Real reflections, commutators, and cross-ratios in complex hyperbolic space

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Abstract. We provide a concrete criterion to determine whether or not two given elements of PU(2, 1) can be written as products of real reflections, with one reflection in common. As an application, we show that the Picard modular groups PU(2, 1, $\mathcal{O}_d$) with $d = 1, 2, 3, 7, 11$ are generated by real reflections up to index 1, 2, 4 or 8.

Contents

1. Introduction

A classical way of studying isometries of a symmetric space $\mathcal{S}$ is to decompose them as products of involutions. The most comfortable situation is to have a class $\mathcal{C}$ of isometric involutions with the following properties:

(1) any two involutions in $\mathcal{C}$ are conjugate in Isom($\mathcal{S}$);

(2) any element of Isom($\mathcal{S}$) can be written as a product $s_1 s_2$, with $s_i \in \mathcal{C}$.

For example, if such a family of involutions exists, describing the fixed points of an isometry $A = s_1 s_2$ of $\mathcal{S}$ amounts to studying the relative position of the fixed point sets of s_1 and s_2 which are totally geodesic subspaces and are isometric to one another. If such a family $\mathcal{C}$ exists, one usually says that Isom($\mathcal{S}$) has involution

length 2 with respect to $\mathcal{C}$. Of course this requirement is too optimistic in general (for example, it fails in Euclidean space of dimension at least 3).

Assuming that a symmetric space has this property, the next question is to decide when two isometries A and B can be decomposed using a common involution. We will call such a pair *decomposable* with respect to $\mathcal{C}$ (see Definition 2.1; the term *linked* is also commonly used, see [BM]). This property simplifies the understanding of the group $\langle A, B \rangle$ as it reduces to studying the relative position of 3 pairwise isometric totally geodesic subspaces.

One of the most elementary cases is that of the Poincaré disk Δ. There are two classes of involutions in $\mathrm{Isom}(\Delta)$, namely *half-turns* and *reflections*. Viewing Δ as the unit disk in $\mathbb{C}$, half-turns are conjugate in $\mathrm{Isom}^+(\Delta)$ to $z \mapsto -z$ and reflections to $z \mapsto \bar{z}$. It is a classical fact that $\mathrm{Isom}^+(\Delta)$ has involution length 2 with respect to reflections, and 3 with respect to half-turns. Moreover, any pair of orientation-preserving isometries of the Poincaré disk can be decomposed in the form

$$A = s_1 s_2 \quad \text{and} \quad B = s_2 s_3, \tag{1.1}$$

with the s_i either all half-turns or all reflections. This makes the description of many properties of the group $\langle A, B \rangle$ easier. For instance, when the s_i are all orientation-reversing, the group $\langle A, B \rangle$ has index two in $\Gamma = \langle s_1, s_2, s_3 \rangle$. In particular, $\langle A, B \rangle$ is discrete if and only if Γ is. The decomposition property we are interested in shares many features with this example, as our involutions are antiholomorphic.

In [BM], Basmajian and Maskit have studied this question for any space form, that is for Euclidean space, real hyperbolic space and the sphere. In particular, they prove that any orientation-preserving isometry is a product of two involutions. They also show for instance that pairs in $\mathrm{Isom}^+(H_\mathbb{R}^n)$ are generically non decomposable when $n \geqslant 4$.

In this paper we study the question of decomposability in the complex hyperbolic plane, which can be seen via a projective model as the unit ball in $\mathbb{C}^2$ equipped with a $\mathrm{PU}(2, 1)$-invariant metric. In fact, $\mathrm{PU}(2, 1)$ is the group of holomorphic isometries of $H_\mathbb{C}^2$; it is the identity component of $\mathrm{Isom}(H_\mathbb{C}^2)$, the other connected component consisting of all antiholomorphic isometries. In particular, $\mathrm{PU}(2, 1)$ has index two in $\mathrm{Isom}(H_\mathbb{C}^2)$ and in this context holomorphicity plays the role of preservation of orientation for space forms. The class of involutions we are interested in consists of antiholomorphic involutions, which are all conjugate in $\mathrm{PU}(2, 1)$ to the map given in affine ball coordinates by

$$\sigma_0 \colon (z_1, z_2) \longmapsto (\overline{z_1}, \overline{z_2}).$$

Clearly, σ_0 fixes pointwise the set of real points of $H_\mathbb{C}^2$, and conjugates of σ_0 by elements of $\mathrm{PU}(2, 1)$ fix pointwise *real planes*, which are (totally real) totally geodesic embedded copies of the Poincaré disk. We will refer to these antiholomorphic involutions as *real reflections*.

It has been known since Falbel and Zocca (see [FZ]) that the involution length of PU(2, 1) with respect to real reflections is 2, and this fact has been generalized to all dimensions by Choi [Cho]. The main question we address in this paper is the following.

When is a pair of elements of PU(2, 1) *decomposable with respect to real reflections*?

We will abbreviate this by saying that a pair $(A, B) \in \mathrm{PU}(2, 1)^2$ is $\mathbb{R}$-*decomposable* (see Definition 2.1). There is no hope that generic pairs of elements in PU(2, 1) are $\mathbb{R}$-decomposable. A rough argument for this is the following dimension count. The group PU(2, 1) has dimension 8, thus the product $\mathrm{PU}(2, 1) \times \mathrm{PU}(2, 1)$ has dimension 16. On the other hand, the set of real planes in $\mathrm{H}^2_{\mathbb{C}}$ has dimension 5 (to see this note that the stabilizer of the set of real points of the ball is PO(2, 1) which has dimension 3). As a real reflection is determined by its fixed real plane, the set of triples of real reflections has dimension 15 (and therefore cannot be diffeomorphic to $\mathrm{PU}(2, 1) \times \mathrm{PU}(2, 1)$). In fact this count leads us to expect that $\mathbb{R}$-decomposability of a pair of isometries is determined by a single (codimension 1) condition, and this will turn out to be the case.

This question has been examined in [W2], where it was proved that, under the assumption that A and B are loxodromic, the pair (A, B) is $\mathbb{R}$-decomposable provided that the trace of the commutator $[A, B]$ is real. However, this result was obtained as a byproduct of a classification of pairs of elements of PU(2, 1) by traces; namely the data $(\mathrm{Tr}\, A, \mathrm{Tr}\, B, \mathrm{Tr}\, AB, \mathrm{Tr}\, A^{-1}B)$ determines the pair (A, B) up to PU(2, 1)-conjugation (modulo an order two indetermination, corresponding to the sign of the imaginary part of $\mathrm{Tr}[A, B]$). The present approach is more natural, and in particular it allows us to remove the assumption that A and B are loxodromic.

Our main result (Theorem 4.1) is the following theorem.

Theorem. *Let* $A, B \in \mathrm{PU}(2, 1)$ *be two isometries not fixing a common point in* $\overline{\mathrm{H}^2_{\mathbb{C}}}$. *Then the pair* (A, B) *is* $\mathbb{R}$-*decomposable if and only if the commutator* $[A, B]$ *has a fixed point in* $\overline{\mathrm{H}^2_{\mathbb{C}}}$ *whose associated eigenvalue is real and positive.*

Note that the eigenvalues of $[A, B]$ do not depend on the choice of lifts of A and B to U(2,1). We now sketch our strategy . Any fixed point p_1 of $[A, B]$ in $\overline{\mathrm{H}^2_{\mathbb{C}}}$ gives rise to a cycle of points

$$p_1 \xrightarrow{B^{-1}} p_2 \xrightarrow{A^{-1}} p_3 \xrightarrow{B} p_4 \xrightarrow{A} p_1. \tag{1.2}$$

A key fact is that a real reflection σ decomposes A and B if and only if it satisfies $\sigma(p_1) = p_3$ and $\sigma(p_2) = p_4$ (Lemma 4.3). Thus we see that the

decomposability of (A, B) is equivalent to a specific symmetry of the above 4-cycle. Now, the existence of this symmetry can be detected via cross-ratios. Goldman has proved that an ideal tetrahedron has this symmetry if and only if the cross-ratio $X(p_2, p_4, p_1, p_3)$ is real and positive (Lemma 7.2.1 of [G]). We extend this fact to quadruples of points in $\overline{H^n_{\mathbb{C}}}$ (Proposition 3.1). The last ingredient is to connect the cross-ratio of the 4-cycle (1.2) to the eigenvalue of $[A, B]$ associated to the fixed point p_1. Denoting this eigenvalue by λ_1, we will see that the product $\lambda_1 \cdot X(p_2, p_4, p_1, p_3)$ is real and positive for any pair (A, B) (this is Relation 4.1). This means that λ_1 is real if and only if $X(p_2, p_4, p_1, p_3)$ is, and both quantities have the same sign, which gives the result. For the sake of completeness we then analyze what happens in the two special cases where λ_1 is negative (Section 4.3), and when A and B have a common fixed point (Section 4.2).

- When λ_1 is negative, the fixed point of $[A, B]$ is always on the boundary of $H^2_{\mathbb{C}}$ and the group $\langle A, B \rangle$ preserves a complex line. This is an example of a maximal representation of the fundamental group of the once punctured torus, in the sense of [BIW].

- To describe the situation when A and B have a common fixed point, the main ingredient is a detailed description of conditions guaranteeing that a real reflection decomposes a given isometry (this is Proposition 2.5). Most cases already appear in the literature, except when A and B are parabolic with a common fixed point. In particular, when A and B are both 3-step unipotent (see the definitions in Section 2.1), we use the relative position of their *invariant fans*, as defined by Goldman and Parker in [GP].

A classical and difficult question in complex hyperbolic geometry is to determine the discreteness or non-discreteness of a given finitely generated subgroup Γ of $PU(2, 1)$, and to obtain a presentation of Γ. Even when discreteness is known from general results (for instance in the case of arithmetic lattices), finding a presentation of the group is a difficult problem. In fact, the most frequent method is to construct a fundamental domain and use the Poincaré Polyhedron theorem. This is a very technical task which requires a detailed understanding of the action of Γ on $H^2_{\mathbb{C}}$.

There are not so many examples of explicit discrete subgroups of $PU(2, 1)$, and most of them are obtained from groups with 2 generators. In this case, the existence of a decomposition as in (1.1) connects an algebraic property of the group (being an index 2 subgroup of a group generated by 3 involutions) to a geometric property (the existence of totally geodesic fixed point sets in a certain configuration). In other words, Γ appears as a *reflection group*. Such decompositions appeared for instance naturally in [FALPAR2] or [W3] which studied certain representations of Fuchsian groups in $PU(2, 1)$. They were also central in the constructions of fundamental domains for Mostow's lattices ([M]) in [DFP], as well as for the new non-arithmetic lattices obtained in [PARPAU], [PAU], [DPP1], and [DPP2].

In all these occurrences, the existence of the real reflections decomposing the generators required some work (part of the detailed geometric construction of the fundamental domains), whereas the concrete criterion given by Theorem 4.1 allows us to easily reprove that the Mostow and Deraux–Parker–Paupert lattices are generated by real reflections. More importantly, the careful analysis of decomposability conditions for parabolic isometries in Proposition 2.5 (c) allows us to show that new lattices are generated by real reflections, namely certain Picard modular groups.

The Picard modular groups $\Gamma_d = \mathrm{SU}(2, 1, \mathcal{O}_d)$ (the subgroup of $\mathrm{SU}(2, 1)$ consisting of matrices with entries in $\mathcal{O}_d$, where d is a positive squarefree integer and $\mathcal{O}_d$ denotes the ring of integers of $\mathbb{Q}[\sqrt{-d}]$) are the simplest kind of arithmetic lattice in $\mathrm{SU}(2, 1)$. However their explicit algebraic or geometric properties such as generators, fundamental domains and presentations are still unknown in all but very few cases. (Much is known however concerning their volumes, see [H], [St], and more generally Prasad's volume formula [Pra]). More specifically, presentations and fundamental domains have been obtained for $\mathrm{PU}(2, 1, \mathcal{O}_d)$ when $d = 3$ (the so-called Eisenstein–Picard modular group) in [FalPar] and when $d = 1$ (the Gauss-Picard modular group) in [FFP]. More recently, generators for $\mathrm{PU}(2, 1, \mathcal{O}_d)$ with $d = 2, 7, 11$ were given in [Z] (these values of d are the ones for which the ring $\mathcal{O}_d$ is Euclidean). Note that, for $d \neq 3$, $\mathrm{SU}(2, 1, \mathcal{O}_d) \simeq \mathrm{PU}(2, 1, \mathcal{O}_d)$ as there are no non-trivial cube roots of unity in $\mathcal{O}_d$.

The Picard modular groups $\mathrm{PU}(2, 1, \mathcal{O}_d)$ are analogous to the Bianchi groups $\mathrm{PGL}(2, \mathcal{O}_d)$ in $\mathrm{PGL}(2, \mathbb{C})$. Bianchi proved in the seminal paper [Bi] that the Bianchi groups are *reflective*, i.e. generated by reflections up to finite index, for $d \leqslant 19$, $d \neq 14, 17$. At the end of the 1980's, Shaiheev extended these results in [Sh], using results of Vinberg, proving in particular that only finitely many of the Bianchi groups are reflective. (The finiteness result now follows from Agol's result, [Ag]). The classification of reflective Bianchi groups was recently completed in [BeMc].

We prove that the Picard modular groups $\mathrm{PU}(2, 1, \mathcal{O}_d)$ with $d = 1, 2, 3, 7, 11$ are reflective; more precisely that they have a subgroup of index 1,2 or 4 which is generated by real reflections (Corollary 5.1).

The paper is organized as follows. We start with some geometric preliminaries in Section 2, then study configurations of points and cross-ratios in Section 3. Section 4 contains the statement and proofs of our main results, which we then apply to various discrete subgroups of $\mathrm{PU}(2, 1)$ in Section 5.

2. Geometric preliminaries

2.1. Complex hyperbolic space and isometries. The standard reference for complex hyperbolic geometry is [G]. For the reader's convenience we include a brief summary of key definitions and facts. Our main result concerns the case of dimension $n = 2$, but the general setup is identical for higher dimensions so we state it for all $n \geqslant 1$.

Distance function. Consider $\mathbb{C}^{n,1}$, the vector space $\mathbb{C}^{n+1}$ endowed with a Hermitian form $\langle \cdot, \cdot \rangle$ of signature $(n, 1)$. Let $V^- = \{Z \in \mathbb{C}^{n,1} | \langle Z, Z \rangle < 0\}$. Let $\pi \colon \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P^n$ denote projectivization. Define $H^n_{\mathbb{C}}$ to be $\pi(V^-) \subset \mathbb{C}P^n$, endowed with the distance d (Bergman metric) given by

$$\cosh^2 \frac{1}{2} d(\pi(X), \pi(Y)) = \frac{|\langle X, Y \rangle|^2}{\langle X, X \rangle \langle Y, Y \rangle}. \tag{2.1}$$

Different choices of Hermitian forms of signature $(n, 1)$ give rise to different models of $H^n_{\mathbb{C}}$. The two most standard choices are the following. First, when the Hermitian form is given by $\langle Z, Z \rangle = |z_1|^2 + \cdots + |z_n|^2 - |z_{n+1}|^2$, the image of V^- under projectivization is the unit ball of $\mathbb{C}^n$, seen in the affine chart $\{z_{n+1} = 1\}$ of $\mathbb{C}P^n$. This model is referred to as the *ball model* of $H^n_{\mathbb{C}}$. Secondly, when $\langle Z, Z \rangle = 2\operatorname{Re}(z_1 \overline{z_{n+1}}) + |z_2|^2 + \cdots + |z_n|^2$, we obtain the so-called *Siegel model* of $H^n_{\mathbb{C}}$, which generalizes the Poincaré upper half-plane. More details on the Siegel model in dimension 2 will be given in the next section.

Isometries. From (2.1) it is clear that $\mathrm{PU}(n, 1)$ acts by isometries on $H^n_{\mathbb{C}}$, where $\mathrm{U}(n, 1)$ denotes the subgroup of $\mathrm{GL}(n + 1, \mathbb{C})$ preserving $\langle \cdot, \cdot \rangle$, and $\mathrm{PU}(n, 1)$ its image in $\mathrm{PGL}(n + 1, \mathbb{C})$. In fact, $\mathrm{PU}(n, 1)$ is the group of holomorphic isometries of $H^n_{\mathbb{C}}$, and the full group of isometries is $\mathrm{PU}(n, 1) \ltimes \mathbb{Z}/2$, where the $\mathbb{Z}/2$ factor corresponds to a real reflection (see below). Holomorphic isometries of $H^n_{\mathbb{C}}$ can be of three types, depending on the number and location of their fixed points. Namely, $g \in \mathrm{PU}(n, 1)$ is

- *elliptic* if it has a fixed point in $H^n_{\mathbb{C}}$;
- *parabolic* if it has (no fixed point in $H^n_{\mathbb{C}}$ and) exactly one fixed point in $\partial H^n_{\mathbb{C}}$;
- *loxodromic* if it has (no fixed point in $H^n_{\mathbb{C}}$ and) exactly two fixed points in $\partial H^n_{\mathbb{C}}$.

Totally geodesic subspaces and related isometries. A *complex k-plane* is a projective k-dimensional subspace of $\mathbb{C}P^n$ intersecting $\pi(V^-)$ non-trivially (so, it is an isometrically embedded copy of $H^k_{\mathbb{C}} \subset H^n_{\mathbb{C}}$). Complex 1-planes are usually called *complex lines*. If $L = \pi(\widetilde{L})$ is a complex $(n-1)$-plane, any $v \in \mathbb{C}^{n+1} - \{0\}$

orthogonal to $\widetilde{L}$ is called a *polar vector* of L. Such a vector satisfies $\langle v, v \rangle > 0$, and we will usually normalize v so that $\langle v, v \rangle = 1$.

A *real k-plane* is the projective image of a totally real $(k + 1)$-subspace W of $\mathbb{C}^{n,1}$, i. e. a $(k + 1)$-dimensional real vector subspace such that $\langle v, w \rangle \in \mathbb{R}$ for all $v, w \in W$. We will usually call real 2-planes simply real planes, or $\mathbb{R}$-planes. Every real n-plane in $H^n_{\mathbb{C}}$ is the fixed-point set of an antiholomorphic isometry of order 2 called a *real reflection* or $\mathbb{R}$-reflection. The prototype of such an isometry is the map given in affine coordinates by $(z_1, \ldots, z_n) \mapsto (\overline{z_1}, \ldots, \overline{z_n})$ (this is an isometry provided that the Hermitian form has real coefficients).

An elliptic isometry g is called *regular* if any of its matrix representatives $A \in \mathrm{U}(n, 1)$ has distinct eigenvalues. The eigenvalues of a matrix $A \in U(n, 1)$ representing an elliptic isometry g have modulus one. Exactly one of these eigenvalues has eigenvectors in V^- (projecting to a fixed point of g in $H^n_{\mathbb{C}}$), and such an eigenvalue will be called *of negative type*. Regular elliptic isometries have an isolated fixed point in $H^n_{\mathbb{C}}$. A non regular elliptic isometry is called *special*. Among the special elliptic isometries are the following two types (which exhaust all special elliptic types when $n = 2$).

(1) A *complex reflection* is an elliptic isometry $g \in \mathrm{PU}(n, 1)$ whose fixed-point set is a complex $(n - 1)$-plane. In other words, any lift such an isometry to U(n,1) has n equal eigenvalues, one of which has negative type.

(2) A *complex reflection in a point* is an elliptic isometry having a lift with n equal eigenvalues, the remaining one being of negative type. In other words, such an isometry is conjugate to some $\lambda \, \mathrm{Id} \in U(n)$, where $U(n)$ is the stabilizer of the origin in the ball model. Complex reflections in a point with order 2 are also called *central involutions*; these are the symmetries that give $H^n_{\mathbb{C}}$ the structure of a symmetric space.

A parabolic isometry is called *unipotent* if it has a unipotent lift in $\mathrm{U}(n, 1)$. If not, it is called *screw-parabolic*, and it can be uniquely decomposed as $g = pe = ep$ with p unipotent and e elliptic (see Theorem 2.2 below). In dimensions $n > 1$, unipotent isometries are either 2-*step* or 3-*step*, according to whether the minimal polynomial of their unipotent lift is $(X - 1)^2$ or $(X - 1)^3$ (see Section 3.4 of [CHGR]).

2.2. Models in dimension 2

2.2.1. The ball model of $H^2_{\mathbb{C}}$.
The ball model of $H^2_{\mathbb{C}}$ arises from the choice of Hermitian form

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

It is classical, and we refer the reader to Chapter 3 of [G]. We only emphasize the following fact: any elliptic isometry of $H_{\mathbb{C}}^2$ is conjugate to one given in ball coordinates by $(z_1, z_2) \mapsto (e^{i\alpha} z_1, e^{i\beta} z_2)$ for some $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$. A matrix representative in SU(2,1) for the latter is

$$E_{(\alpha,\beta)} = \begin{bmatrix} e^{i(2\alpha-\beta)/3} & 0 & 0 \\ 0 & e^{i(2\beta-\alpha)/3} & 0 \\ 0 & 0 & e^{-i(\alpha+\beta)/3} \end{bmatrix}. \tag{2.2}$$

2.2.2. The Siegel model of $H_{\mathbb{C}}^2$. In the presence of parabolic elements, it is very convenient to use the Siegel domain, as the stabilizer of the point at infinity (see below) consists of upper triangle matrices. As mentioned in the previous section, this model corresponds to the Hermitian form given by the matrix

$$H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

In this model, any point $m \in H_{\mathbb{C}}^2$ admits a unique lift to $\mathbb{C}^{2,1}$ of the following form, called its *standard lift*:

$$\mathbf{m} = \begin{bmatrix} (-|z|^2 - u + it)/2 \\ z \\ 1 \end{bmatrix} \quad \text{with } (z, t, u) \in \mathbb{C} \times \mathbb{R} \times]0, \infty[. \tag{2.3}$$

The triple (z, t, u) is called the *horospherical coordinates* of m. The boundary of $H_{\mathbb{C}}^2$ is the level set $\{u = 0\}$, together with the distinguished point at infinity, given by

$$q_\infty \sim \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Level sets $\{u = u_0\}$ with fixed $u_0 > 0$ are called *horospheres based at q_∞*. The boundary $\partial H_{\mathbb{C}}^2 \setminus \{q_\infty\}$ is a copy of the Heisenberg group $\mathfrak{N}$ of dimension 3, with group law given in $[z, t]$ coordinates by

$$[z_1, t_1] \cdot [z_2, t_2] = [z_1 + z_2, t_1 + t_2 + 2\,\mathrm{Im}(z_1 \bar{z}_2)]. \tag{2.4}$$

The stabilizer of q_∞ in SU(2, 1) consists of upper triangular matrices, and is generated by the following 3 types of isometries: Heisenberg translations $T_{[z,t]}$ $((z, t) \in \mathbb{R} \times \mathbb{C})$, Heisenberg rotations R_θ $(\theta \in \mathbb{R}/2\pi\mathbb{Z})$ and Heisenberg dilations D_r $(r > 0)$, where

$$T_{[z,t]} = \begin{bmatrix} 1 & -\bar{z} & -(|z|^2 - it)/2 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.5a}$$

$$R_\theta = \begin{bmatrix} e^{-i\theta/3} & 0 & 0 \\ 0 & e^{2i\theta/3} & 0 \\ 0 & 0 & e^{-i\theta/3} \end{bmatrix}, \tag{2.5b}$$

$$D_r = \begin{bmatrix} r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/r \end{bmatrix}. \tag{2.5c}$$

In Heisenberg coordinates, they correspond respectively to the following transformations

- $T_{[z,t]}$ is the left multiplication by $[z, t]$,

- R_θ is given by $[w, s] \mapsto [e^{i\theta}w, s]$,

- D_r is the Heisenberg dilation $[w, s] \mapsto [rw, r^2s]$.

Note that Heisenberg translations and rotations preserve each horosphere based at q_∞ whereas Heisenberg dilations permute horospheres based at q_∞. We will denote by Isom($\mathfrak{N}$) the non-loxodromic stabilizer of q_∞ in SU(2, 1), which is generated by the $T_{[z,t]}$ and R_θ. The notation Isom($\mathfrak{N}$) comes from the fact it is the isometry group of the *Cygan metric*, which we will not use here (see [FALPAR]). The group Isom($\mathfrak{N}$) consists exactly of those matrices of the form

$$P_{(z,t,\theta)} = T_{[z,t]}R_\theta = \begin{bmatrix} e^{-i\theta/3} & -e^{2i\theta/3}\bar{z} & -e^{-i\theta/3}(|z|^2 - it)/2 \\ 0 & e^{2i\theta/3} & e^{-i\theta/3}z \\ 0 & 0 & e^{-i\theta/3} \end{bmatrix}.$$

It is sometimes more convenient to work with the lift of $P_{(z,t,\theta)}$ to U(2,1) given by

$$\begin{bmatrix} 1 & -\bar{z}e^{i\theta} & -(|z|^2 - it)/2 \\ 0 & e^{i\theta} & z \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.6}$$

In terms of these parameters the various parabolic conjugacy classes are easily described. Any parabolic isometry is conjugate in PU(2, 1) to exactly one of the following:

- $P_{(1,0,0)} = T_{[1,0]}$ if it is 3-step unipotent,
- $P_{(0,\pm 1,0)}$ if it is 2-step unipotent,
- $P_{(0,1,\theta)}$ for some non-zero $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ if it is screw parabolic.

Remark 1. Screw and 2-step unipotent parabolics share the property of preserving a (unique) complex line. In the case of $P_{(0,1,\theta)}$, it is the line polar to the vector $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$. As we will see in the next section, the situation is slightly more complicated for 3-step unipotent parabolics.

Finally, recall from [FalPar] for future reference that the exact sequence

$$1 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{N} \overset{\Pi}{\longrightarrow} \mathbb{C} \longrightarrow 1 \tag{2.7}$$

induces an exact sequence

$$1 \longrightarrow \mathrm{Isom}^+(\mathbb{R}) \longrightarrow \mathrm{Isom}(\mathfrak{N}) \overset{\Pi_*}{\longrightarrow} \mathrm{Isom}^+(\mathbb{C}) \longrightarrow 1. \tag{2.8}$$

Explicitly,

$$\Pi_*(P_{(z,t,\theta)}) = \begin{pmatrix} e^{i\theta} & z \\ 0 & 1 \end{pmatrix},$$

acting on $\mathbb{C}$ by

$$w \longmapsto e^{i\theta}w + z.$$

2.3. The invariant fan of a 3-step unipotent parabolic. The Siegel model is very well adapted to describing the action of parabolic isometries. To do so, we give a few more details on the structure of the boundary of $\mathrm{H}_{\mathbb{C}}^2$. It is equipped with a (CR) contact structure, which is given in Heisenberg coordinates as the kernel of the 1-form

$$\alpha = dt - 2x\,dy + 2y\,dx.$$

We will denote by C_m the contact plane at a point $m \in \partial\mathrm{H}_{\mathbb{C}}^2$. For any real plane R of $\mathrm{H}_{\mathbb{C}}^2$, the intersection $\overline{R} \cap \partial\mathrm{H}_{\mathbb{C}}^2$ is a closed curve, which is called an $\mathbb{R}$-circle (see [G]). These curves are Legendrian: they are everywhere tangent to the contact distribution. The following lemma concerns those $\mathbb{R}$-circles containing the point q_∞, called *infinite $\mathbb{R}$-circles*. We refer to Chapter 4 of [G] for proofs.

Lemma 2.1. (1) *Let R be a real plane containing the point q_∞ of the Siegel model. Then in Heisenberg coordinates, the $\mathbb{R}$-circle ∂R is an affine line in the Heisenberg group, which is contained in the contact plane at any of its points, and the restriction to the boundary of the reflection about R is the (Euclidean) half-turn about this affine line.*

(2) *More precisely, an affine line in the Heisenberg group is an infinite $\mathbb{R}$-circle if and only if it is contained in the contact plane at one of its points.*

Viewing the boundary of $H_{\mathbb{C}}^2$ as the one point compactification of the Heisenberg group $\mathfrak{N} = \mathbb{C} \ltimes \mathbb{R}$ endows the set of complex lines through q_∞ with the stucture of the affine space $\mathbb{C}$. Indeed, for any $z \in \mathbb{C}$ there exists a unique complex line through q_∞ which contains q_∞ and the point $[z, 0]$ (which is polar to the vector $\begin{bmatrix} \bar{z} & 1 & 0 \end{bmatrix}^T$). This induces a projection $\widetilde{\Pi} \colon \overline{H_{\mathbb{C}}^2} \setminus q_\infty \mapsto \mathbb{C}$ whose fibers are the complex lines through q_∞. In restriction to the boundary, this projection is just the vertical projection Π defined in Equation (2.7), which is given in Heisenberg coordinates by

$$\Pi \colon [z, t] \longmapsto [z, 0]. \tag{2.9}$$

A *fan* through q_∞ is the preimage of any affine line in $\mathbb{C}$ under the projection $\widetilde{\Pi}$. In view of (2.9), if L is an affine line in $\mathbb{C}$, the fan $\widetilde{\Pi}^{-1}(L)$ intersects the boundary of $H_{\mathbb{C}}^2$ along the vertical plane containing L in the Heisenberg group. A general fan is the image of a fan through q_∞ by an element of $PU(2, 1)$. These objects were defined by Goldman and Parker in [GoP] (see also Chapter 4 of [G]). As stated in [GoP], fans enjoy a double foliation, by real planes and complex lines. We now make this foliation explicit in the case of a fan F through q_∞ in $H_{\mathbb{C}}^2$, with ∂F projecting to an affine line $L \subset \mathbb{C} \subset \partial H_{\mathbb{C}}^2$.

(1) First, the foliation by complex lines is given by the fibers of $\widetilde{\Pi}$ above the affine line L. In the boundary, this foliation correspond to the foliation of the vertical plane above L by vertical lines.

(2) Consider a point m in the vertical plane ∂F. The contact plane at m intersects ∂F along an affine line $L' = C_m \cap \partial F$ in the Heisenberg group, which projects vertically onto L. By Lemma 2.1, the line L' is the boundary of a real plane. Then ∂F is foliated by the family of lines parallel to L' contained in ∂F. In other words, the foliation of ∂F is obtained by taking all lifts of L to the contact structure. All these lines are boundaries of real planes, and this foliation of ∂F extends inside $H_{\mathbb{C}}^2$ as a foliation of F by real planes.

We can be a bit more explicit.

Lemma 2.2. *Let $L_{w,k}$ be the affine line in $\mathbb{C}$ parametrized by $L_{w,k} = \{w(s + ik), s \in \mathbb{R}\}$, for some unit modulus w and $k \geq 0$. Then the boundary foliation of the fan above $L_{w,k}$ is given by the lines parametrized in Heisenberg coordinates by $L_{t_0} = \{[w(s + ik), t_0 + 2sk], s \in \mathbb{R}\}$.*

Proof. The lines L_{t_0} all project onto L by the vertical projection. A tangent vector to L_{t_0} is given by $(\mathrm{Re}(w), \mathrm{Im}(w), 2k)$. Evaluating the 1-form α on this vector at the point of parameter $s = 1$, and using $|w| = 1$ shows that the line L_{t_0} is in the kernel of α at this point, thus in the contact plane. Therefore L_{t_0} is an $\mathbb{R}$-circle by Lemma 2.1. $\qquad\square$

Our interest in fans comes from the following fact.

Proposition 2.1. *Let P be a 3-step unipotent parabolic isometry of $\mathrm{H}_\mathbb{C}^2$. There exists a unique fan F_P through the fixed point of P such that*

(1) *the fan F_P is P-invariant and*

(2) *every leaf of the foliation of F_P by real planes is P-invariant.*

Moreover, when P fixes q_∞ and thus corresponds to left-translation by $[z,t]$ with $z \neq 0$ in Heisenberg coordinates, the fan F_P is the one above the affine line $L_{w,k}$, where $w = z/|z|$, and $k = t/(4|z|)$.

Given a 3-step unipotent parabolic isometry P, we will refer to the fan F_P as *the invariant fan of P*, even though all the fans corresponding to distinct parallel vertical planes are P-invariant (but their leaves are not P-invariant).

Proof. First assume that $P = T_{[1,0]}$, acting on the Heisenberg group as

$$[z,t] \longmapsto [z+1, t - 2\,\mathrm{Im}(z)].$$

Then every vertical plane $\mathrm{Im}(z) = k$ is globally preserved. The real foliation of the fan corresponding to the vertical plane $\mathrm{Im}(z) = k$ is given by the family of lines $L_{t_0} = \{[s + ik, t_0 + 2sk], s \in \mathbb{R}\}$. Since

$$[1,0] \cdot [s + ik, t_0 + 2sk] = [s + 1 + ik, t_0 + 2(s-1)k],$$

we see that the real foliation of the vertical plane $\mathrm{Im}(z) = k$ is preserved by P if and only if $k = 0$, which gives the result for this normalization of P. The first part of the proposition is then obtained by using the fact that any 3-step unipotent parabolic is conjugate in $\mathrm{PU}(2,1)$ to $T_{[1,0]}$. To check the last part, write $w = z/|z|$ and $k = t/(4|z|)$; then by a direct calculation

$$[z,t] \cdot [w(s + ik, t_0 + 2sk] = [w(|z| + s + ik), t_0 + t + 2sk - 2|z|k]$$
$$= [w(|z| + s + ik), t_0 + 2(|z| + s)k],$$

which proves that the real leaves of the fan above $L_{w,k}$ are preserved. $\square$

In fact the proof of Proposition 2.1 gives us a little more information, which we summarize in the following corollary.

Corollary 2.1. *Let P be a 3-step unipotent parabolic.*

(1) *A real plane is P-invariant if and only if it is a leaf of the real foliation of its invariant fan.*

(2) *P is characterized by its invariant fan F, and its restriction to F.*

For future reference, let us state the following proposition, which is just gathering together what we just exposed.

Proposition 2.2. *Let P be a parabolic isometry in* $\mathrm{PU}(2, 1)$.

(1) *If P is screw parabolic or 2-step unipotent, it has a unique invariant complex line.*

(2) *If P is 3-step unipotent parabolic, then there exists a unique fan $\mathcal{F}$ centered at the fixed point of P which is P-invariant and such that every leaf of the foliation of $\mathcal{F}$ by real planes is P-invariant.*

We will also use the following characterization of commuting parabolic isometries.

Lemma 2.3. *Let P_1 and P_2 be two parabolic isometries fixing q_∞. Then P_1 and P_2 commute if and only if one of the following possibilities occurs.*

(1) *Both P_1 and P_2 are either 2-step unipotent or screw parabolics with the same invariant complex line.*

(2) *Both P_1 and P_2 are 3-step unipotent with the same fixed point and their invariant fans intersect $\partial\,\mathrm{H}^2_{\mathbb{C}}$ along parallel vertical planes.*

Proof. We only prove the second part, the first one being classical. Two 3-step unipotent maps fixing q_∞ are respectively conjugate to the left translations by $[z_1, t_1]$ and $[z_2, t_2]$. Using (2.4), we see that these two translations commute if and only if $z_2\overline{z_1} \in \mathbb{R}$, which is equivalent to saying that their invariant fans are parallel. □

2.4. Eigenvalues and traces. The following classification of conjugacy classes in $\mathrm{U}(n, 1)$ is due to Chen–Greenberg (Theorem 3.4.1 of [CHGR], where the real and quaternionic cases are treated as well).

Theorem 2.2 (Chen–Greenberg). (a) *Any elliptic element is semisimple, with eigenvalues of norm 1. Two elliptic elements are conjugate in* $\mathrm{U}(n, 1)$ *if and only if they have the same eigenvalues and the same eigenvalue of negative type.*

(b) *Any loxodromic element is semisimple, with exactly $n - 1$ eigenvalues of norm 1. Two loxodromic elements are conjugate in* $\mathrm{U}(n, 1)$ *if and only if they have same eigenvalues.*

(c) *Any parabolic element is not semisimple, and all its eigenvalues have norm 1. It has a unique decomposition $g = pe = ep$ with p strictly parabolic and e elliptic. Two parabolic elements are conjugate in* $\mathrm{U}(n, 1)$ *if and only if their strictly parabolic and elliptic components are conjugate.*

(d) *There are three classes of strictly parabolic elements for $n > 1$: two classes of vertical Heisenberg translations, and one class of horizontal ones.*

Note that two elliptic elements may be conjugate in $GL(n + 1, \mathbb{C})$ but not in $U(n, 1)$ (if they have the same eigenvalues but different eigenvalues of negative type). It will be useful for future reference to have matrix representatives in SU(2,1) of the different conjugacy classes of isometries when $n = 2$.

(1) Any elliptic isometry of $H_{\mathbb{C}}^2$ is conjugate to one given in the ball model by the matrix $E_{(\alpha,\beta)}$ given in (2.2).

(2) Any loxodromic isometry of $H_{\mathbb{C}}^2$ is conjugate to one given in the Siegel model by the matrix $D_r R_\theta$ where D_r and R_θ are as in (2.5).

(3) Any parabolic isometry of $H_{\mathbb{C}}^2$ is conjugate to one given in the Siegel model by the matrix $P_{(z,t,\theta)} = T_{[z,t]} R_\theta$. The isometry associated to $P_{(z,t,\theta)}$ is unipotent if and only if $\theta = 0$ and 2-step unipotent if and only if $\theta = 0$ and $z = 0$.

As in the classical case of the Poincaré disc, the isometry type of an isometry is closely related to the trace of a lift to SU(2,1). The characteristic polynomial of a matrix A in SU(2,1) is given by

$$\chi_A(X) = X^3 - z \cdot x^2 + \bar{z} \cdot x - 1, \quad \text{where } z = \operatorname{Tr} A. \qquad (2.10)$$

Computing its discriminant, we obtain

$$f(z) = \operatorname{Res}(\chi_A, \chi_A') = |z|^4 - 8\operatorname{Re}(z^3) + 18|z|^2 - 27. \qquad (2.11)$$

This function provides the following classification of holomorphic isometries via the trace of their lifts to SU(2,1) (see ch. 6 of [G]), which is analogous to the classical SL(2,$\mathbb{C}$) case. Denote by C_3 the set of cube roots of unity in $\mathbb{C}$.

Theorem 2.3 (Goldman). *Let $A \in SU(2, 1)$ and $g \in PU(2, 1)$ the corresponding isometry of $H_{\mathbb{C}}^2$. Then*

- *g is regular elliptic $\iff$ $f(\operatorname{Tr}(A)) < 0$;*
- *g is loxodromic $\iff$ $f(\operatorname{Tr}(A)) > 0$;*
- *g is special elliptic or screw-parabolic $\iff$ $f(\operatorname{Tr}(A)) = 0$ and $\operatorname{Tr}(A) \notin 3C_3$;*
- *g is unipotent or the identity $\iff$ $\operatorname{Tr}(A) \in 3C_3$.*

The null-locus of the polynomial f can be seen in Figure 1.
We now focus on the special case of elements of SU(2,1) having real trace.

Proposition 2.3. *Let $A \in SU(2, 1)$ satisfy $\operatorname{Tr} A \in \mathbb{R}$. Then A has an eigenvalue equal to 1. More precisely,*

- *if A is loxodromic then A has eigenvalues $\{1, r, 1/r\}$ for some $r \in \mathbb{R} \setminus [-1, 1]$;*
- *if A is elliptic then A has eigenvalues $\{1, e^{i\theta}, e^{-i\theta}\}$ for some $\theta \in [0, \pi]$;*
- *if A is parabolic then A has eigenvalues $\{1, 1, 1\}$ or $\{1, -1, -1\}$.*
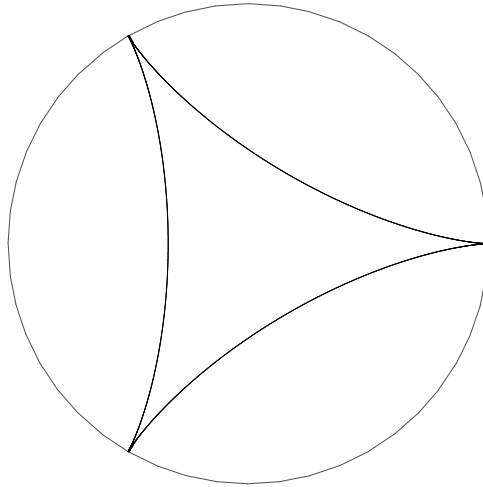
Figure 1. The null-locus of the polynomial f inscribed in the circle of radius 3 centered at the origin.

Proof. If Tr A is real, then χ_A has real coefficients, therefore the eigenvalue spectrum of A is invariant under complex conjugation. The result follows. $\square$

Remark 2. (a) Any loxodromic or parabolic element of SU(2,l) with real trace is conjugate to its inverse; for elliptic isometries this is true under the additional assumption that the eigenvalue 1 has negative type. This follows from Proposition 2.2. An element of a group which is conjugate to its inverse is sometimes called *achiral* or *reversible* in the context of isometry groups. In [GonP], Gongopadhyay and Parker have studied and classified these isometries in PU(n, 1) for all $n \geqslant 1$.

(b) For later use, let us note that matrix representatives of the conjugacy classes of elements of PU(2, 1) having a fixed point in $\overline{H_{\mathbb{C}}^2}$ with associated positive eigenvalue are given by, in the notation of (2.2) and (2.5),

- $E_{(\theta, -\theta)}$ with $\theta \in \mathbb{R}$ (in the ball model) for elliptic classes,
- D_r with $r > 1$ (in the Siegel model) for loxodromic classes, and
- $P_{(z,t,0)}$ with $z \in \mathbb{C}$ and $t \in \mathbb{R}$ (in the Siegel model) for parabolic classes.

2.5. Antiholomorphic isometries. The following lemma is useful when computing with antiholomorphic isometries (see [FalPau] in the elliptic case, where the corresponding matrix was called a *Souriau matrix*). To simplify the statement, we consider a projective model (for example the ball or Siegel model) for which complex conjugation in affine coordinates $\sigma_0 \colon (z_1, \ldots, z_n) \mapsto (\overline{z_1}, \ldots, \overline{z_n})$ is an isometry (i.e. the Hermitian form has real coefficients).

Lemma 2.4. *Let f be an antiholomorphic isometry of $H_\mathbb{C}^n$. Then there exists $M \in U(n, 1)$ such that, for any $m \in H_\mathbb{C}^n$ with lift $\mathbf{m} \in \mathbb{C}^{n+1}$ (denoting $\bar{\mathbf{m}}$ the componentwise complex conjugate of $\mathbf{m}$),*

$$f(m) = M \cdot \bar{\mathbf{m}}.$$

Any matrix $M \in U(n, 1)$ with this property will be called a lift of f.

Proof. $f \circ \sigma_0$ is a holomorphic isometry, i.e. corresponds to an element of $PU(n, 1)$. Any of its lifts $M \in U(n, 1)$ satisfies the required property. $\quad\square$

Definition 2.1. Given a real reflection σ and an isometry $A \in PU(n, 1)$, we say that σ *decomposes* A if $A = \sigma\tau$ for some real reflection τ (equivalently, $A = \tau'\sigma$ where $\tau' = \sigma\tau\sigma$ is also a real reflection).

Given two isometries $A, B \in PU(n, 1)$, we say that the pair (A, B) is $\mathbb{R}$-*decomposable* if there exists a real reflection which decomposes both A and B.

Note that when writing the two isometries A and B as products of real reflections, the order in which the reflections appear is not important. As an application of Lemma 2.4, we obtain a necessary condition on a pair (A, B) to be $\mathbb{R}$-decomposable, namely that the commutator $[A, B]$ must have real trace. Note that the trace of an element of $PU(n, 1)$ is not well-defined in general. However, if A and B are in $PU(n, 1)$, then the matrix $[\widetilde{A}, \widetilde{B}]$ does not depend on the choice of lifts $\widetilde{A}$ and $\widetilde{B}$ made for A and B. This allows us to consider the condition of having real trace for a commutator.

Lemma 2.5. *If a pair $(A, B) \in PU(n, 1) \times PU(n, 1)$ is $\mathbb{R}$-decomposable, then the commutator $[A, B]$ is the square of an anti-holomorphic isometry.*

Proof. If $A = \sigma_1\sigma_2$ and $B = \sigma_2\sigma_3$ with each σ_i a real reflection, then $[A, B] = (\sigma_1\sigma_2)(\sigma_2\sigma_3)(\sigma_2\sigma_1)(\sigma_3\sigma_2) = (\sigma_1\sigma_3\sigma_2)^2$, where $\sigma_1\sigma_3\sigma_2$ is an antiholomorphic isometry. $\quad\square$

Corollary 2.4. *If a pair $(A, B) \in PU(n, 1) \times PU(n, 1)$ is $\mathbb{R}$-decomposable, then $\mathrm{Tr}([A, B]) \in \mathbb{R}$.*

Proof. First note that if f_1 and f_2 are antiholomorphic isometries with lifts M_1 and M_2, then $f_1 \circ f_2$ is holomorphic with lift $M_1\overline{M_2} \in U(n, 1)$. Using this fact together with Lemma 2.5, we see that if (A, B) is $\mathbb{R}$-decomposable, then the commutator can be lifted to $SU(n, 1)$ as $M\overline{M}$, with $M = M_1\overline{M_3}M_2$, where M_k is a lift of σ_k. As a consequence, we see that

$$\overline{\mathrm{Tr}(M\overline{M})} = \mathrm{Tr}(\overline{M}M) = \mathrm{Tr}(M\overline{M}) \in \mathbb{R}. \qquad\square$$

Note that this necessary condition holds in any dimension n. However, to obtain a sufficient condition, we will need Lemma 2.7 below, which is false in dimensions $n > 2$.

Lemma 2.6. *Any antiholomorphic isometry of* $\mathrm{H}^1_{\mathbb{C}}$ *having a fixed point in* $\mathrm{H}^1_{\mathbb{C}}$ *is a real reflection.*

Proof. Let f be an antiholomorphic isometry of $\mathrm{H}^1_{\mathbb{C}}$ with a fixed point p. Then, for any point $q \neq p$, the angles $(f^{-1}(q), p, q)$ and $(q, p, f(q))$ are opposite (because f is antiholomorphic), therefore $f^{-1}(q) = f(q)$ (because f is an isometry), and f is an involution and fixes pointwise a geodesic.   □

Lemma 2.7. *If* g *is an antiholomorphic isometry of* $\mathrm{H}^2_{\mathbb{C}}$ *which exchanges two points of* $\overline{\mathrm{H}^2_{\mathbb{C}}}$, *then* g *is an* $\mathbb{R}$-*reflection.*

Proof. Let p and q be the two points exchanged by g. Then g has a fixed point m on the geodesic (pq) (which is the midpoint of the geodesic segment $[pq]$ when p and q are in $\mathrm{H}^2_{\mathbb{C}}$). Then g stabilizes the complex line C spanned by p and q, as well as the complex line C' orthogonal to C at m. By Lemma 2.6, the restrictions of g to C and C' are real reflections, fixing geodesics γ and γ'. Now γ and γ' are geodesics contained in orthogonal complex lines, therefore they span a Lagrangian plane, which is fixed pointwise by g.   □

Lemma 2.7 is false as soon as $n \geq 3$. Indeed, consider an element $M \in \mathrm{SU}(n,1)$ given in the Siegel model by

$$M = \begin{bmatrix} (0) & & z \\ & U & \\ 1/\bar{z} & & (0) \end{bmatrix},$$

where $z \in \mathbb{C}^{\star}$ and $U \in U(n-1)$ is a matrix such that $U\bar{U} \neq \mathrm{Id}$, that is a non-symmetric matrix in $U(n-1)$ (such matrices only exist when $n > 2$). Then

$$M\bar{M} = \begin{bmatrix} 1 & & (0) \\ & U\bar{U} & \\ (0) & \dots & 1 \end{bmatrix} \neq \mathrm{Id}.$$

Therefore the antiholomorphic isometry assciated with M is not a real reflection, though it exchanges the two points corresponding to the first and last vectors in the canonical basis of $\mathbb{C}^{n+1}$, which are both on the boundary of $\mathrm{H}^n_{\mathbb{C}}$.

The following proposition describes which elements of $\mathrm{PU}(2,1)$ can arise as squares of antiholomorphic isometries.

Proposition 2.4. *Let* $A \in \mathrm{PU}(2,1)$ *be an isometry admitting a lift* $\tilde{A}$ *to* $\mathrm{SU}(2,1)$ *such that for any fixed point A in* $\overline{\mathrm{H}^2_{\mathbb{C}}}$, *the corresponding eigenvalue of* $\tilde{A}$ *is positive. Then A is the square of an antiholomorphic isometry unless A is a 2-step unipotent parabolic isometry.*

Proof. Being the square of an antiholomorphic isometry is preserved by conjugation, therefore we only need to prove the result for representatives of each conjugacy class. We first provide matrices M such that $M\bar{M}$ is equal to the lifts listed in Remark 2(b) in the elliptic and loxodromic cases.

(1) If A is elliptic the following M satisfies the above condition,

$$M = \begin{bmatrix} 0 & e^{i\theta/2} & 0 \\ e^{-i\theta/2} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(2) If A is loxodromic, then we can take

$$M = \begin{bmatrix} \sqrt{r} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{r} \end{bmatrix}.$$

We now examine the case where A is parabolic, thus unipotent in view of the assumptions. Assume that A is unipotent and that $A = \phi^2$ with ϕ an antiholomorphic isometry. First, ϕ has at least one fixed point in the closure of $H_{\mathbb{C}}^2$ and ϕ^2 has exactly one fixed point there. This implies that ϕ has only one fixed point, which is the same as that of A. Conjugating by an element of PU(2, 1), we can assume that this fixed point is q_∞. Consider a lift M of ϕ to SU(2, 1) in the sense of Lemma 2.4. The fact that $M\bar{M}$ is a unipotent map fixing q_∞, which corresponds to the vector $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ implies that M is an upper triangular matrix in SU(2, 1) with unit modulus diagonal entries. Therefore M is of the form $P_{(z,t,\theta)}$. Computing $M\bar{M} = P_{(z,t,\theta)}P_{(\bar{z},-t,-\theta)}$, we obtain that A has a lift to SU(2,1) of the form

$$\begin{bmatrix} 1 & -(\bar{z} + ze^{-i\theta}) & -\bar{z}(z + \bar{z}e^{i\theta}) \\ 0 & 1 & (z + \bar{z}e^{i\theta}) \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.12}$$

This matrix can only be 3-step unipotent or the identity. This proves the result as there is only one PU(2, 1)-conjugacy class of 3-step unipotents. $\square$

2.6. Decomposing isometries. The following proposition summarizes results characterizing which real reflections decompose a given holomorphic isometry. Parts (e1-e3) are Proposition 2.4 of [FALPAU], part (l) is Proposition 4(2) of [W2] (and follows from Proposition 3.1 of [FZ]), and part (p) is new.

Proposition 2.5. *Let $A \in \mathrm{PU}(2, 1)$ and σ a real reflection with fixed $\mathbb{R}$-plane L.*

(e1) *If A is a complex reflection with fixed complex line C, then σ decomposes A if and only if $L \cap C$ is a geodesic.*

(e2) *If A is a complex reflection in a point p_A, then σ decomposes A if and only if $p_A \in L$.*

(e3) *If A is regular elliptic with fixed point p_A and invariant complex lines C_1, C_2, then σ decomposes A if and only if $p_A \in L$ and $L \cap C_i$ is a geodesic for $i = 1, 2$.*

(l) *If A is loxodromic then σ decomposes A if and only if σ exchanges the 2 fixed points of A.*

(p) *If A is parabolic one of the following cases holds.*

  (1) *If A is screw-parabolic or 2-step unipotent with fixed point p and invariant complex line L then σ decomposes A if and only if $\sigma(p) = p$ and $\sigma(L) = L$.*

  (2) *If A is 3-step unipotent with fixed point p and invariant fan F (see Proposition 2.2), then σ decomposes A if and only if $\sigma(p) = p$, $\sigma(F) = F$, and the restriction of σ to ∂F is a half-turn.*

Proof. Parts (e1–e3) are Proposition 2.4 of [FALPAU], part (l) is Proposition 4(2) of [W2] (and follows from Proposition 3.1 of [FZ]). Let us prove part (p).

Denote by p the fixed point of A. Assume that $A = \sigma\sigma'$ where σ' is another real reflection. Because both σ and σ' are involutions, either σ and σ' fix p, or there exists $q \neq p$ in $\partial \mathrm{H}^2_{\mathbb{C}}$ such that σ and σ' both swap p and q. In this case, A would fix two distinct points in $\partial \mathrm{H}^2_{\mathbb{C}}$, which is not possible for a parabolic. Thus the fixed real plane of σ contains p. Replacing p by V and using the fact that V is the unique invariant complex line or invariant fan of A, we obtain that σ and σ' both preserve V.

Conversely, let σ be such a real reflection.

(1) If A is 2-step unipotent or screw parabolic, then V is a complex line. The restriction of σ to V is an involution fixing a boundary point of V. As V is a copy of the Poincaré disk, this implies that the restriction $\sigma_{|V}$ is a symmetry about a geodesic, and therefore L intersects V along a geodesic γ, one of whose endpoints is the fixed point of A. Let us call a the other endpoint of γ, and $b = A(a)$. It is a simple exercise to check that $A \circ \sigma$ exchanges a and b. Therefore $A \circ \sigma$, which is an antiholomorphic isometry, exchanges two points of $\partial \mathrm{H}^2_{\mathbb{C}}$. By Lemma 2.7, it is a real reflection and therefore σ decomposes A.

(2) If A is 3-step unipotent, we may assume by conjugating that $A = T_{[1,0]}$; then the boundary of its invariant fan F_A is the vertical plane $\{y = 0\}$, on which A acts as a horizontal translation by 1. In view of Lemma 2.8, the fixed $\mathbb{R}$-circle of σ is either a horizontal line $\{t = t_0\}$ contained in the plane ∂F_A, or an infinite $\mathbb{R}$-circle orthogonal to this vertical plane.

In the first case $\sigma \circ A$ acts on ∂F_A by $(x, t) \mapsto (x + 1, t_0 - t)$ which is not an involution, and therefore σ cannot decompose A. In the second case, the restriction of σ to ∂F_A is a half-turn, and so is the restriction of $\sigma \circ A$. The latter is also the restriction to ∂F_A of a reflection σ' about a infinite $\mathbb{R}$-circle; then $A = \sigma \circ \sigma'$ by Corollary 2.1, as A and $\sigma \circ \sigma'$ agree on their common invariant fan. $\qquad\square$

The following lemma describes real reflections that preserve a given fan. We state it only for fans through q_∞, as this is all we will need. Let F be a fan through q_∞ and $(L_t)_{t \in \mathbb{R}}$ be the foliation of its boundary by infinite $\mathbb{R}$-circles. Let L be an infinite $\mathbb{R}$-circle not contained in or parallel to ∂F, and L_{t_0} be the unique leaf of ∂F such that $L \cap L_{t_0}$ is nonempty. We will say that L and F are orthogonal whenever L and L_{t_0} are orthogonal in the contact plane at the point $L \cap L_{t_0}$. It is a direct consequence of Lemma 2.1 that the reflection about an infinite $\mathbb{R}$-circle preserves any fan which is orthogonal to it.

Lemma 2.8. *Let F be a fan through q_∞. Let σ be the real reflection about a real plane R. Then σ preserves F if and only R contains the point q_∞ on its boundary and one of the following occurs*:

(1) *the real plane R is one of the leaves of F, in which case σ acts on the vertical plane ∂F as the reflection across the affine line ∂R;*

(2) *the real plane R intersects the Heisenberg group along an affine line orthogonal to the vertical plane ∂F. In that case, it acts on ∂F as a Euclidean half-turn.*

Proof. First, σ fixes the point q_∞. Indeed, as it preserves F, σ preserves globally the (singular) foliation of F by real planes and therefore fixes the intersection of all leaves, which is q_∞. This means that R contains q_∞, and that the corresponding $\mathbb{R}$ circle is infinite. Conjugating by an element of $\mathrm{Isom}(\mathfrak{N})$, we can assume that σ is the real reflection about the real plane $H_\mathbb{R}^2 \cap H_\mathbb{C}^2$, which acts on the boundary by $\sigma([z, t]) = [\bar{z}, -t]$. A vertical plane is σ-invariant if and only if it is the vertical plane $\{y = 0\}$, in which case R is a leaf of the corresponding fan, or a plane orthogonal to the x-axis, which corresponds to the second case. $\qquad\square$

3. Configurations of points and cross-ratios

3.1. Triples of points

Definition 3.1. Given a triple (p_1, p_2, p_3) of distinct points in $H_{\mathbb{C}}^n \cup \partial H_{\mathbb{C}}^n$ and lifts $P_i \in \mathbb{C}^{n,1}$ of the p_i, the ratio

$$T(p_1, p_2, p_3) = \frac{\langle P_1, P_2 \rangle \langle P_2, P_3 \rangle \langle P_3, P_1 \rangle}{\langle P_1, P_3 \rangle \langle P_3, P_2 \rangle \langle P_2, P_1 \rangle}$$

does not depend on the lifts P_i. We will call $T(p_1, p_2, p_3)$ the *triple-ratio* of (p_1, p_2, p_3).

Note that $T(p_1, p_2, p_3)$ is also well-defined if 2 or more of the points are equal in $H_{\mathbb{C}}^n$ (but not in $\partial H_{\mathbb{C}}^n$).

Observe that holomorphic isometries (elements of $PU(n, 1)$) clearly preserve the triple-ratio, whereas for any antiholomorphic isometry g, we have

$$T(g(p_1), g(p_2), g(p_3)) = \overline{T(p_1, p_2, p_3)}.$$

The triple-ratio is related to the classical Cartan angular invariant $\mathbb{A}$ (see [C]) and Brehm's shape invariant σ (see [Br]) for triangles as follows.

- The Cartan angular invariant of three points $p_1, p_2, p_3 \in \partial H_{\mathbb{C}}^n$ is defined as

$$\mathbb{A}(p_1, p_2, p_3) = \arg(-\langle P_1, P_2 \rangle \langle P_2, P_3 \rangle \langle P_3, P_1 \rangle).$$

  It relates to the triple-ratio by

$$T(p_1, p_2, p_3) = e^{2i\,\mathbb{A}(p_1, p_2, p_3)}.$$

- Brehm's shape invariant σ of three points in $H_{\mathbb{C}}^n$ is related to the normalized triple product

$$\widetilde{T}(p_1, p_2, p_3) = \frac{\langle P_1, P_2 \rangle \langle P_2, P_3 \rangle \langle P_3, P_1 \rangle}{\langle P_1, P_1 \rangle \langle P_2, P_2 \rangle \langle P_3, P_3 \rangle}.$$

  Namely, $\sigma = -\operatorname{Re}(\widetilde{T})$. Note that

$$T(p_1, p_2, p_3) = \frac{\widetilde{T}(p_1, p_2, p_3)}{\widetilde{T}(p_1, p_3, p_2)}.$$

We refer the reader to Chapter 7 of [G] for classical properties of the Cartan invariant. Note in particular that the Cartan invariant satisfies $\mathbb{A}(p_1, p_2, p_3) \in [-\pi/2, \pi/2]$, and that $\mathbb{A}(p_1, p_2, p_3) = \pm\pi/2$ (resp. $\mathbb{A}(p_1, p_2, p_3) = 0$) if and only if the three points are contained in a complex line (resp. a real plane). Also, the Cartan invariant classifies triples of pairwise distinct points in $\partial H_{\mathbb{C}}^n$ up to holomorphic isometries.

The following classification of triples of points in $H_{\mathbb{C}}^n$ is due to Brehm ([BR]).

Theorem 3.1 (Brehm). *Let (x_1, x_2, x_3), (y_1, y_2, y_3) be two triples of points in $H_{\mathbb{C}}^n$. There exists $g \in \mathrm{Isom}(H_{\mathbb{C}}^n)$ such that $g(x_i) = y_i$ (for $i = 1, 2, 3$) if and only if*

- $d(x_1, x_2) = d(y_1, y_2)$, $d(x_2, x_3) = d(y_2, y_3)$, $d(x_1, x_3) = d(y_1, y_3)$ *and* $T(x_1, x_2, x_3) = T(y_1, y_2, y_3)$, *in which case g is holomorphic,*

or

- $d(x_1, x_2) = d(y_1, y_2)$, $d(x_2, x_3) = d(y_2, y_3)$, $d(x_1, x_3) = d(y_1, y_3)$ *and* $T(x_1, x_2, x_3) = \overline{T(y_1, y_2, y_3)}$, *in which case g is antiholomorphic.*

In fact Brehm's formulation is slightly different as he considers $-\mathrm{Re}(\widetilde{T})$ instead of $\widetilde{T}$ (so his statement doesn't include our 2 cases). This is equivalent because the norm of $\widetilde{T}$ is determined by the 3 side-lengths.

3.2. The complex cross-ratio. The following definition is due in this form to Goldman ([G]) (following Koranyi and Reimann ([KR])) in the case of boundary points.

Definition 3.2. Let (p_1, p_2, p_3, p_4) be a quadruple of distinct points in $H_{\mathbb{C}}^n \cup \partial H_{\mathbb{C}}^n$. The quantity defined by

$$\mathbf{X}(p_1, p_2, p_3, p_4) = \frac{\langle P_3, P_1 \rangle \langle P_4, P_2 \rangle}{\langle P_4, P_1 \rangle \langle P_3, P_2 \rangle} \tag{3.1}$$

does not depend on the choice of lifts P_i of the p_i's, and is called the *complex cross-ratio* of (p_1, p_2, p_3, p_4).

Note that $\mathbf{X}(p_1, p_2, p_3, p_4)$ is also well-defined when some of the 4 points coincide, as long as at most 2 of them coincide in $\partial H_{\mathbb{C}}^n$.

The complex cross-ratio of boundary points has been studied in detail in [G] (pp. 224–228), to which we refer the reader for more details. As for the triple-ratio, it is a direct observation that holomorphic isometries preserve $\mathbf{X}$ whereas antiholomorphic ones change it to its complex conjugate. One of our main tools will be finding conditions under which such a cross-ratio is real, in the spirit of the following result (Theorem 7.2.1 of [G]).

Theorem 3.2 (Goldman). *Let (p_1, p_2, p_3, p_4) be a quadruple of distinct points in $\partial H_{\mathbb{C}}^n$. Then $\mathbf{X}(p_1, p_2, p_3, p_4)$ is real and positive if and only if there exists a real reflection ϕ such that $\phi: p_1 \leftrightarrow p_2$ and $p_3 \leftrightarrow p_4$.*

Note that, if there exists such a real reflection ϕ, then

$$X(p_1, p_2, p_3, p_4) = \overline{X(\phi(p_1), \phi(p_2), \phi(p_3), \phi(p_4))} = \overline{X(p_2, p_1, p_4, p_3)}.$$

Going back to the definition of X, it is straightforward that $X(p_2, p_1, p_4, p_3) = X(p_1, p_2, p_3, p_4)$, and we see that the condition that $X \in \mathbb{R}$ is indeed necessary. In [G] the assumption that $\mathbf{X} > 0$ is omitted, but it must be added for the following reason. X is related to triple products by

$$X(p_1, p_2, p_3, p_4) = \frac{\langle P_1, P_2 \rangle \langle P_2, P_3 \rangle \langle P_3, P_1 \rangle}{\langle P_1, P_2 \rangle \langle P_2, P_4 \rangle \langle P_4, P_1 \rangle} \cdot \frac{|\langle P_4, P_2 \rangle|^2}{|\langle P_3, P_2 \rangle|^2}. \tag{3.2}$$

Because the Cartan invariant belongs to $[-\pi/2, \pi/2]$, we see that if X is real and negative the two triple products $\langle P_1, P_2 \rangle \langle P_2, P_i \rangle \langle P_i, P_1 \rangle$ ($i = 3, 4$) must have arguments either both equal to $\pi/2$ or both equal to $-\pi/2$. This means that p_3 and p_4 belong to the complex line spanned by p_1 and p_2, and are on the same side of the geodesic $(p_1 p_2)$. See Proposition 2 of [KR] and property 7 on p. 226 of [G]. However, if p_1, p_2, p_3, p_4 are in such a configuration then there cannot exist a real reflection ϕ such that $\phi: p_1 \leftrightarrow p_2$ and $p_3 \leftrightarrow p_4$. Indeed, if a real reflection preserves a complex line then it acts on it by reflection in a geodesic.

The following basic observation will allow us to project orthogonally onto the complex sides of the quadrilateral (p_1, p_2, p_3, p_4).

Lemma 3.1. *Let $p_1, p_2, p_3, p_4 \in \overline{H_{\mathbb{C}}^n}$ with $p_1 \neq p_2$, and let π_{12} denote orthogonal projection onto the complex line L_{12} spanned by p_1 and p_2. Then $\mathbf{X}(p_1, p_2, p_3, p_4) = \mathbf{X}(p_1, p_2, \pi_{12}(p_3), \pi_{12}(p_4))$.*

Proof. Let c_{12} be a polar vector for L_{12}, normalized so that $\langle c_{12}, c_{12} \rangle = 1$. Then π_{12} is the projectivization of the linear projection in $\mathbb{C}^{2,1}$ given by $\Pi_{12}(z) = z - \langle z, c_{12} \rangle c_{12}$. Then, for any point p in $H_{\mathbb{C}}^n \cup \partial H_{\mathbb{C}}^n$,

$$\langle \Pi_{12}(p), p_i \rangle = \langle p - \langle p, c_{12} \rangle c_{12}, p_i \rangle = \langle p, p_i \rangle \ (i = 1, 2).$$

The result follows by substituting p_3 and p_4 in this expression. (Here and whenever it is convenient we will slightly abuse notation by using the same letter for points in $\mathbb{C}P^n$ and their lifts to $\mathbb{C}^{n,1}$; we will however insist that lifts of points inside $H_{\mathbb{C}}^n$ have norm -1 and lifts of points outside $\overline{H_{\mathbb{C}}^n}$ have norm 1.) $\square$

3.3. Cross-ratios and real reflections. Recall that the classical cross-ratio of 4 distinct points in $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ is defined by (see for instance [G])

$$[z_1, z_2; z_3, z_4] = \frac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)}. \tag{3.3}$$

It is invariant under the diagonal action of $PGL(2, \mathbb{C})$, and is real if and only if the 4 points are cocyclic or collinear. Moreover, $[z_1, z_2; z_3, z_4]$ is positive if and only if the pairs (z_1, z_2) and (z_3, z_4) do not separate each other on the common circle/line. The KR cross-ratio generalizes the classical cross-ratio in the following sense.

Lemma 3.2. *If p_1, p_2, p_3, p_4 lie in a common complex line $C \subset \mathbb{C}P^n$, then*

$$\mathbf{X}(p_1, p_2, p_3, p_4) = [\sigma(p_1), \sigma(p_2); p_4, p_3]$$

where σ denotes inversion in the boundary circle of C.

Proof. By applying an element of $PU(n, 1)$, we may assume that the complex line C containing the p_i is the first coordinate axis of $\mathbb{C}^n$ (seen as an affine chart of $\mathbb{C}P^n$) in the ball model. Each point of C has a lift to $\mathbb{C}^{n,1}$ of the form $[z, 0, \ldots, 0, 1]$, and in these coordinates, σ is given by $z \mapsto 1/\bar{z}$. We lift each p_i as a vector $P_i = [z_i, 0, \ldots, 0, 1]^T$ (the standard lift in the ball model), and compute

$$\mathbf{X}(p_1, p_2, p_3, p_4) = \frac{(z_3\bar{z_1} - 1)(z_4\bar{z_2} - 1)}{(z_4\bar{z_1} - 1)(z_3\bar{z_2} - 1)}$$

$$= \frac{(z_3 - 1/\bar{z_1})(z_4 - 1/\bar{z_2})}{(z_3 - 1/\bar{z_2})(z_4 - 1/\bar{z_1})}$$

$$= [\sigma(z_1), \sigma(z_2); z_4, z_3]. \qquad \square$$

From Lemma 3.1 and 3.2 and the properties of the classical cross-ratio we obtain the following reality condition for $\mathbf{X}$.

Proposition 3.1. *Let $p_1, p_2, p_3, p_4 \in \overline{\mathbf{H}_{\mathbb{C}}^n}$ with $p_1 \neq p_2$, and let π_{12} denote orthogonal projection onto the complex line L_{12} spanned by p_1 and p_2. Then*

(1) *the complex cross-ratio $\mathbf{X}(p_1, p_2, p_3, p_4)$ is real and positive if and only if*

- *either the points $p_1, p_2, \pi_{12}(p_3), \pi_{12}(p_4)$ are all equidistant from a geodesic γ in L_{12}, with p_1, p_2 on one side of γ and $\pi_{12}(p_3), \pi_{12}(p_4)$ on the other,*

- *or the p_i are on the boundary of a common complex line and $\{p_1, p_2\}$ does not separate $\{p_3, p_4\}$ on this circle;*

(2) *$\mathbf{X}(p_1, p_2, p_3, p_4)$ is real and negative if and only if p_1, p_2, p_3, p_4 are on the boundary of a common complex line and $\{p_1, p_2\}$ separates $\{p_3, p_4\}$ on this circle.*

Note that the second statement with the p_i on the boundary of $H_{\mathbb{C}}^n$ is one half of Proposition 2 of [KR] and of property 7 on p. 226 of [G]. However the statement of the other half (so, our first statement) is different when some of the p_i are in $H_{\mathbb{C}}^n$.

Proof. Normalize as in Lemma 3.2 so that the complex line L_{12} containing p_1 and p_2 is the first coordinate axis in the ball model of $H_{\mathbb{C}}^n$. Denote by $z_1, \ldots, z_4$ the respective coordinates in this unit disk of the points $p_1, p_2, \pi_{12}(p_3)$, and $\pi_{12}(p_4)$. According to Lemma 3.2 and Proposition 3.1,

$$\mathbf{X}(p_1, p_2, p_3, p_4) \in \mathbb{R}$$
$$\Longleftrightarrow [1/\overline{z_1}, 1/\overline{z_2}; z_4, z_3] \in \mathbb{R} \tag{3.4}$$
$$\Longleftrightarrow 1/\overline{z_1}, 1/\overline{z_2}, z_4, z_3 \text{ lie on a common circle } C \text{ in } \mathbb{C}P^1.$$

Note that z_1, z_2, z_3, z_4 are in the closed unit disk of $\mathbb{C}$, so that $1/\overline{z_1}$ and $1/\overline{z_2}$ are outside the open unit disk. In particular, either C intersects the unit circle in 2 points p and q, or C is the unit circle.

(1) In view of (3.4), if $\mathbf{X}(p_1, p_2, p_3, p_4) < 0$, then (z_1, z_2) and $(1/\overline{z_3}, 1/\overline{z_4})$ separate each other on C. The latter remark tells us that this is only possible when C is the unit circle. Therefore p_1, p_2, p_3 and p_4 all belong to the boundary of L_{12} and the pairs (p_1, p_2) and (p_3, p_4) separate each other on C.

(2) Assume that $\mathbf{X}(p_1, p_2, p_3, p_4) > 0$.

   (a) If C is the unit circle, then $z_i = 1/\overline{z_i}$ for $i = 1, 2$ and thus

   $$\mathbf{X}(p_1, p_2, p_3, p_4) = [z_1, z_2, z_4, z_3],$$

   which is positive if and only if $\{p_1, p_2\}$ do not separate $\{p_3, p_4\}$ in C.

   (b) If C intersects the unit circle in two points p and q, let γ denote the geodesic whose endpoints are p and q. Then $\pi_{12}(z_3)$ and $\pi_{12}(z_4)$ are on a hypercycle with endpoints p and q (the part of C which is inside the unit disk), and z_1, z_2 are on the image of this hypercycle by reflection in γ (this is the image of the other half of C by inversion in the unit circle), see the top part of Figure 2. Therefore $p_1, p_2, \pi_{12}(p_3), \pi_{12}(p_4)$ are all equidistant from γ, with p_1, p_2 on one side of γ and $\pi_{12}(p_3), \pi_{12}(p_4)$ on the other. $\square$

The following result is the analogue of Theorem 3.2 in the case where the four points are inside $H_{\mathbb{C}}^2$.

Theorem 3.3. *Let $p_1, p_2, p_3, p_4 \in H_{\mathbb{C}}^2$ with $p_1 \neq p_2$ or $p_3 \neq p_4$. There exists a real reflection ϕ such that $\phi: p_1 \leftrightarrow p_2$ and $p_3 \leftrightarrow p_4$ if and only if $\mathbf{X}(p_1, p_2, p_3, p_4) > 0$, $d(p_1, p_3) = d(p_2, p_4)$ and $d(p_1, p_4) = d(p_2, p_3)$.*
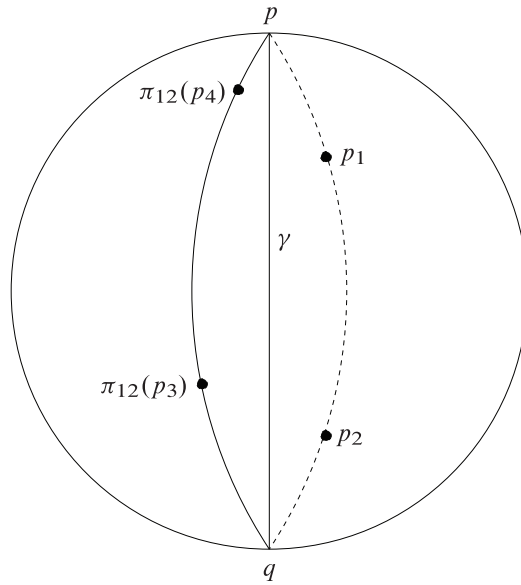
$\mathbf{X}(p_1, p_2, p_3, p_4)$ real
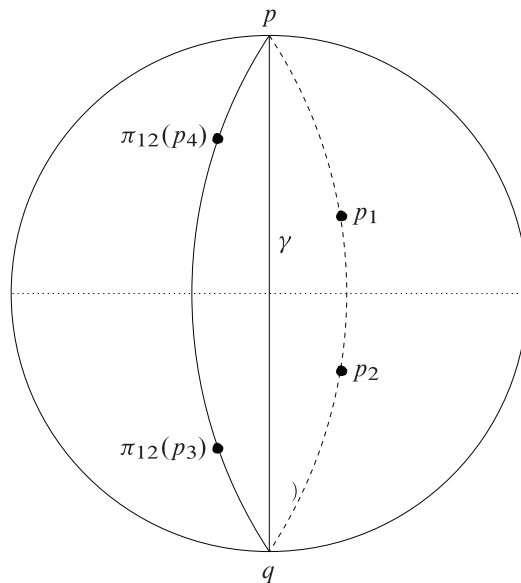
$\mathbf{X}(p_1, p_2, p_3, p_4)$ real, $d(p_1, p_3) = d(p_2, p_4)$, and $d(p_1, p_4) = d(p_2, p_3)$

Figure 2. Projections of quadruples (p_1, p_2, p_3, p_4) to L_{12}.

Proof. Let $p_1, p_2, p_3, p_4 \in H^2_{\mathbb{C}}$ satisfy $\mathbf{X}(p_1, p_2, p_3, p_4) > 0$, $d(p_1, p_3) = d(p_2, p_4)$ and $d(p_1, p_4) = d(p_2, p_3)$. Consider the 2 ordered triples (p_1, p_2, p_3) and (p_2, p_1, p_4); these have equal side-lengths by our assumptions. We now compare the corresponding triple-ratios, $T_1 = T(p_1, p_2, p_3)$ and $T_2 = T(p_2, p_1, p_4)$. Observe that, taking lifts $P_1, \ldots, P_4$ of $p_1, \ldots, p_4$ respectively in $\mathbb{C}^{2,1}$,

$$
\begin{aligned}
T_2 &= \frac{\langle P_2, P_1 \rangle \langle P_1, P_4 \rangle \langle P_4, P_2 \rangle}{\langle P_1, P_2 \rangle \langle P_4, P_1 \rangle \langle P_2, P_4 \rangle} \\
&= \frac{\langle P_2, P_1 \rangle \langle P_3, P_2 \rangle \langle P_1, P_3 \rangle}{\langle P_1, P_2 \rangle \langle P_2, P_3 \rangle \langle P_3, P_1 \rangle} \cdot \frac{\langle P_2, P_3 \rangle \langle P_1, P_4 \rangle}{\langle P_2, P_4 \rangle \langle P_1, P_3 \rangle} \cdot \frac{\langle P_3, P_1 \rangle \langle P_4, P_2 \rangle}{\langle P_3, P_2 \rangle \langle P_4, P_1 \rangle} \\
&= \overline{T_1} \cdot \frac{\overline{\mathbf{X}(p_1, p_2, p_3, p_4)}}{\mathbf{X}(p_1, p_2, p_3, p_4)}.
\end{aligned}
$$

Now by assumption $\mathbf{X}(p_1, p_2, p_3, p_4) > 0$, so that $T_2 = \overline{T_1}$. Then, by Theorem 3.1, there exists an antiholomorphic isometry g sending p_1 to p_2, p_2 to p_1 and p_3 to p_4. But such an isometry must be an $\mathbb{R}$-reflection by Lemma 2.7, assuming that $p_1 \neq p_2$. If $p_1 = p_2$ but $p_3 \neq p_4$, the same argument applies, permuting the indices by the permutation $(13)(24)$. $\qquad \square$

It may seem that we only used the assumption that X was real in the above proof. Recall however that if X is negative, the four points are on the boundary of a complex line, and the two pairs $\{p_1, p_2\}$ and $\{p_3, p_4\}$ separate each other. In that case, as observed above, there cannot exist a real reflection ϕ such that $\phi(p_1) = p_2$ and $\phi(p_3) = p_4$.

4. Commutators, decomposable pairs, and traces

4.1. Main results. Recall that a pair of holomorphic isometries $(A, B) \in \mathrm{PU}(n, 1)^2$ is said to be $\mathbb{R}$-*decomposable* if there exist 3 $\mathbb{R}$-reflections σ_1, σ_2 and σ_3 such that $A = \sigma_1 \sigma_2$ and $B = \sigma_1 \sigma_3$. We are now ready to prove our main result.

Theorem 4.1. *Let $A, B \in \mathrm{PU}(2, 1)$ be two isometries not fixing a common point in $\overline{H^2_{\mathbb{C}}}$. Then the pair (A, B) is $\mathbb{R}$-decomposable if and only if the commutator $[A, B]$ has a fixed point in $\overline{H^2_{\mathbb{C}}}$ whose associated eigenvalue is real and positive.*

Note that the eigenvalues of elements of $\mathrm{PU}(2, 1)$ are not well-defined (up to change of lift in $\mathrm{U}(2, 1)$, or even $\mathrm{SU}(2, 1)$), but the eigenvalues of a commutator are well-defined (the commutator itself is independent of lifts). Using Goldman's classification of isometries by trace and Proposition 2.3, this criterion can be reduced to the following theorem.

Theorem 4.2. *Let $A, B \in \mathrm{PU}(2, 1)$ be two isometries not fixing a common point in $\overline{\mathrm{H}^2_{\mathbb{C}}}$. Then (A, B) is $\mathbb{R}$-decomposable if and only if*

- $[A, B]$ *is loxodromic and* $\mathrm{Tr}[A, B] > 3$,

or

- $[A, B]$ *is unipotent,*

or

- $[A, B]$ *is elliptic,* $\mathrm{Tr}[A, B] \in \mathbb{R}$ *and the eigenvalue 1 of* $[A, B]$ *is of negative type.*

The extra assumption in the elliptic case means that the eigenvalue 1 corresponds to the fixed point of $[A, B]$. The other eigenvalues of $[A, B]$ are then $e^{\pm i\theta}$ for some θ, by the assumption that $\mathrm{Tr}[A, B] \in \mathbb{R}$ and Proposition 2.3.

Proof of Theorem 4.1. Let p_1 be a fixed point of $[A, B]$ in $\overline{\mathrm{H}^2_{\mathbb{C}}}$, P_1 a lift of p_1 in $\mathbb{C}^{2,1}$ and λ_1 the associated eigenvalue, so that $[A, B]P_1 = \lambda_1 P_1$. Consider the cycle of four points defined as follows. Let $P_2 = B^{-1}P_1$, $P_3 = A^{-1}P_2$ and $P_4 = B(P_3)$. First assume for simplicity that these 4 points are all distinct. Then opposite sides of the quadrilateral $(P_1 P_2 P_3 P_4)$ are identified by A and B as on Figure 3.
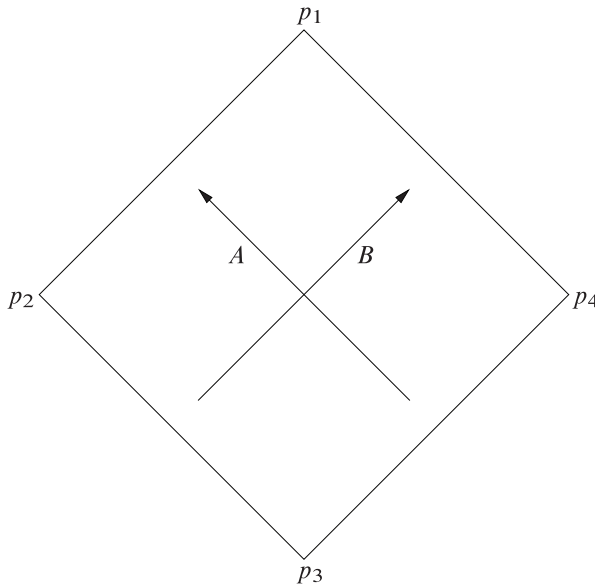

Figure 3. The 4-cycle associated to a fixed point of $[A, B]$.

Note that $AP_4 = [A, B]P_1 = \lambda_1 P_1$. Then

$$
\begin{aligned}
\mathbf{X}(p_2, p_4, p_1, p_3) &= \frac{\langle P_1, P_2 \rangle \langle P_3, P_4 \rangle}{\langle P_3, P_2 \rangle \langle P_1, P_4 \rangle} \\
&= \frac{\langle \lambda_1^{-1} AP_4, AP_3 \rangle \langle P_3, P_4 \rangle}{\langle P_3, P_2 \rangle \langle BP_2, BP_3 \rangle} \\
&= \lambda_1^{-1} \frac{|\langle P_3, P_4 \rangle|^2}{|\langle P_3, P_2 \rangle|^2}.
\end{aligned}
\tag{4.1}
$$

This proves the following:

Lemma 4.1. *The cross ratio* $\mathbf{X}(p_2, p_4, p_1, p_3)$ *is real and positive if and only if* $\lambda_1 \in \mathbb{R}^+$

For the next step we use Theorem 3.2 or 3.3, depending on whether p_1 (and hence all other p_i's) is on $\partial \mathrm{H}_{\mathbb{C}}^2$ or in $\mathrm{H}_{\mathbb{C}}^2$. In the latter case, by construction of the 4 points we have $d(p_1, p_2) = d(p_3, p_4)$ and $d(p_1, p_4) = d(p_2, p_3)$. Therefore Theorem 3.3 tells us:

Lemma 4.2. *There exists a real reflection* ϕ *such that* $\phi(p_1) = p_3$ *and* $\phi(p_2) = p_4$ *if and only if the cross-ratio* $\mathbf{X}(p_2, p_4, p_1, p_3)$ *is real and positive.*

The following lemma concludes the proof of Theorem 4.1.

Lemma 4.3. (A, B) *is* $\mathbb{R}$*-decomposable* $\iff$ *there exists a real reflection* ϕ *such that* $\phi \colon p_1 \leftrightarrow p_3$ *and* $p_2 \leftrightarrow p_4$.

Proof. Indeed, if $A = \sigma_2 \sigma_1$ and $B = \sigma_3 \sigma_1$ then $\sigma_1 = \phi$ satisfies $\phi \colon p_1 \leftrightarrow p_3$ and $p_2 \leftrightarrow p_4$. Conversely, if such a ϕ exists then by Lemma 2.7 above, $A \circ \phi$ and $B \circ \phi$ are real reflections. Indeed they are both antiholomorphic, and $A \circ \phi$ (resp. $B \circ \phi$) exchanges p_1 and p_2 (resp. p_1 and p_4). Therefore (A, B) is $\mathbb{R}$-decomposable.   △

Finally we examine the case where some of the 4 points $p_1, \ldots, p_4$ are equal. The cross-ratio appearing in Lemma 4.1 is well-defined as long as no three of the points are on $\partial \mathrm{H}_{\mathbb{C}}^2$ and equal, in which case all four of them would be equal, contradicting the assumption that A and B do not have a common fixed point. The proofs of Lemmas 4.2 and 4.3 carry through as long as $p_1 \neq p_3$ or $p_2 \neq p_4$.

Now if $p_1 = p_3 = p$ and $p_2 = p_4 = q$, then A and B both exchange p and q (if they are distinct). If p and q are in $\mathrm{H}_{\mathbb{C}}^2$ then A and B both fix the midpoint of the segment $[pq]$ which is again assumed not to be the case. If p and q are on $\partial \mathrm{H}_{\mathbb{C}}^2$ then A and B both have a fixed point on the geodesic line (pq) and act as a half-turn on the complex line spanned by p and q. In that case, on one hand by Proposition 2.5 (al–a3), any real plane containing (pq) decomposes A and B,

therefore (A, B) is R-decomposable. On the other hand, the commutator $[A, B]$ is loxodromic (because it acts by translation along the geodeesic (pq)) and has an eigenvalue equal to 1 (because A and B have a common eigenvector as they both preserve the complex line spanned by p, q), therefore $[A, B]$ has 3 real and positive eigenvalues. In particular, the conclusion of Theorem 4.1 holds in this case as well. $\qquad\square$

4.2. Groups fixing a point. When A and B have a common fixed point in $\overline{\mathrm{H}^2_{\mathbb{C}}}$ the results are the following:

Proposition 4.1. *If $A, B \in \mathrm{PU}(2, 1)$ have a common fixed point in $\mathrm{H}^2_{\mathbb{C}}$ then (A, B) is $\mathbb{R}$-decomposable.*

Proposition 4.1 is the first part of Theorem 2.1 of [FALPAU]; it essentially follows from the fact that, given two complex lines in $\mathbb{C}^2$, there exists a Lagrangian subspace intersecting each of them in a line (see Proposition 2.5 (e1-e3)).

Proposition 4.2. *Let $A, B \in \mathrm{PU}(2, 1)$ have a common fixed point on $\partial \mathrm{H}^2_{\mathbb{C}}$.*

(a) *If A or B is loxodromic then (A, B) is $\mathbb{R}$-decomposable if and only if A and B commute.*

(b) *If A and B are both non-loxodromic and one of them is not 3-step unipotent then (A, B) is $\mathbb{R}$-decomposable.*

(c) *If A and B are both 3-step unipotent then (A, B) is $\mathbb{R}$-decomposable if and only if A and B commute.*

Note that the 3 parts of Proposition 4.2 cover all cases where A and B have a common fixed point on $\partial \mathrm{H}^2_{\mathbb{C}}$, because screw-parabolic isometries have 2-step unipotent part (this follows from the fact that their elliptic and unipotent parts commute in the classification theorem of Chen-Greenberg, our Theorem 2.2(c)).

Proof of Proposition 4.2. (a) First note that if A and B are both loxodromic with a common fixed point, then by Proposition 2.5 (l): (A, B) is $\mathbb{R}$-decomposable if and only if A and B have the same fixed points, that is they commute.

- Assume that one of A, B is loxodromic but not the other, say A is loxodromic and B is parabolic or a complex reflection, and that A and B commute. In particular, A has distinct eigenvalues, therefore B must also be diagonalizable by the assumption that A and B commute. Moreover B fixes the fixed points of A, which means that the fixed line of B contains the axis of A. Therefore any real reflection which decomposes A also decomposes B (by Proposition 2.5 (e2) and (l)). In particular, (A, B) is $\mathbb{R}$-decomposable.

- Conversely, assume that A is loxodromic and B is not, and that (A, B) is $\mathbb{R}$-decomposable. By part (p) of Proposition 2.5, B cannot be parabolic. It is thus a complex reflection. Denote by C the complex line fixed by B, and by p_A, q_A the fixed points of A. Then by Proposition 2.5 (e2) and (l), there exists a real reflection σ with fixed $\mathbb{R}$-plane R such that σ exchanges p_A and q_A, and such that $R \cap C$ is a geodesic. Then σ preserves $\overline{C}$ which contains one of p_A and q_A, therefore $\overline{C}$ also contains the other, and A and B commute.

(b) Assume that A is not 3-step unipotent. Conjugating if necessary, we may assume that the common fixed point is q_∞. Denote by C the invariant complex line of A, and pick a point p in the vertical line ∂C. Any affine line through p contained in the contact plane at p is the boundary of a real plane R such that $R \cap C$ contains p and q_∞. This implies that for any such R, $R \cap C$ is the geodesic connecting p to q_∞, and therefore R decomposes A by part (p) of Proposition 2.5.

- If B preserves a complex line, that is B is a complex reflection or screw parabolic, then its invariant complex line C' is such that $\partial C'$ is a vertical complex line. If $C = C'$ then the result is clear. If not, one of the infinite $\mathbb{R}$-circles through p intersects $\partial C'$ and therefore the corresponding real reflection decomposes B.

- If B is 3-step unipotent then one of the infinite $\mathbb{R}$-circles through p intersects the invariant fan of B orthogonally, therefore the corresponding real reflection decomposes B.

(c) If both A and B are 3-step unipotent then in view of Proposition 2.5 and Lemma 2.8, the pair (A, B) is $\mathbb{R}$-decomposable if and only if there exists a real plane R such that the infinite $\mathbb{R}$-circle ∂R is orthogonal to both invariant fans of A and B. This is equivalent to saying the these two fans are parallel, which by Lemma 2.3 means that A and B commute. $\qquad\square$

4.3. Maximal representations: $\mathbb{C}$-Fuchsian punctured torus groups.

The case where the cross-ratio $\mathbf{X}(p_2, p_4, p_1, p_3)$ is real and negative corresponds to a rigidity phenomenon, giving the following result which holds in all dimensions.

Proposition 4.3. *If $[A, B]$ has a fixed point in $\overline{H^n_\mathbb{C}}$ whose associated eigenvalue is real and negative, then the group $\langle A, B \rangle$ stabilizes a complex line L in $H^n_\mathbb{C}$. Moreover in that case the corresponding fixed point of $[A, B]$ is on the boundary $\partial H^n_\mathbb{C}$, so that $\langle A, B \rangle$ is a $\mathbb{C}$-Fuchsian punctured torus group.*

Proof. Indeed, with the notation of the proof of Theorem 4.1, we have the following variation of lemma 4.1:

$$\mathbf{X}(p_2, p_4, p_1, p_3) \in \mathbb{R}^- \iff \lambda_1 \in \mathbb{R}^-.$$

Now by assumption $\lambda_1 < 0$, so that by Proposition 3.1 p_1, p_2, p_3, p_4 are on the boundary of a common complex line L and the pairs (p_2, p_4) and (p_1, p_3) separate each other. But A sends (p_3, p_4) to (p_2, p_1) and B sends (p_2, p_3) to (p_1, p_4), so A and B both stabilize L. It is then a simple exercise in the hyperbolic plane to check that this combinatorics implies that A and B must both be loxodromic, with axes meeting inside the ideal quadrilateral $Q = (p_1, p_2, p_3, p_4)$. Therefore, Q is disjoint from all its images by elements of the group generated by A and B. This proves that $\langle A, B \rangle$ is discrete. Note that in general this quadrilateral is not a fundamental domain for $\langle A, B \rangle$, as its images only tessellate the complex line L if $[A, B]$ is parabolic. See Figure 4. $\square$
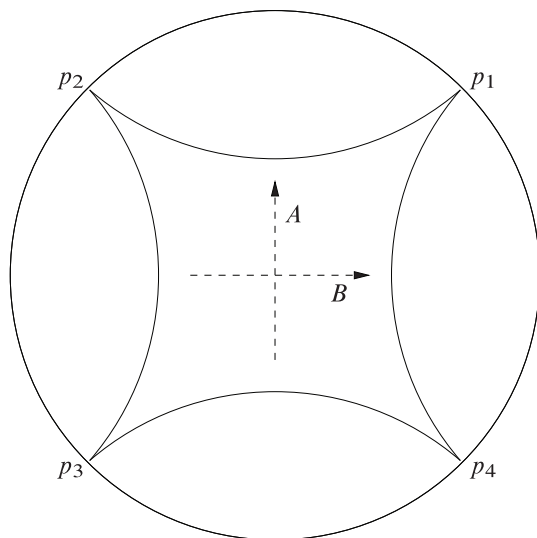

Figure 4. A fundamental domain for the discrete punctured torus group in the case where $\lambda_1 < 0$.

One can also interpret Proposition 4.3 in terms of the Toledo invariant of the corresponding type-preserving representation of the fundamental group of the once-punctured torus. Given a representation ρ of the fundamental group of a surface Σ into $\mathrm{PU}(n, 1)$, the *Toledo invariant* $\tau(\rho)$ is defined as the integral over Σ of the pullback of the Kähler 2-form on $\mathrm{H}^n_{\mathbb{C}}$ by an equivariant map $f \colon \widetilde{\Sigma} \to \mathrm{H}^n_{\mathbb{C}}$:

$$\tau(\rho) = \int_{\Sigma} f^{\star}\omega. \tag{4.2}$$

In the case where Σ is non-compact, one should be careful that this integral is well-defined. This is guaranteed for instance by the condition that the map f has finite energy (see [KM]). The existence of a such a finite-energy equivariant map is guaranteed by the assumption that all peripheral loops on Σ are mapped to parabolics by the representation ([KM]). This assumption being made, it is possible to use an ideal triangulation of $\Sigma = \bigsqcup_i \Delta_i$ to compute the Toledo invariant, as in [GP]. The result is that the Toledo invariant can be written as

$$\tau(\rho) = \sum_i \int_{f(\Delta_i)} \omega, \qquad (4.3)$$

where each integral is computed over any 2-simplex with boundary $f(\partial\Delta_i)$. But the integral of ω over any 2-simplex with boundary $\partial\Delta$ is equal to twice the Cartan invariant of Δ (this is Theorem 7.1.ll of [G]). As a consequence, we see that for a type-preserving representation of the fundamental group of the once-punctured torus, the Toledo invariant is given by

$$\tau(\rho) = 2\left(\mathbb{A}(p_1, p_2, p_3) + \mathbb{A}(p_1, p_3, p_4)\right), \qquad (4.4)$$

with $p_1, \ldots, p_4$ defined as previously. On the other hand, taking arguments in equation (3.2) gives

$$\arg(\mathrm{X}(p_2, p_4, p_1, p_3)) = \mathbb{A}(p_2, p_4, p_1) - \mathbb{A}(p_2, p_1, p_3)$$
$$= \mathbb{A}(p_1, p_2, p_4) + \mathbb{A}(p_1, p_2, p_3).$$

Therefore, if the cross-ratio $\mathrm{X}(p_2, p_4, p_1, p_3)$ is negative, then $\mathbb{A}(p_1, p_2, p_4)$ and $\mathbb{A}(p_1, p_2, p_3)$ must both be equal to either $\pi/2$ or $-\pi/2$. In view of (4.4), this means that $|\tau(\rho)| = 2\pi$. But the Toledo invariant satisfies the Milnor–Wood inequality:

$$|\tau(\rho)| \leq 2\pi(2g - 2 + p), \qquad (4.5)$$

where equality holds if and only if the representation is discrete and preserves a complex line. In that case, the representation is called *maximal* (see [T, BIW]). In the case of the once punctured torus $g = p = 1$, and therefore if $|\tau(\rho)| = 2\pi$ the representation is maximal.

5. Groups generated by real reflections

We now use the criterion from Theorem 4.2 to show that various subgroups of PU(2, 1) are generated by real reflections. More accurately this means that they are the index 2 holomorphic subgroup of a group of isometries generated by real reflections.

5.1. Mostow's lattices and other non-arithmetic lattices in SU(2, 1). Mostow's lattices from [M] (revisited in [DFP]) as well as the new non-arithmetic lattices in SU(2, 1) studied by Deraux, Parker and the first author (see [ParPau], [Pau], [DPP1] and [DPP2]) are all symmetric complex reflection triangle groups. This means that they are generated by 3 complex reflections R_1, R_2 and R_3 which are in a symmetric configuration in the sense that there exists an isometry J of order 3 such that $JR_iJ^{-1} = R_{i+1}$ (with i mod 3). These groups are in fact contained (with index 1 or 3, depending on the parameters) in the group Γ generated by R_1 and J.

It was shown in [DFP] that Mostow's lattices are generated by real reflections, and in [DPP2] this is extended to all symmetric complex reflection triangle groups. In both cases though, one finds an explicit real reflection which decomposes both holomorphic generators R_1 and J, which requires knowing explicit geometric properties of the group. Now the existence of such a real reflection follows immediately from the following consequence of Theorem 4.1.

Proposition 5.1. *If $R \in \mathrm{PU}(2, 1)$ is a complex reflection or a complex reflection in a point, then for any $A \in \mathrm{PU}(2, 1)$, the pair (R, A) is $\mathbb{R}$-decomposable.*

Proof. First assume that R is a complex reflection about a complex line L. Then $[R, A] = RR'$, where $R' = AR^{-1}A^{-1}$ is a complex reflection conjugate to R^{-1}. We denote by L' its fixed complex line. The extensions of L and L' to $\mathbb{C}P^2$ intersect at a unique point $p \in \mathbb{C}P^2$, unless $L = L'$, in which case $[R, A] = \mathrm{Id}$ and the result follows from Theorem 4.2. In general, p is fixed by both R and R', thus by $[R, A]$; we distinguish 2 cases, depending on whether p is in $\overline{\mathrm{H}^2_{\mathbb{C}}}$ or outside of $\overline{\mathrm{H}^2_{\mathbb{C}}}$.

(1) First assume that $p \in \overline{\mathrm{H}^2_{\mathbb{C}}}$. Consider lifts of R and R' to SU(2,1) such that the lift of R has eigenvalues $e^{2i\phi}, e^{-i\phi}, e^{-i\phi}$ (where the rotation angle of R is 3ϕ), with p corresponding to a $e^{-i\phi}$-eigenvector. Likewise, the lift of R' has eigenvalues $e^{-2i\phi}, e^{i\phi}, e^{i\phi}$, with p corresponding to a $e^{i\phi}$-eigenvector of R'. Then p is a fixed point of $[R, A] = RR'$ in $\overline{\mathrm{H}^2_{\mathbb{C}}}$ with corresponding eigenvalue 1. The result follows from Theorem 4.1.

(2) If p is outside of $\overline{\mathrm{H}^2_{\mathbb{C}}}$, then L and L' are ultraparallel and p is polar to their common perpendicular line, which we denote by $\widetilde{L}$. The isometries R and R' act on $\widetilde{L}$ by rotation through angles 3ϕ and -3ϕ respectively. It is an elementary fact from plane hyperbolic geometry that the product of two elliptic elements with opposite rotation angles and distinct fixed points must be hyperbolic (it follows for instance from Lemma 7.38.2 of [Be]). Therefore RR' is loxodromic; moreover its eigenvalue of positive type is 1 (it is the product of those of R and R' which are inverse of one another). Therefore its eigenvalue spectrum must be either $\{r, 1, 1/r\}$ or $\{-r, 1, -1/r\}$ for some

$r > 0$. Geometrically, this means that the rotation angle of the loxodromic isometry $[R, A]$ is 0 or π. By continuity, the rotation angle of $[R, A]$ must be the same for any value ϕ and any relative position of L and L' (with the restriction that they are ultraparallel). It is 0 when $L = L'$ (as R and A are inverse of one another in this case). This means that $[R, A]$ has only positive eigenvalues, and the result follows from Theorem 4.1.

Now assume that R is a complex reflection about a point. As above, we write $[R, A] = RR'$ with $R' = AR^{-1}A^{-1}$. The eigenvalues of R (respectively R') are $\{e^{i\phi}, e^{i\phi}, e^{-2i\phi}\}$ (resp. $\{e^{-i\phi}, e^{-i\phi}, e^{2i\phi}\}$), with $e^{2i\phi}$ (resp. $e^{-2i\phi}$) of negative type. This means that R (resp. R') acts on any complex line through its fixed point as a rotation of angle 3ϕ (resp. -3ϕ). Consider the complex line $\widetilde{L}$ spanned by the fixed points of R and R'. The action of R and R' on $\widetilde{L}$ is the same as in the second item above, and this leads to the same conclusion. $\square$

5.2. Groups with more than two generators. Applying the criterion from Theorem 4.1 to a 2-generator subgroup of $\mathrm{PU}(2, 1)$ is completely straightforward. For subgroups generated by more elements (e.g. Picard modular groups) one needs to be a bit more careful, as being generated by real reflections is stronger than having generators which are pairwise $\mathbb{R}$-decomposable. The following observation gives a way to bridge this gap, however its hypotheses are in general too restrictive and in practice we will need some more work to show that a given group is generated by real reflections.

Lemma 5.1. *Let* Γ *be a subgroup of* $\mathrm{PU}(2, 1)$ *generated by* $A_1, \dots, A_k$. *If there exist real reflections* $\sigma_1, \dots, \sigma_{k+1}$ *such that*

(a) $A_i = \sigma_i \sigma_{i+1}$ *for* $1 \leqslant i \leqslant k$, *or*

(b) $A_i = \sigma_1 \sigma_{i+1}$ *for* $1 \leqslant i \leqslant k$

then Γ *has index 2 in* $\widehat{\Gamma} = \langle \sigma_1, \dots, \sigma_{k+1} \rangle$. *In particular such a* Γ *is generated by real reflections.*

Proof. In each case, pairwise products of the σ_i are in Γ, therefore Γ is the index 2 holomorphic subgroup of $\widehat{\Gamma}$. $\square$

5.3. Picard modular groups. We denote $\Gamma_d = \mathrm{SU}(2, 1, \mathcal{O}_d)$ the subgroup of $\mathrm{SU}(2, 1)$ consisting of matrices with entries in $\mathcal{O}_d$, where d is a positive squarefree integer and $\mathcal{O}_d$ denotes the ring of integers of $\mathbb{Q}[\sqrt{-d}]$. Recall that $\mathcal{O}_d = \mathbb{Z}[i\sqrt{d}]$ when $d \equiv 1, 2 \pmod 4$ and $\mathcal{O}_d = \mathbb{Z}[\frac{1+i\sqrt{d}}{2}]$ when $d \equiv 3 \pmod 4$. The groups Γ_d are usually called *Picard modular groups*. Denote by $\Gamma_\infty(d)$ the stabilizer of q_∞ in Γ_d and $I_0 \in \Gamma_d$ given by

$$I_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{5.1}$$

The isometry I_0 is a complex reflection of order 2 about the complex line polar to the vector $v_0 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$.

Elements of $\Gamma_\infty(d)$ are upper triangular matrices with units of $\mathcal{O}_d$ on the diagonal. In particular, $\Gamma_\infty(d)$ contains no Heisenberg dilations (see (2.5)); in other words $\Gamma_\infty(d) < \mathrm{Isom}(\mathfrak{N})$. Restricting the exact sequence (2.8)

$$1 \longrightarrow \mathrm{Isom}^+(\mathbb{R}) \longrightarrow \mathrm{Isom}(\mathfrak{N}) \xrightarrow{\Pi_*} \mathrm{Isom}^+(\mathbb{C}) \longrightarrow 1$$

to the subgroup $\Gamma_\infty(d) < \mathrm{Isom}(\mathfrak{N})$ gives

$$1 \longrightarrow \Gamma_\infty^v(d) \longrightarrow \Gamma_\infty(d) \xrightarrow{\Pi_*} \Gamma_\infty^h(d) \longrightarrow 1, \qquad (5.2)$$

with $\Gamma_\infty^v(d) < \mathrm{Isom}^+(\mathbb{R})$ and $\Gamma_\infty^h(d) < \mathrm{Isom}^+(\mathbb{C})$ (denoting as usual $\mathrm{Isom}^+(X)$ the group of orientation-preserving isometries of a space X). The following complete description of the subgroups $\Gamma_\infty^h(d)$ and $\Gamma_\infty^v(d)$ was given in [FFPN] for $d \neq 1, 3$ (see [FFP] for $d = 1$ and [FALPAR] for $d = 3$).

Lemma 5.2 ([FFPN]). *For $d \geqslant 1$, denote as above $\Gamma_\infty^v(d)$, resp. $\Gamma_\infty^h(d)$, the vertical, resp. horizontal, part of $\Gamma_\infty(d)$ appearing in the exact sequence (5.2). Then*

(a) $\Gamma_\infty^v(d) = \langle T_0 \rangle$, *with $T_0 = T_{[0,\sqrt{d}]}$ as in (2.5);*

(b) $\Gamma_\infty^h(d) = \mathrm{Isom}^+(\mathcal{O}_d)$ *if $d \equiv 3 \ (mod\ 4)$ $(d \neq 3)$; $\Gamma_\infty^h(d)$ has index 2 in $\mathrm{Isom}^+(\mathcal{O}_d)$ if $d \equiv 1, 2 \ (mod\ 4)$ $(d \neq 1)$.*

Proof. We sketch the proof given in [FFPN]. First, note that an element $P_{(z,t,\theta)}$ of $\mathrm{Isom}(\mathfrak{N})$ is in the kernel of Π_* if and only if $z = \theta = 0$, see (2.8). This gives part (a). Next, when $d \neq 1, 3$, the only units of $\mathcal{O}_d$ are ± 1 and thus the rotational part of an element $P_{(z,t,\theta)} \in \Gamma_\infty(d)$ must satisfy $\theta = 0$ or $\theta = \pi$. In particular the subgroup of $\Gamma_\infty(d)$ formed by Heisenberg translations has index two. The same observation holds for the translation subgroup $\mathrm{Isom}^+(\mathcal{O}_d)$, which is an extension of its translation subgroup Δ_d^t by the central involution $w \mapsto -w$. The strategy used in [FFPN] is to try to lift the generators of Δ_d^t to $\Gamma_\infty(d)$. The existence of such a lift can be verified using the explicit form of the map Π_* given (2.8).

(I) Assume that $d \equiv 3 \pmod 4$ (and $d \neq 3$) . Then $\Delta^t = \langle \widehat{T}_1, \widehat{T}_2 \rangle$ with

$$\widehat{T}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \widehat{T}_2 = \begin{pmatrix} 1 & (1 + i\sqrt{d})/2 \\ 0 & 1 \end{pmatrix}.$$

Both $\widehat{T}_1$ and $\widehat{T}_2$ lift to $\Gamma_\infty(d)$, and their lifts T_1 and T_2 satisfy $[T_2, T_1] = T_0$ which generates $\Gamma_\infty^v(d)$. Therefore, $\langle T_1, T_2 \rangle$ has index 2 in $\Gamma_\infty(d)$, and $\Gamma_\infty^h(d) = \langle \widehat{T}_1, \widehat{T}_2 \rangle = \mathrm{Isom}^+(\mathcal{O}_d)$.

(II) Assume that $d \equiv 2 \pmod 4$. Then $\Delta^t = \langle \widehat{T}_1, \widehat{T}_2 \rangle$ with

$$\widehat{T}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \widehat{T}_2 = \begin{pmatrix} 1 & i\sqrt{d} \\ 0 & 1 \end{pmatrix}.$$

Now $\widehat{T}_2$ lifts to $\Gamma_\infty(d)$, $\widehat{T}_1$ does not but $\widehat{T}_1^2$ does. This implies that $\Gamma_\infty^h(d) = \langle \widehat{T}_1^2, \widehat{T}_2 \rangle$. The lifts T_2 and T_1' satisfy $[T_2, T_1'] = T_0^4$ which generates an index 4 subgroup of $\Gamma_\infty^v(d)$. Therefore, $\langle T_1', T_2 \rangle$ has index 8 in $\Gamma_\infty(d)$.

(III) Assume that $d \equiv 1 \pmod 4$ (and $d \neq 1$). Then $\Delta^t = \langle \widehat{T}_1, \widehat{T}_2 \rangle$ with

$$\widehat{T}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \widehat{T}_2 = \begin{pmatrix} 1 & i\sqrt{d} \\ 0 & 1 \end{pmatrix}.$$

Now $\widehat{T}_1$ and $\widehat{T}_2$ do not lift to $\Gamma_\infty(d)$, but $\widehat{T}_1^2$ and $\widehat{T}_2^2$ do. This implies that $\Gamma_\infty^h(d) = \langle \widehat{T}_1^2, \widehat{T}_2^2 \rangle$, which has index 4 in $\langle \widehat{T}_1, \widehat{T}_2 \rangle$. The lifts T_1' and T_2' satisfy $[T_2', T_1'] = T_0^4$ which generates an index 4 subgroup of $\Gamma_\infty^v(d)$. Therefore, $\langle T_1', T_2' \rangle$ has index 16 in $\Gamma_\infty(d)$. $\qquad \square$

Remark 3. In cases (II) and (III) above, the translation directions of T_1 and T_2 are orthogonal, and so are the translation directions of the generating pairs of $\Gamma_\infty^h(d)$. In case (I) this is not true. However, the two elements $\widehat{T}_1$ and $\widehat{T}_2^2 \widehat{T}_1^{-1}$ have this property and generate an index two subgroup of $\Gamma_\infty^h(d)$. Therefore in view of the previous discussion if we make the following changes:

- in case (I), $(\widehat{T}_1, \widehat{T}_2) \mapsto (\widehat{T}_1, \widehat{T}_2^2 \widehat{T}_1^{-1})$,

- in case (II), $(\widehat{T}_1, \widehat{T}_2) \mapsto (\widehat{T}_1^2, \widehat{T}_2)$,

- in case (III), $(\widehat{T}_1, \widehat{T}_2) \mapsto (\widehat{T}_1^2, \widehat{T}_2^2)$,

we obtain finite index subgroups of $\Gamma_\infty^h(d)$ generated by pairs of translations that lift to $\Gamma_\infty(d)$ and have orthogonal directions. We will denote the lifts of the new $\widehat{T}_1$ and $\widehat{T}_2$ by T_1 and T_2, and from now on we will only consider the group $\langle T_1, T_2 \rangle$.

The central involution $w \mapsto -w$ lifts to $\Gamma_\infty(d)$ as the transformation given by

$$R_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{5.3}$$

In particular, $\Gamma_\infty(d)$ is generated by R_1 and its Heisenberg translation subgroup. Note that R_1 is a complex reflection of order 2 about the complex line polar to the vector $v_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$.

Lemma 5.3. (1) *The real reflection σ_0 about the standard real plane $\mathbb{R}^2 \cap \mathrm{H}^2_\mathbb{C}$ decomposes R_1, I_0 and T_2.*

(2) *The real reflection $R_1\sigma_0$ decomposes T_1.*

Proof. (1) Clearly, σ_0 fixes the polar vector to the mirrors of R_1 and I_0. Therefore it decomposes these two transformations. Indeed the two transformations $I_0\sigma_0$ and $R_1\sigma_0$ fix the vectors v_0 and v_1 respectively and thus preserve the corresponding complex lines. They are therefore real reflections, by Lemmas 2.6 and 2.7. On the other hand, T_2 is a lift to $\Gamma_\infty(d)$ of a translation by a multiple of $i\sqrt{d}$. This implies that T_2 is a Heisenberg translation by $[ik\sqrt{d}, t_2]$, where k is an integer and t_2 is chosen so that $T_2 \in \Gamma_\infty(d)$. In particular, this implies that the invariant fan of T_2 intersects the boundary of $\mathrm{H}^2_\mathbb{C}$ along a vertical plane of the type $\{x = x_2\}$, for some $x_2 \in \mathbb{R}$. The fixed $\mathbb{R}$-circle of σ_0 is the x-axis of the Heisenberg group, and is therefore orthogonal to this vertical plane. This implies that σ_0 decomposes T_2 by Proposition 2.5.

(2) A direct computation shows that in horospherical coordinates $R_1\sigma_0$ acts as $(z, t, u) \mapsto (-\bar{z}, -t, u)$. Its restriction to the boundary is thus the reflection about the infinite $\mathbb{R}$-circle $\{(iy, 0), y \in \mathbb{R}\}$. As T_1 is the Heisenberg translation by some $[k, it_1]$, where $k \in \mathbb{Z}$, its invariant fan intersects the boundary along a vertical plane $\{y = y_1\}$ for some $y_1 \in \mathbb{R}$, and is therefore orthogonal to the fixed $\mathbb{R}$-circle of $R_1\sigma_0$. Again, by Proposition 2.5, this implies that $R_1\sigma_0$ decomposes T_1. $\square$

We will denote the corresponding real reflections as follows:

$$\sigma_1 = R_1\sigma_0, \quad \sigma_2 = T_2\sigma_0, \quad \sigma_4 = I_0\sigma_0, \quad \sigma_3 = T_1\sigma_1 = T_1R_1\sigma_0.$$

Proposition 5.2. *For all $d \geqslant 1$, the subgroup $\langle \Gamma_\infty(d), I_0 \rangle$ of Γ_d is generated by real reflections up to finite index.*

Proof. Consider the group $\Gamma_\sigma = \langle \sigma_j, j = 0 \ldots 4 \rangle$. By construction Γ_σ contains the group $\langle T_1, T_2, R_1, I_0 \rangle$ which has finite index in $\langle \Gamma_\infty(d), I_0 \rangle$. On the other hand, we claim that all pairwise products of the σ_i are in $\langle \Gamma_\infty(d), I_0 \rangle$. Indeed, each of the σ_i is of the form $A_i\sigma_0$ with $A_i \in \langle \Gamma_\infty(d), I_0 \rangle$, therefore $\sigma_i\sigma_j = A_i\sigma_0A_j\sigma_0 = A_i\overline{A_j}$. The result follows by noting that $\Gamma_\infty(d)$ and I_0 are preserved under complex conjugation of entries. $\square$

Note that from the previous discussion, denoting by T_1, T_2 the lifts to $\Gamma_\infty(d)$ of generators of the orthogonal reflection subgroup of $\mathrm{Isom}^+(\mathcal{O}_d)$ (or their square when applicable), the index of $\langle T_1, T_2, R_1, I_0 \rangle$ in $\langle \Gamma_\infty(d), I_0 \rangle$ is 2 when $d \equiv 3 \pmod 4$ (and $d \neq 3$), 4 when $d \equiv 2 \pmod 4$, and 8 when $d \equiv 1 \pmod 4$ (and $d \neq 1$). For $d = 3$ the index is 6 and for $d = 1$ it is 16 (as there is an additional rotational part of order 3, 2 respectively). However in the latter cases we can recover the rotational part in the real reflection group by replacing R_1 in the definition of σ_1 by the appropriate complex reflection of order 4 or 6.

For $d = 1, 2, 3, 7, 11$ it is known from [FALPAR], [FFP], and [Z] that

$$\langle \Gamma_\infty(d), I_0 \rangle = \Gamma_d,$$

giving the following corollary.

Corollary 5.1. *For $d = 3, 7, 11$ (respectively 2 and 1), $\Gamma_d = \mathrm{PU}(2, 1, \mathcal{O}_d)$ has a subgroup of index 2 (respectively 4 and 8) which is generated by real reflections.*

Note that for $d = 3$, the full group Γ_3 is in fact generated by real reflections by Proposition 5.1, as it was shown in [FALPAR] (Proposition 5.11) that $\Gamma_3 = \langle J, R_1 \rangle$ with J regular elliptic of order 3 and R_1 a complex reflection of order 6.

Question. *Are all Picard modular groups $\mathrm{PU}(2, 1, \mathcal{O}_d)$ generated by real reflections up to finite index*?

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