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Cubulation of Gromov–Thurston manifolds

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Abstract. In this article we prove that the fundamental group of certain manifolds, introduced by Gromov and Thurston [GT87] and obtained by branched cyclic covering over arithmetic manifolds, acts geometrically on a CAT(0) cube complex. We show in particular that these groups are linear over $\mathbb{Z}$.

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1. Introduction

The main result of this paper concerns the fundamental group of some negatively curved manifolds, introduced by Gromov and Thurston in [GT87]. Gromov and Thurston constructed infinitely many manifolds which can be equipped with a Riemannian metric with negative sectional curvature less or equal to -1, arbitrarily close to -1, but do not admit any Riemannian metric of constant curvature. For the construction, they consider cyclic ramified coverings over a certain hyperbolic manifold V; an arithmetic hyperbolic manifold of "simple type." The branch locus is a totally geodesic submanifold $K \subset V$ of codimension 2, obtained as the transverse intersection of two codimension-1 totally geodesic submanifolds. It is important to consider ramified covering of arbitrary large degrees to make sure that, among those manifolds, infinitely many do not have a constant curvature metric. On the other hand, the more the normal injectivity radius of K in V is large, the more the sectional curvature of the metric will be close to -1. A manifold $\widehat{V}$ obtained by this construction will be called here a *Gromov–Thurston manifold.*

Our first result is the following theorem.

Theorem 1. *Let $\widehat{V}$ be a Gromov–Thurston manifold. Then $\pi_1(\widehat{V})$ is cubical.*

A group is *cubical* if it acts geometrically on a CAT(0) cube complex (see Definition 7 below). Now a cubical group is said to be *special* – following Haglund and Wise in [HW08] – if the quotient of the CAT(0) cube complex by the group avoids some global hyperplane pathologies (most importantly hyperplanes have to embed in the quotient), see Section 4 for a precise definition. Similarly a cube complex that avoids these pathologies will be said to be *special*. Finally a group is said to be *virtually special* if it contains a finite index special subgroup. Being virtually special has many consequences: virtually special groups inject in $\mathrm{GL}(n, \mathbb{Z})$ for a certain $n \in \mathbb{Z}$, have separable quasi-convex subgroups (see [HW08]), are virtually large (see [HW08]), are virtually bi-ordonnable (see [HW08] and [DT92]), etc. Recently Agol [AGM12] proved that every hyperbolic cubical group is virtually special. Since the fundamental group of a Gromov–Thurston manifold is hyperbolic, using Agol's Theorem [AGM12], Theorem 3 below follows from Theorem 1. Nevertheless, we will prove virtual specialness without using Agol's Theorem.

To prove Theorem 1, we will introduce a notion of ramified coverings of cube complexes (see Definition 14). We shall then prove the following theorem.

Theorem 2. *Cyclic ramified coverings of special cube complexes are special cube complexes.*

We shall then deduce our main result from Theorems 1 and 2.

Theorem 3. *Let $\hat{V}$ be a Gromov–Thurston manifold. Then $\pi_1(\hat{V})$ is virtually special.*

Many examples of compact hyperbolic manifolds with constant curvature equal to -1 have virtually special fundamental groups. For example, hyperbolic surfaces are virtually special. In dimension 3, works of Kahn–Markovic, Wise and Agol imply that fundamental groups of compact hyperbolic 3-manifolds are virtually special [AGM12]. Moreover, in every dimension $n \geq 4$, Bergeron, Haglund and Wise [BHW11] have shown that "simple type" arithmetic hyperbolic manifolds also have a cubical virtually special fundamental group. We will use this to prove Theorem 1. As a consequence of Theorem 3 we can deduce the following corollary.

Corollary 4. Fundamental groups of Gromov–Thurston manifolds are linear (over $\mathbb{Z}$), contain a finite index bi-ordonnable subgroup, have separable quasi-convex subgroups, are virtually large, etc...

We will in fact prove a more general version of Theorem 1 (Theorem 5 below). Let $V = \Gamma \backslash \mathbb{H}^n$ be a hyperbolic manifold, with $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$. Denote by $p\colon \mathbb{H}^n \to V$ the covering map. Suppose that there exist two transverse totally geodesic submanifolds V_1 and V_2 of V, and suppose that both of them separate V. We can construct a cyclic ramified covering in the following way. Each submanifold V_i, $i = 1, 2$, splits V into two disjoint submanifolds V_i^+ and V_i^- of V, with boundary V_i with two different orientations. The intersection $W = V_1^+ \cap V_2$ is a hypersurface of V with boundary. Let $\overline{V}$ be the manifold obtained by cutting V along W. The boundary of $\overline{V}$ is a disjoint union of two copies of W with opposite orientations. For $k \in \mathbb{N}^*$, consider the manifold $\widehat{V}$ obtained by cyclically gluing k copies of $\overline{V}$ along the copies of W according to their orientations. Then the natural projection $\widehat{V} \to V$ is a *cyclic ramified covering* of degree k of V above $\partial W = V_1 \cap V_2$.

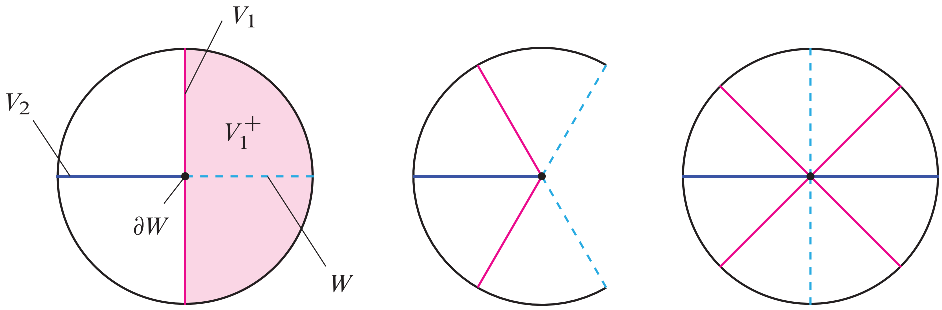


Figure 1. A cyclic ramified covering of degree 2 above ∂W.

In Gromov–Thurston's construction, V is an arithmetic manifold of "simple type," as well as V_1 and V_2. By passing to a finite cover, these two submanifolds satisfy the properties described above. Then Gromov and Thurston construct $\widehat{V}$, a cyclic ramified covering of degree k above the intersection of V_1 and V_2.

An arithmetic manifold of "simple type" contains many immersed compact hypersurfaces. Choose a finite number of them and denote by $\mathcal{H}$ the collection of their preimages in $\mathbb{H}^n$. A general construction of Sageev associates a CAT(0) cube complex that is "dual" to the Γ-invariant collection of hyperplanes $\mathcal{H}$ in $\mathbb{H}^n$. The fundamental group $\Gamma = \pi_1(V)$ acts cocompactly on the resulting cube complex. By choosing enough immersed compact hypersurfaces to start with, one can moreover ensure that the action of Γ on the "dual" cube complex is proper. We then say that V is π_1*-cubulated* by $\mathcal{H}$. We provide details for this construction in Section 2.

Finally we say that a collection of hyperplanes is *generic* if every pair of intersecting hyperplanes in the collection have transverse intersection and if the intersection between three different hyperplanes is either empty or of codimension 3.

Theorem 5. *Let V be an oriented hyperbolic compact manifold, and let V_1 and V_2 be two totally geodesic separating submanifolds of V with transverse non-empty intersection. Let $k \geq 1$ be an integer and let $\widehat{V}$ be the cyclic ramified covering of V of degree k above $V_1 \cap V_2$. Assume furthermore that V is π_1-cubulated by a collection $\mathcal{H}$ of hyperplanes in $\mathbb{H}^n$, such that the reunion of $\mathcal{H}$ with the set of preimages of V_1 and V_2 under p forms a generic collection of hyperplanes of $\mathbb{H}^n$. Then the fundamental group $\pi_1(\widehat{V})$ is cubical.*

We recall Sageev's construction – specialized to our context – in the next section. In Section 3 we prove Theorem 5, in Section 4 we prove Theorem 2. Finally, in Section 5 we prove Theorems 1 and 3 by using Theorems 5 and 2.

2. Cubulation

2.1. Some definitions

Definition 6. A *cube complex* is a CW-complex, such that each cell is a metric Euclidean cube $[0,1]^n$, and gluing maps are isometries between subcubes, i.e. cubes obtained by restricting certain coordinates to 0 or 1.

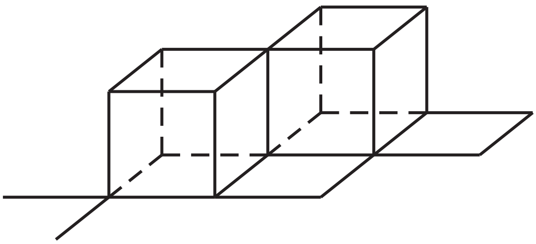


Figure 2. A cube complex.

We shall always equip a cube complex with the metric induced by the Euclidean metric on the cubes. As such it makes sense to speak of a CAT(0) cube complex; we refer to [BH99] for a general study of CAT(0) spaces.

Definition 7. A group is said to be *cubical* if it acts *geometrically*, *i.e. properly* and *cocompactly*, on a CAT(0) cube complex.

2.2. Sageev's construction. In [HP98] Haglund and Paulin introduce the notion of a wallspace. Generalizing a construction of Sageev [Sag95], Nica [Nic04], Chatterji and Niblo [CN05] have then shown how to associate to any wallspace a "dual" CAT(0) cube complex in such a way that a group acting on the wallspace (as defined in [HP98]) also acts on the dual CAT(0) cube complex. We now review this construction in the particular case at hand.

Let V be a closed hyperbolic manifold and let $W_1, W_2, \dots, W_\ell$ be immersed closed codimension-1 totally geodesic submanifolds in V. Assume that these manifolds intersect each other in a generic way. Any lift of W_i to $\mathbb{H}^n$ is a hyperplane of $\mathbb{H}^n$, splitting $\mathbb{H}^n$ into two connected components. Let $\mathcal{H}$ be the collection of all these hyperplanes and let $\mathcal{S}$ be the set of connected components of the space $\mathbb{H}^n \setminus \bigcup_{H\in\mathcal{H}} H$. Each hyperplane H of $\mathcal{H}$ provides a natural bipartition (a so-called *wall*) of $\mathcal{S}$. We call each element of this partition a *halfspace* of H.

We now construct a cube complex Y as follows. We first describe its 1-skeleton $Y^{(1)}$.

A vertex σ of Y is a collection of halfspaces of $\mathcal{S}$ such that

- for all $H \in \mathcal{H}$, exactly one of the two halfspaces of H belongs to σ;
- for all halfspaces A and B, $(A \subset B$ and $A \in \sigma) \implies B \in \sigma$.

Put an edge between two vertices σ and τ if and only if $|\sigma\Delta\tau| = 2$, i.e. if and only if σ and τ share exactly the same halfspace for every hyperplane of $\mathcal{H}$ except for one.

We finally construct Y from $Y^{(1)}$ by adding a n-cube each time one sees the 1-skeleton of a n-cube in $Y^{(1)}$.

The cube complex Y can be complicated. However to each connected component x of $\mathcal{S}$ it corresponds a vertex σ of Y:

$$\sigma = \{A \text{ halfspace: } x \in A\}.$$

It is a simple exercise – that we leave to the reader – to check that σ satisfies the two conditions above. Now two such vertices are adjacent if and only if the associated connected components are separated by a unique hyperplane of $\mathcal{H}$. This defines a (connected) subgraph $\mathcal{G}$ of $Y^{(1)}$; we denote by X' the square subcomplex of Y obtained from $\mathcal{G}$ by gluing a square each time one sees the 1-skeleton of a square in $\mathcal{G}$. The resulting square complex X' is (of course) connected. We define X to be the connected component of X' in Y; it is the cube complex dual to the collection $\mathcal{H}$ of walls of $\mathbb{H}^n$.[1]

2.3. A square complex in $\mathbb{H}^n$. The hyperplanes of $\mathcal{H}$ induce a cellulation of $\mathbb{H}^n$. And since the collection $\mathcal{H}$ is supposed to be *generic* the 2-skeleton of the dual cellulation is a square complex. We realize it in $\mathbb{H}^n$ as follows.

Definition 8. Let $P_{\mathbb{H}^n}$ be the square complex dual to the cellulation of $\mathbb{H}^n$ by $\mathcal{H}$: choose a vertex in each connected component of $\mathcal{S}$, join every pair of vertices of two adjacent components by a geodesic segment, and for every face K of codimension 2, glue a 2-disc along the four geodesic segments surrounding K.

[1] Beware that in general $X^{(1)}$ *strictly* contains $\mathcal{G} = (X')^{(1)}$!

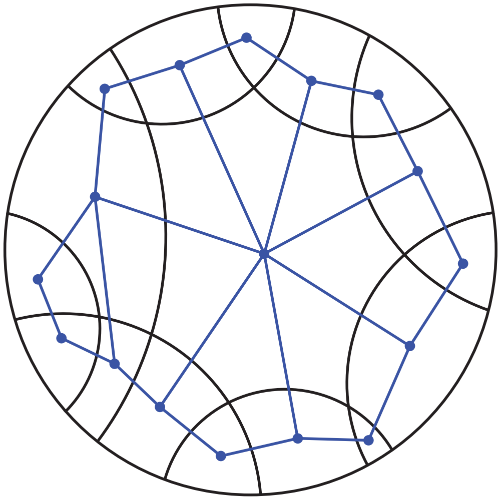


Figure 3. A cube complex.

Lemma 9. *The square complexes $P_{\mathbb{H}^n}$ and X' are combinatorially equivalent.*

Proof. By construction the 1-skeleton of $P_{\mathbb{H}^n}$ and X' are combinatorially equivalent: both set of vertices identified with the set $\mathcal{S}$ of connected components of $\mathbb{H}^n \smallsetminus \bigcup_{H \in \mathcal{H}} H$ and two vertices are adjacent if and only if they are separated in $\mathbb{H}^n$ by a unique hyperplane of $\mathcal{H}$. Finally, both X' and $P_{\mathbb{H}^n}$ are obtained from their 1-skeleton by adding a square every time one sees its 1-skeleton, i.e. when there are four connected component $a, b, c, d \in \mathcal{S}$, and two hyperplanes H and H' in $\mathcal{H}$ such that a and b, c and d are separated by H, and b and c, d and a are separated by H'. ☐

2.4. Cubulation of Γ. The natural action of $\Gamma = \pi_1(V)$ on $\mathcal{H}$ induces an action on X too. Moreover, the collection $\mathcal{H}$ being finite modulo Γ one can prove that Γ acts cocompactly on X; see [Sag97].[2] The following lemma gives a criterion on the set of hyperplanes $W_1, \dots, W_\ell$ to ensure that Γ also acts properly on X (see e.g. [DuF12], Chapter I).

Lemma 10. *Suppose there exists a number m such that every pair of points x, y of $\mathbb{H}^n$ at distance $d(x, y) \geqslant m$ is separated by some hyperplane in $\mathcal{H}$. Then the action of Γ on X is proper.*

In our case, one can even prove that this is in fact an equivalence.

Let C be the quotient of X by Γ. The group Γ also acts on $P_{\mathbb{H}^n}$. Let P_V be the quotient of $P_{\mathbb{H}^n}$ by Γ.

[2] Here we use that $\mathbb{H}^n$ is (Gromov-)hyperbolic and that – being totally geodesic – the hyperplane in $\mathcal{H}$ are quasiconvex.

As $P_{\mathbb{H}^n}$ is the 2-skeleton of the complex dual to the cellulation of $\mathbb{H}^n$ by $\mathcal{H}$, the complex $P_{\mathbb{H}^n}$ is simply connected.

Lemma 11. *The square complex P_V injects combinatorially into C, and this injection induces an isomorphism of fundamental groups.*

Proof. By construction, the combinatorial equivalence $P_{\mathbb{H}^n} \simeq X'$ proved in lemma 9 is Γ-equivariant. The complex $P_{\mathbb{H}^n}$ identifies Γ-equivariantly with a subcomplex of X. Quotienting by Γ identifies P_V with a subcomplex of C. Since $P_{\mathbb{H}^n}$ is simply connected, the inclusion $P_V \hookrightarrow C$ induces an isomorphism of fundamental groups. □

2.5. Hyperplanes in cube complexes

Definition 12. A *midcube* of a k-cube $[0,1]^k$ is a $(k-1)$-cube obtained by fixing one of the coordinates at $\frac{1}{2}$. In a CAT(0) cube complex X, a *hyperplane* H is a connected subspace of X such that the intersection of H with every cube of X is either a *midcube* or the empty set. In a non positively curved cube complex C, a *hyperplane* is the projection of a hyperplane of the universal cover of C onto C. It immerses in C.

Let X be a CAT(0) cube complex and $Y \subset X$ a subcomplex. We will abusively call hyperplanes of Y the traces of the hyperplanes of X on Y.

Going back to the situations of the preceding paragraphs, there exists a natural bijection between $\mathcal{H}$ and the set of hyperplanes of the dual CAT(0) cube complex X. A hyperplane in $\mathcal{H}$ can be associated with a hyperplane of the square complex $P_{\mathbb{H}^n}$, and to a hyperplane of X' using the isomorphism between $P_{\mathbb{H}^n}$ and X'. Finally one can extend this hyperplane to a hyperplane of X. This map is well defined and induces a natural bijection between hyperplanes of $\mathcal{H}$ and hyperplanes of X.

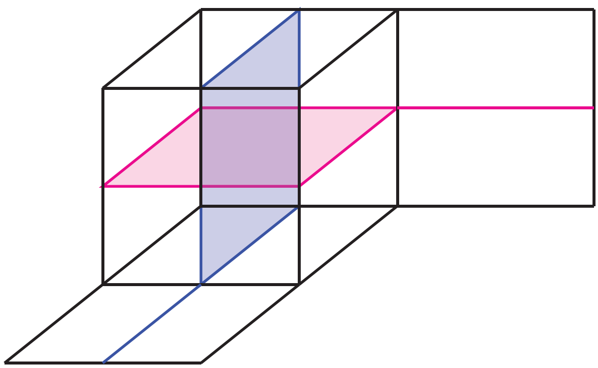


Figure 4. Two hyperplanes in a cube complex.

We shall finally need the following definition.

Definition 13. Let $C_k = [0,1]^k$ be a Euclidean cube. The *cubical barycentric subdivision* of C^k is the subdivision of C^k along its hyperplanes. The cube C^k then becomes a cube complex composed of 2^k k-cubes glued together along hyperplanes of C^k. In general, the *cubical barycentric subdivision* of a cube complex C is the cube complex described as the union of the *cubical barycentric subdivision* of each of its cubes.

3. Proof of Theorem 5

3.1. Construction. To prove Theorem 5 we shall first construct a ramified covering $\widehat{C}$ of the cube complex C. Next – in Section 3.2 – we will prove that the fundamental groups of $\widehat{V}$ and $\widehat{C}$ are isomorphic. And finally we will show – see Lemma 25 – that $\widehat{C}$ is locally CAT(0).

Definition 14. Let C be a cube complex and k be an integer. Assume that C contains two separating subspaces C_1 and C_2, both of them being unions of disjoint hyperplanes of C. Each C_i, for $i = 1, 2$ splits C into two parts: fix a base point x_0 in $C \setminus (C_1 \cup C_2)$; we let C_i^+ (resp. C_i^-) be the set of $x \in C$ such that every path from x to x_0 cuts C_i an even (resp. odd) number of times. Let $\overline{C}$ be C cut along $\mathrm{Int}(C_1^+ \cap C_2)$. We define $\widehat{C}$ by cyclically gluing k copies of $\overline{C}$ along copies of $\mathrm{Int}(C_1^+ \cap C_2)$. We call the complex $\widehat{C}$ a *ramified covering* of degree k of C.

A ramified covering as described above is not a cube complex. Nevertheless, passing to the cubical barycentric subdivision of C puts a cube complex structure on the corresponding subdivision of $\overline{C}$.

Proof of Theorem 5. The manifold V is π_1-cubulated by a finite collection of closed immersed codimension-1 submanifolds $W_1, W_2, \ldots, W_\ell$. Let $\mathcal{H}'$ be the set of all hyperplanes of $\mathbb{H}^n$ lifting the W_i's. And let $\mathcal{H}$ be the reunion of $\mathcal{H}'$ with the set of hyperplanes of $\mathbb{H}^n$ lifting the connected components of V_1 and V_2. As each V_i is a closed codimension-1 submanifold of V, it follows that the results of the preceding section apply to $\mathcal{H}$. Let X be the CAT(0) cube complex dual to $\mathcal{H}$. The fundamental group Γ of V acts properly[3] and cocompactly on X. Denote by C the quotient of X by Γ.

To construct a ramified covering $\widehat{C}$ of C, as in Definition 14, we first need to define C_1 and C_2 (as unions of hyperplanes of C). The bijection between hyperplanes of $\mathcal{H}$ and hyperplanes of X (see the end of Section 2) induces a

[3] Indeed, adding hyperplanes to $\mathcal{H}'$, can only improve the necessary condition for properness of Lemma 10.

bijection between hyperplanes of V and and hyperplanes of C. We then define C_i ($i = 1, 2$) to be the union of all the hyperplanes of C in bijection with the connected components of V_i. We now show that C_1 and C_2 separate C.

Let $p_c\colon X \to C$ is the covering map given by the quotient of X by Γ. The subspace C_1 separates C if and only if $X \smallsetminus p_c^{-1}(C_1)$ has a Γ-invariant bicoloration. Denote by p the projection $\mathbb{H}^n \to V$. As V_1 separates V, then $\mathbb{H}^n \smallsetminus p^{-1}(V_1)$ has a Γ-invariant bicoloration, and so does the complex $P_{\mathbb{H}^n} \smallsetminus (P_{\mathbb{H}^n} \cap p^{-1}(V_1))$. This complex can be seen as a subcomplex of $X \smallsetminus p_c^{-1}(C_1)$. For each point x of $X \smallsetminus p_c^{-1}(C_1)$ consider an element $x_0 \in P_{\mathbb{H}^n}$ and a path xx_0 in X. Choose for x the color of x_0 if the path xx_0 crosses $p^{-1}(C_1)$ an even number of times, and the other color if it crosses $p^{-1}(C_1)$ an odd number of times. This choice is well defined because $p^{-1}(C_1)$ is a union of hyperplanes of X, which separates X. So does $p^{-1}(C_1)$. By the same argument, C_2 separates C. To complete the proof of Theorem 5, it remains to prove that

(1) the groups $\pi_1(\widehat{V})$ and $\pi_1(\widehat{C})$ are isomorphic and

(2) the ramified covering $\widehat{C}$ of C is locally CAT(0).

The first statement is proved in Proposition 15 below and the second statement is proved in Proposition 25 below. □

3.2. Fundamental groups. The goal of this Section is to prove the following proposition.

Proposition 15. *The groups $\pi_1(\widehat{V})$ and $\pi_1(\widehat{C})$ are isomorphic.*

To compute the fundamental group $\pi_1(\widehat{V})$ (respectively $\pi_1(\widehat{C})$) we will use a different construction of $\widehat{V}$ (respectively $\widehat{C}$).

Let $N_o(V_1 \cap V_2)$ be an open tubular neighborhood of $V_1 \cap V_2$ in V. Let $V^0 = V \smallsetminus N_o(V_1 \cap V_2)$. V^0 is a submanifold of V with a boundary isomorphic to $(V_1 \cap V_2) \times \mathbb{S}^1$. Consider

$$\theta_V\colon \pi_1(V^0) \to \mathbb{Z},$$

such that for any loop l of V^0, $\theta_V(l)$ is the algebraic intersection number between l and $V_1^+ \cap V_2$, and let π be projection $\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$ where k is the degree of the ramification $\widehat{V}$ over V. Denote by $\widehat{V}^0$ the covering of V^0 associated with the group $\mathrm{Ker}(\pi \circ \theta_V)$. In restriction to the boundary $(V_1 \cap V_2) \times \mathbb{S}^1$, this covering is a k-cyclic covering on the first factor and trivial on the second one. Then the manifold $\widehat{V}$ is obtained by gluing a product $(V_1 \cap V_2) \times D$ to $\widehat{V}^0$, where D is a disk, along the boundary isomorphic to $(V_1 \cap V_2) \times \mathbb{S}^1$.

We will compute the fundamental group of $\widehat{V}$ using this construction. It will be a quotient of the subgroup $\mathrm{Ker}(\pi \circ \theta_V)$ of $\pi_1(V^0)$. Fix a base point x_0 in V^0. To any connected component K_i of $V_1 \cap V_2$ we associate a loop γ_i as follows.

Choose a path p_i from x_0 to a point x_i in the boundary of K_i then choose a loop l_i based in x_i which turns once around K_i. Define γ_i as the concatenation $p_i l_i p_i^{-1}$. Note that $\theta_V(\gamma_i) = \pm 1$. Each γ_i represents an element of $\operatorname{Ker}(\pi \circ \theta_V)$ (again abusively denoted γ_i). By the Seifert-Van Kampen Theorem recursively applied to the union of $\widehat{V}^0$ and for each $K_i \times D$ one gets

$$\pi_1(\widehat{V}) = \operatorname{Ker}(\pi \circ \theta_V)/\langle\langle \gamma_1^k, \dots, \gamma_p^k \rangle\rangle.$$

We can construct $\widehat{C}$ in the same way. In the cube complex C define a *tubular neighborhood* $N_o(C_1 \cap C_2)$ of $C_1 \cap C_2$ as the interior of the union of every cube which has a non trivial intersection with $C_1 \cap C_2$. Remark that it is isomorphic to the product $(C_1 \cap C_2) \times \square$, with $\square$ the interior of a square. Let $C^0 = C \smallsetminus N_o(C_1 \cap C_2)$. The complex $(C_1 \cap C_2) \times \partial\square$ is called the boundary of C^0. Define $\theta_C : \pi_1(C^0) \to \mathbb{Z}$ in analogy with θ_V by counting intersections of a loop with $C_1^+ \cap C_2$. Let $\widehat{C}^0$ be the covering corresponding to the subgroup $\operatorname{Ker}(\pi \circ \theta_C)$. The preimage of the boundary of C^0 under this map is isomorphic to a product $(C_1 \cap C_2) \times C_{4k}$ where C_{4k} is a cyclic graph with $4k$ edges. Call this complex the boundary of $\widehat{C}^0$. The complex $\widehat{C}$ is obtained by gluing the product of $(C_1 \cap C_2)$ with a $4k$-gon to $(C_1 \cap C_2) \times C_{4k}$.

Now we can calculate the fundamental group of $\widehat{C}$ in analogy with the calculation of the fundamental group of $\widehat{V}$. We simply need to define loops passing once around every connected component of $N_o(C_1 \cap C_2)$.

The link between the two complexes will be the subspace P_V^0 described below. The cellulation of V by the W_i's and by V_1 and V_2 induces a cellulation of V^0 by restricting the W_i's, V_1 and V_2 to V^0 and by adding several $n-2$ cells on the boundary. Then we define P_V^0 as the cube complex dual to this cellulation of V^0. It can also be described as the complex obtained by removing every square intersecting $V_1 \cap V_2$ from P_V, indeed the 1-skeleton of the two complexes dual to the cellulations of V and V^0 is the same, the only cells of codimension 2 that are in V and not in V^0 are the ones given by $V_1 \cap V_2$. Finally according to Lemma 11 P_V^0 identifies simultaneously with a subspace of V^0 and of C^0.

Proof of Proposition 15. The square complex P_V^0 is a subspace of V^0 and the inclusion induces an isomorphism of fundamental groups by definition. We will prove in Proposition 16 below that the inclusion of P_V^0 in C^0 induces an isomorphism of fundamental groups. Note that V^0 and C^0 have isomorphic fundamental groups. For every connected component of $V_1 \cap V_2$, choose γ_i included in P_V^0 and denote by θ the restriction of θ_V and θ_C to P_V^0. Therefore

$$\operatorname{Ker}(\pi \circ \theta_V) = \operatorname{Ker}(\pi \circ \theta) = \operatorname{Ker}(\pi \circ \theta_C),$$

and

$$\pi_1(\widehat{V}) = \operatorname{Ker}(\pi \circ \theta_V)/\langle\langle \gamma_1^k, \dots, \gamma_p^k \rangle\rangle = \operatorname{Ker}(\pi \circ \theta_C)/\langle\langle \gamma_1^k, \dots, \gamma_p^k \rangle\rangle = \pi_1(\widehat{C}).$$

$\square$

Proposition 16. *The inclusion of P_V^0 into C^0 induces an isomorphism of fundamental groups.*

Denote by $p_c\colon X \to C$ and by X^0 the preimage of C^0 by p_c.

Proof. Let $P^0_{\mathbb{H}^n}$ be the 2-complex dual to the cellulation of $\mathbb{H}^n \smallsetminus p^{-1}(V_1 \cap V_2)$ by hyperplanes of $\mathcal{H}$. It can be seen as a subcomplex of $P_{\mathbb{H}^n}$. The inclusion of $P_{\mathbb{H}^n}$ in X is Γ-equivariant, and to prove the proposition it suffices to show that the inclusion of $P^0_{\mathbb{H}^n}$ in X^0 is a π_1-isomorphism.

$$\begin{array}{ccc} P^0_{\mathbb{H}^n} & \hookrightarrow & X^0 \\ \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \Gamma} \\ P^0_V & \hookrightarrow & C^0 \end{array}$$

□

Proposition 17. *The inclusion of $P^0_{\mathbb{H}^n}$ into X^0 induces an isomorphism of fundamental groups.*

Proof. Let $x_0 \in P^0_{\mathbb{H}^n}$. The fundamental group of $\mathbb{H}^n \smallsetminus p^{-1}(V_1 \cap V_2)$ is an infinite free group generated by a loop for every connected component of $p^{-1}(V_1 \cap V_2)$. Consider the following system of generators: one can choose the loops in $P^0_{\mathbb{H}^n}$ because it is the 2-skeleton of the dual cellulation. For every connected component K_i of $p^{-1}(V_1 \cap V_2)$ let l_i be a loop of $P^0_{\mathbb{H}^n}$ described as a boundary of a square of $P_{\mathbb{H}^n} \smallsetminus P^0_{\mathbb{H}^n}$ associated with K_i. For each vertex of l_i there exists a path from x_0 to this vertex which does not cross the same hyperplane of X twice, as described in Lemma 18, and for one of the four vertices y of l_i the path does not cross either the two hyperplanes of $P_{\mathbb{H}^n}$ which form K_i. Denote by a this path and take $\alpha_i = a l_i a^{-1}$ as a generator of $\pi_1(P^0_{\mathbb{H}^n})$ associated with K_i. We prove in Proposition 20 that the fundamental group of X^0 is an infinite free group generated by the α_i. Then this inclusion induces a π_1-isomorphism. □

We will use combinatorial loops on the 1-skeleton of X^0. The combinatorial loops can be seen as loops of the 1-skeleton of X. If we choose a vertex, this loop is uniquely determined by the sequence of hyperplanes successively dual to the edges of this loop. As X is CAT(0), a path in the 1-skeleton of X is a geodesic in X^1 if and only if the associated sequence of hyperplanes of X does not contain the same hyperplane twice.

Lemma 18. *For each pair of vertices (x, y) of $P^0_{\mathbb{H}^n}$ there exists an edge path from x to y which crosses each hyperplane of $P_{\mathbb{H}^n}$ at most once.*

Proof. To a pair (x, y) of vertices of $P_{\mathbb{H}^n}$ we associate the pair of connected components of $\mathbb{H}^n$ separated by hyperplanes of $\mathcal{H}$ which contains respectively x and y. Choose a point for each of these components in $\mathbb{H}^n$, and consider a geodesic between them. The sequence of hyperplanes of $\mathcal{H}$ crossed by this geodesic gives a path of edges of $P_{\mathbb{H}^n}$ which crosses each hyperplane at most once, i.e. the path is a geodesic of the 1-skeleton of $P_{\mathbb{H}^n}$. □

Lemma 19. *Let γ be a combinatorial path in X^0 with the following sequence of dual hyperplanes of X: $ABH_1 \dots H_n A$ such that for all $i = 1, 2, \dots, n$, $H_i \neq A, B$, and A, B are not simultaneously in the preimage of C_1 and C_2 under p_c. Then γ is fixed-end-point homotopic in X^0 to the path associated with the sequence $BAH_1 \dots H_n A$.*

Proof. We will show that the first two edges of this path border a square of X^0. As defined in Section 2, a vertex of X is a choice of halfspace for every hyperplane of $\mathcal{H}$, such that if a halfspace $\vec{C}$ is included in another one $\vec{D}$ and if $\vec{C}$ belongs to a certain vertex then $\vec{D}$ belongs to this vertex too. We will say that $\overleftarrow{D}$ and $\vec{C}$ are *compatible* if they can belong to a same vertex of X.

Consider the three first vertices v_1, v_2 and v_3 of the path γ and the two walls $\bar{A} = \{\vec{A}, \overleftarrow{A}\}$ and $\bar{B} = \{\vec{B}, \overleftarrow{B}\}$ associated with the hyperplanes A and B. Suppose that v_1 contains the halfspaces $\vec{A}$ and $\vec{B}$. As v_1 and v_2 are separated by an edge dual to A, then v_2 contains halfspaces $\overleftarrow{A}$ and $\vec{B}$. Moreover v_3 contains the halfspaces $\overleftarrow{A}$ and $\overleftarrow{B}$. Consider the collection s of halfspaces composed with $\vec{A}$, $\overleftarrow{B}$ and every halfspace which simultaneously belongs to vertices v_1, v_2 and v_3. To prove that s is a vertex of X we will show that every pair of hyperplanes of s is compatible. The last vertex of γ contains $\vec{A}$ and $\overleftarrow{B}$ because for all $i = 1, 2, \dots, n$, $H_i \neq A, B$. The collection of halfspaces associated with v_1, v_2 and v_3 show that halfspaces $\vec{A}$, $\vec{B}$, $\overleftarrow{A}$ and $\overleftarrow{B}$ are compatible with all halfspaces shared by v_1, v_2 and v_3, and that the pairs $\vec{A}$ and $\vec{B}$, $\overleftarrow{A}$ and $\vec{B}$, $\overleftarrow{A}$ and $\overleftarrow{B}$ are compatible. Hence s is a vertex of X, and the three first vertices of γ, s, and the edges between them describe the boundary of a square of X. Since A and B are not simultaneously preimages of C_1 or C_2 the square is still in X^0. □

Proposition 20. *The fundamental group $\pi_1(X^0)$ of X^0 is an infinite free group generated by $\{\alpha_i\}$.*

Lemma 21. *Let K_i be a connected component of a preimage of $C_1 \cap C_2$ in X, let l be an oriented loop obtained as the boundary of a square of X with a non trivial intersection with K_i, and let z be a vertex of l. Then there exists a loop $\alpha_i' = a' l a'^{-1}$ homotopic to α_i (or α_i^{-1}), with a' geodesic in the 1-skeleton of X^0 between x_0 and z.*

Proof. The intersection between two hyperplanes has a natural cube complex structure. The neighborhood of K_i in X is a product $W \times \square$ with $\square$ a square and W a cube complex isomorphic to K_i. The neighborhood of K_i in X^0 will be $W \times \partial(\square)$. There exists a loop $\alpha_i = a l_i a^{-1}$, one of the generators of $\pi_1(P^0_{\mathbb{H}^n})$, such that l_i turns around K_i. Then there exists $w_1,\ w_2$ in W such that $w_1 \times \partial(\square) = l$ and $w_2 \times \partial(\square) = l_i$. Denote by $w_2 \times \bullet$ the vertex of l_i which is the last vertex of a. Since W is connected, choose a path w' in W from w_1 to w_2, and consider $w = w' \times \bullet$ a path in X^0. Denote by c the concatenation of a and w.

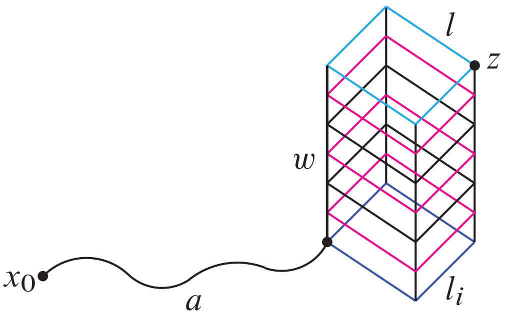


Figure 5

We will construct a path c' homotopic to c such that c' is a geodesic in the 1-skeleton of X, i.e. such that c' does not cross the same hyperplane of X twice. The path c does not cross any hyperplane that is a preimage of C_1 or C_2. Indeed, by construction, neither does a, nor does w because C_1 and C_2 do not self-intersect (because V_1 and V_2 do not self-intersect). Suppose that c is not geodesic and consider the two nearest edges dual to the same hyperplane H. The subpath of c between these edges is not dual to H and we can apply Lemma 19 several times, until the two edges are next to each other. Then one can delete the two edges by homotopy. By recurrence on these pairs of hyperplanes, we obtain a path c' from x_0 to l homotopic to c which crosses each hyperplane of X at most once. Then consider a' the path obtained from c' to z by adding one or two edges of l. These additional edges are dual to the preimage of C_1 or C_2, a' is still geodesic in the 1-skeleton of X, and $a' l a'^{-1}$ is homotopic to $a l_i a^{-1}$. □

Proof of Proposition 20. First we prove that $\pi_1(X^0)$ is generated by $\{\alpha_i\}$. For each combinatorial loop γ, denote by $|\gamma|$ the length of the loop, then consider

$$L\colon \pi_1(X^0) \longrightarrow \mathbb{N}$$

$$L(\gamma) = \min\{|\gamma'|,\ \gamma \sim \gamma'\}.$$

We will use a recursive argumentation on the length L of homotopy classes of loops. If $L(\gamma) = 0$ then γ is homotopic to x_0. Suppose that every loop of length strictly less than N is generated by $\{\alpha_i\}$, and let β be a loop of X^0, such that $L(\beta) = N$. A loop of X^0 can be described as a concatenation of loops of type blb^{-1}, with l the boundary of a square of the neighbourhood of the preimage of a connected component of $C_1 \cap C_2$. If β is not exactly one of these loops then β is a concatenation of at least 2 such loops of length less than N, and by recurrence β is generated by $\{\alpha_i\}$. Suppose now that $\beta = blb^{-1}$. Applying Lemma 21 to l and z, the last vertex of b, there exists a loop $\alpha_i' = a'la'^{-1}$ homotopic to $\alpha_i^{\pm 1}$ such that a' is minimal between x_0 and z.

$$\beta = blb^{-1} = ba'^{-1}a'la'^{-1}a'b^{-1} = (ba'^{-1})a'la'^{-1}(a'b^{-1}).$$

Fix $j = |b|$, then $N = |\beta| = 2j + 4$. As a' is a geodesic with the same endpoints as b then $|a'| \leq j$. Finally, $L(ba'^{-1}) \leq |ba'^{-1}| \leq 2j < N$, and by recurrence ba'^{-1} is generated by α_i. Furthermore, as $a'la'^{-1}$ is homotopic to α_i, β is also generated by $\{\alpha_i\}$.

To see that $\pi_1(X^0)$ is a free group, we will construct an injective morphism from $\pi_1(X^0)$ to an infinite free group.

Every hyperplane of C_2 in C is divided into several different hyperplanes in C^0. Let E be the set of all lifts of these hyperplanes in X^0. For every $e \in E$, H_e will be the hyperplane of X that contains e [4]. Consider the infinite free group $\mathbb{F}_\infty$ generated by E.

The elements of E are hyperplanes of X^0, we choose an orientation of each hyperplane of X which induces an orientation of hyperplanes of E. Consider the map m such that if γ is a combinatorial loop of X^0 then $m(\gamma)$ is the word in $E^{\pm}$ obtained by juxtaposing hyperplanes of X^0 crossed by γ, to the power of ± 1 depending on the orientation. The map m induces the morphism

$$h\colon \pi_1(X^0) \longrightarrow \mathbb{F}_\infty.$$

Homotopies between combinatorial loops can be described as a succession of elementary homotopies: going on an edge and coming back is homotopic to the identity, and for every square of X^0 an edge of its boundary is homotopic to the path which runs along the three other edges of the boundary. Then m is well defined modulo these elementary homotopies.

The map h is injective. Assume that γ is a combinatorial loop of X^0 such that $h(\gamma) = 1$. Since γ is a loop, every hyperplane of X crosses γ an even number of times. If γ does not intersect any hyperplanes of $\{H_e, e \in E\}$ then the hyperplanes dual to γ are not in $p_c^{-1}(C_2)$ and we use Lemma 19 to gather and eliminate two by two those hyperplanes dual to γ by elementary homotopies in the following way. Each hyperplane of X^0 dual to γ appears an even number of times in the sequence of hyperplanes dual to γ. Consider two repetitions of the same hyperplane as closed as possible. Then every hyperplane contained between

[4] The map $e \mapsto H_e$ is not injective

those two repetitions will appear once. If there is not a hyperplane between the two repetitions then γ is homotopic to a loop given by the same sequence minus the two occurrences of this hyperplane. If there are some hyperplanes between the repetitions, we can use Lemma 19 several times, since no hyperplane of $p^{-1}(C_2)$ belongs to the sequence, bringing us back to the previous case. Then the homotopy class of γ is trivial in $\pi_1(X^0)$.

Now suppose that one of the hyperplanes K of X, crossed by γ, is associated with an element of E. Since $h(\gamma) = 1$, the word $m(\gamma)$ contains a subword ee^{-1} with $e \in E$. One can assume that $K = H_e$. Then the enumeration of hyperplanes *of* X^0 dual to γ contains a sequence $e, h_1, \ldots, h_n, e$, where the h_i's do not belong to E. Denote by c the subpath of γ associated with this sequence. For every $i \in \{1, \ldots, n\}$ denote W_i the hyperplane of X that contains h_i. For every $i \in \{1, \ldots, n\}$, as h_i is not in E, then W_i is not contained in $p_c^{-1}(C_2)$. Furthermore if W_i is contained in $p_c^{-1}(C_1)$ then the h_i's do not intersect e, because e is a connected component of $p_c^{-1}(C_2)$ cut along $p_c^{-1}(C_1)$. Therefore there exists an even number of h_j's such that $W_i = W_j$. If such a pair of elements $W_i = W_j \subset p_c^{-1}(C_1)$ exists then by applying Lemma 19 several times to the sentence $H_i, H_{i+1}, \ldots, H_j$ which does not contain hyperplanes dual to $p_c^{-1}(C_2)$ we get $j = i + 1$, and $H_i H_i$ is homotopic to a point. Finally, applying Lemma 19 several times to the sentence $H_e, H_1, \ldots, H_n, H_e$ with $H_i \not\subset p_c^{-1}(C_1)$, one can reduce the number of ee^{-1} associated with γ. □

3.3. Ramified covering over a locally CAT(0) cube complex is locally CAT(0). We want to prove that if $\widehat{C}$ is a ramified covering of a locally CAT(0) cube complex C constructed above the intersection of two unions of hyperplanes (see Definition 14), then $\widehat{C}$ is locally CAT(0). We will prove this with a more general definition of a ramified covering.

Definition 22. Let C be a cube complex and L be a subcomplex of C. A cube complex $\widehat{C}$ is a *general ramified covering* of C above L if there exists a combinatorial map $f\colon \widehat{C} \to C$ and a subcomplex $\widehat{L}$ of $\widehat{C}$ such that

- $f_{|\widehat{L}} : \widehat{L} \xrightarrow{\simeq} L$ and
- $f_{|(\widehat{C} \setminus \widehat{L})} : (\widehat{C} \setminus \widehat{L}) \to (C \setminus L)$ is a cover.

Definition 23. A subcomplex L of a cube complex C is *locally convex* if for every cube Q of C the subcomplex $L \cap Q$ is either a unique face of Q or the whole cube Q.

We will use the following characterization of being locally CAT(0) in a cube complex, see [BH99].

Proposition 24. *A cube complex C is locally* CAT(0) *if and only if for every $v \in C^0$,* link(v, C) *is a simplicial flag complex.*

Proposition 25. *Let $(\widehat{C}, \widehat{L})$ be a general ramified covering of a cube complex (C, L), with C locally* CAT(0) *and L locally convex in C. Then $\widehat{C}$ is locally* CAT(0).

Proof. First, we prove that for every vertex v of $\widehat{C}$, the complex link$(v, \widehat{C})$ is simplicial , and not multi-simplicial, i.e. link$(v, \widehat{C})$ is totally determined by its boundary. Suppose that v is not in $\widehat{L}$, then a small ball around $f(v)$ is homeomorphic to a small ball around v, so the $link$ is the same. As C is locally CAT(0), then link$(v, \widehat{C})$ is simplicial. Now suppose that v belongs to $\widehat{L}$. Consider two $(k-1)$-simplices of link$(v, \widehat{C})$ for $k \geq 2$ sharing the same boundary. Denote by Q_1 and Q_2 the two k-cubes of $\widehat{C}$ associated with these $(k-1)$-simplices. As the two simplices have the same boundary, then Q_1 and Q_2 are glued along subcubes of codimension 1 containing v. Let f be the projection associated with the ramified covering $\widehat{C} \to C$. As f is combinatorial then C_1 and C_2 are projected on k-cubes of C. As C is locally CAT(0) then C_1 and C_2 are projected onto the same cube. The map f induces an isomorphism between $\widehat{L}$ and L, and L is convex in C. Denote by P_i, $i \in \{1, 2, \ldots, k-1\}$, codimension 1 subcube of Q. If every P_i belongs to $\widehat{L}$ then $Q \subset L$ and $Q_1 = Q_2$. If there exists $i \in \{1, 2, \ldots, k-1\}$ such that P_i does not belong to $\widehat{L}$, the restriction of f to $(Q_1 \cup Q_2) \setminus \widehat{L}$ is a covering on its image of degree 1, because the preimage of $f(P_i)$ is P_i. Then Q_1 and Q_2 are equal.

It remains to see that for every vertex $v \in \widehat{C}$, the simplicial complex link$(v, \widehat{C})$ is flag. Let v be a vertex of C and $e_1, \ldots, e_p$ be two by two connected vertices of link$(v, \widehat{C})$. The function f projects two such vertices on two different vertices of link$(f(v), C)$ because link$(f(v), C)$ is simplicial and not multi-simplicial. As C is locally CAT(0) link$(f(v), C)$ is flag and there exists a p-simplex of link$(f(v), C)$ which has $f([e_i, e_j])$ as 1-skeleton. The $(p+1)$-cube associated with this p-simplex lifts to a cube with a 0-skeleton of exactly $\{e_1, \ldots, e_p\}$. □

4. The cyclic ramified covering of a special cube complex is special

We will use the following characterization of being special.

Definition 26. (See [HW08]) A cube complex is *special* if it does not contain pathologies of hyperplanes such as self-intersection (see Figure 6), self-osculation (see Figure 7) or inter-osculation (see Figure 8), and furthermore if its hyperplanes are two-sided i.e. a neighborhood is homeomorphic to the product of the hyperplane with an interval.

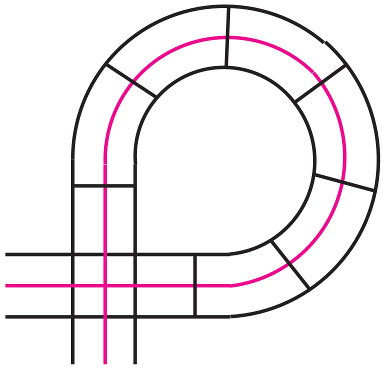


Figure 6. A self-intersecting hyperplane.

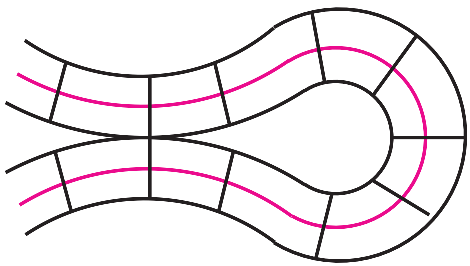


Figure 7. A self-osculating hyperplane.

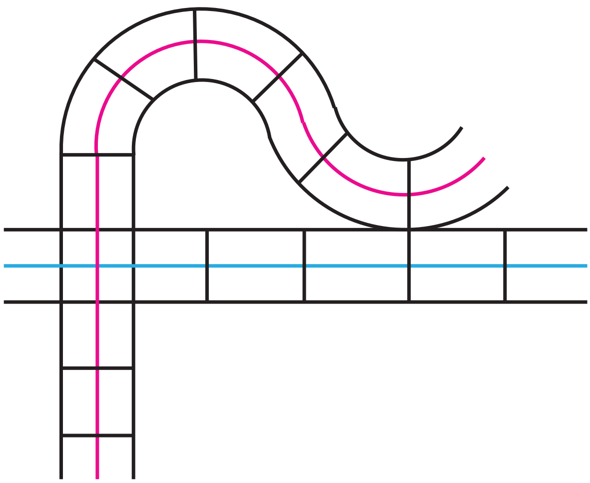


Figure 8. Two inter-osculating hyperplanes.

Proposition 27. *Let C be a special cube complex and let C_1 and C_2 be two separating unions of hyperplanes of C. Then the ramified covering $\widehat{C}$ (in the sense of Definition 14) of the cubical barycentric subdivision C' of C above $C_1 \cap C_2$ is special.*

Lemma 28. *Let C be a cube complex and let C' be the cubical barycentric subdivision of C. If C is special then C' is special.*

Proof. The neighborhood of a hyperplane of C' is the half neighborhood of a certain hyperplane of C. Then a pathology of a hyperplane of C' would imply a pathology of the associated hyperplanes of C. $\square$

Proof of the Proposition 27. We will use the notations from Definition 14, and denote by f the projection $\widehat{C} \to C'$. We show that if there is a pathology of hyperplanes as defined in 26 in $\widehat{C}$ then there is a pathology in C'.

A hyperplane of $\widehat{C}$ is a union of hyperplanes of some $\overline{C'}^i$'s (k copies of the cubical barycentric subdivision of $\overline{C}$ glued together to form $\widehat{C}$) which coincides on the boundary between $\overline{C'}^i$ and $\overline{C'}^{i+1}$ for every i. The projection of all different pieces of this hyperplane on C' forms a unique hyperplane of C'. If $\widehat{H}$ is not a two-sided hyperplane of $\widehat{C}$, then there is a loop in $\widehat{H}$ such that the parallel transport of a vector orthogonal to $\widehat{H}$ along this loop gives a vector of the opposite direction. Then the projection of the loop has the same property, and the projection of the hyperplane to C is not two-sided.

The map f projects a self-intersecting hyperplane of $\widehat{C}$ on a self-intersecting hyperplane of C'. This prevents the first pathology from occurring.

Suppose now that there exists two hyperplanes $\widehat{H}$ and $\widehat{K}$ of $\widehat{C}$ which are inter-osculating. Let $\hat{e}_1$ and $\hat{e}_2$ be two edges of $\widehat{C}$ respectively transverse to $\widehat{H}$ and $\widehat{K}$ which share a vertex $\hat{v}$. Let e_1, e_2, H and K be the images of $\hat{e}_1$, $\hat{e}_2$, $\widehat{H}$ and $\widehat{K}$ under f. If $e_1 = e_2$, then f projects $\widehat{H}$ and $\widehat{K}$ on a unique self-intersecting hyperplane, contradicting the fact that C' is special. If e_1 and e_2 are two osculating edges of C', then H and K are two inter-osculating hyperplanes of C'.The case where e_1 and e_2 form the corner of a square Q remains to be seen (see figure 8). In this case, Q has two different lifts $\widehat{Q}_1$ and $\widehat{Q}_2$ which contain respectively $\hat{e}_1$ and $\hat{e}_2$. As there is a unique lift of the vertex $v = f(\hat{v})$ in the union of squares $\widehat{Q}_1 \cup \widehat{Q}_2$, the vertex v belongs to the ramification locus. However, as there are two lifts of e_i in $\widehat{Q}_1 \cup \widehat{Q}_2$, the square Q is not included in K. As the cubical neighborhood of $C_1 \cap C_2$ is isomorphic to the cube complex $C_1 \cap C_2 \times \square$, where $\square$ is a square, and as the square Q of the cubical barycentric subdivision C' of C is not totally included in $C_1 \cap C_2$, the square Q is a quarter of a square of C given by a point times $\square$ in the neighborhood of $C_1 \cap C_2$. We can assume that e_i belongs to C_i. If not, switch the notation of e_1 and e_2. Consider the path $\hat{\gamma}$ obtained by the concatenation of a path $\hat{\gamma}_H$ of $\widehat{H}$ from the center of $\widehat{Q}_1$ to the intersection of $\widehat{H}$ and $\widehat{K}$, and a path $\hat{\gamma}_K$ of $\widehat{K}$ from this intersection to the center of $\widehat{Q}_2$. Let γ, γ_H and γ_K be the image of $\hat{\gamma}$, $\hat{\gamma}_H$ and $\hat{\gamma}_K$ under the projection f. The path γ is a loop of C'. Since $\hat{\gamma}$ itself is not a loop the (algebraic) intersection number of γ with $C_1^+ \cap C_2$ has to be different from zero, otherwise γ would lift to a loop in the ramified cover. We will reach a contradiction by showing that in fact γ has a trivial intersection with $C_1^+ \cap C_2$. Actually, H and K are disjoint from C_2 and C_1. For example e_1 is an edge of C', contained in an edge of C dual to the hyperplane C_2 in C. The intersection of every cube D of C with C_2 is obtained by setting one coordinate to $\frac{1}{2}$, and the intersection with H is obtained by setting the same coordinate to $\frac{1}{4}$ or $\frac{3}{4}$, so H is a disjoint copy of C_2. We can use exactly the same reasoning to prove that K is disjoint from C_1. Now, there are two possibilities depending on whether K is totally included in C_1^- or in C_1^+. In the first case, as $\gamma_H \subset H$ which is disjoint from C_2, and $\gamma_K \subset K \subset C_1^-$, the loop γ never crosses $C_1^+ \cap C_2$. Now, if K is totally included in C_1^+, as C_2 separates C, the algebraic intersection number of the loop γ with C_2 is zero. Since H is disjoint from C_2, the path $\gamma_H \subset H$ does not cross C_2. Then intersections of γ and C_2 are in γ_K and the intersection number of γ_K with C_2 is zero. As γ_K is included in C_1^+, the intersection number of the loop γ with $C_1^+ \cap C_2$ is zero, and we have a contradiction.

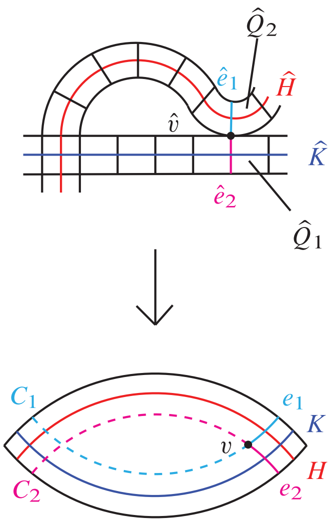


Figure 9. When e_1 and e_2 are two adjacent edges of a square.

Let $\widehat{H}$ be a self-osculating hyperplane of $\widehat{C}$. Denote by $\hat{e}_1$ and $\hat{e}_2$ two distinct edges, transverse to $\widehat{H}$, sharing a vertex $\hat{v}$. Denote by e_1, e_2, v and H the image of $\hat{e}_1$, $\hat{e}_2$, $\widehat{V}$ and $\widehat{H}$ under f. If $e_1 \neq e_2$ then H will be a self-osculating or a self-intersecting hyperplane of C'. If $e_1 = e_2$, then v will belong to the ramification locus and $e_1 \subset C_1^+ \cap C_2$. A path γ from the center of $\hat{e}_1$ to the center of $\hat{e}_2$ is sent onto a loop of C' which crosses $C_1^+ \cap C_2$ non trivially. As H is transverse to e_1 H is a copy of a hypeplane of C_1, so H is totally included in C_1^+, and as C_2 separates C', $f(\gamma)$ has to cross $C_1^+ \cap C_2$ trivially. □

5. Gromov–Thurston manifolds

The main result relies on the following Theorem of [BHW11].

Theorem 29. *"Simple type" arithmetic manifolds are cubical and virtually special.*

Let V be a "Simple type" arithmetic manifold, containing many (immersed) compact totally geodesic codimension 1 submanifolds. The fundamental group of V acts cocompactly on the cube complex associated with a finite number of such submanifolds. To show the last Theorem, Gromov and Thurston chose these submanifolds such that the action of $\pi_1(V)$ on the dual cube complex is proper, using the following criteria. let $\mathcal{H}$ be a collection of lifts of a finite number of hyperplanes $W_1, \ldots, W_\ell$ of V described as above. For every pair of point x, y, denote by $d_{\mathcal{H}}$ the number of elements of $\mathcal{H}$ witch separate x from y. If $d_{\mathcal{H}}$ and the usual distance on $\mathbb{H}^n$ are quasi-isometric then the action of $\pi_1(V)$ on the dual cube complex is proper.

To choose such a collection of hyperplanes in [BHW11], the authors use the fact that hyperplanes of $\mathbb{H}^n$ which project to a compact submanifold of V are dense in the set of hyperplanes of $\mathbb{H}^n$ (see p. 6 of [BHW11]). Later, they prove that by passing to a finite cover of V, the quotient of the cube complex obtained using this construction by the group $\pi_1(V)$ is special.

Let V be an arithmetic manifold. By passing to a finite cover, Gromov and Thurston constructed two totally geodesic manifolds, V_1 and V_2, which separate V, and then consider $\widehat{V}$ a ramified covering of V above $V_1 \cap V_2$. Let us prove that $\widehat{V}$ is virtually special.

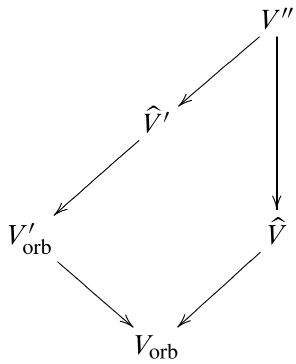
Proof of Theorem 1. In the proof of Proposition 2.1 in [BHW11], the argument of density for choosing $W_1, \ldots, W_\ell$ allows to suppose furthermore that the intersections between hyperplanes of $\mathcal{H}$ are generic. Then let V' be a finite cover of V such that the quotient of the cube complex dual to $\mathcal{H}$ by $\pi_1(V')$ is special. Denote by $\widehat{V}'$ the cyclic ramified cover of degree k of V' above the intersection of the preimage of V_1 and V_2 in V'. By Theorem 5, $\pi_1(\widehat{V}')$ is cubical, and furthermore by Proposition 27, it is special. We will show that there exists a finite cover of $\widehat{V}'$ and $\widehat{V}$. Then the fundamental group of $\widehat{V}$ will be virtually special. To see this we will use covering orbifold theory.

Let V_{orb} be an orbifold. Its underlying space is V and its singular locus is $V_1 \cap V_2$. The orbifold structure is given by the following maps. If x belongs to

$V \smallsetminus V_1 \cap V_2$, choose a neighborhood sufficiently small which does not intersect the singular locus and take the identity on V. If $x \in V_1 \cap V_2$, a small tubular neighborhood of x is isomorphic to $D^2 \times]-1,1[^{n-2}$. Then choose the quotient of $D^2 \times]-1,1[^{n-2}$ by the action of $\mathbb{Z}/k\mathbb{Z}$ on D^2.

Let V'_{orb} be an orbifold with V' as the underlying space obtained by pulling back under the covering map $p: V' \to V$ the orbifold structure of V_{orb}. For every point x of V', a local map will be the composition of p and of a map of V' at $p(x)$. The projection $V'_{\mathrm{orb}} \to V_{\mathrm{orb}}$ is a covering orbifold. Ramified covering $\widehat{V}$ is a covering orbifold of V_{orb}. Indeed, far from the singular locus, the projection $\widehat{V} \to V$ is the identity, and on a neighborhood of $V_1 \cap V_2$ this projection is a quotient by the cyclic group $\mathbb{Z}/k\mathbb{Z}$. By the same reasoning, the manifold $\widehat{V}'$ is a covering orbifold of V'.

Denote by $\widetilde{V}_{\mathrm{orb}}$ the universal covering orbifold of V_{orb}. There exists two groups G_1 and G_2 such that $\widehat{V}' = \widetilde{V}_{\mathrm{orb}}/G_1$ and $\widehat{V} = \widetilde{V}_{\mathrm{orb}}/G_2$. Note that G_1 is special. Consider now $V'' = \widetilde{V}_{\mathrm{orb}}/G_1 \cap G_2$. V'' is a manifold and a (classical) cover of $\widehat{V}$ since $\widehat{V}$ is a manifold. Furthermore, as G_1 and G_2 have a finite index in G, the group $G_1 \cap G_2$ has a finite index in G_2, and $G_1 \cap G_2$ is special as a subgroup of G_1. Then $\pi_1(\widehat{V}) = G_2$ is virtually special.



□

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