\mathrm{BS}(1,2)$. As in the proof of Theorem 2, the case where $S$ contains two hyperbolic elements with different axis immediately gives the lower bound of $\omega(\mathrm{BS}(1,2), S) \geq 2 > \beta$. We thus only have to treat the corresponding elliptic case, that is, there exists a positive hyperbolic element $x \in S$ with axis $L_x$ and an elliptic element $z \in S$ such that $z(L_x) \neq L_x$.

As observed in the elliptic case of the proof of Theorem 2 the intersection of $L_x$ with the fixed point set of $z$ is nonempty. Upon conjugating the generating set $S$, we can suppose that the lowest vertex on $L_x$ fixed by $z$ is $A$, which implies that $z$ belongs to $A$. Since $z$ does not fix the direct descendants $tA$ and $atA$ it must be an odd power of $A$.

Consider the action of $a$ on the second generation of descendants of $A$, that is $t^2A, tatA, at^2A$ and $atatA$. The action has order four, mapping $t^2A \mapsto at^2A \mapsto a^2t^2A = tatA \mapsto atatA \mapsto a^2tatA = t^2A$. The action of $z$, as an odd power of $A$ is thus necessarily equal to the action of $a$ or $a^{-1}$ on these second generation descendants. It follows that $xA, zx^2A$ and $z^{-1}x^2A$ are leaves of a tree rooted at $A$, and hence $x, zx^2, z^{-1}x^2$ generate a free monoid by the ping pong Lemma 6. Since these elements have lengths 1, 3 and 3 respectively, we can invoke Lemma 7 to conclude that the grow rate of $\mathrm{BS}(1,2)$ with respect to $S$ is greater or equal to the greatest and unique real root of $x^3 - x^2 - 2$. Finally, Lemma 8 gives

$$\omega(\mathrm{BS}(1,2), S) \geq \omega(\mathrm{BS}(1,2), \{a, t\}),$$

which finishes the proof of the theorem. □

## 5. The lamplighter group $\mathbb{Z} \wr \mathbb{Z}$

The groups $\mathcal{L}_n = (\mathbb{Z}/n\mathbb{Z}) \wr \mathbb{Z}$ are factor groups of the wreath product $\mathbb{Z} \wr \mathbb{Z}$. Actually, the following nice fact is also true.

**Proposition 11.** *The groups* $\mathrm{BS}(1,n)$ *are factor groups of the group* $\mathbb{Z} \wr \mathbb{Z}$.

*Proof.* As seen above, the groups $\mathbb{Z} \wr \mathbb{Z}$ and $\mathrm{BS}(1,n)$ can be presented as

$$\mathbb{Z} \wr \mathbb{Z} = \langle a, t \mid [a, t^k a t^{-k}] = 1,\ k \in \mathbb{N} \rangle, \tag{5.1}$$

$$\mathrm{BS}(1,n) = \langle a, t \mid t a t^{-1} = a^n \rangle. \tag{5.2}$$

According to (5.2), for every positive $k$ the element $t^k a t^{-k}$ is a power of $a$, hence it commutes with $a$, so the corresponding relation in (5.1) holds true. □

We will see below that $\lim_{k\to\infty}(\omega(\mathrm{BS}(1,2k+1),\{a,t\})) = 1+\sqrt{2} = \omega(\mathbb{Z} \wr \mathbb{Z}, \{a,t\})$, which is some further evidence for the fact that $\mathbb{Z} \wr \mathbb{Z}$ is a limit of the groups $\mathrm{BS}(1,n)$ in the marked groups space topology (see [13, Theorem 2]).

The next lemma will be needed to prove Corollary 3.

**Lemma 12.** *The limit* $\lim\limits_{k\to\infty} \omega_k$ *exists, and it is equal to* $1+\sqrt{2}$.

*Proof.* From Lemma 10 and the definition of $\omega_k$ we know that $\omega_k$ is a single positive root of the polynomial $T_k(x)$, and $(1+\sqrt{5})/2 < \omega_k < 1+\sqrt{2}$ for every $k \geq 1$. Then the reciprocal polynomial $R_k(x) = 1 - x - 2x^2 - \cdots - 2x^k - 2x^{k+1}$ has a single positive root $1/\omega_k$ which belongs to the interval $I = (1/3, 2/3)$. Consequently the polynomial

$$R_k'(x) = (1-x)R_k = (1-x)^2 - 2x^2(1-x^k) = 1 - 2x - x^2 + 2x^{k+2}$$

also has two positive roots: 1 and $1/\omega_k$. Obviously, for $k \to \infty$ the polynomials $2x^{k+2}$ uniformly converge to the zero function on the enlarged interval $I' = (1/4, 3/4)$. For this reason the roots $1/\omega_k$ of $R_k'(x)$ on $I$ converge to the root of the polynomial $1 - 2x - x^2$ on $I$, and the latter root is equal to $\sqrt{2} - 1 = 1/(1+\sqrt{2})$, which proves the lemma. □

*Proof of Corollary 3.* We use Parry's formula (3.2) to compute the series $\Sigma(x)$ for the growth function $\mathbb{Z} \wr \mathbb{Z}$ with respect to the generating set $\{a,t\}$:

$$\Sigma(x) = \frac{(1-x^2)^3(1+x^2)}{(1-x-x^2-x^3)^2(1-2x-x^2)}.$$

All the roots of the numerator lie on the unit circle, while the denominator has only two roots inside the unit circle, whose reciprocals are $\alpha = 1+\sqrt{2}$ and $\beta \approx 1.839$. Hence, $\omega(\mathbb{Z} \wr \mathbb{Z}, \{a,t\}) = 1+\sqrt{2}$.

Now we will show that $\Omega(\mathbb{Z} \wr \mathbb{Z}) = 1+\sqrt{2}$. We already know that $\Omega(\mathbb{Z} \wr \mathbb{Z}) \leq 1+\sqrt{2}$. Suppose that $\Omega(\mathbb{Z} \wr \mathbb{Z}) = 1+\sqrt{2}-\varepsilon$, where $\varepsilon > 0$. As any group $\mathcal{L}_p$ is a factor group of the group $\mathbb{Z} \wr \mathbb{Z}$, then for any prime $p$ we have $\Omega(\mathcal{L}_p) \leq 1+\sqrt{2}-\varepsilon$ which contradicts the equality $\lim\limits_{p\to\infty} \Omega(\mathcal{L}_p) = \lim\limits_{k\to\infty} \omega_k = 1+\sqrt{2}$ proven in Lemma 12. □

## References

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