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Groups, Geometry, and Dynamics

Limit directions for Lorentzian Coxeter systems

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Abstract. Every Coxeter group admits a geometric representation as a group generated by reflections in a real vector space. In the projective representation space of a Coxeter group, limit directions arising from a point are accumulation points of the orbit of this point. In particular, limit directions arising from roots are called limit roots. Recent studies show that limit roots lie on the isotropic cone of the representation space. In this paper, we study limit directions of Coxeter groups arising from *any* point when the representation space is a Lorentz space. We prove that the limit roots are the only light-like limit directions, and characterize the limit roots using eigenvectors of infinite-order elements. Then we describe the structure of space-like limit directions in terms of the projective Coxeter arrangement. Some non-Lorentzian cases are also discussed.

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Keywords. Coxeter groups, Lorentz space, limit set, Coxeter arrangement, infinite root systems, fractal.

Contents

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1. Introduction

In this article, we study actions of infinite Coxeter groups on Lorentz spaces. Vectors in a Lorentz space are partitioned by the light cone into 3 different types: time-like, light-like and space-like vectors. The time-like and light-like vectors are used to model the hyperbolic space and its boundary. Group actions on this part of Lorentz space are well-understood in the framework of hyperbolic geometry. The present investigation on actions of Coxeter groups takes the entire Lorentz space, in particular the space-like part, into consideration and reveals behaviors that did not get noticed previously.

Every Coxeter group has a linear representation as a reflection group acting on a real vector space endowed with a canonical bilinear form; see [4] or [15]. The unit basis vectors of the representation space are called simple roots. The vectors in the orbits of simple roots under the action of the Coxeter group are called roots. The roots form the root system associated to the Coxeter system. Every root corresponds to a unique orthogonal reflecting hyperplane. The set of reflecting hyperplanes is called the Coxeter arrangement.

For infinite Coxeter groups, Vinberg introduced more general representations using different bilinear forms for the representation space [29]. We use the notation $(W, S)_B$ to denote a Coxeter system (W, S) associated with a matrix B that determines the bilinear form used to represent W. We call $(W, S)_B$ a geometric Coxeter system. When the bilinear form has signature $(n-1, 1)$, where n is the rank of (W, S), the representation space is a Lorentz space. In this case, we say that $(W, S)_B$ is a Lorentzian Coxeter system.

Let $(W, S)_B$ be a geometric Coxeter system. The Coxeter group W acts linearly on the representation space V, therefore one may also consider its action on the corresponding projective space $\mathbb{P}V$. A point of $\mathbb{P}V$ is a limit direction of $(W, S)_B$ arising from a base point $\hat{\mathbf{x}}_0 \in \mathbb{P}V$ if it is the limit of an injective sequence of points in the orbit of $\hat{\mathbf{x}}_0$. When the base point $\hat{\mathbf{x}}_0$ is the direction of a simple root, the limit directions arising from $\hat{\mathbf{x}}_0$ are called limit roots. The notion of limit roots was introduced and studied in [12]. Properties of limit roots for infinite Coxeter systems were investigated in a series of papers. Limit roots lie on the isotropic cone of the bilinear form associated to the representation space [12, Theorem 2.7]. The convex cone spanned by limit roots is the imaginary cone [10, Theorem 5.4]; see also [16, Lemma 5.8]. The relations between limit roots and the imaginary cone are further investigated in [11].

For irreducible Coxeter systems, the limit roots arise from the direction of any root or from any limit root [11, Theorem 3.1]. For Lorentzian Coxeter systems, the limit roots arise from any time-like direction [13, Theorem 3.3]. This result is obtained by interpreting the Coxeter group as a Kleinian group acting on the hyperbolic space, i.e. the time-like part of the projective Lorentz space. Theorem 2.6 of the present paper asserts that the limit roots are the only light-like limit directions, hence they also arise from any light-like direction.

Furthermore, the limit roots also arise from the direction of any weight associated to the Coxeter system [7, Theorem 3.4]. Theorem 2.4 below summarizes known results on base points whose orbits under the action of the Coxeter group accumulate at limit roots. Note that some of the weights and all of the roots are space-like vectors. By the work of Calabi and Markus [5], the action of an infinite group on the space-like part of the projective Lorentz space (also called de Sitter space) can not be discrete, so there is no reason to expect the accumulation points of the orbit of a space-like direction to be contained in the light-cone. Existence of such directions (e.g. roots and space-like weights) motivates us to study limit directions of Coxeter groups arising from an *arbitrary* base point $\widehat{\mathbf{x}}_0 \in \mathbb{P}V$ in more detail.

Based on the classification of Lorentzian transformations according to their eigenvalues, our first main result introduces a spectral perspective for limit roots involving infinite-order elements of the group.

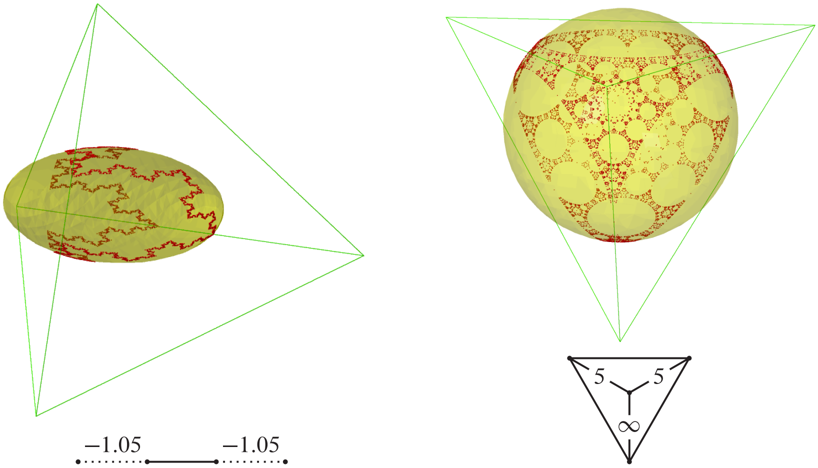
Theorem 1.1. *Let E_∞ be the set of directions of non-unimodular eigenvectors for infinite-order elements of a Lorentzian Coxeter system $(W, S)_B$. The set of limit roots E_Φ of $(W, S)_B$ is the closure of E_∞, that is*

$$E_\Phi = \overline{E_\infty}.$$

This spectral description provides a natural way to compute limit roots from the Coxeter group and its geometric action through eigenspaces of infinite-order elements. The second author implemented a package in Sage to do such computations [19, 28]. In the previous articles [12, 11, 13, 7], figures showed approximations of the limit roots using roots or weights. In contrast, Figure 1 presents two examples from our computations that reveal limit roots precisely.

Theorem 1.1 can be derived from the Kleinian group interpretation [13] or from the minimality of E_Φ under the action of W, see [11]. However, in the hope of further generalizations to non-Lorentzian Coxeter systems, our proof uses a different approach. The key of our proof is Theorem 2.6, an interesting result in its own right, which asserts for Lorentzian Coxeter systems that any light-like limit direction is a limit root. However, for non-Lorentzian Coxeter systems, there may be isotropic limit directions that are not limit roots, as shown in Example 3.12.

In Section 2.4, we relate limit roots to the study of infinite reduced words and infinite biclosed sets for Lorentzian Coxeter systems. This suggests a possible connection between two conjectures through limit roots: a conjecture of Lam and Pylyavskyy, stating that the limit weak order is a lattice [20, Conjecture 10.3], and a conjecture of Dyer, stating that the extended weak order is a complete ortholattice [9, Conjecture 2.5]. The limit weak order extends the usual weak order to infinite reduced words, whereas the extended weak order extends the usual weak order to infinite biclosed sets. The relations between limit weak order and extended weak order in general deserve to be explored in more detail.



(a) Some 30080 limit roots of the geometric Coxeter system with the shown Coxeter graph. They are obtained from infinite-order elements of length 3 and 4 and their conjugates with elements of length 1 to 9.

(b) Some 28019 limit roots of the geometric Coxeter system with the shown Coxeter graph. They are obtained from infinite-order elements of length 2, 3 and 4 and their conjugates with elements of length 1 to 5.

Figure 1. Limit roots of two Lorentzian Coxeter systems of rank 4 visualized in the affine space spanned by the simple roots. The vertices of the tetrahedra represent the simple roots, and the ellipsoids represent the light cone; see context of equation (1) for a detailed explanation. The Coxeter graphs follow Vinberg's convention; see Remark 2.2 for a detailed description.

We propose limit roots as a potential tool for the investigation of boundaries of Coxeter groups. This idea is supported by some previous studies: on the one hand, Lam and Thomas [21] proved that blocks of infinite reduced words induce a partition of the boundary of the Davis complex; on the other hand, Hosaka [14] proved that limit points of a Coxeter group arising through elements of infinite order are dense in the boundary of the Davis complex. Visibly, Hosaka's result is very similar to our Theorem 1.1.

While proving Theorem 1.1, we found space-like limit directions. A referee of [7] pointed out that space-like limit directions exist by a result of Calabi and Markus [5]. Our second main result describes the set of limit directions for Lorentzian Coxeter systems in terms of the projective Coxeter arrangement, i.e. the infinite arrangement of reflecting hyperplanes in the projective representation space.

Theorem 1.2. *Let E_V be the set of limit directions of a Lorentzian Coxeter system $(W, S)_B$ and $\mathcal{L}_{\text{hyp}}$ be the union of codimension-2 space-like intersections of the projective Coxeter arrangement associated to $(W, S)_B$. The set of limit directions E_V is "sandwiched" between $\mathcal{L}_{\text{hyp}}$ and its closure, that is*

$$\mathcal{L}_{\text{hyp}} \subsetneq E_V \subseteq \overline{\mathcal{L}_{\text{hyp}}}.$$

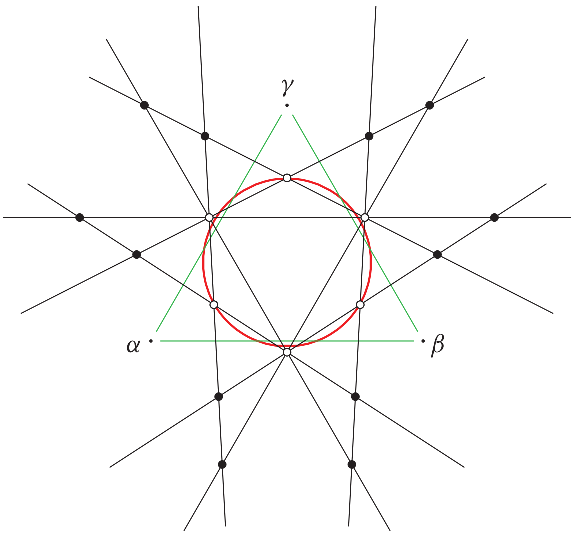


Figure 2. Some reflecting hyperplanes in the projective Coxeter arrangement of the universal Coxeter group of rank 3 associated with a bilinear form where $c_{ij} = 1.1$ whenever $i \neq j$; see Section 2.1. Codimension-2 space-like intersections are marked with dots. By Theorem 1.2, the intersections in $\mathcal{L}_{\text{hyp}}$ are limit directions. The six intersections in white are weights.

The hyperplane arrangement $\mathcal{L}_{\text{hyp}}$ involved in Theorem 1.2 is infinite and non-discrete. The readers are warned not to confuse "union of codimension-2 space-like intersections" with "set of space-like points on codimension-2 intersections." Figure 2 shows part of the Coxeter arrangement of a universal Coxeter group and some codimension-2 space-like intersections in $\mathcal{L}_{\text{hyp}}$.

While Theorem 1.2 has a combinatorial flavour, a stronger relation holds. The unimodular subspace of an infinite-order element is the set of its unimodular eigenvectors. Let $\mathcal{U}_{\text{hyp}}$ be the union of space-like projective unimodular subspaces of infinite-order elements, then the set of limit directions satisfies (see Section 4.4)

$$\mathcal{U}_{\text{hyp}} \sqcup E_\Phi \subseteq E_V \subseteq \overline{\mathcal{U}_{\text{hyp}}}.$$

Notably, space-like weights are all limit directions, as marked by white points in Figures 2 and 12. Moreover, roots may also be limit directions. In Section 4.4, we discuss the possibility of equalities on either side, and point out certain unexplained linear dependencies among limit directions in $\mathcal{U}_{\text{hyp}}$; see Figure 12.

The present paper is organized as follows. In Section 2, we recall the geometric representations of Coxeter systems, define the notion of limit directions, and review some results on limit roots. Then we prove that limit roots are the only light-like limit directions for Lorentzian Coxeter systems, and study the relation between infinite reduced words and limit roots. In Section 3, we recall spectral properties of Lorentz transformations, and prove Theorem 1.1. In the last part of Section 3, we give an example of non-Lorentzian Coxeter system, for which some isotropic limit directions are not limit roots. In Section 4, we define and study the projective Coxeter arrangement, and prove Theorem 1.2. Finally, some open problems are discussed.

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2. Limit roots of Lorentzian Coxeter systems

2.1. Lorentzian Coxeter systems. An n-dimensional *Lorentz space* $(V, \mathcal{B})$ is a vector space V associated with a bilinear form $\mathcal{B}$ of signature $(n-1, 1)$. A linear transformation on V that preserves the bilinear form $\mathcal{B}$ is called a *Lorentz transformation*. The group of Lorentz transformations is called *Lorentz group* and denoted by $O_{\mathcal{B}}(V)$. In a Lorentz space, a vector $\mathbf{x}$ is *space-like* (resp. *time-like, light-like*) if $\mathcal{B}(\mathbf{x}, \mathbf{x})$ is positive (resp. negative, zero). The set of light-like vectors $Q = \{\mathbf{x} \in V \mid \mathcal{B}(\mathbf{x}, \mathbf{x}) = 0\}$ forms a cone called the *light cone*. The following proposition characterizes the totally-isotropic subspaces of Lorentz spaces.

Proposition 2.1 ([6, Theorem 2.3]). *Let $(V, \mathcal{B})$ be a Lorentz space and $\mathbf{x}, \mathbf{y} \in Q$ be two light-like vectors. Then $\mathcal{B}(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = c\mathbf{y}$ for some $c \in \mathbb{R}$.*

Let (W, S) be a finitely generated *Coxeter system*, where S is a finite set of generators and the *Coxeter group* W is generated by S with the relations $(st)^{m_{s,t}} = e$, where $s, t \in S$, $m_{s,s} = 1$ and $m_{s,t} = m_{t,s} \geq 2$ or $= \infty$ if $s \neq t$. The cardinality $|S| = n$ is the *rank* of the Coxeter system (W, S). The Coxeter system is *universal* if $m_{s,t} = \infty$ whenever $s \neq t$. For an element $w \in W$, the *length* $\ell(w)$ of w is the smallest natural number k such that $w = s_1 s_2 \ldots s_k$ for $s_i \in S$.

We refer the readers to [4, 15] for more detail. We associate a matrix B to (W, S) as follows:

$$B_{s,t} = \begin{cases} -\cos(\pi/m_{s,t}) & \text{if } m_{s,t} < \infty, \\ -c_{s,t} & \text{if } m_{s,t} = \infty, \end{cases}$$

for $s, t \in S$, where $c_{s,t}$ are chosen arbitrarily with $c_{s,t} = c_{t,s} \geq 1$. We call the Coxeter system (W, S) associated with the matrix B a *geometric Coxeter system* and denote it by $(W, S)_B$.

Remark 2.2. The Coxeter graphs in the current paper follow Vinberg's convention that encodes both the Coxeter system (W, S) and the matrix B. Two vertices s, t are not connected if $B_{s,t} = 0$, connected by a solid edge with no label if $B_{s,t} = -1/2$, by a solid edge with label $m_{s,t}$ if $-1/2 > B_{s,t} \geq -1$, and by a dotted edge with label $-c_{s,t}$ if $B_{s,t} = -c_{s,t} < -1$.

Let V be a real vector space of dimension n, equipped with a basis $\Delta = \{\alpha_s\}_{s \in S}$. The matrix B defines a bilinear form $\mathcal{B}$ on V by $\mathcal{B}(\alpha_s, \alpha_t) = \alpha_s^\intercal B \alpha_t$ for $s, t \in S$. For a vector $\alpha \in V$ such that $\mathcal{B}(\alpha, \alpha) \neq 0$, we define the reflection σ_α

$$\sigma_\alpha(\mathbf{x}) := \mathbf{x} - 2\frac{\mathcal{B}(\mathbf{x}, \alpha)}{\mathcal{B}(\alpha, \alpha)}\alpha \quad \text{for all } \mathbf{x} \in V.$$

The homomorphism $\rho\colon W \to \mathrm{GL}(V)$ sending s to σ_{α_s} is a faithful geometric representation of the Coxeter group W. We refer the readers to [17, Chapter 1] and [12, Section 1] for more details. In the following, we will write $w(x)$ in place of $\rho(w)(x)$.

A geometric Coxeter system $(W, S)_B$ is of *finite type* if B is positive definite. In this case, W is a finite group, and can be represented as a spherical reflection group. A geometric Coxeter system $(W, S)_B$ is of *affine type* if the matrix B is positive semi-definite but not definite. In this case, the group W can be represented as a reflection group in Euclidean space. If the matrix B has signature $(n-1, 1)$, the pair $(V, \mathcal{B})$ is an n-dimensional Lorentz space, and W acts on V as a reflection subgroup of the Lorentz group. In this case, we say that the geometric Coxeter system $(W, S)_B$ is *Lorentzian* and, by abuse of language, that W is a *Lorentzian Coxeter group*. See [7, Remark 2.2] for further discussions on terminologies.

We may also pass to the *projective space* $\mathbb{P}V$, i.e. the topological space of 1-dimensional subspaces of V. For a non-zero vector $\mathbf{x} \in V \setminus \{0\}$, let $\widehat{\mathbf{x}} \in \mathbb{P}V$ denote the line passing through $\mathbf{x}$ and the origin. The group action of W on V by reflection induces a *projective action* of W on $\mathbb{P}V$ as $w \cdot \widehat{\mathbf{x}} = \widehat{w(\mathbf{x})}$, for $w \in W$ and $\mathbf{x} \in V$. For a set $X \subset V$, the corresponding projective set is $\widehat{X} := \{\widehat{\mathbf{x}} \in \mathbb{P}V \mid \mathbf{x} \in X\}$. The *projective light cone* is denoted by $\widehat{Q}$.

Let $h(\mathbf{x})$ denote the sum of the coordinates of $\mathbf{x}$ in the basis Δ, and call it the *height* of the vector $\mathbf{x}$. The hyperplane $\{\mathbf{x} \in V \mid h(\mathbf{x}) = 1\}$ is the affine subspace $\mathsf{aff}(\Delta)$ spanned by the basis of V. It is useful to identify the projective space $\mathbb{P}V$ with the affine subspace $\mathsf{aff}(\Delta)$ plus a *projective hyperplane at infinity*. For a vector $\mathbf{x} \in V$, if $h(\mathbf{x}) \neq 0$, $\widehat{\mathbf{x}}$ is identified with the vector

$$\mathbf{x}/h(\mathbf{x}) \in \mathsf{aff}(\Delta). \tag{1}$$

Otherwise, if $h(\mathbf{x}) = 0$, the direction $\widehat{\mathbf{x}}$ is identified with a point at infinity. For a basis vector $\alpha \in \Delta$, the affine picture of $\hat{\alpha}$ is α itself. In fact, if $h(\mathbf{x}) \neq 0$, $\widehat{\mathbf{x}}$ is identified with the intersection of $\mathsf{aff}(\Delta)$ with the straight line passing through $\mathbf{x}$ and the origin. The projective light cone $\widehat{Q}$ is projectively equivalent to a sphere. The affine subspace $\mathsf{aff}(\Delta)$ is practical for visualizing $\mathbb{P}V$ and developing geometric intuitions.

Given a topological space X and a subset $Y \subseteq X$, a point $x \in X$ is an *accumulation point* of Y if every neighborhood of x contains a point of Y different from x. Let G be a group acting on X, then a point $x \in X$ is a *limit point* of G if x is an accumulation point of the orbit $G(x_0)$ for some *base point* $x_0 \in X$. Equivalently, x is a limit point of G, if there is a base point $x_0 \in X$ and a sequence of elements $(g_k)_{k\in\mathbb{N}} \in G$ such that the sequence of points $(g_k(x_0))_{k\in\mathbb{N}}$ is injective and converges to x as $k \to \infty$. In this case, we say that x is a limit point of G acting on X *arising from* the base point x_0 *through* the sequence $(g_k)_{k\in\mathbb{N}}$. We now define the main object of study of the present paper.

Definition 2.3. *Limit directions* of a geometric Coxeter system $(W, S)_B$ are limit points of W acting on $\mathbb{P}V$. The set of limit directions is denoted by E_V. In other words,

$$E_V = \{\widehat{\mathbf{x}} \in \mathbb{P}V \mid \text{there is a } \widehat{\mathbf{x}}_0 \in \mathbb{P}V \text{ and an injective sequence } (w_i \cdot \widehat{\mathbf{x}}_0)_{i\in\mathbb{N}} \\ \text{in the orbit } W \cdot \widehat{\mathbf{x}}_0 \text{ such that } \lim_{i\to\infty} w_i \cdot \widehat{\mathbf{x}}_0 = \widehat{\mathbf{x}}\}.$$

2.2. Limit roots. We call the basis vectors in Δ the *simple roots*. Let $\Phi = W(\Delta)$ be the orbit of Δ under the action of W, then the vectors in Φ are called *roots*. The pair (Φ, Δ) is a *based root system*. The roots Φ are partitioned into *positive roots* $\Phi^+ = \mathsf{cone}(\Delta) \cap \Phi$ and *negative roots* $\Phi^- = -\Phi^+$. The *depth* of a positive root $\gamma \in \Phi^+$ is the smallest integer k such that $\gamma = s_1 s_2 \ldots s_{k-1}(\alpha)$, for $s_i \in S$ and $\alpha \in \Delta$.

Let V^* be the dual vector space of V with dual basis Δ^*. If the bilinear form $\mathcal{B}$ is non-singular, which is the case for Lorentz spaces, V^* can be identified with V, and $\Delta^* = \{\omega_s\}_{s\in S}$ can be identified with a set of vectors in V such that

$$\mathcal{B}(\alpha_s, \omega_t) = \delta_{s,t},$$

where $\delta_{s,t}$ is the Kronecker delta function. Vectors in Δ^* are called *fundamental weights*, and vectors in the orbit $\Omega := W(\Delta^*)$ are called *weights*.

Limit roots were introduced in [12] to study infinite root systems associated to infinite Coxeter groups. They are the accumulation points of the projective roots $\widehat{\Phi}$ in $\mathbb{P}V$, i.e. limit directions arising from projective roots. In other words, the set of limit roots is defined as follows

$$E_\Phi = \{\widehat{\mathbf{x}} \in \mathbb{P}V \mid \text{there is an injective sequence } (\gamma_i)_{i\in\mathbb{N}} \in \Phi \text{ such that } \lim_{i\to\infty} \widehat{\gamma}_i = \widehat{\mathbf{x}}\}.$$

Limit roots are on the *isotropic cone* $\{\widehat{\mathbf{x}} \in \mathbb{P}V \mid \mathcal{B}(\mathbf{x},\mathbf{x}) = 0\}$ [12, Theorem 2.7(ii)]. The cone over the limit roots is the imaginary cone [10, Theorem 5.4]. Limits roots are also limit directions arising from different base points, as summarized in the following theorem and schematized in Figure 3.

Theorem 2.4. *The limit roots of a geometric Coxeter system* $(W,S)_B$ *are the limit directions arising from* [11, Theorem 3.1]

(i) *any simple root,*

(ii) *any limit root,*

(iii) *any projective root.*

Moreover, if $(W,S)_B$ *is Lorentzian, limit roots are limit directions arising from*

(iv) *any time-like direction*; *see* [13, Theorem 3.3],

(v) *any projective weight*; *see* [7, Theorem 3.4],

(vi) *any light-like direction, by Theorem* 2.6 *of this paper.*

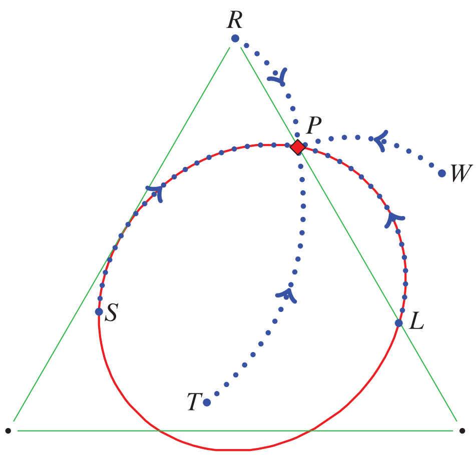


Figure 3. Schematic picture showing a limit root P arising from different base points: time-like direction (T), light-like direction (S), projective weight (W), projective root (R) or another limit root (L). The elliptic-shape is the projective light cone, and the triangle is the convex hull of the projective simple roots.

Remark 2.5. In this paper, we require the simple roots to be a basis for a based root system. However, the linear independence of the simple roots is never used in the arguments. So the results of this paper are all valid in the more general setting of [12, 10, 11], where the simple roots only needs to be positively independent but not necessarily linearly independent.

2.3. Light-like limit directions. By Theorem 2.4, the set of limit roots E_Φ is contained in E_V. The following theorem states that limit roots are the only light-like limit directions of Lorentzian Coxeter groups.

Theorem 2.6. *For a Lorentzian Coxeter system $(W, S)_B$, consider a sequence $(w_k)_{k\in\mathbb{N}} \in W$ and a base point $\mathbf{x}_0 \in V$. If $(w_k \cdot \widehat{\mathbf{x}}_0)_{k\in\mathbb{N}}$ is injective and converges to a limit direction $\widehat{\mathbf{x}}$ in the projective light cone $\widehat{Q}$, then $\widehat{\mathbf{x}}$ is a limit root.*

Proof. Choose a sequence of simple roots $(\alpha_k)_{k\in\mathbb{N}}$ such that $(w_k \cdot \hat{\alpha}_k)_{k\in\mathbb{N}}$ is an injective sequence of projective roots. Since Δ is a finite set, the sequence $(\alpha_k)_{k\in\mathbb{N}}$ visits a certain simple root, say $\alpha \in \Delta$, infinitely often. By passing to a subsequence, we may assume that $\alpha_k = \alpha$ for all $k \in \mathbb{N}$. Since $\mathbb{P}V$ is compact, by passing again to a subsequence, we may assume that $(w_k \cdot \hat{\alpha})$ converges to a limit root $\hat{\beta} \in \widehat{Q}$.

Assume that $\mathbf{x}_0$ is not light-like, then $|h(w_k(\mathbf{x}_0))|$ tends to infinity since

$$0 = \mathcal{B}(\widehat{\mathbf{x}}, \widehat{\mathbf{x}}) = \lim_{k\to\infty} \mathcal{B}(w_k \cdot \widehat{\mathbf{x}}_0, w_k \cdot \widehat{\mathbf{x}}_0) = \lim_{k\to\infty} \frac{\mathcal{B}(\mathbf{x}_0, \mathbf{x}_0)}{h(w_k(\mathbf{x}_0))^2}.$$

While $\mathcal{B}(w_k(\alpha), w_k(\mathbf{x}_0)) = \mathcal{B}(\alpha, \mathbf{x}_0)$ is constant, the height $h(w_k(\alpha))$ tends to infinity (see the proof of [12, Theorem 2.7]). Therefore

$$\mathcal{B}(\widehat{\mathbf{x}}, \hat{\beta}) = \lim_{k\to\infty} \mathcal{B}(w_k \cdot \widehat{\mathbf{x}}_0, w_k \cdot \alpha) = \lim_{k\to\infty} \frac{\mathcal{B}(\mathbf{x}_0, \alpha)}{h(w_k(\mathbf{x}_0))h(w_k(\alpha))} = 0.$$

Since $\widehat{\mathbf{x}}$ and $\hat{\beta}$ are both in the projective light cone, we have $\widehat{\mathbf{x}} = \hat{\beta}$ by Proposition 2.1. The limit direction $\widehat{\mathbf{x}}$ is therefore a limit root. This argument does not depend on the choice of base point $\mathbf{x}_0 \notin Q$. So if $\widehat{\mathbf{y}} \in \widehat{Q}$ is another limit direction arising from $\mathbf{y}_0 \notin Q$ through the same sequence $(w_k)_{k\in\mathbb{N}}$, we have $\widehat{\mathbf{x}} = \widehat{\mathbf{y}} = \hat{\beta} \in E_\Phi$.

If $\mathbf{x}_0$ is light-like, it can be decomposed as a linear combination of a time-like vector $\mathbf{x}_0'$ and a space-like vector $\mathbf{x}_0''$; see Figure 4 for an illustration of this case. Under the action of the sequence $(w_k)_{k\in\mathbb{N}}$, the time-like component $(w_k \cdot \widehat{\mathbf{x}}_0')_{k\in\mathbb{N}}$ converges to a limit root $\hat{\beta} \in \widehat{Q}$; see [13, Theorem 3.3]. If the space-like component $(w_k \cdot \widehat{\mathbf{x}}_0'')_{k\in\mathbb{N}}$ does not accumulate at the light cone, the norm of $w_k(\mathbf{x}_0'')$ is bounded because the action of W preserves the bilinear form. In this case, we have

$$\lim_{k\to\infty} w_k \cdot \widehat{\mathbf{x}}_0 = \lim_{k\to\infty} w_k \cdot \widehat{\mathbf{x}}_0' = \hat{\beta} \in E_\Phi.$$

If the space-like component also converges to the light cone, then its direction $(w_k \cdot \hat{\mathbf{x}}_0'')_{k\in\mathbb{N}}$ also converges to the limit root $\hat{\beta}$. So the sequence $(w_k \cdot \hat{\mathbf{x}}_0)_{k\in\mathbb{N}}$, being the direction of a light-like linear combination of the two components, must converge to the same limit root $\hat{\beta}$. □

As a consequence, limit directions arising from light-like directions are limit roots, as mentioned in Theorem 2.4(vi).

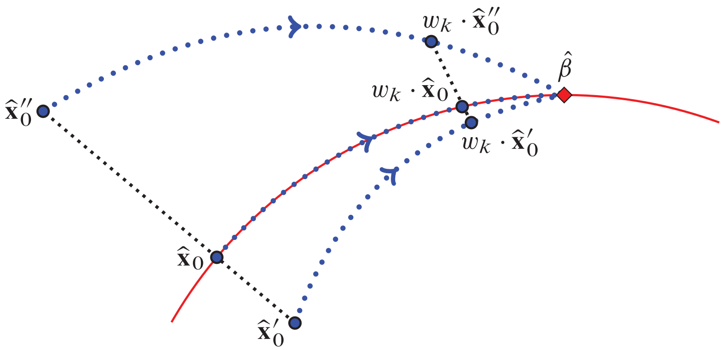


Figure 4. Illustration for the proof of Theorem 2.6 in the case where the base point $\mathbf{x}_0$ is light-like.

Corollary 2.7 (of the proof). *Limit roots arising from different base points but through the same sequence are the same.*

Therefore, when we talk about limit roots, we need not specify the base points from which they arise.

Remark 2.8. If the geometric Coxeter system $(W, S)_B$ is not Lorentzian, limit roots are in general not the only limit directions on the isotropic cone. An example will be given in Example 3.12 of Section 3.3.

2.4. Infinite reduced words. Let $s_1 s_2 \cdots s_k$ be a reduced expression of an element $w_k \in W$. The *inversion set* $\mathsf{inv}(w_k) \subset \Phi^+$ is the set of positive roots in the form of $s_1 \cdots s_{i-1}(\alpha_{s_i})$, where $1 \le i \le k-1$. This set is independent of the choice of reduced expression for w_k [3, Section 1.4]. An *infinite reduced word* is an infinite sequence $\mathbf{w} = s_1 s_2 \ldots$ of generators in S such that every finite prefix $w_k = s_1 s_2 \ldots s_k$ is a reduced expression; see [20, Section 4.2] and [21]. The *inversion set* $\mathsf{inv}(\mathbf{w}) \subset \Phi^+$ of an infinite reduced word $\mathbf{w}$ is the set of positive roots in the form of $w_{k-1}(\alpha_{s_k})$, $k \in \mathbb{N}$. To prove the following theorem, we need the notion of biclosed sets of roots. A subset A of roots in Φ^+ is *closed* if for two roots $\alpha, \beta \in A$, any root that is a positive combination of α and β is also in A.

A subset A is *biclosed* if both A and its complement in Φ^+ are closed. Finite biclosed sets of Φ^+ are in bijections with the inversion sets of elements of W; see [24, Proposition 1.2]. We refer the readers to [18, Chapter 2] for more details on the relation between biclosed sets and the study of limit roots.

Theorem 2.9. *Let* $\mathbf{w}$ *be an infinite reduced word of a Lorentzian Coxeter system* $(W,S)_B$. *The projective inversion set* $\widehat{\mathsf{inv}(\mathbf{w})}$ *of an infinite reduced word* $\mathbf{w}$ *has a limit root as its unique accumulation point.*

Proof. As a set of positive root, the accumulation points of $\widehat{\mathsf{inv}(\mathbf{w})}$ are limit roots. For the sake of contradiction, assume that $\widehat{\mathsf{inv}(\mathbf{w})}$ accumulates at two distinct limit roots $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$. Then for any neighborhood $N_x \ni \widehat{\mathbf{x}}$ and $N_y \ni \widehat{\mathbf{y}}$, we can find two projective roots $\hat{\alpha} \in N_x \cap \widehat{\mathsf{inv}(\mathbf{w})}$ and $\hat{\beta} \in N_y \cap \widehat{\mathsf{inv}(\mathbf{w})}$. Moreover, there is a positive integer $k > 0$ such that α and β are contained in $\mathsf{inv}(w_k)$. However, since the interior of the projective light cone (time-like part) is strictly convex, we can pick N_x and N_y small enough so that the segment $[\hat{\alpha}, \hat{\beta}]$ intersects $\widehat{Q}$. In this case, the reflections in α and β generate an infinite dihedral group, so $\mathsf{inv}(w_k)$ can not be finite and closed at the same time. □

We can therefore associate a unique limit root to each infinite reduced word $\mathbf{w}$ and denote it by $\hat{\gamma}(\mathbf{w})$.

Corollary 2.10. *The limit root* $\hat{\gamma}(\mathbf{w})$ *arises through the sequence of prefixes* $(w_k = s_1 \dots s_k)_{k\in\mathbb{N}}$ *of* $\mathbf{w}$.

Proof. At least one generator $s \in S$ appears in $\mathbf{w}$ infinitely many times, so we can take from $\mathsf{inv}(\mathbf{w})$ an injective subsequence $(w_k(\alpha_s)_{k\in\mathbb{N}})$ such that $s_{k+1} = s$ for all $k \in \mathbb{N}$. Then $\hat{\gamma}(\mathbf{w})$ is the limit root arising from α_s through the sequence (w_k). By Corollary 2.7, the same limit root arises through the same sequence from any projective root. We then conclude that $\hat{\gamma}(\mathbf{w})$ arises through the sequence $(w_k)_{k\in\mathbb{N}}$. □

Consider two infinite reduced words $\mathbf{w}$ and $\mathbf{w}'$ of a general Coxeter system (W,S). In [21], $\mathbf{w}$ and $\mathbf{w}'$ are said to be in the same *block* if $\mathsf{inv}(\mathbf{w})$ and $\mathsf{inv}(\mathbf{w}')$ differ by finitely many roots, and it is shown that the blocks induce a partition of the Tits boundary of the Davis complex. If (W,S) is Lorentzian, Theorem 2.9 and Corollary 2.10 imply that $\hat{\gamma}(\mathbf{w}) = \hat{\gamma}(\mathbf{w}')$ if $\mathsf{inv}(\mathbf{w})$ and $\mathsf{inv}(\mathbf{w}')$ share infinitely many roots. In Figure 5, we show the inversion set of an infinite Coxeter element and the corresponding sequence of chambers in the Coxeter complex. The sequence of chambers correspond to a geodesic ray in the Cayley graph of (W,S), which is the 1-skeleton of the Davis complex. This suggests that limit roots may be used as a geometric picture for the boundaries of Coxeter groups. For this, it would be interesting to find an equivalence relation on infinite reduced words such

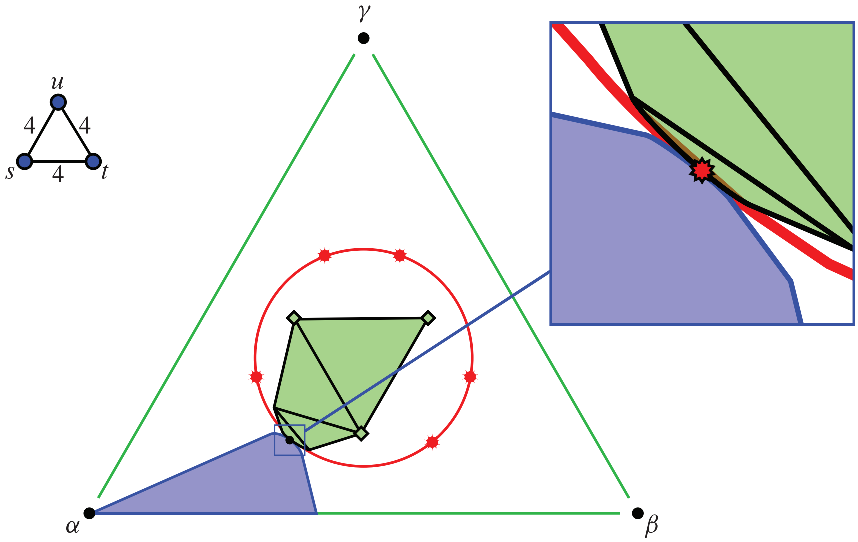


Figure 5. The inversion set of the infinite reduced word $(stu)^\infty$ represented by the convex hull of its roots (blue), and the corresponding sequence of chambers (green) inside the Tits cone. They are disjoint and share a unique point on their boundary: the limit root given by the dominant eigendirection of stu. The other star shape points correspond to the other Coxeter elements.

that two words $\mathbf{w}$ and $\mathbf{w}'$ are equivalent if and only if $\hat{\gamma}(\mathbf{w}) = \hat{\gamma}(\mathbf{w}')$. For example, consider the infinite reduced words $(st)^\infty$ and $(ts)^\infty$ of the infinite dihedral group of affine type ($c_{s,t} = 1$). Their inversion sets are disjoint, but they correspond to a same limit root. In this case, it would make more sense if $(st)^\infty$ and $(ts)^\infty$ are considered as equivalent.

We also notice potential relations between limit roots and two conjectures. On the one hand, Lam and Pylyavskyy conjectured that the *limit weak order*, i.e. the finite and infinite inversion sets ordered by inclusion, for the affine Coxeter group $\tilde{A}_n$ forms a lattice; see [20, Conjecture 10.3]. On the other hand, Dyer conjectured that the *extended weak order*, i.e. the biclosed sets ordered by inclusion, forms a complete ortholattice; see [9, Conjecture 2.5]. In view of Theorem 2.9, it seems reasonable to use the notion of limit roots to unify both conjectures. Namely, one verifies that an infinite inversion set $\mathsf{inv}(\mathbf{w})$ is biclosed in the affine and Lorentzian case, otherwise it would contradict the biclosedness of the inversion set of a certain finite prefix of $\mathbf{w}$. The difference between the two conjectures lies in the fact that there are many biclosed sets that are neither finite nor cofinite, yet are not infinite inversion sets; see [18, Figure 2.11] for an example. The relations between the two conjectures should be made clear and deserve better attention.

3. Spectra of elements of Lorentzian Coxeter groups

3.1. Spectra of Lorentz transformations. In a Lorentz space $(V, \mathcal{B})$, a subspace U of V is *space-like* if its non-zero vectors are all space-like, *light-like* if U contains some non-zero light-like vector but no time-like vector, or *time-like* if U contains some time-like vector. Two vectors $\mathbf{x}, \mathbf{y} \in V$ are said to be orthogonal if $\mathcal{B}(\mathbf{x}, \mathbf{y}) = 0$. For a vector $\mathbf{x} \in V$, we define its orthogonal hyperplane

$$H_{\mathbf{x}} = \{\mathbf{y} \in V \mid \mathcal{B}(\mathbf{x}, \mathbf{y}) = 0\}.$$

We see that $H_{\mathbf{x}}$ is space-like (resp. light-like, time-like) if and only if $\mathbf{x}$ is time-like (resp. light-like, space-like). For a subspace U of V, its orthogonal companion is defined as

$$U^{\perp} = \{\mathbf{y} \in V \mid \mathcal{B}(\mathbf{y}, \mathbf{x}) = 0 \text{ for all } \mathbf{x} \in U\}.$$

Note that if U is lightlike, $U \cap U^{\perp} \neq \emptyset$ and $U + U^{\perp} \neq V$.

To study the eigenvalues and eigenvectors of Lorentz transformations, it is useful to work in the complexification $V_{\mathbb{C}} = V \oplus iV$. A vector $\mathbf{z} \in V_{\mathbb{C}}$ can be written as $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in V$. We call $\mathbf{x}$ (resp. $\mathbf{y}$) the *real part* (resp. *imaginary part*) of $\mathbf{z}$, and the vector $\bar{\mathbf{z}} = \mathbf{x} - i\mathbf{y}$ represents its *conjugate vector*. If $\mathbf{y} \neq 0$, we say that $\mathbf{z}$ is *space-like* (resp. *light-like*, *time-like*) if the subspace spanned by $\mathbf{x}$ and $\mathbf{y}$ is space-like (resp. light-like, time-like). The bilinear form $\mathcal{B}$ on V is viewed as the restriction of a sesquilinear form on $V_{\mathbb{C}}$ defined by requiring in addition that

$$\mathcal{B}(\lambda \mathbf{z}_1, \mu \mathbf{z}_2) = \lambda \bar{\mu} \mathcal{B}(\mathbf{z}_1, \mathbf{z}_2)$$

for $\mathbf{z}_1, \mathbf{z}_2 \in V_{\mathbb{C}}$ and $\lambda, \mu \in \mathbb{C}$. Then the action of the Lorentz group $O_{\mathcal{B}}(V)$ naturally extends to $V_{\mathbb{C}}$. Again, two vectors $\mathbf{z}_1, \mathbf{z}_2 \in V_{\mathbb{C}}$ are *orthogonal* if $\mathcal{B}(\mathbf{z}_1, \mathbf{z}_2) = 0$.

Remark 3.1. The bilinear form $\mathcal{B}$ associated to V can also be viewed as the restriction of a bilinear form on $V_{\mathbb{C}}$, as in [26, Chapter III]. This is algebraically more natural, while a sesquilinear form is geometrically more natural.

A non-zero vector $\mathbf{z} \in V_{\mathbb{C}}$ is an eigenvector of a Lorentz transformation $\phi \in O_{\mathcal{B}}(V)$ if $\phi(\mathbf{z}) = \lambda \mathbf{z}$ for some $\lambda \in \mathbb{C}$. An eigenvalue λ is *unimodular* if $|\lambda| = 1$, in which case a λ-eigenvector $\mathbf{z}$ is also said to be an unimodular eigenvector of ϕ. If $\mathbf{x} \in V$ is an eigenvector of ϕ, we call $\widehat{\mathbf{x}} \in \mathbb{P}V$ an *eigendirection* of ϕ.

The following proposition gathers some basic facts about eigenvectors.

Proposition 3.2. *Let ϕ be a Lorentz transformation and* $\mathbf{z}$ *be a λ-eigenvector of ϕ, then*

(i) $\bar{\mathbf{z}}$ *is an eigenvector of ϕ with eigenvalue $\bar{\lambda}$,*

(ii) $\mathbf{z}$ *is an eigenvector of w^k, $k \in \mathbb{N}$, with eigenvalue λ^k,*

(iii) $\mathbf{z}$ *is an eigenvector of w^{-1} with eigenvalue λ^{-1},*

(iv) *let $\varphi \in O_{\mathcal{B}}(V)$, then $\varphi(\mathbf{z})$ is an eigenvector of $\varphi\phi\varphi^{-1}$ with eigenvalue λ.*

Proposition 3.3. *Let $\mathbf{z}_1$ and $\mathbf{z}_2$ be λ- and μ-eigenvectors of $\phi \in O_{\mathcal{B}}(V)$, respectively. If $\lambda\bar{\mu} \neq 1$, then $\mathcal{B}(\mathbf{z}_1, \mathbf{z}_2) = 0$.*

Proof. Since w preserves the bilinear form, we have

$$\mathcal{B}(\mathbf{z}_1, \mathbf{z}_2) = \mathcal{B}(\phi(\mathbf{z}_1), \phi(\mathbf{z}_2)) = \lambda\bar{\mu}\mathcal{B}(\mathbf{z}_1, \mathbf{z}_2).$$

So $\mathcal{B}(\mathbf{z}_1, \mathbf{z}_2) = 0$ because $\lambda\bar{\mu} \neq 1$. □

In the following propositions, we classify Lorentz transformations into three types. Such a classification is present in many references, often in the language of Möbius transformations or hyperbolic isometries; see for instance [2, Chapter 4, Theorem 1.6], [25, Section 4.7], [17, Proposition 4.5.1] and [27, Section 7.8]. Our formulation is adapted from [26, Chapter III], which deals with Lorentz space and is suitable for our use. See also discussions in [6, Section 3.3] for a geometric insight.

Proposition 3.4 ([26, Section 3.7]). *Lorentz transformations are partitionned into three types*:

- ***elliptic transformations*** *are diagonalizable, and have only unimodular eigenvalues*;
- ***parabolic transformations*** *have only unimodular eigenvalues, but are not diagonalizable*;
- ***hyperbolic transformations*** *are diagonalizable and have exactly one pair of simple, real, non-unimodular eigenvalues, namely $\lambda^{\pm 1}$ for some $|\lambda| > 1$.*

Proposition 3.5 ([26, Section 3.7-3.9]). *The two non-unimodular eigendirections of a hyperbolic transformation are light-like, while its unimodular eigenvectors are all space-like.*

Proposition 3.6 ([26, Section 3.10]). *The Jordan form of a parabolic transformation ϕ contains a unique Jordan block of size* 3, *corresponding to the eigenvalue $\varepsilon = 1$ or -1. The $(n-2)$-dimensional* ***real*** *subspace U_ϕ spanned by eigenvectors of ϕ is light-like. The* 1*-dimensional light-like subspace of U_ϕ is an ε-eigendirection. The minimal polynomial $f(x)$ such that $f(\phi)$ annihilates $U_\phi^\perp$ is $(x-\varepsilon)^2$.*

3.2. Spectral interpretation of limit roots. Let $(W,S)_B$ be a Lorentzian Coxeter system. In this section, we consider the limit directions arising through sequences in the form of $(w^k)_{k\in\mathbb{N}}$ for some $w \in W$. We say that w is an *elliptic* (resp. *parabolic, hyperbolic*) element of W if its corresponding transformation $\rho(w)$ is an elliptic (resp. parabolic, hyperbolic) transformation.

Theorem 3.7. *Let $(W,S)_B$ be a Lorentzian Coxeter system, then an element $w \in W$ is of finite order if and only if w is an elliptic Lorentz transformation.*

Proof. Assume that $w^k = e$ for some $k < \infty$. Since the minimal polynomial of w divides $x^k - 1$, its roots are all distinct and unimodular. Hence w is diagonalizable with only unimodular eigenvalues, i.e. w is elliptic. For the sake of contradiction, assume that w is an elliptic element but of infinite order. So the eigenvalues of w are all unimodular but are not all roots of unity. Let the sequence $(w^k)_{k\in\mathbb{N}}$ act on a simple root $\alpha \in \Delta$. We conclude from Kronecker's theorem that α is a limit point of W acting on the Lorentz space. This contradicts the discreteness of the root system [17, Lemma 1.2.5]. □

Consequently, whenever w is of infinite order, it is either parabolic or hyperbolic. Then Theorem 1.1 follows directly from the fact that the set of limit roots equals the limit set of the Coxeter group regarded as a Kleinian group acting on the hyperbolic space [13, Theorem 1.1]. For the relation between fixed points and limit sets of Kleinian groups; see for instance [22, Lemma 2.4.1(ii)].

However, we provide here a different proof for the following reasons. First, in our proof of Theorem 3.8 and 3.10, the behavior of infinite-order elements will be analysed in detail. This will be useful in the proof of Theorem 1.2. Second, Calabi and Markus [5] proved that a group acting discretely on the space-like part of $\mathbb{P}V$ must be finite. In fact, as shown by Theorem 1.2, limit directions arising from an arbitrary space-like base point is not necessarily light-like. Therefore, one should regard the fact that limit roots lie on the light cone as an unusual phenomenon. As mentioned in the introduction, our proof tries to rely minimally on hyperbolic geometry, in the hope of a deeper insight and further generalizations to non-Lorentzian infinite Coxeter systems. In particular, concepts involved in the statement of Theorem 1.1 are all well-defined for general infinite Coxeter systems. The non-Lorentzian cases are discussed in detail in Section 3.3.

For a Lorentzian Coxeter group W, we denote by W_∞ the set of elements of infinite order, by W_{par} the set of parabolic elements, by and W_{hyp} the set of hyperbolic elements. Then $W_\infty = W_{\text{par}} \sqcup W_{\text{hyp}}$. Given an element $w \in W_\infty$, the $(n-2)$-dimensional *real* subspace spanned by unimodular eigenvectors of w is called the *unimodular subspace* of w, and denoted by U_w.

Theorem 3.8. *The light-like eigendirection* $\widehat{\mathbf{x}}$ *of a parabolic element* $w \in W_{\mathrm{par}}$ *is a limit root of* W.

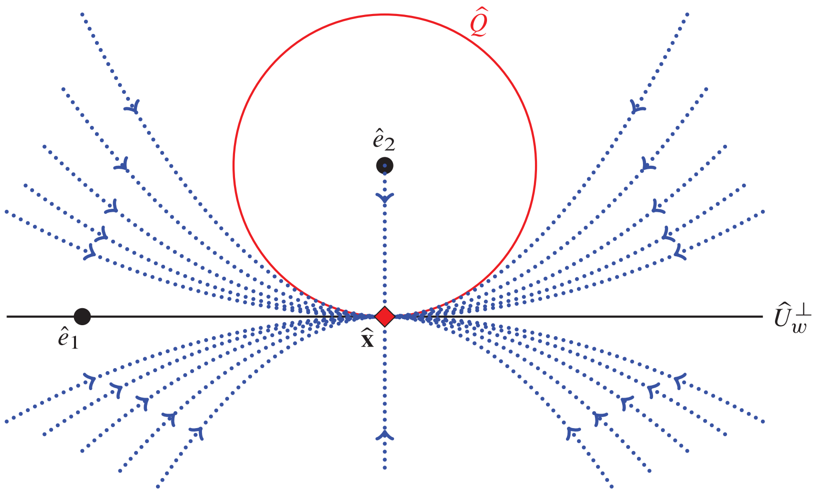


Figure 6. The dynamics of a parabolic element as described in the proof of Theorem 3.8.

Proof. Let $\mathbf{x}$ be the light-like eigenvector of w with eigenvalue $\varepsilon = 1$ or -1. By Proposition 3.6, there are two vectors $\mathbf{e}_1, \mathbf{e}_2 \notin U_w$ such that

$$w(\mathbf{e}_2) - \varepsilon\mathbf{e}_2 = \mathbf{e}_1, \quad w(\mathbf{e}_1) - \varepsilon\mathbf{e}_1 = \mathbf{x}.$$

By Proposition 3.6, we know that $\mathbf{x} \in U_w^{\perp} \cap U_w$, $\mathbf{e}_1 \in U_w^{\perp}$, while $\mathbf{e}_2 \notin U_w^{\perp} + U_w$. Let $\mathbf{y} \in V$ be a real vector, it can be decomposed into

$$\mathbf{y} = a\mathbf{x} + b\mathbf{e}_1 + c\mathbf{e}_2 + \mathbf{y}^{\circ} \tag{2}$$

where $\mathbf{y}^{\circ} \in U_w \setminus U_w^{\perp}$, or 0 if $\mathbf{x}$, $\mathbf{e}_1$ and $\mathbf{e}_2$ span V. Under the action of w^k, we have

$$w^k(\mathbf{y}) = a_k\mathbf{x} + b_k\mathbf{e}_1 + c_k\mathbf{e}_2 + w^k(\mathbf{y}^{\circ}). \tag{3}$$

where the coefficients

$$a_k = a\varepsilon^k + bk\varepsilon^{k-1} + c\tbinom{k}{2}\varepsilon^{k-2},$$

$$b_k = b\varepsilon^k + ck\varepsilon^{k-1},$$

$$c_k = c\varepsilon^k.$$

The term $w^k(\mathbf{y}^\circ)$ has bounded norm since $\mathbf{y}^\circ$ is contained in the unimodular subspace U_w. As long as $\mathbf{y} \notin U_w$, we have $b \neq 0$ or $c \neq 0$, then $b_k = o(a_k)$ and $c_k = o(b_k)$, i.e. a_k dominates b_k and c_k, and b_k dominates c_k, as k tends to infinity. Consequently, the direction of the sequence $(w^k(\mathbf{y}))_{k\in\mathbb{N}}$ converges to the direction of $\mathbf{x}$. That is, $\hat{\mathbf{x}}$ is a limit direction arising from $\hat{\mathbf{y}}$ through the sequence $(w^k)_{k\in\mathbb{N}}$.

Since Δ spans the vector space V, there is a simple root $\alpha \notin U_w$. Using α as the base point, we conclude that $\hat{\mathbf{x}}$ is a limit root. The dynamics of a parabolic element is illustrated in Figure 6. □

Remark 3.9. We see in the proof that the sequence $(w^k(\mathbf{y}))_{k\in\mathbb{N}}$ is *asymptotically tangent* to $U_w^\perp$ as k tends to infinity, in the sense that the coefficient for the component $\mathbf{e}_1 \in U_w^\perp$ dominates the components that are not in $U_w^\perp$ ($\mathbf{e}_2$ and $\mathbf{y}^\circ$).

Theorem 3.10. *Let $w \in W_{\mathrm{hyp}}$ be a hyperbolic element. The two light-like eigendirections of w are limit roots and the projective unimodular subspace $\widehat{U}_w$ is contained in the set of limit directions of W.*

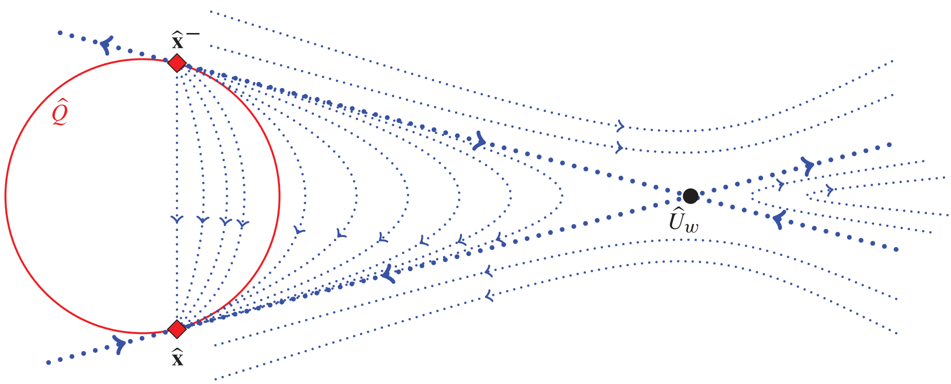


Figure 7. The dynamics of a hyperbolic element as described in the proof of Theorem 3.10.

Proof. From Proposition 3.4, the element w possesses a light-like eigenvector $\mathbf{x}$ which is non-unimodular. By replacing w with w^{-1} if necessary, we may assume that the eigenvalue λ corresponding to $\mathbf{x}$ is greater than 1 in absolute value. Let $\mathbf{x}^-$ be an eigenvector of w with eigenvalue λ^{-1}. Since w is diagonalizable, there exists an eigenbasis of $V_\mathbb{C}$ consisting of $\mathbf{x}$, $\mathbf{x}^-$ and $n-2$ unimodular eigenvectors of w. Then any real vector $\mathbf{y} \in V$ can be decomposed into

$$\mathbf{y} = a^+\mathbf{x} + a^-\mathbf{x}^- + \mathbf{y}^\circ, \tag{4}$$

where $\mathbf{y}^\circ \in U_w$ is orthogonal to $\mathbf{x}$ and $\mathbf{x}^-$. By Proposition 3.2ii, we have

$$w^k(\mathbf{y}) = a^+\lambda^k\mathbf{x} + a^-\lambda^{-k}\mathbf{x}^- + w^k(\mathbf{y}^\circ).$$

Using $\mathbf{y}$ as the base point, three cases are possible.

Case 1. $a^+ \neq 0$, so $\mathbf{y} \notin H_{\mathbf{x}^-}$. The coefficient $|a^+\lambda^k|$ diverges to infinity while the coefficient $|a^-\lambda^{-k}|$ tends to 0. Besides, the term $w^k(\mathbf{y}^\circ)$ has a bounded norm since $\mathbf{y}^\circ \in U_w$. Therefore, as k tends to infinity, the direction of the sequence $(w^k(\mathbf{y}))_{k\in\mathbb{N}}$ converges to the direction of $\mathbf{x}$. So $\widehat{\mathbf{x}}$ is a limit direction arising from $\widehat{\mathbf{y}}$ through the sequence $(w^k)_{k\in\mathbb{N}}$.

Case 2. $a^+ = 0$ and $\mathbf{y}^\circ \neq 0$, so $\mathbf{y} \in H_{\mathbf{x}^-}$ but $\widehat{\mathbf{y}} \neq \widehat{\mathbf{x}}^-$. Again, the coefficient $|a^-\lambda^{-k}|$ tends to 0, so $\mathbf{y}^\circ \in U_w$ is an accumulation point of the set $\{w^k(\mathbf{y}) \mid k \in \mathbb{N}\}$ by Kronecker's theorem. Consequently, $\widehat{\mathbf{y}}^\circ \in U_w$ is a limit direction arising from $\widehat{\mathbf{y}}$ through some subsequence of $(w^k)_{k\in\mathbb{N}}$.

Case 3. $a^+ = 0$, $\mathbf{y}^\circ = 0$ and $a^- \neq 0$, so $\widehat{\mathbf{y}} = \widehat{\mathbf{x}}^-$. The sequence of vectors $(w^k(\mathbf{y}))$ converges to 0, while the direction of $w^k \cdot \widehat{\mathbf{y}}$ remains $\widehat{\mathbf{x}}^-$ for all $k \in \mathbb{N}$. The sequence $(w^k \cdot \widehat{\mathbf{x}})$ visits only one point in $\mathbb{P}V$, so no limit direction arises.

Since Δ spans the vector space V, there is a simple root $\alpha \notin H_{\mathbf{x}^-}$. Using α as the base point, we conclude from Case 1 that $\widehat{\mathbf{x}}$ is a limit root. The dynamics of a hyperbolic element is illustrated in Figure 7. □

For a Lorentzian Coxeter system $(W,S)_B$, let E_{par} (resp. E_{hyp}) be the set of light-like eigendirections of elements in W_{par} (resp. W_{hyp}). We now prove the following theorem, which is equivalent to Theorem 1.1 since $E_\infty = E_{\text{hyp}} \sqcup E_{\text{par}}$. It can be derived from the minimality of E_Φ under the action of W [11, Theorem 3.1(b)]. We provide here a self-contained proof using the results in Section 2.3.

Theorem 3.11. *For a Lorentzian Coxeter system $(W,S)_B$, the set of light-like eigenvectors E_{par} (if not empty) and E_{hyp} are dense in the set of limit roots E_Φ.*

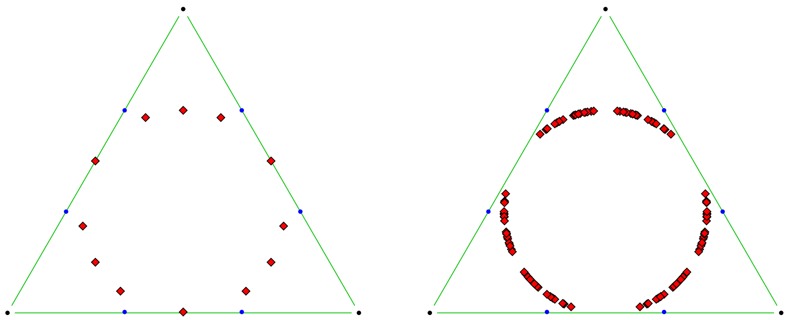
Proof. We first prove that E_{hyp} is not empty. Since the group is infinite, the set of limit root E_Φ is not empty. Let $\widehat{\mathbf{x}}$ be a limit root, since Δ spans V, there is a simple root $\alpha \in \Delta$ such that $\mathcal{B}(\alpha, \mathbf{x}) \neq 0$, and the reflection in α gives a limit root $\widehat{\mathbf{y}}$ different from $\widehat{\mathbf{x}}$. As in the proof of Theorem 2.9, we may take two projective roots $\hat{\alpha}$ and $\hat{\beta}$ respectively close to $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$, such that the segment $[\hat{\alpha}, \hat{\beta}]$ intersect the light cone at two points. Then the product of the reflections in α and β give an hyperbolic element in W.

Let $\hat{\gamma} \in E_\Phi$ be a limit root obtained from an injective sequence $(\hat{\gamma}_k)_{i\in\mathbb{N}}$ of projective roots. By passing to a subsequence, we may assume that $\hat{\gamma}$ is obtained from an injective sequence $(g_k(\alpha))_{i\in\mathbb{N}}$, where α is a fixed simple root in Δ and $g_k \in W$.

Let $\widehat{\mathbf{z}} \in E_{\text{hyp}}$ be a light-like eigendirection of a hyperbolic element $w \in W_{\text{hyp}}$. By Proposition 3.2iv, $g_k \cdot \widehat{\mathbf{z}}$ is a light-like eigendirection of the hyperbolic element $g_k w g_k^{-1}$. So the sequence $(g_k \cdot \widehat{\mathbf{z}})_{k\in\mathbb{N}}$ is a sequence of limit roots in E_{hyp}. By compactness, we may assume that $(g_k \cdot \widehat{\mathbf{z}})$ converges. By Theorem 2.6, its limit is a limit root.

The last step consists of applying Corollary 2.7 to the sequence $(g_k \cdot \widehat{\mathbf{z}})_{k\in\mathbb{N}}$ to prove that $\hat{\gamma}$ is the limit. For this, it remains to prove that the sequence contains infinitely many distinct points. By [12, Proposition 2.15] and [7, Remark 2.3], we may assume that the Lorentzian Coxeter system $(W, S)_B$ is irreducible, so W acts irreducibly on the representation space V; see [15, Proposition 6.3] and [29, Lemma 14]. Using this irreducible action, we can find a basis $\mathbf{b}_1, \dots, \mathbf{b}_n$ of V in the orbit $W(\mathbf{z})$. By the injectivity of the sequence g_k, there exists a basis vector $\mathbf{b}_i$ such that $(g_k \cdot \widehat{\mathbf{b}}_i)_{k\in\mathbb{N}}$ contains an injective subsequence. Taking this $\mathbf{b}_i$ as the vector $\mathbf{z}$ above finishes the proof.

The same arguments work, *mutatis mutandis,* for parabolic elements. □



(a) The 12 light-like eigendirections of parabolic elements of length smaller or equal to 6.

(b) The 126 light-like eigendirections of hyperbolic elements of length smaller or equal to 6.

Figure 8. Light-like eigendirections of some elements of the rank-3 universal Coxeter group of affine type, seen in the affine space spanned by the simple roots.

Examples of light-like eigendirections of parabolic and hyperbolic elements are illustrated in Figure 8 for the rank 3 universal Coxeter group of affine type. Observe that their number and distribution are quite different.

Hosaka [14] proves that, if a Coxeter group acts geometrically on a CAT(0) space (e.g. Davis complex), then the limit points of W arising through $(w^k)_{k\in\mathbb{N}}$, $w \in W_\infty$, is dense in the boundary of X. The similarity between Hosaka's result and Theorem 1.1 gives another motivation for further investigations on the relation between limit roots and the boundaries of Coxeter groups, as we discussed in Section 2.4.

3.3. Non-Lorentzian Coxeter systems. The density of E_{hyp} in E_Φ can also be derived from a result of Conze and Guivarc'h [8]. In that paper, a linear transformation is said to be *proximal* if there exists a *simple* eigenvalue which is strictly greater than any other eigenvalue in absolute value. This is the case for hyperbolic transformations of Lorentzian Coxeter systems. An eigenvector with this eigenvalue is called *dominant eigenvector*, so E_{hyp} is in fact the set of dominant eigendirections. By [8, Proposition 2.4], $\overline{E_{\mathrm{hyp}}}$ is the only minimal set of the action of W. Then by [11, Theorem 3.1(b)], this set is E_Φ. For a non-Lorentzian infinite Coxeter system (W, S), it remains true that the set of dominant eigendirections is dense in the set of limit roots. It is proved in [17, Section 6.5] that an element $w \in W$ is proximal if it is not contained in any proper parabolic subgroup of W.

However, an arbitrary element w of infinite order is not necessarily proximal, even if w has non-unimodular eigenvalues. Let λ denotes the eigenvalue of w with largest absolute value, it is possible that the geometric multiplicity of λ is not 1. The real subspace U spanned by λ-eigenvectors of w is *totally isotropic*, meaning that $\mathcal{B}(\mathbf{x}, \mathbf{y}) = 0$ for any $\mathbf{x}, \mathbf{y} \in U$. Then any direction $\hat{\mathbf{x}}$ in $\widehat{U}$ is an isotropic limit direction, but only those in $\mathsf{conv}(\Delta)$ are limit roots. Therefore, Theorem 1.1 and 2.6 do not generalize to other infinite Coxeter systems.

Example 3.12. In Figure 9, we show an example inspired by [12, Example 5.8] and [10, Example 7.12 and 9.18]. It is the Coxeter graph of an irreducible Coxeter group, associated with a non-degenerate bilinear form of signature $(3, 2)$. The element $s_1 s_2 s_4 s_5$ is of infinite order with a simple eigenvalue 1, and two non-unimodular eigenvalues $7 \pm 4\sqrt{3}$ each of multiplicity 2. Any direction in the 2-dimensional $(7 + 4\sqrt{3})$-eigenspace is a limit direction.

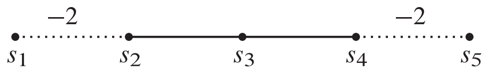


Figure 9. The Coxeter graph of a non-Lorentzian irreducible Coxeter group for which the limit roots are not the only limit directions on the isotropic cone.

4. Coxeter arrangement and limit directions

In this section, we characterize the set of limit directions of a Lorentzian Coxeter system in terms of the Coxeter arrangement in projective space.

4.1. Projective Lorentzian Coxeter arrangement. Let $(W, S)_B$ be a Lorentzian Coxeter system and (Φ, Δ) be the associated root system. The linear subspaces H_γ orthogonal to a positive root $\gamma \in \Phi^+$ is time-like and fixed by the reflection σ_γ. The *projective Coxeter arrangement* is the set of projective subspaces

$$\mathcal{H} = \{\widehat{H}_\gamma \mid \gamma \in \Phi^+\}.$$

Clearly, $\mathcal{H}$ is invariant under the action of W. For a linear subspace U of V, the set

$$\mathcal{H}^U = \{\widehat{H} \cap \widehat{U} \mid \widehat{H} \in \mathcal{H}\}$$

is a projective hyperplane arrangement in $\widehat{U}$. The connected components of the complement of a projective arrangement are called *chambers*. Let $\mathcal{I}$ be the set of non-empty intersections of the hyperplanes in $\mathcal{H}$, including the projective hyperplanes themselves. Then the chambers of $\mathcal{H}$, together with the chambers of $\mathcal{H}^I$ for all $I \in \mathcal{I}$, are called *cells* of $\mathcal{H}$. The projective space $\mathbb{P}V$ is therefore decomposed into cells, and we denote the set of cells by Σ.

For two cells $C, C' \in \Sigma$, we say that C' is a *face* of C if $C' \in \overline{C}$, and write $C' \leq C$. This defines a partial order on Σ. The *support* of a cell C is defined as

$$\mathsf{supp}(C) = \bigcap_{\substack{\gamma \in \Phi^+ \\ C \in \widehat{H}_\gamma}} \widehat{H}_\gamma,$$

while the support of a chamber of $\mathcal{H}$ is $\mathbb{P}V$. The *dimension* of a cell is defined as the dimension of its support. The *codimension* of a cell is defined similarly. Cells of dimension 0 are called *vertices*. Cells of positive dimensions are open in their supports. A cell is said to be *space-like* (resp. *light-like*, *time-like*) if its support is a space-like (resp. light-like, time-like) projective subspace. So chambers of $\mathcal{H}$ are time-like cells of codimension 0. Chambers of $\mathcal{H}^H$ for $H \in \mathcal{H}$ are called *panels*, and they are time-like cells of codimension 1.

For any chamber C of $\mathcal{H}$ that contains light-like directions, the *projective Tits cone* $\mathcal{T}$ is the union of the orbit $W \cdot C$. $\mathcal{T}$ is invariant under the action of W. In the Lorentzian case, $\mathcal{T}$ is the projective cone over the set of weights Ω, and contains the projective light cone $\widehat{Q}$ [23, Corollary 1.3].

Remark 4.1. A Lorentzian hyperplane arrangement is infinite and not discrete. Unlike finite hyperplane arrangements, the union of hyperplanes in a Lorentzian hyperplane arrangement is in general not a closed set.

4.2. Unimodular subspaces. Let $\alpha, \beta \in \Phi$ be two positive roots. If the segment $[\hat{\alpha}, \hat{\beta}]$ intersect the projective light cone $\widehat{Q}$ transversally (i.e. $\mathcal{B}(\alpha, \beta) < -1$), then $\sigma_\alpha \sigma_\beta$ is a hyperbolic transformation. If the segment $[\hat{\alpha}, \hat{\beta}]$ is tangent to $\widehat{Q}$ (i.e. $\mathcal{B}(\alpha, \beta) = -1$), then $\sigma_\alpha \sigma_\beta$ is a parabolic transformation. In either case, we know from [12, Section 4] that the limit roots of the subgroup generated by σ_α and σ_β are the points in $\widehat{Q} \cap [\hat{\alpha}, \hat{\beta}]$. By Theorem 3.10, these are the light-like eigendirections of $\sigma_\alpha \sigma_\beta \in W_\infty$. The unimodular subspace of $\sigma_\alpha \sigma_\beta$ is clearly $H_\alpha \cap H_\beta$. We define

$$\mathcal{L}_{\text{hyp}} = \bigcup_{\substack{\alpha,\beta \in \Phi^+ \\ \mathcal{B}(\alpha,\beta) < -1}} \widehat{H}_\alpha \cap \widehat{H}_\beta, \quad \mathcal{L}_{\text{par}} = \bigcup_{\substack{\alpha,\beta \in \Phi^+ \\ \mathcal{B}(\alpha,\beta) = -1}} \widehat{H}_\alpha \cap \widehat{H}_\beta.$$

Furthermore, we define the unions of projective unimodular subspaces for parabolic, hyperbolic, and infinite-order elements

$$\mathcal{U}_{\text{par}} = \bigcup_{w \in W_{\text{par}}} \widehat{U}_w, \quad \mathcal{U}_{\text{hyp}} = \bigcup_{w \in W_{\text{hyp}}} \widehat{U}_w, \quad \mathcal{U} = \mathcal{U}_{\text{par}} \cup \mathcal{U}_{\text{hyp}} = \bigcup_{w \in W_\infty} \widehat{U}_w.$$

We have clearly $\mathcal{L}_{\text{hyp}} \subset \mathcal{U}_{\text{hyp}}$ and $\mathcal{L}_{\text{par}} \subset \mathcal{U}_{\text{par}}$. The following theorem concerns a reversed inclusion.

Theorem 4.2. *The projective unimodular subspace $\widehat{U}_w$ of an element of infinite order $w \in W_\infty$ is included in $\overline{\mathcal{L}_{\text{hyp}}}$. In other words,*

$$\mathcal{U} \subseteq \overline{\mathcal{L}_{\text{hyp}}}.$$

Proof. We set a natural map Λ from the set of *distinct* pairs of light-like directions to the set of codimension-2 projective subspaces of $\widehat{V}$, both equipped with the induced Grassmanian topology. This map sends a pair of *distinct* light-like directions $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) \in \widehat{Q}^2$ to the codimension-2 projective subspace $\Lambda(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) := \widehat{H}_{\mathbf{x}} \cap \widehat{H}_{\mathbf{y}}$ and a direct verification shows that this map is continuous. For a hyperbolic element $w \in W_{\text{hyp}}$, let $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}})$ be its two non-unimodular eigendirections, then $\Lambda(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) = \widehat{U}_w$. Since $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ are limit roots, we can find two sequences of roots $(\alpha_k)_{k\in\mathbb{N}}$ and $(\beta_k)_{k\in\mathbb{N}}$ such that $\hat{\alpha}_k$ converges to $\widehat{\mathbf{x}}$ and $\hat{\beta}_k$ converges to $\widehat{\mathbf{y}}$. The sequence of segments $[\hat{\alpha}_k, \hat{\beta}_k]$ eventually intersect the projective light cone $\widehat{Q}$ at two limit roots, say $\widehat{\mathbf{x}}_k$ and $\widehat{\mathbf{y}}_k$. These two limit roots determine a unimodular projective subspace $\widehat{U}_k = \Lambda(\widehat{\mathbf{x}}_k, \widehat{\mathbf{y}}_k) \in \mathcal{L}_{\text{hyp}}$. The two sequences of limit roots $(\widehat{\mathbf{x}}_k)_{k\in\mathbb{N}}$ and $(\widehat{\mathbf{y}}_k)_{k\in\mathbb{N}}$ converge to $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ respectively. By the continuity of Λ, $\widehat{U}_k$ converges to $\widehat{U}_w$ as k tends to infinity. See Figure 10 for an illustration.

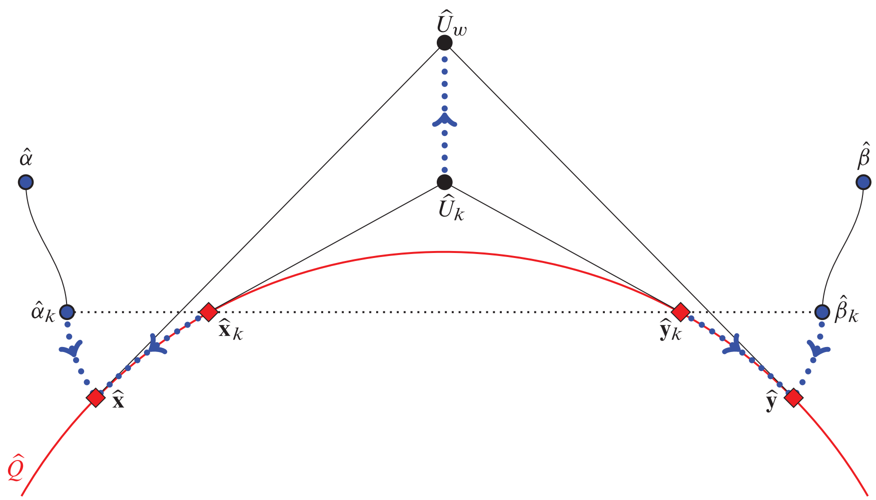


Figure 10. Illustration for the proof of Theorem 4.2 in the case of hyperbolic elements at step k.

For a parabolic element $w \in W_{\text{par}}$, let $\widehat{\mathbf{x}} \in \widehat{U}_w$ be the light-like eigendirection of w, then $\widehat{U}_w + \widehat{U}_w^{\perp} = \widehat{H}_{\mathbf{x}}$ is the codimension-1 hyperplane that is tangent to $\widehat{Q}$ at $\widehat{\mathbf{x}}$. See Figure 11 for an illustration. Let $W_{\mathbf{x}} \ni w$ be the stabilizer subgroup of $\widehat{\mathbf{x}}$. It is generated by reflections in the positive roots on $H_{\mathbf{x}}$ [10, Lemma 1.10]

$$\Phi_{\mathbf{x}} = \{\gamma \in \Phi^+ \mid \mathcal{B}(\gamma, \widehat{\mathbf{x}}) = 0\} \subset H_{\mathbf{x}}.$$

Since the restriction of $\mathcal{B}$ on $H_{\mathbf{x}}$ is positive semi-definite with a radical of dimension 1, $W_{\mathbf{x}}$ is an irreducible affine Coxeter group, and $\widehat{\mathbf{x}}$ is the only limit root; see [12, Corollary 2.16]. Furthermore, we claim that $\Phi_{\mathbf{x}} \not\subset U_w$, otherwise we have $U_w^{\perp} \subseteq U_w$ and $H_{\mathbf{x}} \subseteq U_w$, which is not possible.

Since $\widehat{\mathbf{x}} \in \operatorname{conv}(\widehat{\Phi}_{\mathbf{x}})$, we can find two positive roots $\alpha, \beta \in H_{\mathbf{x}}$ on different sides of U^w, so they have opposite signs for the coefficient b in the decomposition (2). Recall that, under the action of $(w^k)_{k\in\mathbb{N}}$, the coefficients b_k dominates c_k in equation (3) as k tends to infinity. Therefore, the sequences $(\hat{\alpha}_k = w^k \cdot \hat{\alpha})_{k\in\mathbb{N}}$ and $(\hat{\beta}_k = w^k \cdot \hat{\beta})_{k\in\mathbb{N}}$ not only converge to $\widehat{\mathbf{x}}$, but are also asymptotically tangent to $\widehat{U}_w^{\perp}$ at $\widehat{\mathbf{x}}$; see Remark 3.9.

Since the simple roots Δ spans V and the Coxeter group W is Lorentzian, there is a simple root δ such that the segment $[\hat{\delta}, \widehat{\mathbf{x}}]$ intersects $\widehat{Q}$ at two points (including $\widehat{\mathbf{x}}$ itself). We now construct two sequences of projective roots $(\hat{\alpha}'_k = w^k \sigma_\alpha \cdot \hat{\delta})_{k\in\mathbb{N}}$ and $(\hat{\beta}'_k = w^k \sigma_\beta \cdot \hat{\delta})_{k\in\mathbb{N}}$. Both sequences converge to $\widehat{\mathbf{x}}$ and are asymptotically tangent to $\widehat{U}_w^{\perp}$ at $\widehat{\mathbf{x}}$. The sequence of segments $[\hat{\alpha}'_k, \hat{\beta}'_k]$ eventually intersect $\widehat{Q}$ at two limit roots, say $\widehat{\mathbf{x}}_k$ and $\widehat{\mathbf{y}}_k$, which determine a unimodular projective subspace $U_k = \Lambda(\widehat{\mathbf{x}}_k, \widehat{\mathbf{y}}_k) \in \mathcal{L}_{\text{hyp}}$. The two sequences of limit roots $(\widehat{\mathbf{x}}_k)_{k\in\mathbb{N}}$ and $(\widehat{\mathbf{y}}_k)_{k\in\mathbb{N}}$ both converge to $\widehat{\mathbf{x}}$, and are asymptotically tangent to $\widehat{U}_w^{\perp}$ at $\widehat{\mathbf{x}}$. So $\widehat{U}_k$ converges to the orthogonal companion of $\widehat{U}_w^{\perp}$, which is $\widehat{U}_w^{\perp\perp} = \widehat{U}_w$, as k tends to infinity. □

4.3. Limit directions. By Theorem 3.10, we have $\mathcal{L}_{\text{hyp}} \subset \mathcal{U}_{\text{hyp}} \subset E_V$. Notably, while limit roots arise from projective roots and projective weights, it is possible for a projective root to be a limit direction, and space-like projective weights are all limit directions.

In this part, we prove the other inclusion of Theorem 1.2, namely that $E_V \subseteq \overline{\mathcal{L}_{\text{hyp}}}$. We will need the following two lemmas.

Lemma 4.3 (Selberg's lemma, [25, Section 7.5]). *Every finitely generated subgroup G of* $\mathrm{GL}(n, \mathbb{C})$ *has a torsion-free normal subgroup of finite index.*

Lemma 4.4. *Let G be a group acting on a vector space X, and H be a subgroup of G of finite index. Then the set of limit points of H is equal to the set of limit points of G.*

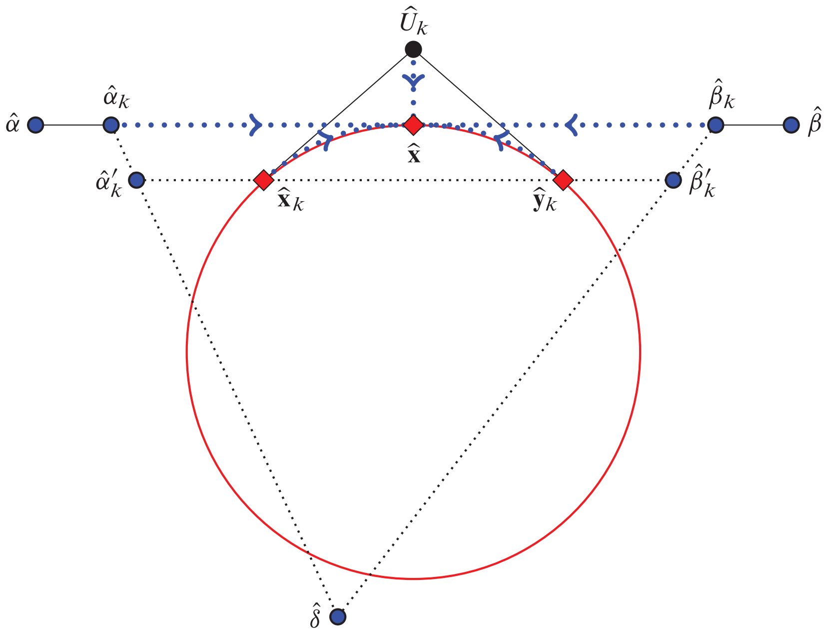


Figure 11. Illustration for the proof of Theorem 4.2 in the case of parabolic elements at step k.

Proof. A limit point of H is a limit point of G. Conversely, let $x \in X$ be a limit point of G arising from a base point $x_0 \in X$ through the sequence $(g_k)_{k\in\mathbb{N}} \in G$. Since H is of finite index, by passing to a subsequence if necessary, we may assume that the sequence $(g_k)_{k\in\mathbb{N}}$ is contained in a single coset of H. That is, there is a sequence $(h_k)_{k\in\mathbb{N}} \in H$ and a fixed element $g \in G$ such that $g_k = h_k g$ for all $k \in \mathbb{N}$. Then x is a limit point of H arising from the point $g(x_0)$ through the sequence $(h_k)_{k\in\mathbb{N}}$. □

Theorem 4.5. *The set of limit direction of a Lorentzian Coxeter system is included in* $\overline{\mathcal{L}_{\text{hyp}}}$*,*

$$E_V \subseteq \overline{\mathcal{L}_{\text{hyp}}}.$$

Proof. Assume that some $\hat{\mathbf{x}} \notin \overline{\mathcal{L}_{\text{hyp}}}$ is a limit direction. By Selberg's lemma, there exists a subgroup of W of finite index whose only element of finite order is the identity. By Lemma 4.4, the limit direction $\hat{\mathbf{x}}$ arises through a sequence of infinite-order elements. By the definition of limit point, for any neighborhood N of $\hat{\mathbf{x}}$, there is infinitely many elements $w \in W_\infty$ such that $(w \cdot N) \cap N \neq \varnothing$.

We claim that $\hat{\mathbf{x}}$ is not in the Tits cone $\mathcal{T}$. If $\hat{\mathbf{x}}$ is in the interior of $\mathcal{T}$, there is a neighborhood N of $\hat{\mathbf{x}}$ such that $(w \cdot N) \cap N = \varnothing$ for any element $w \in W_\infty$; see [1, Exercise 2.90]. If $\hat{\mathbf{x}}$ is on the boundary of $\mathcal{T}$, the stabilizer of $\hat{\mathbf{x}}$ is infinite, so there is an element $w \in W_\infty$ such that $\hat{\mathbf{x}} \in \widehat{U}_w \subset \overline{\mathcal{L}_{\text{hyp}}}$, contradicting our assumption.

Let $C \in \Sigma$ be the cell of $\mathcal{H}$ containing $\hat{\mathbf{x}}$. Since $\hat{\mathbf{x}} \notin \overline{\mathcal{L}_{\text{hyp}}}$, C does not intersect U_w for any $w \in W_\infty$. We now prove that $(w \cdot C) \cap C = \varnothing$ for any element $w \in W_\infty$.

Assume that $w \in W_\infty$ is an element of infinite order such that $(w \cdot C) \cap C \neq \varnothing$. As $\mathcal{H}$ is invariant under the action of W, we must have $w \cdot C = C$. Since $C \cap \widehat{U}_w = \varnothing$, there must be a vertex v of $\overline{C}$ such that $v \notin \widehat{U}_w$. From the proof of Theorem 3.8 and 3.10, we see that the sequence $(w^k \cdot v)_{k \in \mathbb{N}}$ converges to a light-like eigendirection of w. Because C is invariant under the action of w, the vertex v must be itself a light-like eigendirection of w. We then claim that v is the only vertex of C that is not in $\widehat{U}_w$. Otherwise, if $u \notin \widehat{U}_w$ is another vertex of C, then $u \in \widehat{Q}$ and the segment $[u, v]$ is inside the light cone $\widehat{Q}$, so $\hat{\mathbf{x}} \in C$ is in the interior of the Tits cone $\mathcal{T}$, which has been proved to be impossible. Moreover, w must be hyperbolic; otherwise if w is parabolic, we have $C \subset \widehat{U}_w \subset \overline{\mathcal{L}_{\text{hyp}}}$, contradicting our assumption.

To summarize, we have proved that w is hyperbolic, and C is in the projective subspace spanned by the unimodular subspace $\widehat{U}_w$ and a light-like eigendirection v of w. From the proof of Theorem 3.10, we know that C is in the light-like subspace $\widehat{H}_v$. We now prove that this situation is again not possible.

Let W_C be the stabilizer subgroup of C. It is generated by reflections in the positive roots $\Phi_C = \{\gamma \in \Phi^+ \mid C \subset H_\gamma\}$. These roots lie on the orthogonal companion of $\operatorname{supp}(C)$, which is light-like and tangent to $\widehat{Q}$ at v. The restriction of the bilinear form $\mathcal{B}$ on the subspace spanned by Φ_C is positive semi-definite, so W_C is an affine Coxeter group. We then conclude that there is an element $w' \in W_{\text{par}}$ such that $C \in \widehat{U}_{w'} \subset \overline{\mathcal{L}_{\text{hyp}}}$, contradicting our assumption.

We have proved that $(w \cdot C) \cap C = \varnothing$ for all $w \in W_\infty$. Since $\hat{\mathbf{x}} \in C$ and C is open in $\operatorname{supp}(C)$, we have found a neighborhood N of $\hat{\mathbf{x}}$ such that $(w \cdot N) \cap N = \varnothing$ for all $w \in W_\infty$. Therefore, $\hat{\mathbf{x}}$ can not be a limit direction. □

4.4. Open problems on limit directions. We proved that the set E_V of limit directions is located between the set $\mathcal{L}_{\text{hyp}}$ and its closure. In fact, by Theorem 3.10 and Section 4.2, a stronger result can be obtained:

$$\mathcal{U}_{\text{hyp}} \sqcup E_\Phi \subseteq E_V \subseteq \overline{\mathcal{U}_{\text{hyp}}}.$$

Figure 12 shows some unimodular eigenvectors for infinite order elements for the same geometric Coxeter system as in Figure 2. Many of the eigenvectors are not in the intersections of the Coxeter arrangement, but can be approximated by intersections. Interestingly, the eigenvectors seem to obey certain linear dependences which are not present in the hyperplane arrangement. For example, the unimodular eigenspaces of the elements $ststu, stu, sutu, tusu, tsu$ and $tstsu$ lie on a hyperplane which is not a reflecting hyperplane. It would be interesting to study these linear dependences of unimodular eigenvectors in relation with the structure of the Coxeter group.

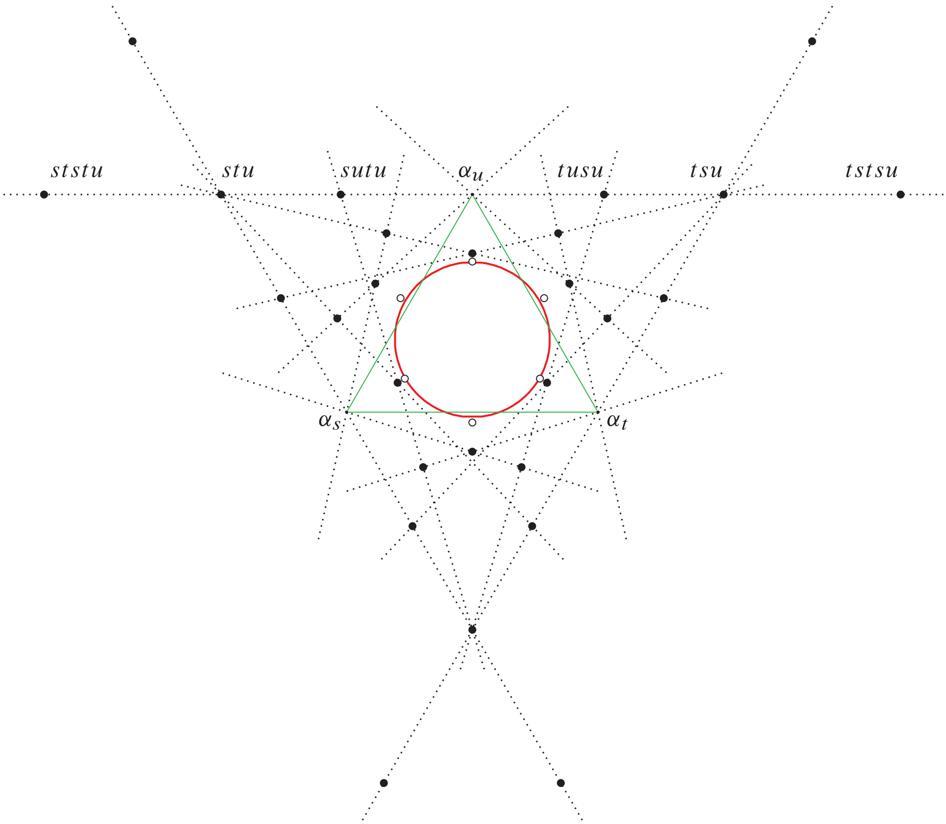


Figure 12. The affine representation of some space-like limit directions of the universal Coxeter group of rank 3 associated with a bilinear form where $c_{ij} = 1.1$ whenever $i \neq j$, same as in Figure 2. The generators of the Coxeter group are s, t and u, while α_s, α_t and α_u in the figure are the corresponding simple roots. The dots are unimodular eigenvectors of hyperbolic elements of length ≤ 5. The six eigenvectors marked in white are also weights. Some unimodular eigenspaces are labeled by the corresponding hyperbolic elements. The dotted lines show some linear dependences between limit directions verified with Sage.

Problem 4.6. *Prove or disprove the following equalities*:

$$E_V = \mathcal{U}_{\text{hyp}} \sqcup E_\Phi, \tag{5}$$

$$E_V = \overline{\mathcal{U}_{\text{hyp}}}. \tag{6}$$

Only one of the equality may be true. In fact, in the case where $E_\Phi = \widehat{Q}$ (see for instance Figure 5 and Figure 8 of [12] and Figure 8 of the present paper), $\mathcal{U}_{\text{hyp}}$ is the union of countably many codimension-2 subspaces, while $\overline{\mathcal{U}_{\text{hyp}}}$ consists of all space-like and light-like directions, so $\mathcal{U}_{\text{hyp}}$ is a proper subset of $\overline{\mathcal{U}_{\text{hyp}}}$. Then equation (6) would imply counterintuitively that every non-time-like direction is a limit direction.

In the boundary $\partial(\mathcal{U}_{\text{hyp}}) = \overline{\mathcal{U}_{\text{hyp}}} \setminus \mathcal{U}_{\text{hyp}}$, we know that the limit roots $E_\Phi \subset \partial(\mathcal{U}_{\text{hyp}})$ are limit directions. Since every limit direction in $\mathcal{U}_{\text{hyp}}$ arises through a sequence of the form $(w^k)_{k\in\mathbb{N}}$ for $w \in W_\infty$, equation (5) is equivalent to the following conjecture

Conjecture 4.7. *Every space-like limit direction arises through a sequence of the form $(w^k)_{k\in\mathbb{N}}$ for $w \in W_\infty$.*

To prove equation (6), it suffices to prove that $E_{\mathbf{x}}$ is closed. Let $E_{\mathbf{x}}$ be the set of limit directions arising from a fixed base point $\widehat{\mathbf{x}} \in \mathbb{P}V$. Since $E_{\mathbf{x}}$ is the set of accumulation points of $W \cdot \widehat{\mathbf{x}}$, it is clear that E_V is closed and invariant under the action of W. In particular, $E_{\mathbf{x}} = E_{\mathbf{y}}$ if $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ lie in a same orbit of W, i.e. $\widehat{\mathbf{x}} = w \cdot \widehat{\mathbf{y}}$ for some $w \in W$. However, the set of limit directions, being the infinite union

$$E_V = \bigcup_{\widehat{\mathbf{x}}\in\mathbb{P}V} E_{\mathbf{x}} = \bigcup_{\widehat{\mathbf{x}}\in\mathbb{P}V/G} E_{\mathbf{x}},$$

may not be closed in general.

Since the set of limit roots E_Φ is a minimal set of W, we have $E_\Phi \subseteq E_{\mathbf{x}}$ for all $\widehat{\mathbf{x}} \in \mathbb{P}V$. In Section 2, we have seen that $E_{\mathbf{x}} = E_\Phi$ if $\widehat{\mathbf{x}}$ is a time-like, light-like, projective root or a projective weight. Using the argument in the proof of Theorem 4.5, we can prove that $E_{\mathbf{x}} = E_\Phi$ if $\widehat{\mathbf{x}}$ is in the Tits cone $\mathcal{T}$. Define the set $F_\Phi = \{\widehat{\mathbf{x}} \in \mathbb{P}V \mid E_{\mathbf{x}} = E_\Phi\}$.

Problem 4.8. *Is F_Φ a closed set*?

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