Groups Geom. Dyn. 11 (2017), 291–310
DOI 10.4171/GGD/397

Groups, Geometry, and Dynamics

Finitely presented groups and the Whitehead nightmare

Daniele Ettore Otera and Valentin Poénaru

Abstract. We define a "nice representation" of a finitely presented group Γ as being a non-degenerate essentially surjective simplicial map f from a "nice" space X into a 3-complex associated to a presentation of Γ, with a strong control over the singularities of f, and such that X is WGSC (*weakly geometrically simply connected*), meaning that it admits a filtration by simply connected and compact subcomplexes. In this paper we study such representations for a very large class of groups, namely QSF (*quasi-simply filtered*) groups, where QSF is a topological tameness condition of groups that is similar to, but weaker than, WGSC. In particular, we prove that any QSF group admits a WGSC representation which is locally finite, equivariant and whose double point set is closed.

Mathematics Subject Classification (2010). 57M05; 57M10; 57N35.

Keywords. Finitely presented groups, weak geometric simple connectivity, quasi-simple filtration, universal covering spaces, singularities.

1. Introduction

The present paper deals with finitely presented groups satisfying a rather mild tameness condition, the QSF *property* introduced and studied by Brick and Mihalik in [1]. Roughly speaking, a space is QSF (*quasi-simply filtered*) if any compact subspace of it can be "approximated" by an (abstract) simply connected compact space; in particular this means that a QSF space admits a "quasi-filtration" by compact and simply connected subspaces. We may also give for it the following equivalent definition (that is actually a theorem of the first author (D. Otera) with L. Funar [5]): the finitely presented group Γ is QSF if and only if there exists a smooth compact manifold M such that $\pi_1 M = \Gamma$ and whose universal cover $\widetilde{M}$ is *weakly geometrically simply connected* (WGSC), meaning that $\widetilde{M}$ admits a filtration by compact and simply connected submanifolds. But one should keep in mind that this simpler condition is *not* group-presentation invariant, while the original definition of Brick and Mihalik, which we will define properly in the next section, is. This is actually one of the important virtues of the concept QSF.

Although usually a (topological) presentation of a finitely presented group Γ is just any finite simplicial complex K such that $\pi_1 K = \Gamma$, in this present paper, we will be very choosy for them (see Section 2). Actually, all along the paper, a *presentation* of Γ will *always* be a particular 3-dimensional complex obtained by a suitable thickening of a 2-complex (or, in other words, a compact 3-manifold with *singularities*, locally as (the wedge of three lines)$\times\mathbb{R}^2$; this is enough for catching all finitely presented groups.) Let us denote by $M(\Gamma)$ such a 3-complex with $\pi_1 M(\Gamma) = \Gamma$, and by $\widetilde{M(\Gamma)}$ its universal cover (we will not recall anymore they are 3-dimensional spaces).

Our basic tool for dealing with finitely presented groups will be the notion of WGSC-REPRESENTATION, which we will define formally in the next section. It will suffice to say, for right now, that contrary to the more usual group representations which, for a group Γ, take the general form "$\Gamma \to$ something," our REPRESENTATIONS, which we will always write in capital letters, take the dual form "some space X with special features $\xrightarrow{f} \widetilde{M(\Gamma)}$" (and note that the universal covering space $\widetilde{M(\Gamma)}$ is the same thing as the group Γ, up to quasi-isometry).

This triple, $X \xrightarrow{f} \widetilde{M(\Gamma)}$, is endowed with the following properties: X is a simplicial complex which is *weakly geometrically simply connected* (WGSC), f is a *non-degenerate* simplicial map, meaning that $f(d\text{-simplex}) = d\text{-simplex}$, and, furthermore, the map f is *zippable*, by which we intend that the "smallest" equivalence relation on X which is compatible with f and which is also such that the quotient space immerses into $\widetilde{M(\Gamma)}$, via the obviously induced map, is the trivial equivalence relation induced by f itself, namely: $x \sim y \iff f(x) = f(y)$. In other words, what zippability means is that the "cheapest" way to kill all the *singularities* of f (that are the points $x \in X$ where f is not locally an embedding, i.e. the non-immersive points of X), is to kill all the double points of f; and this will actually happen via folding maps.

Here are some additional explanations concerning this definition, which was given now rather informally, and which will be restated rigorously in the next section. First of all, the notion of *weak geometric simple connectivity* (WGSC) has been introduced by L. Funar (see [3]), and also studied by him and the first author (D. Otera), in the context of geometric group theory (see [5]).

Definition 1. A locally finite simplicial complex X is said to be *weakly geometrically simply connected* (WGSC), if it has an exhaustion by finite (compact) and simply connected subcomplexes $K_1 \subset K_2 \subset \cdots \subset X = \bigcup_i K_i$.

But when it comes to groups, WGSC is not the more appropriate condition, since it is not presentation independent, and the good one is the QSF property introduced by Brick and Mihalik [1] (see also [5] and [9]).

Definition 2. A locally compact simplicial complex X is QSF (i.e. *quasi-simply filtered*) if for any compact subcomplex $k \subset X$ there is a simply-connected compact (abstract) complex K endowed with an inclusion $k \xrightarrow{j} K$ and with a simplicial map $K \xrightarrow{f} X$ satisfying the *"Dehn condition"*: $M_2(f) \cap j(k) = \emptyset$ (where $M_2(f) \subset K$ denotes the set of double points of f), and entering in the following commutative diagram:

$$\begin{tikzcd} k \arrow[rr, hook, "j"] \arrow[dr, "i"'] & & K \arrow[dl, "f"] \\ & X & \end{tikzcd} \tag{1}$$

where i is the canonical injection.

Then, being QSF means that the space X can just be "approximated" by a filtration of simply connected and compact subspaces; but what one gains is very valuable, since, unlike WGSC, QSF turns out to be a group theoretical, presentation-independent notion: if K_1, K_2 are two presentations (i.e. presentation complexes) for the same finitely presented group Γ, then $\widetilde{K}_1 \in$ QSF $\iff$ $\widetilde{K}_2 \in$ QSF (see [1]), and in such a case Γ is said QSF. Recently, in the group theoretical context, L. Funar and the first author (D. Otera) [5] have proved that if Γ has a presentation K_1 such that $\widetilde{K}_1 \in$ QSF, then it also has a presentation K_2 such that $\widetilde{K}_2 \in$ WGSC.

Coming back to the non-degeneracy of f, note that this means, among other things, that the dimension of the REPRESENTATION space X, source of f, is restricted to dim $X \leq 3$; and the only serious cases are actually dim $X = 2$ and dim $X = 3$, each interesting in its own right.

So, we will speak about 2^d- and 3^d-REPRESENTATIONS, and the capital letters should remind the reader that we are not talking about the mundane group representations, where the dimension of the representation means quite a different thing. Retain also that our WGSC-REPRESENTATIONS $X \xrightarrow{f} \widetilde{M(\Gamma)}$ are sort of *resolutions* of $\widetilde{M(\Gamma)}$ into a WGSC space X.

With all these things, here is our main result, whose statement will become more precise (see Theorem 2) in the next sections, after the REPRESENTATIONS are more formally defined.

Theorem 1. *For any finitely presented QSF group Γ, there exists a 2^d-WGSC-REPRESENTATION $X^2 \xrightarrow{f} \widetilde{M(\Gamma)}$, where the simplicial complex X^2 is locally-finite and WGSC, such that*

(i) *both* $f(X^2) \subset \widetilde{M(\Gamma)}$ *and the double point set*

$$M_2(f) = \{x \in X^2 \mid \sharp\{f^{-1}(f(x))\} > 1\} \subset X^2$$

are ***closed*** *subsets*;

(ii) *moreover, one can get an* X^2 *with a free* Γ*-action* $\Gamma \times X^2 \to X^2$ *such that* f *is* ***equivariant****, i.e.* $f(\gamma x) = \gamma x$ *for all* $\gamma \in \Gamma$, $x \in X^2$.

Definition 3. A REPRESENTATION satisfying condition (i) above will be called *easy*.

Thus, we may also rephrase Theorem 1 as follows: finitely presented QSF groups admit easy WGSC-REPRESENTATIONS.

Since WGSC is a weak and simplified version of the more known GSC (*geometric simple connectivity*) concept, which stems from differential topology, and which concerns handle decompositions without handles of index 1 (for more on this important notion see e.g. [3, 5, 6, 8, 12, 18]), we propose, at least informally, the following definition and the associated conjecture:

Definition 4. A finitely presented group Γ is *easy* (or *easily-representable*) if it admits a 2-dimensional GSC-REPRESENTATION $X^2 \xrightarrow{f} \widetilde{M(\Gamma)}$ (namely with a GSC X^2) which is easy, in the sense just defined (i.e. with closed $f(X) \subset \widetilde{M(\Gamma)}$ and $M_2(f) \subset X^2$).

Conjecture 1. *All finitely presented* QSF *groups are easy* (*i.e. easily-representable*).

Remark 1.1. (1) The converse implication is already a theorem proved by the authors in [7].

(2) The second author (V. Poénaru) has developed a program [17, 18, 19] aiming to prove that *all* finitely presented groups are QSF.

(3) From papers like [12, 13, 15], it can be extracted a proof of the following general form: if a finitely presented group satisfies a nice geometric condition (like e.g. Gromov-hyperbolicity, almost-convexity, automaticity, combability etc.), then it is easy.

Coming back now to the REPRESENTATIONS, with which this paper deals, they were already present in the paper [14], where homotopy 3-spheres Σ^3 were REPRESENTED. Then, in [12, 13, 15, 16], universal covering spaces of compact 3-manifolds have been REPRESENTED, while in [20], REPRESENTATIONS of the classical *Whitehead manifold* Wh^3 [25] were investigated and what was found there, was that for the simplest and most natural 2^d-REPRESENTATIONS of it, $X^2 \xrightarrow{f} Wh^3$, the $M_2(f) \subset X^2$ is *not* a closed subset. (Hence, the $M_2(f)$ being closed can be viewed as on obstruction for a complex to admit a cocompact free action of an infinite group).

Note also that the notion of REPRESENTATION of a group Γ, $X \xrightarrow{f} \widetilde{M(\Gamma)}$, has a-priori nothing group-theoretical about it, except that it allows the possibility of a free action $\Gamma \times X \longrightarrow X$, with an *equivariant* f, i.e. $f(gx) = gf(x)$; point (ii) in Theorem 1 brings this option to life.

In the next section we will state more formally, and with more details, what the paper actually proves. Then Theorem 1 will appear as a piece of some bigger, more comprehensive statement. This will deal with 3-dimensional REPRESENTATIONS too, and then the "*Whitehead nightmare*" appearing in the title of this paper will be explained too.

Acknowledgments. We wish to thank Louis Funar, David Gabai, Frédéric Haglund, and Barry Mazur for very useful conversations and remarks. In particular Louis Funar helped us to fix up a first version of our paper. We are also very grateful to the referee for his corrections and for his appropriate and constructive suggestions to improve the readability of the paper.

The first author (D. Otera) was funded by a grant from the Research Council of Lithuania (Researcher teams' project No. MIP-046/2014/LSS-580000-446). He also thanks the Mathematics Department of the University of Paris-Sud 11 for the hospitality during research visits.

2. Definitions and statements of the results

We will give now, with full details, the definition of the REPRESENTATIONS for finitely presented groups Γ, which were only very informally presented in the last section.

To begin with, like in [17], we consider (topological) presentations for Γ which are singular compact 3-manifolds with non-empty boundary, denoted by $M(\Gamma)$. The structure of such an $M(\Gamma)$ is very simple (see e.g. [7]). Start with a compact 3-dimensional handlebody of some appropriate genus g, call it H; this embodies the generators of the group Γ. Then 2-handles are attached to H, embodying the relations of Γ. Explicitly, the attaching zones are given by an *immersion*

$$\sum_{j=1}^{k} (S_j^1 \times [0,1]) \xrightarrow{\phi} \partial H, \tag{2}$$

which injects on each individual $S_j^1 \times I$, the double points coming from (singular) little squares $S \in \partial H$, where $\phi(S_l^1 \times I)$ and $\phi(S_m^1 \times I)$, for $m \neq l$, go through each other. These *immortal singularities* S are the points where $M(\Gamma)$ fails to be a 3-manifold.

Now we are ready to give the precise and formal definition of WGSC-REPRESENTATIONS for finitely presented groups, leaving more details and comments just after the definition.

Definition 5. A WGSC-REPRESENTATION of a finitely presented group Γ is a simplicial map

$$X \xrightarrow{f} \widetilde{M(\Gamma)}, \tag{3}$$

where $\widetilde{M(\Gamma)}$ is the universal cover of $M(\Gamma)$, which satisfies the following list of conditions:

the space X is a countable simplicial complex which is *not* necessarily assumed to be locally-finite; but it is assumed to be *weakly geometrically simply connected* (WGSC); (3-1)

the simplicial map f is *non-degenerate*, which also means that $\dim X \leq 3$. Hence, once the meaningless case $\dim X = 1$ is discarded, we are left with the two meaningful cases $\dim X = 2$ and $\dim X = 3$, namely with 2- and 3-dimensional REPRESENTATIONS; (3-2)

the equality $\Psi(f) = \Phi(f)$ holds (see the explanation here below), and in this case we say that f is *zippable*; (3-3)

the map f is "essentially surjective," which means the following: if $\dim X = 3$, then $\overline{\operatorname{Im} f} = \widetilde{M(\Gamma)}$, and if $\dim X = 2$, then $\widetilde{M(\Gamma)} = \overline{\operatorname{Im} f} + \{\text{cells of dimension 2 and 3}\}$. (3-3)

Here, some remarks and details are needed. First of all, concerning (3-3) above, consider a non-degenerate simplicial map $g \colon A \to B$, like, for instance, our map f from (3); for any such a map we define the set of *mortal singularities*, $\operatorname{Sing}(g) \subset A$, as being the set of those points $x \in A$, at which g fails to be immersive. There are two interesting equivalence relations on A, in this context. To begin with, we have the trivial one $\Phi(g) \subset A \times A$, where $(x, y) \in \Phi(g) \iff g(x) = g(y)$. Then (and see here [11, 17] for more details) there is the following more subtle equivalence relation $\Psi(g) \subset \Phi(g)$, which is defined as follows (and it can be proved that this definition makes sense, see [11]): $\Psi(g) \subset A \times A$ is the "smallest" equivalence relation compatible with g, which kills all the mortal singularities, i.e. which is such that in the following diagram the map g_1 is an immersion (i.e. $\operatorname{Sing}(g_1) = \emptyset$)

$$\begin{tikzcd}
A \arrow[rr, "g"] \arrow[dr, "\pi"'] & & B \arrow[dl, "g_1"] \\
& A/\Psi(g). &
\end{tikzcd}$$

It can be shown that there is a *uniquely* well-defined equivalence relation $\Psi(g)$ (constructed via a sequence of *folding maps*) with the properties listed above, and that it has the additional property that the following induced map is *surjective*

$$\pi_1(A) \xrightarrow{(g_1)_*} \pi_1(A/\Psi(g)).$$

Details concerning the equivalence relations Ψ and Φ can be found in [11, 17].

Remark 2.1. It should be stressed that, the general definition of REPRESENTATION $X \overset{f}{\to} Y$ is such that the object Y which is REPRESENTED, automatically comes with $\pi_1 Y = 0$.

For any finitely presented group, it can be shown that REPRESENTATIONS always exist [17]; but usually, the simplest REPRESENTATIONS which one stumbles upon *fail* to be locally finite. On the other hand, in this paper, only REPRESENTATIONS of groups with a *locally-finite* X will be considered.

Many other objects can be REPRESENTED, provided they are simply connected. The definition is always exactly the same, but what is special when one represents groups, which comes automatically with the canonical action $\Gamma \times \widetilde{M(\Gamma)} \longrightarrow \widetilde{M(\Gamma)}$, is that there is then the possibility that the REPRESENTATION $X \xrightarrow{f} \widetilde{M(\Gamma)}$ may be *equivariant*, meaning that there may be a second free action $\Gamma \times X \longrightarrow X$, coming with $f(\gamma x) = \gamma f(x)$ for all $\gamma \in \Gamma,\ x \in X$.

Without any additional assumption on the triple $X \xrightarrow{f} \widetilde{M(\Gamma)}$ from (3) above, there is a *metric structure*, well-defined up to *quasi-isometry*, which permeates this whole story. Chose any Riemannian metric on $M(\Gamma)$, and what we mean by this is the following. On each individual 3-dimensional handle H_i^λ of $M(\Gamma)$, a Riemannian metric is given and, whenever two handles are incident, it is required that the induced metrics on the intersection should coincide. Then, using the non trivial free group action $\Gamma \times \widetilde{M(\Gamma)} \longrightarrow \widetilde{M(\Gamma)}$, the arbitrarily chosen Riemannian metric on $M(\Gamma)$ lifts to an equivariant metric on $\widetilde{M(\Gamma)}$. Finally, one lifts this metric on X, via the non-degenerate map $f : X \to \widetilde{M(\Gamma)}$. Thus, X becomes a metric space and, up to quasi-isometry, this metric on X is *canonical*, i.e. independent of the original choice of Riemannian metric on $M(\Gamma)$.

Let us fix now a compact *fundamental domain* $\delta \subset \widetilde{M(\Gamma)}$, such that $\widetilde{M(\Gamma)} = \bigcup_{\gamma \in \Gamma} \gamma\delta$. In a similar vein, we consider "*large fundamental domains*" $\Delta \subset X$, and a locally finite decomposition of X into such domains, $X = \bigcup_{j \in J} \Delta_j$, where J is some countable set of indices. Since there is no group action on X (in the general case, at least), what we will ask now from the compact pieces Δ_j above, apart from the obvious condition that their interiors should be disjoined, is the existence of two positive constants $C_2 > C_1 > 0$ such that we should have

$$C_1 \leq \|\Delta_j\| \leq C_2, \quad \text{for all } j \in J. \tag{3-5}$$

Here $\|\Delta_j\|$ is the diameter of Δ_j. Our large fundamental domains Δ_j could be, for instance, maximal dimensional cells of the cell-decomposition of the representation space X occurring in (3), satisfying the metric condition (3-5), when $j \to \infty$. The next Theorem 2, stated below, has two parts corresponding to the dimension of X, in a 3-dimensional REPRESENTATION this is $X = X^3$, while in a 2-dimensional REPRESENTATION it is $X = X^2$. In both cases we have also *immortal singularities*, $\operatorname{Sing}\big(\widetilde{M(\Gamma)}\big) \subset \widetilde{M(\Gamma)}$, and *mortal singularities*, $\operatorname{Sing}(f) \subset X$.

At least in the 2^d case, we will want to be a bit more specific about the singularity issues, and so, when it comes to the 2-dimensional part of the Theorem 2 stated below, the following condition will be imposed too

> the set of mortal singularities $\operatorname{Sing}(f) \subset X^2$ is *discrete* and, at each $x \in \operatorname{Sing}(f)$, there is the following local model. (3-6)

There is an open neighborhood $P = P_1 \cup P_2$ of x in X^2 and an embedding $\mathbb{R}^3 \longrightarrow \widetilde{M(\Gamma)}$ (which, a priori, might happily go through $\operatorname{Sing} \widetilde{M(\Gamma)}$), through which $P \xrightarrow{f} \widetilde{M(\Gamma)}$ factorizes. At the source X^2, the P_1, P_2 are two planes $\mathbb{R}^2$ glued along a half-line $[0, \infty)$ with $x = 0$, x being here our mortal singularity.

In the diagram below

$$\begin{tikzcd}
P \arrow[rr, "j"] \arrow[dr, "f"'] & & \mathbb{R}^3 \arrow[dl] \\
& \widetilde{M(\Gamma)} &
\end{tikzcd}$$

each $j|_{P_1}, j|_{P_2}$ injects, the two being transverse. So, there is a double line in $M_2(f)$ starting at the mortal singularity x. This is a local model already used by the second author (V. Poénaru) in [14], where, according to a suggestion of Barry Mazur, these singularities were called "*undrawable.*"

For our REPRESENTATION (3), we will also assume that

$$f(\operatorname{Sing}(f)) \cap \operatorname{Sing}(\widetilde{M(\Gamma)}) = \emptyset. \tag{3-7}$$

But, at the later stages in the zipping of f, this condition may be violated. Then, besides the $\operatorname{Sing}(f) \subset X^2$, there is also another set of immortal singularities, $\operatorname{Sing}(X^2) \subset X^2 - \operatorname{Sing}(f)$, which is also *discrete*. This comes with the inclusion $f(\operatorname{Sing}(X^2)) \subset \operatorname{Sing}(\widetilde{M(\Gamma)})$. At the points $x \in \operatorname{Sing}(X^2)$, there are no local factorizations

$$\begin{tikzcd}
& V \subset \mathbb{R}^3 & \\
X^2 \supset U \arrow[ur] \arrow[rr, "f"'] & & \widetilde{M(\Gamma)}, \arrow[ul]
\end{tikzcd}$$

and it is their absence which makes the $x \in \mathrm{Sing}(X^2)$ be an immortal singularity, never to be killed by the zipping. But, in purely topological terms, and forgetting about f, at one immortal singularity $x \in X^2$, the X^2 looks exactly alike as at a mortal singularity. This ends our digression on $\mathrm{Sing}(f)$.

We are now ready to state with all details the main result of the present paper.

Theorem 2 (main theorem). (1) (3^d*-part*) *For any finitely presented* QSF *group* Γ, *there exists a locally finite* 3^d*-*WGSC-REPRESENTATION, $X^3 \xrightarrow{f} \widetilde{M(\Gamma)}$, *such that the following conditions are satisfied for any* $\gamma \in \Gamma$:

$$\text{there is a free action } \Gamma \times X^3 \longrightarrow X^3, \text{ and } f \text{ is } \textbf{equivariant}; \tag{4}$$

there is a constant $C = C(\|\gamma\|) > 0$ *depending on the word-length of* γ *such that*

$$\#\{\Delta_i \mid f(\Delta_i) \cap \gamma\delta \neq \emptyset\} < C. \tag{5}$$

In particular, any given domain $\gamma\delta \subset \widetilde{M(\Gamma)} = \bigcup_{\gamma\in\Gamma} \gamma\delta$ *downstairs, can only be hit* ***finitely many*** *times by the image of a large domain* $\Delta \subset X^3 = \bigcup_{j\in J} \Delta_j$ *from upstairs.*

(2) (2^d*-part*) *For any finitely presented* QSF *group* Γ, *there exists a locally finite* 2^d*-*WGSC-REPRESENTATION $X^2 \xrightarrow{f} \widetilde{M(\Gamma)}$ *which is both* ***equivariant***, *like in* (4), *and which also satisfies the following condition*:

both

$$f(X^2) \subset \widetilde{M(\Gamma)}$$

and

$$M_2(f) = \{x \in X^2 \mid \sharp\{f^{-1}(f(x))\} > 1\} \subset X^2 \tag{6}$$

are ***closed.***

Remark 2.2. For a generic 3^d-REPRESENTATION $X^3 \xrightarrow{f} \widetilde{M(\Gamma)}$, one normally finds the following situation, at the opposite pole with respect to our (5) above, and which, in papers like [16], the second author (V. Poénaru) has called the *Whitehead nightmare*

$$\#\{\Delta_i \colon f(\Delta_i) \cap \gamma\delta \neq \emptyset\} = \infty, \quad \text{for all } \gamma \in \Gamma. \tag{5*}$$

Our present Whitehead nightmare under discussion, should remind the reader of the basic structure of the classical Whitehead manifold Wh^3, see [25] (whence the name of our nightmare), of the *Casson Handle* [4], or of the *gropes* of M. Freedman and F. Quinn [2].

So, the first part of our Theorem means that QSF finitely presented groups can *avoid* the Whitehead nightmare, and this is what the title of the present paper refers to.

The 2^d counterpart of the Whitehead nightmare (5*) is the following condition

$$M_2(f) \subset X^2 \text{ is } \textit{not} \text{ closed.} \tag{6*}$$

This *is* the generic situation for 2^d-REPRESENTATIONS and one has to start by living with it and look at the accumulation pattern of $M_2(f)$ inside X^2, all this being studied in [17, 18, 19].

Remark 2.3. (1) The representation spaces X occurring in the two points above are, of course, distinct spaces, although not quite totally unrelated, as we shall see.

(2) Equation (5) in Theorem 2 can also be replaced by the following variant: there exist *equivariant triangulations* for $\widetilde{M(\Gamma)}$ and for X^3, and also a constant C' such that, for any simplex $\sigma \subset \widetilde{M(\Gamma)}$, we should have

$$\#\{\text{simplexes } S \subset X^3 \colon f(S) \cap \sigma \neq \emptyset\} < C'. \tag{5-bis}$$

3. Preliminaries lemmas

We give now the beginning of the proof of Theorem 2. Some technicalities will be postponed until the next section. Since Γ is QSF, this also means that $\widetilde{M(\Gamma)} \in$ QSF (because $M(\Gamma)$ is a compact 2-complex associated to a presentation of Γ). Since our 3-dimensional complex $\widetilde{M(\Gamma)}$ has singularities, we prefer to replace it by a smooth, albeit higher dimensional, object.

Let $\mathcal{R}$ be a *resolution* of the singularities of $\widetilde{M(\Gamma)}$ induced by a resolution of $M(\Gamma)$ (and see here [14], or better, our recent joint work [7], where all this issue is explained in a context which is very much akin to the present one). Given a choice of $\mathcal{R}$, we get a smooth 4-manifold

$$\Theta^4(M(\Gamma), \mathcal{R}), \tag{7}$$

and, as soon as one takes the product with B^m, for $m \geq 1$, and one goes to $\Theta^4(M(\Gamma), \mathcal{R}) \times B^m$, then the $\mathcal{R}$-dependence is washed away, and everything becomes then *canonical*. In particular, there is now a free action of Γ on $\Theta^4(\widetilde{M(\Gamma)}, \mathcal{R}) \times B^m$, for $m \geq 1$, and one has that

$$(\Theta^4(\widetilde{M(\Gamma)}, \mathcal{R}) \times B^n)/\Gamma = \Theta^4(M(\Gamma), \mathcal{R}) \times B^n. \tag{8}$$

We take now $n = m + 4 \geq 5$, and then we get the manifold

$$M^n \overset{\text{DEF}}{=} \Theta^4(\widetilde{M(\Gamma)}, \mathcal{R}) \times B^{n-4} \equiv \text{the universal cover of } (\Theta^4(M(\Gamma), \mathcal{R}) \times B^{n-4}). \tag{9}$$

This M^n is a smooth non-compact manifold, of very large boundary. Also, because $\Gamma \in$ QSF, we also have $M^n \in$ QSF.

Lemma 1. *If $N >> n$, then the manifold $W^p \overset{DEF}{=} M^n \times B^N$, for $p = n + N$, is* WGSC.

Proof. The proof of this lemma is rather standard (see e.g. [3, 5]). Hence we omit it. □

Lemma 2. *It suffices to prove Theorem 2 in the case of one-ended groups.*

Proof. When $e(\Gamma) = 0$, then Γ is finite, $\widetilde{M(\Gamma)}$ is compact, and the canonical REPRESENTATION id: $\widetilde{M(\Gamma)} \to \widetilde{M(\Gamma)}$ (i.e. with $X = \widetilde{M(\Gamma)}$ and $f = \mathrm{id}$) satisfies our main theorem for Γ.

When $e(\Gamma) = 2$, then we have a very good explicit description of Γ (as finite extension of $\mathbb{Z}$), with which the main theorem for Γ is easily proved, directly.

Finally, when $e(\Gamma) = \infty$, we need to appeal to the celebrated theorem of J. Stallings (see [24, 10]), which tells us that Γ is gotten by amalgamation from one or two groups G with $e(G) = 1$ and a finite group F (with $e(F) = 0$). Now, $\widetilde{M(G)}$ contains (a lot of) copies of $\widetilde{M(F)}$. For each G with $e(G) = 1$, assuming that Theorem 2 holds for one-ended groups, we have a WGSC-REPRESENTATION $f : X \to \widetilde{M(G)}$ like in the main theorem and it may be assumed that, for each $\widetilde{M(F)} \subset \widetilde{M(G)}$, the map $f|_{f^{-1}\widetilde{M(F)}}$ is the identity map.

Now, we got the $\widetilde{M(\Gamma)}$ by taking infinitely many copies of the G's, each coming with its WGSC-REPRESENTATION $f_i : X_i \to \widetilde{M(G)_i}$ like in the main theorem, and then $\widetilde{M(\Gamma)}$ is an infinite tree-like union of these $\widetilde{M(G)}_i$'s, glued along the common $\widetilde{M(F)}$'s.

With these things the following map

$$\bigcup_{f^{-1}\widetilde{M(F)}} X_i \xrightarrow{\cup_i f_i} \bigcup_{\widetilde{M(F)}} \widetilde{M(G)_i} = \widetilde{M(\Gamma)} \tag{10}$$

is a WGSC-REPRESENTATION of Γ satisfying Theorem 2. □

Now, Lemma 1 tells us that there is an exhaustion by compact, simply-connected, codimension zero submanifolds, each embedded in the interior of the next

$$K_1 \subset K_2 \subset \cdots \subset W^p = \bigcup_1^\infty K_i. \tag{11}$$

While, by Lemma 2, we can suppose $e(\Gamma) = 1$. Hence, W^p has one end too, and so the sets

$$(\partial K_i - \partial W) \text{ are disjoined,} \tag{12}$$

and, for each i,

$$\text{both } (\partial K_i - \partial W) \text{ and } (K_{i+1} - K_i) \text{ are connected.} \tag{13}$$

Lemma 3. *Given* Γ*, we can chose our presentation* $M(\Gamma)$ *so that, for any desingularization* $\mathcal{R}$*, the smooth 4-manifold* $Y^4 \overset{DEF}{=} \Theta^4\big(M(\Gamma), \mathcal{R}\big)$ *is parallelizable and there is a smooth submersion*

$$Y^4 \xrightarrow{\phi_1} \mathbb{R}^4. \tag{14}$$

Proof. Along each singular square $S \subset \mathrm{Sing}(M(\Gamma))$, the $M(\Gamma)$ has three smooth branches

$$U_1 \subset H (= \text{the } 3^d \text{ handlebody}), \quad U_2 \subset D^2_{j_1} \times [0,1], \quad U_3 \subset D^2_{j_2} \times [0,1],$$

coming with $S = S^1_{j_1} \times [0,1] \cap S^1_{j_2} \times [0,1] \subset \partial H$, where $\partial D_j = S^1_j$ (see (2)).

Each of the $U_1 \cup U_2$ and $U_1 \cup U_3$ is a smooth 3-manifold, and, for each $x \in S$, there is a canonical identification $T_x(U_1 \cup U_2) = T_x(U_1 \cup U_3)$, defining the $T_x\big(M(\Gamma)\big)$ for $x \in S$. For the smooth points of $M(\Gamma)$ this tangent space is obvious.

Sublemma 1. *For each* Γ*, we can chose the* $M(\Gamma)$ *so that there is a smooth submersion into the Euclidean* 3*-space*

$$M(\Gamma) \xrightarrow{\psi_0} \mathbb{R}^3. \tag{15}$$

Proof. Start with an arbitrary chosen presentation for our Γ

$$M(\Gamma)_0 = H \cup \sum_{j=1}^{k} D^2_j \times [0,1]$$

where each $D^2_j \times [0,1]$ is glued to H via the $\phi|_{S^1_j \times [0,1]}$ in (2), with, of course, $S^1_j = \partial D^2_j$. Next, take any embedding $H \subset \mathbb{R}^3$, the standard one if one wants, but it does not matter. If $\phi(S^1_j \times [0,1]) \subset \mathbb{R}^3$ extends now to a submersion, we are ok, in the sense that our $H \subset \mathbb{R}^3$ extends to a submersion of $H \cup D^2_j \times [0,1]$. If *not*, we can change the embedding $S^1_j \times [0,1] \subset H$ by letting it spiral around H so that now we get a regular homotopy class $\phi|S^1_j \times [0,1] \longrightarrow \mathbb{R}^3$ which does extend to an immersion $D^2_j \times [0,1] \longrightarrow \mathbb{R}^3$.

This process can be performed in such a way that the homotopy class of $\sum_i^k S^1_j \longrightarrow H$ should stay unchanged. Of course, more singularities S get created, the $S^1_j \times [0,1]$'s are only immersed and not embedded, but all this is ok. △

Sublemma 1 provides us with a smooth field of frames

$$F^3(x) \in \{\text{Frames of } T_x(M(\Gamma))\} \simeq \mathrm{SO}(3), \tag{16}$$

for each $x \in M(\Gamma)$. We consider now the composite map

$$M(\Gamma) \xrightarrow{\phi_0} \mathbb{R}^3 = \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4 = \mathbb{R}^3 \times (-\infty < t < +\infty), \tag{17}$$

starting from which, any desingularization $\mathcal{R}$ of $M(\Gamma)$

$$\{U_2, U_3\} \xrightarrow{\mathcal{R}_S} \{s, n\}, \tag{18}$$

produces a smooth immersion

$$M(\Gamma) \xrightarrow{\Phi_0} \mathbb{R}^4, \tag{19}$$

simply by pushing the s-branch in (18) towards $t = +1$ and the n-branch towards $t = -1$. With this (as explained in [7, 14]), we have

$$\begin{aligned} Y^4 \overset{\text{DEF}}{=} \Theta^4(M(\Gamma), \mathcal{R}) = \{&\text{the } 4^d \text{ smooth regular neighborhood of } M(\Gamma), \\ &\text{induced by } \Phi_0\}. \end{aligned} \tag{20}$$

At each $x \in M(\Gamma)$, the $(\Phi_0)_* F^3(x)$ (with F^3 like in (16)) is a 3-frame of the tangent space $T_{\Phi_0(x)}\mathbb{R}^4$. By adding appropriately a fourth orthogonal vector, we can complete this 3-frame into an oriented 4-frame. This is then a trivialization of the tangent space $T(Y^4)|_{M(\Gamma)}$, which then easily induces a parallelization for Y^4. Hence, via the h-principle for immersions and/or submersions (which in this particular case boils down to the standard Smale–Hirsh theory), it follows that there exists our smooth submersion ϕ_1. This ends Lemma 3. ☐

When the ϕ_1 of (14) is extended to a larger version of Y^4,

$$Y_1^4 \overset{\text{DEF}}{=} Y^4 \cup \big(\partial Y^4 \times [0, 1)\big) \supset Y^4$$

we get a locally finite *affine structure* on the extension Y_1^4 of Y_4, i.e. a Riemannian (not necessarily complete) metric with sectional curvature $K = 0$. There exists also a second structure on Y^4, namely a *foliated structure*, to be described next. Both the affine and the foliated structures are compatibles with the natural Diff structure of Y^4.

Let $L_3 = \partial Y^4$ and let consider the following natural retraction r coming from (20):

$$\begin{tikzcd} L^3 \subset Y^4 \arrow[r, "r"] & M(\Gamma). \end{tikzcd} \tag{21}$$

$$\begin{tikzcd} L^3 \arrow[r, bend right, "r|_{L^3}"'] & M(\Gamma) \end{tikzcd}$$

Lemma 4. (1) *The map $r|_{L^3}$ is simplicially non-degenerate, and, outside of some very simple fold-type singularities, it is an immersion into $M(\Gamma)$.*

(2) *There is an isomorphism*

$$(Y^4, L^3) \simeq \Big(M(\Gamma) \bigcup_{L^3 \times \{0\}} L^3 \times [0, 1], L^3 \times \{1\}\Big), \tag{22}$$

where the map $r|_{L^3 \times \{0\}}$ is used for glueing together $M(\Gamma)$ and $L^3 \times [0, 1]$.

The proof is trivial, and we left it to the reader. Lemma 4 tells us that Y^4 admits a codimension-one *foliation* $\mathcal{F}$, given by

$$Y^4 = \bigcup_{t\in[0,1]} L_t^3,$$

where, for $t > 0$, we have $L_t^3 = L^3$ and where $L_0^3 \equiv M(\Gamma)$ is the unique *singular leaf.*

Returning now to the affine structure which ϕ_1, equation (14) induces on $Y_1^4 \supset Y^4$, we endow $\mathbb{R}^4$ with a very fine affine triangulation, which we afterwards pull back on Y_1^4, so that $L^3 = \partial Y^4$ becomes a polyhedral hypersurface. Next, with an appropriate $N_1 \in \mathbb{Z}_+$, in the context of the Lemma 1, we have that

$$M^n \times B^N = (\widetilde{Y}^4 \times B^{N_1}) = \widetilde{Y^4 \times B^{N_1}}$$

and $B^{N_1} = [0,1]^{N_1}$ has its own canonical affine structure, putting now affine structures on $Y^4 \times B^{N_1}$ and on $M^n \times B^N$ (here M^n is like in (9)).

Remember that Lemma 1 tells us that there is a WGSC cell decomposition of $M^n \times B^N$, call it $H(0)$. Without any loss of generality, there is an affine triangulation Θ of $Y^4 \times B^{N_1}$ such that $H(0) \equiv \widetilde{\Theta} \overset{\text{DEF}}{=} \{$the lift of Θ from $Y^4 \times B^{N_1}$ to $\widetilde{Y^4 \times B^{N_1}}\}$.

Our strategy will be now to work downstairs, at the level of $Y^4 \times B^{N_1} \xrightarrow{\pi} Y^4$ and use only *admissible subdivisions* for our cell-decompositions (by admissible we mean subdivisions which are baricentric or stellar or Siebenmann bisections [22]). When we will lift these things, afterwards, at the level $M^n \times B^N = \widetilde{Y^4 \times B^{N_1}}$, equivariance will be automatic, the admissible condition, which is local, is verified upstairs too, and there it will preserve the WGSC property which $H(0) = \widetilde{\Theta}$ initially had.

We will be interested now in 3- and 4-dimensional skeleta of the triangulation Θ of $Y^4 \times B^{N_1}$. For notational convenience, we denote them by Z^ϵ, for $\epsilon = 3$ or 4. These come with maps

$$Z^4 \xrightarrow{F=\pi|_{Y^4}} Y^4 \quad \text{and} \quad Z^3 \xrightarrow{f=r\circ F} M(\Gamma). \tag{23}$$

Lemma 5. *After a small perturbation of the* 0*-skeleton,* $\Theta^{(0)}$*, of* Θ*, followed by a global isotopic perturbation of* Θ*, which leaves it affine, we can make so that the maps*

$$Z^4 \xrightarrow{F} Y^4 \quad \textit{and} \quad Z^3 \supset F^{-1}\partial Y^4 \xrightarrow{F|_{F^{-1}\partial Y^4}} \partial Y^4 \tag{24}$$

are non-degenerate simplicial surjections, the restrictions of which, on each simplex, are affine.

Proof. The proof is left to the reader (see e.g. the argument analogous to this one in [12]). ☐

So, in the context of (24), we have now two affine triangulations, $\Theta(Z^4)$ and $\Theta(Y^4)$, connected by a simplicial non-degenerate map F.

We introduce now a second class of triangulations, compatible with the same differential structure as the $\Theta(Y^4)$, but related now to the foliation $\mathcal{F}$ too. These triangulations are denoted $\Theta_{\mathcal{F}}(Y^4)$, and will be subjected to the following conditions:

$M(\Gamma)$ is a subcomplex of $\Theta_{\mathcal{F}}(Y^4)$; (24-1)

there is a distinguished, quite dense, set of leaves, all subcomplexes of $\Theta_{\mathcal{F}}(Y^4)$, (24-2)

$$L_0^3 = M(\Gamma),\ L_1^3, L_2^3, \ldots, L_q^3 = L^3 \times \{1\} = \partial Y^4,$$

such that every 4-simplex σ^4 of $\Theta_{\mathcal{F}}(Y^4)$ rests on two consecutive distinguished leaves L_i^3, L_{i+1}^3;

the 3-simplexes of $\Theta_{\mathcal{F}}(Y^4)$ are all essentially parallel to $\mathcal{F}$, always transversal to the fibers of the retraction r (from (21)), and such that $r|\sigma^3$ injects. (24-3)

Moreover, it is assumed that the triangulation $\Theta_{\mathcal{F}}(Y^4)|_{M(\Gamma)}$ is sufficiently fine so that $r(\sigma^3)$ is a subcomplex.

In the context of $\Theta_{\mathcal{F}}$ we will have $\mathcal{F}$-*admissible* subdivisions

$$\Theta_{\mathcal{F}}(Y^4) \xrightarrow[\text{subdivisions}]{\mathcal{F}\text{-admissible}} \Theta^1_{\mathcal{F}}(Y^4) \tag{25}$$

which are both admissible and respect the conditions (24-1) to (24-3), with a possibly denser, bigger subset of distinguished leaves.

Lemma 6. *Once both $\Theta(Y^4)$ and $\Theta_{\mathcal{F}}(Y^4)$ are given, there exist then admissible, respectively $\mathcal{F}$-admissible, subdivisions for each of them, yielding isomorphic cell-decompositions, like in the diagram below* (*where all the vertical arrows are subdivisions*)

$$\begin{tikzcd}
 & \Theta_{\mathcal{F}}(Y^4) \arrow[d] & Y^4 \arrow[l, "\textit{cell dec.}"'] \arrow[r, "\textit{cell dec.}"] & \Theta(Y^4) \arrow[d] & \Theta(Z^4) \arrow[l, "F"'] \arrow[d] \\
M(\Gamma) & \Theta^1_{\mathcal{F}} \arrow[l, "r"'] & & \Theta^1(Y^4) \arrow[ll, "\textit{isomorphism } \mathcal{I}"'] & \Theta^1(Z^4) \arrow[l, "F^1"']
\end{tikzcd} \tag{26}$$

where both F and F^1 in the diagram are simplicial and non-degenerate.

Proof. Both $\Theta(Y^4)$ and $\Theta_{\mathcal{F}}(Y^4)$ are compatible with the same Diff structure on Y^4, and, via the smooth Hauptvermutung, they have isomorphic subdivisions. From there on, one uses Siebenmann's cellulations and his very transparent version of the old Alexander lemma [22]. □

3.1. Proof of the main theorem, part 1. By taking the universal cover of the lower long composite arrow in (26), we get the following map

$$X^3 \overset{\text{DEF}}{=} \{\text{the 3-skeleton of the universal cover of } \Theta^1(Z^4)\} \xrightarrow{f} \widetilde{M(\Gamma)}, \tag{27}$$

where $f \overset{\text{DEF}}{=} (r \circ \mathcal{I} \circ F^1)^\sim$, which has the following features:

since both F^1 and r are non-degenerate, so is f; (27-1)

we have started from $H(0) = \widetilde{\Theta}$ which was wgsc and, from there on, all the subdivisions were admissible: this implies that X^3 is also wgsc; (27-2)

the map f is surjective and, moreover, it admits the section (27-3)

$$s \colon \widetilde{M(\Gamma)} \longrightarrow X^3$$

(see (24-1)), which is such that $f|_{s(\widetilde{M(\Gamma)})} = \text{id}$.

From this point on, there is a standard argument showing that $\Psi(f) = \Phi(f)$ (and see here e.g. the proof of Lemma 2.8 in [12] too). In a nutshell, this argument is the following. Assume $\Psi(f) \subsetneq \Phi(f)$, then the induced map

$$X^3/\Psi(f) \longrightarrow \widetilde{M(\Gamma)}$$

would have singularities, which is a contradiction.

So, by now, we have already shown that (27) is an equivariant, wgsc 3^d-representation of $\widetilde{M(\Gamma)}$. It remains to check equation (5) of Theorem 2, or, equivalently, (5-bis).

Since the fibers of $Y^4 \times B^{N_1} \xrightarrow{r \circ \pi} M(\Gamma)$ are compact, then so are also those of $\Theta^1(Z^4) \xrightarrow{r \circ \mathcal{I} \circ F^1} M(\Gamma)$ and of

$$\text{the 3-skeleton of } \Theta^1(Z^4) \xrightarrow{r \circ \mathcal{I} \circ F^1} M(\Gamma). \tag{28}$$

This means that, in the context of (28), for any 3-simplex σ^3 of $M(\Gamma)$, the inverse image consists of a finite number of 3-simplexes, this number being clearly uniformly bounded.

By equivariance, the same is true for

$$X^3 \xrightarrow{f} \widetilde{M(\Gamma)},$$

and point (1) in Theorem 2 is by now proved.

3.2. Proof of the main theorem, part 2. We want to move now from the 3-dimensional REPRESENTATION (27) to a 2-dimensional one

$$X^2 \xrightarrow{r} \widetilde{M(\Gamma)}, \tag{29}$$

which should be WGSC, equivariant, and also satisfying (6).

The general idea is that, for the passage (27) $\Longrightarrow$ (29), there is a similar step in [18], and the techniques used there can be adapted here too. Hence, we will only give here the main lines of the argument. Since we want to have equivariance, we will work downstairs at level $M(\Gamma)$, taking universal coverings in the end. From the lower line in (26), we pick now the map

$$Y^3 \overset{\text{DEF}}{=} \text{The 3-skeleton of } \Theta^1(Z^4) \xrightarrow{g \overset{\text{DEF}}{=} r \circ \mathcal{I} \circ F^1} M(\Gamma) \quad (\text{where } Y^3 = X^3/\Gamma), \tag{30}$$

choosing to read Y^3 like a singular handlebody decomposition (see here [17, 18, 21]). For each 3-handle of our Y^3 of (30), there are three mutually orthogonal, not everywhere well-defined foliations

$$\mathcal{F}_0(\text{BLUE}), \quad \mathcal{F}_1(\text{RED}), \quad \mathcal{F}_2(\text{BLACK}). \tag{31}$$

Each 3-handle is endowed with the three foliations, but, $\mathcal{F}_\lambda$(COLOR) is *natural* for the handles of index λ. There, it is essentially a product foliation of copies of the lateral surface of the handle in question, namely ∂(cocore)$\times$ core. The reader is invited to look at the figures in [21]. The paper [21] was written, of course, in the non-singular context of $\widetilde{M}^3$ rather then of $\widetilde{M(\Gamma)}$, but, for these individual handles, the story is the same. In [21], $\widetilde{M}^3$ was non-singular and the three foliations were global, although of course not everywhere well-defined. While here, our $\widetilde{M(\Gamma)}$ is singular and the 3^d equivariant context corresponds to [17] rather than to [21]. Each individual handle has now its own three foliations. We can use these foliations, like in [21], in order to got from the 3^d-REPRESENTATION (27) to the 2^d-REPRESENTATION (29).

Since the map (30) is devoid of any pathology at infinity, we can afford to work with usual compact handles H^λ, of index λ and dimension 3. For each of these handles H^λ we consider now a very dense 2-skeleton, which uses only finitely many leaves of the foliations (31).

Putting these things together, we get a simplicial non-degenerate map

$$h \colon Y^2 \longrightarrow M(\Gamma) \tag{32}$$

about which the following items may be assumed without any loss of generality:

we have subsets $\mathrm{Sing}(h) \subset Y^2, \mathrm{Sing}(Y^2) \subset Y^2 - \mathrm{Sing}(h)$, just like in (3-6) and (3-7), and, outside $\mathrm{Sing}(h)$, the double points of h, $M_2(h) \subset Y^2$, are transversal intersection points; (32-1)

the image $h(Y^2) \subset M(\Gamma)$ is very dense, that is, the complement $M(\Gamma) - h(Y^2)$ consists of a disjoint union of three copies of $\mathbb{R}^3_+$ glued along their common $\partial \mathbb{R}^3_+ = \mathbb{R}^2$. (32-2)

Next, we take the universal cover of (32),

$$X^2 \overset{\text{DEF}}{=} \widetilde{Y}^2 \xrightarrow{\tilde{h}} \widetilde{M(\Gamma)}. \tag{33}$$

Here is what we can say about (33). Our (33) is automatically equivariant and, since the X^3 in (27) was WGSC, so is our present X^2 too, since it is essentially the 2-skeleton of it. The fact that in the context of (27) we had $\Psi(f) = \Phi(f)$, together with the fact that X^2 is very dense, make that in the context of (33) we also have $\Psi(\tilde{h}) = \Phi(\tilde{h})$, so that (33) is an equivariant WGSC 2^d-REPRESENTATION, for which local finiteness should be obvious.

Locally, (33) is exactly like (32), where Y^2 is a finite complex. Hence, equation (6) in Theorem 2 follows automatically. Theorem 2 is by now completely proved.

References

[1] S. G. Brick and M. L. Mihalik, The QSF property for groups and spaces. *Math. Z.* **220** (1995), no. 2, 207–217. Zbl 0843.57003 MR 1355026

[2] M. H. Freedman and F. Quinn, *Topology of 4-manifolds.* Princeton Mathematical Series, 39. Princeton University Press, Princeton, N.J., 1990. Zbl 0705.57001 MR 1201584

[3] L. Funar and S. Gadgil, On the geometric simple connectivity of open manifolds. *Int. Math. Res. Not.* **2004** (2004), no. 24, 1193–1248. Zbl 1088.57016 MR 2040183

[4] L. Guillou and A. Marin (eds.), *A la recherche de la topologie perdue.* I. Du côté de chez Rohlin. II. Le côté de Casson. Progress in Mathematics, 62. Birkhäuser Boston, Boston, MA, 1986. Zbl 0597.57001 MR 0900243

[5] L. Funar and D. E. Otera, On the WGSC and QSF tameness conditions for finitely presented groups. *Groups Geom. Dyn.* **4** (2010), no. 3, 549–596. Zbl 1218.57012 MR 2653975

[6] D. E. Otera, Topological tameness conditions of spaces and groups: Results and developments. *Lith. Math. J.* **56** (2016), no. 3, 357–376. Zbl 06631570 MR 3530224

[7] D. E. Otera and V. Poénaru, "Easy" representations and the QSF property for groups. *Bull. Belg. Math. Soc. Simon Stevin* **19** (2012), no. 3, 385–398. Zbl 1277.57005 MR 3027350

[8] D. E. Otera, V. Poénaru, and C. Tanasi, On geometric simple connectivity. *Bull. Math. Soc. Sci. Math. Roumanie* (*N.S.*) **53(101)** (2010), no. 2, 157–176. Zbl 1212.57004 MR 2667976

[9] D. E. Otera, F. G. Russo, and C. Tanasi, On 3-dimensional wgsc inverse-representations of groups. *Acta Math. Hungar.* **151** (2017), no. 2, 379–390. MR 3620839

[10] V. Poénaru, *Groupes discrets*. Lecture Notes in Mathematics, 421. Springer-Verlag, Berlin etc., 1974. Zbl 0295.57002 MR 0407155

[11] V. Poénaru, On the equivalence relation forced by the singularities of a nondegenerate simplicial map. *Duke Math. J.* **63** (1991), no. 2, 421–429. Zbl 0738.57008 MR 1115115

[12] V. Poénaru, Killing handles of index one stably and π_1^∞. *Duke Math. J.* **63** (1991), no. 2, 431–447. Zbl 0738.57009 MR 1115116

[13] V. Poénaru, Almost convex groups, Lipschitz combing, and π_1^∞ for universal covering spaces of closed 3-manifolds. *J. Differential Geom.* **35** (1992), no. 1, 103–130. Zbl 0777.57001 MR 1152227

[14] V. Poénaru, The collapsible pseudo-spine representation theorem. *Topology* **31** (1992), no. 3, 625–656. Zbl 0777.57013 MR 1174263

[15] V. Poénaru, Geometry "à la Gromov" for the fundamental group of a closed 3-manifold M^3 and the simple connectivity at infinity of $\widetilde{M}^3$. *Topology* **33** (1994), no. 1, 181–196. Zbl 0830.57010 MR 1259521

[16] V. Poénaru, π_1^∞ and simple homotopy type in dimension 3. Contemporary Math. AMS, **239** (1999), 1–28. In H. Nencka (ed.), *Low-dimensional topology.* (University of Madeira, Funchal, 1998.) American Mathematical Society, Providence, R.I., 1999, 1–28. Zbl 0933.57017 MR 1701672

[17] V. Poénaru, Equivariant, locally finite inverse representations with uniformly bounded zipping length, for arbitrary finitely presented groups. *Geom. Dedicata* **167** (2013), 91–121. Zbl 1354.20027 MR 3128772

[18] V. Poénaru, Geometric simple connectivity and finitely presented groups. Preprint 2014. arXiv:1404.4283 [math.GT]

[19] V. Poénaru, All finitely presented groups are QSF. Preprint 2014. arXiv:1409.7325 [math.GT]

[20] V. Poénaru and C. Tanasi, Representations of the Whitehead manifold Wh^3 and Julia sets. *Ann. Fac. Sci. Toulouse Math.* (6) **4** (1995), no. 3, 655–694. Zbl 0878.57015 MR 1607516

[21] V. Poénaru and C. Tanasi, Equivariant, almost-arborescent representations of open simply-connected 3-manifolds; a finiteness result. *Mem. Amer. Math. Soc.* **169** (2004), no. 800, 89pp. Zbl 1052.57017 MR 2045642

[22] L. C. Siebenmann, Les bissections expliquent le thórème de Reidemeister–Singer. Un retour aux sources. *Ann. Fac. Sci. Toulouse Math.* (6) **24** (2015), no. 5, 1025–1056. Zbl 1350.57025 MR 3485325

[23] S. Smale, On the structure of manifolds. *Amer. J. Math.* **84** (1962), 387–399. Zbl 0109.41103 MR 0153022

[24] J. R. Stallings, *Group theory and three-dimensional manifolds.* Yale Mathematical Monographs. 4. New Haven-London: Yale University Press (1971). A James K. Whittemore Lecture in Mathematics given at Yale University, 1969. Yale Mathematical Monographs, 4. Yale University Press, New Haven, CT, and London, 1971. Zbl 0241.57001 MR 0415622

[25] J. H. C. Whitehead, A certain open manifold whose group is unity. *Q. J. Math., Oxf. Ser.* **6** (1935), 268-279. JFM 61.0607.01 Zbl 0013.08103 MR 0013.08103

Received November 4, 2015

Daniele Ettore Otera, Mathematics and Informatics Institute of Vilnius University,
Akademijos st. 4, LT-08663, Vilnius, Lithuania

e-mail: daniele.otera@mii.vu.lt

e-mail: daniele.otera@gmail.com

Valentin Poénaru, Professor Emeritus at Université Paris-Sud 11,
Département de Mathématiques, Bâtiment 425, 91405 Orsay, France

e-mail: valpoe@hotmail.com