

Full groups of Cuntz–Krieger algebras and Higman–Thompson groups

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Abstract. In this paper, we will study representations of the continuous full group Γ_A of a one-sided topological Markov shift (X_A, σ_A) for an irreducible matrix A with entries in $\{0, 1\}$ as a generalization of Higman–Thompson groups V_N , $1 < N \in \mathbb{N}$. We will show that the group Γ_A can be represented as a group Γ_A^{tab} of matrices, called A -adic tables, with entries in admissible words of the shift space X_A , and a group Γ_A^{PL} of right continuous piecewise linear functions, called A -adic PL functions, on $[0, 1]$ with finite singularities.

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Contents

1	Introduction	500
2	Preliminaries	503
3	Realization of \mathcal{O}_A on $L^2([0, 1])$	506
4	A -adic PL functions	510
5	A -adic Tables	513
6	Isomorphisms among Γ_A , Γ_A^{tab} and Γ_A^{PL}	519
7	A realization of Γ_A as A -adic PL functions	521
8	Generalizations of other Thompson groups	527
	References	530

1. Introduction

In 1960's, R. J. Thompson has initiated a study of finitely presented simple infinite groups. He has discovered first two such groups in [25]. They are now known as the groups V_2 and T_2 . G. Higman has generalized the group V_2 to infinite family of finitely presented infinite groups. One of such families are groups written V_N , $1 < N \in \mathbb{N}$ which are called the Higman–Thompson groups. They are finitely presented and their commutator subgroups are simple. Their abelianizations are trivial if N is even, and \mathbb{Z}_2 if N is odd. K. S. Brown has extended the groups V_N to triplets of infinite families $F_N \subset T_N \subset V_N$, $1 < N \in \mathbb{N}$, and proved that each of the groups is finitely presented ([1]). The Higman–Thompson group V_N is known to be represented as the group of right continuous piecewise linear functions $f: [0, 1) \rightarrow [0, 1)$ having finitely many singularities such that all singularities of f are in $\mathbb{Z}[\frac{1}{N}]$, the derivative of f at any non-singular point is N^k for some $k \in \mathbb{Z}$ and $f(\mathbb{Z}[\frac{1}{N}] \cap [0, 1)) = \mathbb{Z}[\frac{1}{N}] \cap [0, 1)$ (cf. [25]). See [2] for general reference on these groups.

V. Nekrashevych [20] has shown that the Higman–Thompson group V_N appears as a certain subgroup of the unitary group of the Cuntz algebra \mathcal{O}_N . The second named author has observed in [17, Remark 6.3] that the subgroup is nothing but the continuous full group Γ_N of \mathcal{O}_N , which is also realized as the topological full group of the associated groupoid. Such full groups have arisen from a study of orbit equivalence of symbolic dynamics ([8]).

Recently the authors have studied full groups of the Cuntz–Krieger algebras and full groups of the groupoids coming from shifts of finite type. The first named author has studied the normalizer groups of the canonical maximal abelian C^* -subalgebras in the Cuntz–Krieger algebras which are called the continuous full groups from the view point of orbit equivalences of topological Markov shifts and classification of C^* -algebras ([8], [9], etc.), and showed that the continuous full groups are complete invariants for the continuous orbit equivalence classes of the underlying topological Markov shifts ([11], more generally [17]). The second named author has studied the continuous full groups of more general étale groupoids ([15], [16], [17], etc.), and called them the topological full groups of étale groupoids. He has proved that if an étale groupoid is minimal, the topological full group of the groupoid is a complete invariant for the isomorphism class of the groupoid. He has also shown that if a groupoid comes from a shift of finite type, the topological full group is of type F_∞ and in particular finitely presented. He has furthermore obtained that the topological full groups for shifts of finite type are simple if and only if its homology group $H_0(G_A)$ of the groupoid G_A is 2-divisible, and that its commutator subgroups are always simple. We have obtained the following results on the group Γ_A for the topological Markov shift (X_A, σ_A) defined by an irreducible square matrix with entries in $\{0, 1\}$.

Theorem 1.1 ([11], [13], [17]). *Let A and B be irreducible, not any permutation matrices with entries in $\{0, 1\}$. The following conditions are equivalent:*

- (1) *The one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent;*
- (2) *the étale groupoids G_A and G_B are isomorphic;*
- (3) *the groups Γ_A and Γ_B are isomorphic;*
- (4) *the Cuntz–Krieger algebras \mathcal{O}_A and \mathcal{O}_B are isomorphic and $\det(\text{id} - A) = \det(\text{id} - B)$.*

Suppose that A is an $N \times N$ matrix and B is an $M \times M$ matrix. It is well-known that the Cuntz–Krieger algebras \mathcal{O}_A and \mathcal{O}_B are isomorphic if and only if there exists an isomorphism Φ of groups from $\mathbb{Z}^N / (\text{id} - A^t)\mathbb{Z}^N$ to $\mathbb{Z}^M / (\text{id} - B^t)\mathbb{Z}^M$ such that $\Phi(u_A) = u_B$ where u_A and u_B are the classes of the vectors $[1, \dots, 1]$ ([24]). Hence the isomorphism classes of the groups Γ_A are completely classified in terms of the underlying matrices A , so that there exist an infinite family of finitely presented infinite simple groups of the form Γ_A .

In this paper, we will study representations of the group Γ_A for an irreducible matrix A with entries in $\{0, 1\}$ as a generalization of the Higman–Thompson groups V_N , $1 < N \in \mathbb{N}$. The group Γ_A has been originally defined as the group of homeomorphisms τ on the shift space X_A of a topological Markov shift (X_A, σ) such that

$$\sigma_A^{k_\tau(x)}(\tau(x)) = \sigma_A^{l_\tau(x)}(x), \quad x \in X_A, \tag{1.1}$$

for some continuous functions $k_\tau, l_\tau: X_A \rightarrow \mathbb{Z}_+$ (it is written $[\sigma_A]$ in the earlier papers [8], [10]). If the matrix A is the $N \times N$ -matrix whose entries are all 1's, the group Γ_A coincides with the Higman–Thompson group V_N of order N .

We will introduce a notion of A -adic PL (piecewise linear) function which is a right continuous bijective piecewise linear function on the interval $[0, 1)$ associated with the matrix A to represent an element of the group Γ_A . Let $1 < \beta \in \mathbb{R}$ be the Perron–Frobenius eigenvalue of A . Let us denote by $\mathbb{Z}[\frac{1}{\beta}, \beta]$ the set of β -adic rationals which is defined by

$$\mathbb{Z}[\frac{1}{\beta}, \beta] = \left\{ \frac{a_0 + a_1\beta + a_2\beta^2 + \dots + a_n\beta^n}{\beta^n} \mid a_0, a_1, \dots, a_n \in \mathbb{Z} \right\}.$$

Then the group of A -adic PL functions on $[0, 1)$ is realized as a subgroup of right continuous bijective piecewise linear functions f on $[0, 1)$ having finitely many singularities such that all singularities of f are in $\mathbb{Z}[\frac{1}{\beta}, \beta]$, the derivative of f at any non-singular point is β^k for some $k \in \mathbb{Z}$ and $f(\mathbb{Z}[\frac{1}{\beta}, \beta] \cap [0, 1)) \subset \mathbb{Z}[\frac{1}{\beta}, \beta] \cap [0, 1)$. See Section 4 for the precise definition. We also introduce a notion of A -adic table in order to represent elements of Γ_A which is a matrix

$$\begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ \nu(1) & \nu(2) & \cdots & \nu(m) \end{bmatrix}$$

with entries in admissible words $\nu(i), \mu(i), i = 1, \dots, m$ of the one-sided topological Markov shift (X_A, σ_A) satisfying certain properties. We may define an equivalence relation of the A -adic tables, and a product structure in the set Γ_A^{tab} of the equivalence classes of A -adic tables which makes it a group. We will show the following theorem which is a generalization of a well-known result for the Higman–Thompson groups. Assume that A is an irreducible and non permutation matrix with entries in $\{0, 1\}$.

Theorem 1.2 (Theorem 6.3). *There exist canonical isomorphisms of discrete groups among the continuous full group Γ_A , the group Γ_A^{tab} of the equivalence classes of A -adic tables, and the group Γ_A^{PL} of A -adic PL functions on $[0, 1)$, that is*

$$\Gamma_A \cong \Gamma_A^{\text{tab}} \cong \Gamma_A^{\text{PL}}.$$

Let $1 < \beta \in \mathbb{R}$ be the Perron–Frobenius eigenvalue of A . For $\tau \in \Gamma_A$, we put $d_\tau(x) = l_\tau(x) - k_\tau(x)$, $x \in X_A$ for the continuous functions k_τ, l_τ satisfying (1.1). We define the derivative D_τ of τ as a real valued continuous function on X_A :

$$D_\tau(x) = \beta^{d_\tau(x)}, \quad x \in X_A.$$

We know that D_τ satisfies the following law of derivative:

$$D_{\tau_2 \circ \tau_1} = D_{\tau_1} \cdot (D_{\tau_2} \circ \tau_1), \quad D_{\tau^{-1}} = (D_\tau \circ \tau^{-1})^{-1}$$

for $\tau, \tau_1, \tau_2 \in \Gamma_A$ (Proposition 7.9).

The continuous full group Γ_A is isomorphic to the group Γ_A^{PL} of all A -adic PL functions on $[0, 1)$ by the above theorem. We will show that $\tau \in \Gamma_A$ is realized as an A -adic PL function on $[0, 1)$ in the following way, where X_A is endowed with lexicographic order.

Theorem 1.3 (Theorem 7.10). *There exists an order preserving continuous surjection $\rho_A: X_A \rightarrow [0, 1]$ from the shift space X_A of a one-sided topological Markov shift (X_A, σ_A) to the closed interval $[0, 1]$ such that for any element $\tau \in \Gamma_A$, there exists an A -adic PL function f_τ and a finite set $S_\tau \subset X_A$ satisfying the following properties:*

- (i) $f_\tau(\rho_A(x)) = \rho_A(\tau(x))$ for $x \in X_A \setminus S_\tau$,
- (ii) $\frac{df_\tau}{dt}(\rho_A(x)) = D_\tau(x)$ for $x \in X_A \setminus S_\tau$.

In [1], K. S. Brown has extended the groups $V_N, 1 < N \in \mathbb{N}$ to triplets $F_N \subset T_N \subset V_N$ of infinite discrete groups. In the final section, we will generalize the triplet to the triplet $F_A \subset T_A \subset \Gamma_A$ of infinite discrete groups.

Throughout the paper, we denote by \mathbb{N} and by \mathbb{Z}_+ the set of positive integers and the set of nonnegative integers, respectively.

2. Preliminaries

Let $A = [A(i, j)]_{i,j=1}^N$ be an $N \times N$ matrix with entries in $\{0, 1\}$, where $1 < N \in \mathbb{N}$. Then A is said to be irreducible if for every pair $(i, j), i, j = 1, \dots, N$, there exists $k \in \mathbb{N}$ such that $A^k(i, j) \geq 1$. If $A^m = \text{id}$ for some $m \in \mathbb{N}$, then A is called a permutation matrix. Throughout the paper, we assume that A is irreducible and not any permutations. We denote by X_A the shift space

$$X_A = \{(x_n)_{n \in \mathbb{N}} \in \{1, \dots, N\}^{\mathbb{N}} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{N}\}$$

of the right one-sided topological Markov shift for A . It is a compact Hausdorff space in natural product topology. The shift transformation σ_A on X_A defined by $\sigma_A((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}$ is a continuous surjection on X_A . The topological dynamical system (X_A, σ_A) is called the (right one-sided) topological Markov shift for A . Since A is assumed to be irreducible and not any permutations, the shift space X_A is homeomorphic to a Cantor discontinuum.

A word $\mu = (\mu_1, \dots, \mu_m)$ for $\mu_i \in \{1, \dots, N\}$ is said to be admissible for X_A if μ appears somewhere in some element x in X_A . The length of μ is m and denoted by $|\mu|$. We denote by $B_m(X_A)$ the set of all admissible words of length m . For $m = 0$ we denote by $B_0(X_A)$ the empty word \emptyset . We put $B_*(X_A) = \bigcup_{m=0}^{\infty} B_m(X_A)$ the set of admissible words of X_A . For two words $\mu = (\mu_1, \dots, \mu_m) \in B_m(X_A), \nu = (\nu_1, \dots, \nu_n) \in B_n(X_A)$, we denote by $\mu\nu$ the word $(\mu_1, \dots, \mu_m, \nu_1, \dots, \nu_n)$. For a word $\mu = (\mu_1, \dots, \mu_m) \in B_m(X_A)$, the cylinder set $U_\mu \subset X_A$ is defined by

$$U_\mu = \{(x_n)_{n \in \mathbb{N}} \in X_A \mid x_1 = \mu_1, \dots, x_m = \mu_m\}.$$

We put

$$\Gamma_k^+(\mu) = \{(\eta_1, \dots, \eta_k) \in B_k(X_A) \mid (\mu_1, \dots, \mu_m, \eta_1, \dots, \eta_k) \in B_{m+k}(X_A)\},$$

for $k \in \mathbb{Z}_+$, and

$$\Gamma_\infty^+(\mu) = \{(x_n)_{n \in \mathbb{N}} \in X_A \mid (\mu_1, \dots, \mu_m, x_1, x_2, \dots) \in X_A\}$$

and $\Gamma_*^+(\mu) = \bigcup_{k=1}^{\infty} \Gamma_k^+(\mu)$ which is called the follower set of μ . For two words $\mu, \nu \in B_*(X_A)$, we see that $\Gamma_*^+(\mu) = \Gamma_*^+(\nu)$ if and only if $\Gamma_\infty^+(\mu) = \Gamma_\infty^+(\nu)$.

A homeomorphism τ on X_A is said to be a *cylinder map* if there exist two families

$$\begin{aligned} \mu(i) &= (\mu_1(i), \mu_2(i), \dots, \mu_{k_i}(i)) \in B_{k_i}(X_A), & i &= 1, \dots, m, \\ \nu(i) &= (\nu_1(i), \nu_2(i), \dots, \nu_{l_i}(i)) \in B_{l_i}(X_A), & i &= 1, \dots, m, \end{aligned}$$

of words such that

$$U_{v(i)} \cap U_{v(j)} = U_{\mu(i)} \cap U_{\mu(j)} = \emptyset \quad \text{for } i \neq j, \tag{2.1}$$

$$\bigcup_{i=1}^m U_{v(i)} = \bigcup_{i=1}^m U_{\mu(i)} = X_A, \tag{2.2}$$

$$\Gamma_*^+(v(i)) = \Gamma_*^+(\mu(i)) \quad \text{for } i = 1, \dots, m, \tag{2.3}$$

and

$$\begin{aligned} \tau(v_1(i), v_2(i), \dots, v_{l_i}(i), x_{l_i+1}, x_{l_i+2}, \dots) \\ = (\mu_1(i), \mu_2(i), \dots, \mu_{k_i}(i), x_{l_i+1}, x_{l_i+2}, \dots) \end{aligned} \tag{2.4}$$

for $(x_{l_i+1}, x_{l_i+2}, \dots) \in \Gamma_\infty^+(v(i))$ and $i = 1, \dots, m$. It is easy to see that the set of cylinder maps forms a subgroup of the group $\text{Homeo}(X_A)$ of all homeomorphisms on X_A .

Definition 2.1. The *continuous full group* Γ_A of (X_A, σ_A) is defined as the group of cylinder maps on X_A .

For a cylinder map $\tau \in \Gamma_A$, define continuous functions $k_\tau, l_\tau: X_A \rightarrow \mathbb{Z}_+$ by

$$\begin{aligned} k_\tau(x) &= k_i \quad \text{for } x \in U_{\mu(i)}, \\ l_\tau(x) &= l_i \quad \text{for } x \in U_{v(i)}, \end{aligned}$$

so that they satisfy

$$\sigma_A^{k_\tau(x)}(\tau(x)) = \sigma_A^{l_\tau(x)}(x) \quad \text{for all } x \in X_A. \tag{2.5}$$

Conversely a homeomorphism τ satisfying the equality (2.5) for some continuous functions $k_\tau, l_\tau: X_A \rightarrow \mathbb{Z}_+$ gives rise to a cylinder map (cf. ([11])).

The Cuntz–Krieger algebra \mathcal{O}_A for the matrix A has been defined in [5] as the universal C^* -algebra generated by N partial isometries S_1, \dots, S_N subject to the relations:

$$\sum_{j=1}^N S_j S_j^* = 1, \quad S_i^* S_i = \sum_{j=1}^N A(i, j) S_j S_j^*, \quad i = 1, \dots, N. \tag{2.6}$$

The algebra \mathcal{O}_A is known to be the unique C^* -algebra subject to the above relations. For a word $\mu = (\mu_1, \dots, \mu_k)$ with $\mu_i \in \{1, \dots, N\}$, we denote the product $S_{\mu_1} \cdots S_{\mu_k}$ by S_μ . Then $S_\mu \neq 0$ if and only if $\mu \in B_*(X_A)$. Let $C^*(S_\mu S_\mu^*, \mu \in B_*(X_A))$ be the C^* -subalgebra of \mathcal{O}_A generated by the projections of the form $S_\mu S_\mu^*, \mu \in B_*(X_A)$, which we denote by \mathcal{D}_A . It is isomorphic to the commutative C^* -algebra $C(X_A)$ of all complex valued continuous functions on

X_A through the correspondence $S_\mu S_\mu^* \in \mathcal{D}_A \leftrightarrow \chi_\mu \in C(X_A)$ where χ_μ denotes the characteristic function on X_A for the cylinder set U_μ for $\mu \in B_*(X_A)$. We will identify $C(X_A)$ with the subalgebra \mathcal{D}_A of \mathcal{O}_A . It is well-known that the algebra \mathcal{D}_A is maximal abelian in \mathcal{O}_A ([5, Remark 2.18]). We denote by $U(\mathcal{O}_A)$ and $U(\mathcal{D}_A)$ the group of unitaries in \mathcal{O}_A and the group of unitaries in \mathcal{D}_A , respectively. The normalizer $N(\mathcal{O}_A, \mathcal{D}_A)$ of \mathcal{D}_A in \mathcal{O}_A is defined by

$$N(\mathcal{O}_A, \mathcal{D}_A) = \{u \in U(\mathcal{O}_A) \mid u\mathcal{D}_A u^* = \mathcal{D}_A\}.$$

The étale groupoid G_A for the topological Markov shift (X_A, σ_A) is given by

$$G_A = \{(x, n, y) \in X_A \times \mathbb{Z}_+ \times X_A \mid \text{there exist } k, l \in \mathbb{Z}_+; \\ n = k - l, \sigma_A^k(x) = \sigma_A^l(y)\}.$$

The topology of G_A is generated by the sets

$$\{(x, k - l, y) \in G_A \mid x \in V, y \in W, \sigma_A^k(x) = \sigma_A^l(y)\}$$

for open sets $V, W \subset X_A$ and $k, l \in \mathbb{Z}_+$. Two elements $(x, n, y), (x', n', y') \in G_A$ are composable if and only if $y = x'$ and the product and the inverse are given by

$$(x, n, y) \cdot (x', n', y') = (x, n + n', y'), \quad (x, n, y)^{-1} = (y, -n, x).$$

The unit space $G_A^{(0)}$ is defined by $\{(x, 0, x) \mid x \in X_A\}$, which is identified with X_A . The range map, source map $r, s: G_A \rightarrow G^{(0)}$ are defined by $r(x, n, y) = x, s(x, n, y) = y$ respectively. A subset $U \subset G_A$ is called a G_A -set if $r|_U, s|_U$ are injective. For an open G_A -set U , denote by π_U the homeomorphism $r \circ (s|_U)^{-1}$ from $s(U)$ to $r(U)$. The topological full group $[[G_A]]$ of G_A is defined by the group of all homeomorphisms π_U for some compact open G_A -set U such that $s(U) = r(U) = G^{(0)}$ (see [17]). The groupoid C^* -algebra $C_r^*(G_A)$ of the groupoid G_A is nothing but the Cuntz–Krieger algebra \mathcal{O}_A and the commutative C^* -algebra $C(G_A^{(0)})$ on the unit space $G_A^{(0)}$ is \mathcal{D}_A . The topological full group $[[G_A]]$ of the étale groupoid G_A for the topological Markov shift (X_A, σ_A) is naturally identified with the continuous full group Γ_A ([17]).

Lemma 2.2. *For $\tau \in \Gamma_A$, there exist $u_\tau \in N(\mathcal{O}_A, \mathcal{D}_A)$ and $\mu(i), \nu(i) \in B_*(X_A)$, $i = 1, \dots, m$ such that*

- (1) $u_\tau = \sum_{i=1}^m S_{\mu(i)} S_{\nu(i)}^*$ and
 - (a) $S_{\nu(i)}^* S_{\nu(i)} = S_{\mu(i)}^* S_{\mu(i)}, \quad i = 1, \dots, m,$
 - (b) $\sum_{i=1}^m S_{\nu(i)} S_{\nu(i)}^* = \sum_{i=1}^m S_{\mu(i)} S_{\mu(i)}^* = 1;$
- (2) $f \circ \tau^{-1} = u_\tau f u_\tau^*$ for $f \in \mathcal{D}_A$.

Proof. Since τ is a cylinder map, there exist two families of words $\mu(1), \dots, \mu(m)$ and $\nu(1), \dots, \nu(m)$ satisfying (2.1)–(2.4). Hence we have

$$\sum_{i=1}^m S_{\nu(i)} S_{\nu(i)}^* = \sum_{i=1}^m S_{\mu(i)} S_{\mu(i)}^* = 1, \quad S_{\nu(i)}^* S_{\nu(i)} = S_{\mu(i)}^* S_{\mu(i)}, \quad i = 1, \dots, m.$$

By putting $u_\tau = \sum_{i=1}^m S_{\mu(i)} S_{\nu(i)}^*$ we see that u_τ belongs to $N(\mathcal{O}_A, \mathcal{D}_A)$ and satisfies $\chi_{U_\eta} \circ \tau^{-1} = u_\tau \chi_{U_\eta} u_\tau^*$ for all $\eta \in B_*(X_A)$ where χ_{U_η} is identified with $S_\eta S_\eta^*$, so that $f \circ \tau^{-1} = u_\tau f u_\tau^*$ for all $f \in \mathcal{D}_A$. \square

As in [8, Theorem 1.2], [15, Proposition 5.6], there exists a short exact sequence

$$1 \longrightarrow U(\mathcal{D}_A) \longrightarrow N(\mathcal{O}_A, \mathcal{D}_A) \longrightarrow \Gamma_A \longrightarrow 1$$

that splits.

It has been proved by the second named author [17] that the homology group $H_0(G_A)$ of the groupoid G_A is isomorphic to the K_0 -group

$$K_0(\mathcal{O}_A) = \mathbb{Z}^N / (\text{id} - A^t)\mathbb{Z}^N$$

of the C^* -algebra \mathcal{O}_A . He has proved that the group Γ_A is simple if and only if $H_0(G_A)$ is 2-divisible. He has also proved that Γ_A is finitely presented and its commutator subgroup $D(\Gamma_A)$ is always simple. As the group Γ_A is non-amenable ([10], [17]), we see

Theorem 2.3 ([17]). *The group Γ_A is a countably infinite, non-amenable, finitely presented discrete group. It is simple if and only if the group $\mathbb{Z}^N / (\text{id} - A^t)\mathbb{Z}^N$ is 2-divisible.*

It has been shown that for two irreducible square matrices A and B , the groups Γ_A and Γ_B are isomorphic if and only if the C^* -algebras \mathcal{O}_A and \mathcal{O}_B are isomorphic and $\det(\text{id} - A) = \det(\text{id} - B)$ ([13]). Hence the family $\{\Gamma_A\}$ of our groups supply us many mutually non-isomorphic countably infinite, non-amenable, finitely presented simple groups.

3. Realization of \mathcal{O}_A on $L^2([0, 1])$

The Higman–Thompson group V_N , $1 < N \in \mathbb{N}$ is represented as the group of right continuous piecewise linear bijective functions $f: [0, 1) \rightarrow [0, 1)$ having finitely many singularities such that all singularities of f are in $\mathbb{Z}[\frac{1}{N}]$, the derivative of f at any non-singular point is N^k for some $k \in \mathbb{Z}$ and $f(\mathbb{Z}[\frac{1}{N}] \cap [0, 1)) = \mathbb{Z}[\frac{1}{N}] \cap [0, 1)$. In order to represent our group Γ_A as a group of piecewise linear functions on $[0, 1)$, we will represent the algebra \mathcal{O}_A on the Hilbert space H of

the square integrable functions $L^2([0, 1])$ on $[0, 1]$ with respect to the Lebesgue measure in the following way. We note that the essentially bounded measurable functions $L^\infty([0, 1])$ act on H by left multiplication.

Since A is irreducible and not any permutations, its Perron–Frobenius eigenvalue written β is greater than one. By Ruelle’s Perron-Frobenius theory for Markov chains, there uniquely exists a faithful Borel probability measure φ on X_A satisfying the equality

$$\int_{x \in X_A} g(x) d\varphi(\sigma_A(x)) = \beta \int_{x \in X_A} g(x) d\varphi(x), \quad g \in C(X_A) \tag{3.1}$$

(see [22]). Under the identification between $C(X_A)$ and the C^* -subalgebra \mathcal{D}_A of \mathcal{O}_A , the probability measure φ on X_A is regarded as a continuous linear functional on \mathcal{D}_A , which is still denoted by φ . Let $\lambda_A: \mathcal{D}_A \rightarrow \mathcal{D}_A$ be the positive operator defined by $\lambda_A(g) = \sum_{i=1}^N S_i^* g S_i$ for $g \in \mathcal{D}_A$. Since the characteristic function χ_μ on X_A for the cylinder set of an admissible word $\mu \in B_*(X_A)$ is regarded as the projection $S_\mu S_\mu^*$ in \mathcal{D}_A , the identity (3.1) implies

$$\varphi(\lambda_A(g)) = \beta \varphi(g), \quad g \in \mathcal{D}_A, \tag{3.2}$$

so that the equality

$$\sum_{j=1}^N A(i, j) \varphi(S_j S_j^*) = \beta \varphi(S_i S_i^*), \quad i = 1, \dots, N \tag{3.3}$$

holds. Put $p_j = \varphi(S_j S_j^*)$, $j = 1, \dots, N$. The equality (3.3) means that the vector $\begin{bmatrix} p_1 \\ \vdots \\ p_N \end{bmatrix}$ is a unique normalized positive eigenvector for the Perron–Frobenius eigenvalue β . For $i, j = 1, 2, \dots, N$, put $p_{ij} = \varphi(S_i S_j S_j^* S_i^*)$ so that

$$p_{ij} = \frac{1}{\beta^2} \varphi(S_j^* S_i^* S_i S_j) = \frac{1}{\beta^2} A(i, j) \varphi(S_j^* S_j) = \frac{1}{\beta} A(i, j) p_j.$$

We set for $i, j = 1, 2, \dots, N$,

$$p(0) = 0, \quad p(i) = \sum_{k=1}^i p_k, \quad q(0, 0) = q(i, 0) = 0, \quad q(i, j) = \sum_{k=1}^j p_{ik}$$

and define the intervals I_i, I_{ij} in $[0, 1)$ by

$$I_i = [p(i - 1), p(i)), \tag{3.4}$$

$$I_{ij} = [p(i - 1) + q(i, j - 1), p(i - 1) + q(i, j)). \tag{3.5}$$

The latter interval I_{ij} is empty if $A(i, j) = 0$. We set

$$\begin{aligned} l(I_i) &= p(i-1), & r(I_i) &= p(i), \\ l(I_{ij}) &= p(i-1) + q(i, j-1), & r(I_{ij}) &= p(i-1) + q(i, j), \end{aligned}$$

so that

$$I_i = [l(I_i), r(I_i)), \quad I_{ij} = [l(I_{ij}), r(I_{ij})).$$

Lemma 3.1. *Keep the above notations.*

(i) $[0, 1) = \bigsqcup_{i=1}^N I_i$: disjoint union.

(ii) $I_i = \bigsqcup_{j=1}^{N_i} I_{ij}$: disjoint union.

Proof. (i) is clear.

(ii) Let $N_i = \text{Max}\{j = 1, \dots, N \mid A(i, j) = 1\}$. As we have

$$q(i, N_i) = \sum_{k=1}^{N_i} p_{ik} = \frac{1}{\beta} \sum_{k=1}^{N_i} A(i, k) p_k = p_i,$$

the equality $p(i-1) + q(i, N_i) = p(i)$ holds so that $r(I_{i, N_i}) = r(I_i)$. As the intervals $I_{ij}, I_{ij'}$ are disjoint for $j \neq j'$, one easily sees that $I_i = \bigsqcup_{j=1}^{N_i} I_{ij} = \bigsqcup_{j=1}^N I_{ij}$. \square

We define right continuous functions f_A, g_1, \dots, g_N in the following way. The function $f_A: [0, 1) \rightarrow [0, 1)$ is defined by

$$f_A(x) = \beta(x - l(I_{ij})) + l(I_j) \quad \text{for } x \in I_{ij}$$

so that f_A is linear on I_{ij} with slope β and $f_A(I_{ij}) = I_j$. We set

$$J_i = \bigcup_{\substack{j=1, \dots, N \\ A(i, j)=1}} I_j.$$

The function $g_i: J_i \rightarrow I_i$ for each $i = 1, \dots, N$ is defined by

$$g_i(x) = \frac{1}{\beta}(x - l(I_j)) + l(I_{ij}) \quad \text{for } x \in I_j \text{ with } A(i, j) = 1$$

so that g_i is linear on I_j for $A(i, j) = 1$ with slope $\frac{1}{\beta}$ and $g_i(I_j) = I_{ij}$, $g_i(J_i) = I_i$. The following lemma is direct.

Lemma 3.2. *For $i = 1, \dots, N$, we have*

- (i) $f_A(g_i(x)) = x$ for $x \in J_i$,
- (ii) $g_i(f_A(x)) = x$ for $x \in I_i$.

For a measurable subset E of $[0, 1)$, denote by χ_E the multiplication operator on H of the characteristic function of E . Define the bounded linear operators T_{f_A} , T_{g_i} , $i = 1, \dots, N$ on H by

$$(T_{f_A}\xi)(x) = \xi(f_A(x)), \quad (T_{g_i}\xi)(x) = \chi_{J_i}(x)\xi(g_i(x)), \quad \text{for } \xi \in H, x \in [0, 1).$$

The following lemma is straightforward:

Lemma 3.3. *Keep the above notations. We have*

- (i) $T_{f_A}^* = \frac{1}{\beta} \sum_{i=1}^N T_{g_i}$,
- (ii) $T_{f_A}^* T_{f_A} = \frac{1}{\beta} \sum_{i=1}^N \chi_{J_i}$,
- (iii) $T_{g_i}^* T_{g_i} = \beta \chi_{I_i}$ for $i = 1, \dots, N$ and hence $\sum_{i=1}^N T_{g_i}^* T_{g_i} = \beta 1$,
- (iv) $T_{g_i} T_{g_i}^* = \beta \chi_{J_i}$ for $i = 1, \dots, N$.

We define the operators s_i , $i = 1, \dots, N$ on H by setting

$$s_i = \frac{1}{\sqrt{\beta}} T_{g_i}^*, \quad i = 1, \dots, N.$$

By the above lemma, we have

Proposition 3.4. *The operators s_i , $i = 1, \dots, N$ are partial isometries such that*

$$s_i s_i^* = \chi_{I_i}, \quad s_i^* s_i = \chi_{J_i}, \quad i = 1, \dots, N.$$

Hence they satisfy the relations

$$\sum_{j=1}^N s_j s_j^* = 1, \quad s_i^* s_i = \sum_{j=1}^N A(i, j) s_j s_j^*, \quad i = 1, \dots, N.$$

Therefore the correspondence $S_i \rightarrow s_i$, $i = 1, \dots, N$ gives rise to an isomorphism from the Cuntz–Krieger algebra \mathcal{O}_A onto the C^* -algebra $C^*(s_1, \dots, s_N)$ on H .

4. A -adic PL functions

By Proposition 3.4, we may represent \mathcal{O}_A on H by identifying S_i with s_i for $i = 1, \dots, N$. In this section, we will define PL (piecewise linear) functions on $[0, 1)$ associated to the topological Markov shift (X_A, σ_A) . For $\mu = (\mu_1, \dots, \mu_n) \in B_n(X_A)$, define

$$l(\mu) = \sum_{\substack{v \in B_n(X_A) \\ v < \mu}} \varphi(S_v S_v^*), \quad r(\mu) = l(\mu) + \varphi(S_\mu S_\mu^*)$$

where $\mu < v$ means $v_1 = \mu_1, \dots, v_k = \mu_k$ and $v_{k+1} < \mu_{k+1}$ for some k . Put the interval

$$I_\mu = [l(\mu), r(\mu)).$$

The following lemma is clear.

Lemma 4.1. *For each $n \in \mathbb{N}$ we have*

- (i) $I_\mu \cap I_\nu = \emptyset$ for $\mu, \nu \in B_n(X_A)$ with $\mu \neq \nu$,
- (ii) $\bigcup_{\mu \in B_n(X_A)} I_\mu = [0, 1)$.

For $\mu = (\mu_1, \dots, \mu_n) \in B_n(X_A)$, we note that the following equalities hold

$$\varphi(S_\mu S_\mu^*) = \frac{1}{\beta^n} \varphi(S_\mu^* S_\mu) = \frac{1}{\beta^n} \varphi(S_{\mu_n}^* S_{\mu_n}) = \frac{1}{\beta^n} \sum_{j=1}^N A(\mu_n, j) p_j. \quad (4.1)$$

For $i, j = 1, \dots, N$ with $A(i, j) = 1$, we apply (4.1) for $\mu = i$, (i, j) so that

$$l(i) = \sum_{j < i} \varphi(S_j S_j^*) = \sum_{j=1}^{i-1} p_j = p(i-1),$$

$$r(i) = l(i) + \varphi(S_i S_i^*) = p(i-1) + p_i = p(i)$$

and

$$\begin{aligned} l(i, j) &= \sum_{(\mu_1, \mu_2) < (i, j)} \varphi(S_{\mu_1} S_{\mu_2} S_{\mu_2}^* S_{\mu_1}^*) \\ &= \sum_{(\mu_1, \mu_2) < (i, j)} p_{\mu_1 \mu_2} \\ &= \sum_{\mu_1=1}^{i-1} \sum_{\mu_2=1}^N p_{\mu_1 \mu_2} + \sum_{\mu_2=1}^{j-1} p_{i \mu_2} \\ &= \sum_{\mu_1=1}^{i-1} \sum_{\mu_2=1}^N A(\mu_1, \mu_2) \frac{1}{\beta} p_{\mu_2} + q(i, j-1) \\ &= p(i-1) + q(i, j-1), \end{aligned}$$

$$\begin{aligned} r(i, j) &= l(i, j) + \varphi(S_i S_j S_j^* S_i^*) \\ &= p(i - 1) + q(i, j - 1) + p_{ij} \\ &= p(i - 1) + q(i, j). \end{aligned}$$

Hence we see that

$$[l(i), r(i)] = [p(i - 1), p(i)] = I_i,$$

the interval defined in (3.4);

$$[l(i, j), r(i, j)] = [p(i - 1) + q(i, j - 1), p(i - 1) + q(i, j)] = I_{ij},$$

the interval defined in (3.5).

Lemma 4.2. *For $\mu = (\mu_1, \dots, \mu_m) \in B_m(X_A)$, we have*

$$f_A(I_\mu) = I_{\mu_2 \cdots \mu_m}$$

and hence

$$f_A^{m-1}(I_\mu) = I_{\mu_m} (= [l(\mu_m), r(\mu_m)]).$$

Proof. The algebra \mathcal{O}_A is represented on H by identifying S_i with s_i for $i = 1, \dots, N$. We then see

$$S_\mu S_\mu^* = \chi_{I_\mu} \quad \text{and} \quad \lambda_A(S_\mu S_\mu^*) = \chi_{f_A(I_\mu)}.$$

Since $S_{\mu_1}^* S_{\mu_1} \geq S_{\mu_2} S_{\mu_2}^*$, we have

$$\begin{aligned} \lambda_A(S_\mu S_\mu^*) &= S_{\mu_1}^* S_{\mu_1} S_{\mu_2} \cdots S_{\mu_m} S_{\mu_m}^* \cdots S_{\mu_2}^* S_{\mu_1}^* S_{\mu_1} \\ &= S_{\mu_2} \cdots S_{\mu_m} S_{\mu_m}^* \cdots S_{\mu_2}^* \end{aligned}$$

so that $\chi_{I_{\mu_2 \cdots \mu_m}} = \chi_{f_A(I_\mu)}$. □

Lemma 4.3. *For $\mu = (\mu_1, \dots, \mu_m) \in B_m(X_A)$, $\nu = (\nu_1, \dots, \nu_n) \in B_n(X_A)$, the condition $S_\mu^* S_\mu = S_\nu^* S_\nu$ implies*

$$\frac{r(\mu) - l(\mu)}{r(\nu) - l(\nu)} = \beta^{n-m}. \tag{4.2}$$

Proof. Since $r(\mu) - l(\mu) = \varphi(S_\mu S_\mu^*) = \frac{1}{\beta^m} \varphi(S_\mu^* S_\mu)$ and similarly $r(\nu) - l(\nu) = \frac{1}{\beta^n} \varphi(S_\nu^* S_\nu)$, the condition $S_\mu^* S_\mu = S_\nu^* S_\nu$ implies (4.2). □

Lemma 4.4. For $\mu = (\mu_1, \dots, \mu_m) \in B_m(X_A)$, $\nu = (\nu_1, \dots, \nu_n) \in B_n(X_A)$, the following five conditions are equivalent:

- (i) $\Gamma_*^+(\mu) = \Gamma_*^+(\nu)$,
- (ii) $S_\mu^* S_\mu = S_\nu^* S_\nu$,
- (iii) $S_{\mu_m}^* S_{\mu_m} = S_{\nu_n}^* S_{\nu_n}$,
- (iv) $f_A^m(I_\mu) = f_A^n(I_\nu)$,
- (v) $f_A(I_{\mu_m}) = f_A(I_{\nu_n})$.

Proof. For $\mu = (\mu_1, \dots, \mu_m) \in B_m(X_A)$, the identities

$$\chi_{f_A^m(I_\mu)} = \chi_{f_A(I_{\mu_m})} = \lambda_A(S_{\mu_m} S_{\mu_m}^*) = S_{\mu_m}^* S_{\mu_m} = S_\mu^* S_\mu$$

hold. They imply the desired assertion. \square

Definition 4.5. (i) For a word $\nu \in B_*(X_A)$, an interval $[x_1, x_2]$ in $[0, 1)$ is said to be an *A-adic interval* for ν if $x_1 = l(\nu)$ and $x_2 = r(\nu)$.

(ii) A rectangle $I \times J$ in $[0, 1) \times [0, 1)$ is said to be an *A-adic rectangle* if both the intervals I, J are *A-adic intervals* for some words $\nu \in B_n(X_A)$, $\mu \in B_m(X_A)$, respectively such that

$$I = [l(\nu), r(\nu)), \quad J = [l(\mu), r(\mu)),$$

and

$$f_A^n(I) = f_A^m(J).$$

(iii) For two partitions

$$0 = x_0 < x_1 < \dots < x_{m-1} < x_m = 1,$$

$$0 = y_0 < y_1 < \dots < y_{m-1} < y_m = 1$$

of $[0, 1)$, put

$$I_p = [x_{p-1}, x_p), \quad J_p = [y_{p-1}, y_p) \quad \text{for } p = 1, 2, \dots, m.$$

The partition $I_p \times J_q$, $p, q = 1, \dots, m$ of $[0, 1) \times [0, 1)$ is said to be an *A-adic pattern of rectangles* if there exists a permutation σ on $\{1, 2, \dots, m\}$ such that the rectangles $I_p \times J_{\sigma(p)}$ are *A-adic rectangles* for all $p = 1, 2, \dots, m$.

For an *A-adic pattern of rectangles* above, the slopes of its diagonals

$$s_p = \frac{y_{\sigma(p)} - y_{\sigma(p)-1}}{x_p - x_{p-1}}, \quad p = 1, 2, \dots, m,$$

are said to be *rectangle slopes*.

Definition 4.6. A piecewise linear function f on $[0, 1)$ is called an *A-adic PL function* if f is a right continuous bijection on $[0, 1)$ such that there exists an *A-adic pattern of rectangles* $I_p \times J_p$, $p = 1, 2, \dots, m$ where $I_p = [x_{p-1}, x_p)$, $J_p = [y_{p-1}, y_p)$, $p = 1, \dots, m$ with a permutation σ on $\{1, 2, \dots, m\}$ such that

$$f(x_{p-1}) = y_{\sigma(p)-1}, \quad f_-(x_p) = y_{\sigma(p-1)+1}, \quad p = 1, 2, \dots, m,$$

where $f_-(x_p) = \lim_{h \rightarrow 0^+} f(x_p - h)$, and f is linear on $[x_{p-1}, x_p)$ with slope $\frac{y_{\sigma(p)} - y_{\sigma(p)-1}}{x_p - x_{p-1}}$ for $p = 1, 2, \dots, m$.

Lemma 4.7. *The composition of two A-adic PL functions and the inverse function of an A-adic PL function are also A-adic PL functions.*

By the above lemma, the set of *A-adic PL functions* forms a group under compositions of functions.

Definition 4.8. We denote by Γ_A^{PL} the group of *A-adic PL functions*.

The following proposition is immediate by definition of *A-adic PL functions*.

Proposition 4.9. *An A-adic PL function naturally gives rise to an A-adic pattern of rectangles, whose rectangle slopes are the slopes of the A-adic PL function. Conversely, an A-adic pattern of rectangles gives rise to an A-adic PL function by taking its diagonal lines of the rectangles.*

5. A-adic Tables

For two words $\mu = (\mu_1, \dots, \mu_m) \in B_m(X_A)$, $\nu = (\nu_1, \dots, \nu_n) \in B_n(X_A)$ with $U_\mu \cap U_\nu = \emptyset$, we write $\mu < \nu$ if $\mu_1 = \nu_1, \dots, \mu_k = \nu_k$ and $\mu_{k+1} < \nu_{k+1}$ for some k . Nekrashevych in [20] has introduced a notion of table to represent elements of the Higman–Thompson group V_N . We will generalize the Nekrashevych’s notion to a notion of *A-adic table* in order to represent elements of the continuous full group Γ_A .

Definition 5.1. An *A-adic table* is a matrix T

$$T = \begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ \nu(1) & \nu(2) & \cdots & \nu(m) \end{bmatrix}$$

for $\mu(i), \nu(i) \in B_*(X_A)$, $i = 1, \dots, m$ such that

- (a) $\Gamma_*^+(\nu(i)) = \Gamma_*^+(\mu(i))$, $i = 1, \dots, m$,
- (b) $X_A = \bigsqcup_{i=1}^m U_{\nu(i)} = \bigsqcup_{i=1}^m U_{\mu(i)}$: disjoint unions.

Since the words $v(i), i = 1, \dots, m$ satisfy $U_{v(i)} \cap U_{v(j)} = \emptyset$ for $i \neq j$, we may reorder them such as $v(1) < v(2) < \dots < v(m)$. As the above two conditions (a) and (b) are equivalent to the conditions (a) and (b) in Lemma 2.2 (1) respectively, we have

Lemma 5.2. *For an element $\tau \in \Gamma_A$, let words $\mu(i), v(i), i = 1, \dots, m$ and the unitary $u_\tau = \sum_{i=1}^m S_{\mu(i)} S_{v(i)}^*$ satisfy the conditions (1) and (2) in Lemma 2.2. Then the matrix*

$$T = \begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ v(1) & v(2) & \cdots & v(m) \end{bmatrix}$$

is an A -adic table.

The A -adic table T above is called a representation of τ . It is also called that T represents τ .

For an A -adic table $T = \begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ v(1) & v(2) & \cdots & v(m) \end{bmatrix}$ and $i = 1, 2, \dots, m$, let $\eta(i, j) \in B_*(X_A), j = 1, \dots, n_i$ be a family of (possibly empty) words satisfying the following three conditions:

- (i) $\eta(i, 1) < \eta(i, 2) < \dots < \eta(i, n_i)$,
- (ii) $\eta(i, j) \in \Gamma_*^+(v(i))$ for $j = 1, \dots, n_i$,
- (iii) $U_{v(i)} = \bigcup_{j=1}^{n_i} U_{v(i)\eta(i,j)}$.

Since $\Gamma_*^+(v(i)) = \Gamma_*^+(\mu(i))$, one has $\eta(i, j) \in \Gamma_*^+(\mu(i))$ and

$$U_{\mu(i)} = \bigcup_{j=1}^{n_i} U_{\mu(i)\eta(i,j)}.$$

Put

$$v(i, j) = v(i)\eta(i, j), \quad \mu(i, j) = \mu(i)\eta(i, j), \quad j = 1, \dots, n_i, \quad i = 1, \dots, m. \quad (5.1)$$

Then the $2 \times m$ matrix

$$\begin{bmatrix} \mu(1, 1) & \cdots & \mu(1, n_1) & \mu(2, 1) & \cdots & \mu(2, n_2) & \cdots & \mu(m, 1) & \cdots & \mu(m, n_m) \\ v(1, 1) & \cdots & v(1, n_1) & v(2, 1) & \cdots & v(2, n_2) & \cdots & v(m, 1) & \cdots & v(m, n_m) \end{bmatrix}$$

is an A -adic table, which is called an *expansion* of T . Let us denote by \approx the equivalence relation in the A -adic tables generated by the expansions. This means that two A -adic tables

$$T = \begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ v(1) & v(2) & \cdots & v(m) \end{bmatrix}, \quad T' = \begin{bmatrix} \mu'(1) & \mu'(2) & \cdots & \mu'(m') \\ v'(1) & v'(2) & \cdots & v'(m') \end{bmatrix},$$

are equivalent and written $T \approx T'$ if there exists a finite sequence T_1, T_2, \dots, T_k of A -adic tables such that $T = T_1, T' = T_k$ and T_i is an expansion of T_{i+1} , or T_{i+1} is an expansion of T_i .

Lemma 5.3. *For $\tau, \tau' \in \Gamma_A$, let T, T' be A -adic tables representing τ, τ' respectively. Then $\tau = \tau'$ if and only if $T \approx T'$.*

Proof. Let T, T' be the matrices

$$T = \begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ \nu(1) & \nu(2) & \cdots & \nu(m) \end{bmatrix}, \quad T' = \begin{bmatrix} \mu'(1) & \mu'(2) & \cdots & \mu'(m') \\ \nu'(1) & \nu'(2) & \cdots & \nu'(m') \end{bmatrix}.$$

Suppose that T' is an expansion of T . We write T' as

$$\begin{bmatrix} \mu(1, 1) & \cdots & \mu(1, n_1) & \mu(2, 1) & \cdots & \mu(2, n_2) & \cdots & \mu(m, 1) & \cdots & \mu(m, n_m) \\ \nu(1, 1) & \cdots & \nu(1, n_1) & \nu(2, 1) & \cdots & \nu(2, n_2) & \cdots & \nu(m, 1) & \cdots & \nu(m, n_m) \end{bmatrix}$$

where $\mu(i, j)$ and $\nu(i, j)$ are words for $\eta(i, j)$ as in (5.1). The homeomorphisms τ and τ' on X_A are induced by the unitaries u_T and $u_{T'}$ defined by

$$u_T = \sum_{i=1}^m S_{\mu(i)} S_{\nu(i)}^* \quad \text{and} \quad u_{T'} = \sum_{i=1}^{m'} S_{\mu'(i)} S_{\nu'(i)}^*$$

such as $f \circ \tau^{-1} = \text{Ad}(u_\tau)(f)$ and $f \circ \tau'^{-1} = \text{Ad}(u_{\tau'})(f)$ for $f \in C(X_A) = \mathcal{D}_A$. As

$$S_{\mu(i)} S_{\nu(i)}^* = \sum_{j=1}^{n_i} S_{\mu(i)} S_{\eta(i,j)} S_{\eta(i,j)}^* S_{\nu(i)}^* = \sum_{j=1}^{n_i} S_{\mu(i,j)} S_{\nu(i,j)}^*,$$

we have

$$u_\tau = \sum_{i=1}^m S_{\mu(i)} S_{\nu(i)}^* = \sum_{i=1}^m S_{\mu(i,j)} S_{\nu(i,j)}^* = u_{\tau'}$$

so that $\tau = \tau'$.

Conversely, suppose that $\tau = \tau'$. Let

$$K' = \text{Max}\{|\nu'(i)| \mid 1 \leq i \leq m'\}, \quad L' = \text{Max}\{|\mu'(i)| \mid 1 \leq i \leq m'\}.$$

There exist admissible words $\eta(i, j) \in B_*(X_A)$, $j = 1, \dots, n_i$, $i = 1, \dots, m$ such that

- (a) $\eta(i, 1) \prec \eta(i, 2) \prec \cdots \prec \eta(i, n_i)$,
- (b) $\eta(i, j) \in \Gamma_*^+(\nu(i))$,
- (c) $|\nu(i)\eta(i, j)| \geq K'$, $|\mu(i)\eta(i, j)| \geq L'$,
- (d) $U_{\nu(i)} = \bigsqcup_{j=1}^{n_i} U_{\nu(i)\eta(i,j)}$, $U_{\mu(i)} = \bigsqcup_{j=1}^{n_i} U_{\mu(i)\eta(i,j)}$.

Put

$$v(i, j) = v(i)\eta(i, j), \quad \mu(i, j) = \mu(i)\eta(i, j), \quad j = 1, \dots, n_i, \quad i = 1, \dots, m,$$

and

$$T^\eta = \begin{bmatrix} \mu(1, 1) & \cdots & \mu(1, n_1) & \mu(2, 1) & \cdots & \mu(2, n_2) & \cdots \\ v(1, 1) & \cdots & v(1, n_1) & v(2, 1) & \cdots & v(2, n_2) & \cdots \\ & & & \cdots & & \mu(m, 1) & \cdots & \mu(m, n_m) \\ & & & \cdots & & v(m, 1) & \cdots & v(m, n_m) \end{bmatrix}.$$

Hence T^η is an expansion of T . We will compare T^η and T' . Put

$$F_k = \{(i, j) \mid v(i, j) < v'(k)\}, \quad k = 1, \dots, m'.$$

Since $|v(i, j)| \geq K'$, $|\mu(i, j)| \geq L'$, one has

$$v'(k) = \bigcup_{(i, j) \in F_k} v(i, j).$$

Since $|v(i, j)| \geq |v'(k)|$, there exist $\eta'(k, (i, j)) \in B_*(X_A)$ such that

$$v(i, j) = v'(k)\eta'(k, (i, j)) \quad \text{for } (i, j) \in F_k.$$

As $\tau = \tau'$, we have

$$\tau(\chi U_{v(i, j)}) = \chi U_{\mu(i)\eta(i, j)} = \chi U_{\mu(i, j)} = \tau'(\chi U_{v(i, j)}) = \chi U_{\mu'(k)\eta'(k, (i, j))}$$

so that

$$\mu(i, j) = \mu'(k)\eta'(k, (i, j)) \quad \text{for } (i, j) \in F_k.$$

This implies that T^η is an expansion of T' to prove that T is equivalent to T' . \square

We denote by $[T]$ the equivalence class of an A -adic table T . For $\tau \in \Gamma_A$, denote by T_τ an A -adic table representing τ . The preceding lemma says that its equivalence class $[T_\tau]$ does not depend on the choice of T_τ representing τ . An A -adic table $\begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ v(1) & v(2) & \cdots & v(m) \end{bmatrix}$ presenting $\tau \in \Gamma_A$ is said to be *reduced* if it has a minimal length m in the set of A -adic tables presenting τ . Recall that for a word $\mu = (\mu_1, \dots, \mu_k) \in B_*(X_A)$, we write $\Gamma_1^+(\mu) = \{j \in \{1, \dots, N\} \mid A(\mu_k, j) = 1\}$. The following lemma is obvious.

Lemma 5.4. *For an A -adic table $T = \begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ v(1) & v(2) & \cdots & v(m) \end{bmatrix}$ and $i = 1, \dots, m$, let $\Gamma_1^+(\mu(i)) = \{\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_{n_i}}\}$ such that $\alpha_{i_1} < \alpha_{i_2} < \cdots < \alpha_{i_{n_i}}$. Put the words*

$$\begin{aligned} \mu(i, 1) &= \mu(i)\alpha_{i_1}, & \mu(i, 2) &= \mu(i)\alpha_{i_2}, & \dots, & \mu(i, n_i) &= \mu(i)\alpha_{i_{n_i}}, \\ v(i, 1) &= v(i)\alpha_{i_1}, & v(i, 2) &= v(i)\alpha_{i_2}, & \dots, & v(i, n_i) &= v(i)\alpha_{i_{n_i}}. \end{aligned}$$

Then the A -adic table T'_i obtained from T by replacing $\mu(i)$ with $\mu(i, 1), \dots, \mu(i, n_i)$, and $\nu(i)$ with $\nu(i, 1), \dots, \nu(i, n_i)$ such that

$$T'_i = \begin{bmatrix} \mu(1) & \cdots & \mu(i-1) & \mu(i, 1) & \cdots & \mu(i, n_i) & \mu(i+1) & \cdots & \mu(m) \\ \nu(1) & \cdots & \nu(i-1) & \nu(i, 1) & \cdots & \nu(i, n_i) & \nu(i+1) & \cdots & \nu(m) \end{bmatrix}$$

is equivalent to T .

For an A -adic table $T = \begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ \nu(1) & \nu(2) & \cdots & \nu(m) \end{bmatrix}$, define the range depth $R(T)$ and the domain depth $D(T)$ by

$$R(T) = \text{Max}\{|\mu(i)| \mid 1 \leq i \leq m\}, \quad D(T) = \text{Max}\{|\nu(i)| \mid 1 \leq i \leq m\}.$$

By using the above lemma recursively, we know the following lemma.

Lemma 5.5. *Let*

$$T = \begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ \nu(1) & \nu(2) & \cdots & \nu(m) \end{bmatrix}$$

be an A -adic table.

(i) *For a positive integer $M \geq D(T)$, there exists an A -adic table*

$$T' = \begin{bmatrix} \mu'(1) & \mu'(2) & \cdots & \mu'(m') \\ \nu'(1) & \nu'(2) & \cdots & \nu'(m') \end{bmatrix}$$

such that

$$T' \approx T \quad \text{and} \quad \{\nu'(i) \mid i = 1, \dots, m'\} = B_M(X_A).$$

(ii) *For a positive integer $M \geq R(T)$, there exists an A -adic table*

$$T'' = \begin{bmatrix} \mu''(1) & \mu''(2) & \cdots & \mu''(m'') \\ \nu''(1) & \nu''(2) & \cdots & \nu''(m'') \end{bmatrix}$$

such that

$$T'' \approx T \quad \text{and} \quad \{\mu''(i) \mid i = 1, \dots, m''\} = B_M(X_A).$$

Let T_1, T_2 be two A -adic tables. Take M such that $M \geq D(T_1), R(T_2)$. By the preceding lemma, there exist A -adic tables

$$T'_1 = \begin{bmatrix} \mu'_1(1) & \mu'_1(2) & \cdots & \mu'_1(p) \\ \nu'_1(1) & \nu'_1(2) & \cdots & \nu'_1(p) \end{bmatrix}, \quad T'_2 = \begin{bmatrix} \mu'_2(1) & \mu'_2(2) & \cdots & \mu'_2(q) \\ \nu'_2(1) & \nu'_2(2) & \cdots & \nu'_2(q) \end{bmatrix}$$

such that $T'_1 \approx T_1$ and $T'_2 \approx T_2$ and

$$|\nu'_1(1)| = \cdots = |\nu'_1(p)| = |\mu'_2(1)| = \cdots = |\mu'_2(q)| = M.$$

Hence we have $p = q = |B_M(X_A)|$. One may reorder $\nu'_1(i), \mu'_2(i)$ such as

$$\nu'_1(1) < \nu'_1(2) < \cdots < \nu'_1(p), \quad \mu'_2(1) < \mu'_2(2) < \cdots < \mu'_2(q)$$

so that

$$\nu'_1(1) = \mu'_2(1), \quad \nu'_1(2) = \mu'_2(2), \quad \dots, \quad \nu'_1(p) = \mu'_2(q).$$

Define the product $T'_1 \circ T'_2$ by the A -adic table

$$T'_1 \circ T'_2 = \begin{bmatrix} \mu'_1(1) & \mu'_1(2) & \cdots & \mu'_1(p) \\ \nu'_2(1) & \nu'_2(2) & \cdots & \nu'_2(p) \end{bmatrix}.$$

It is easy to see that $T'_1 \circ T'_2$ is an A -adic table. It is straightforward to see that the equivalence class $[T'_1 \circ T'_2]$ does not depend on the choice of representatives T'_1 of $[T'_1]$ and T'_2 of $[T'_2]$. Hence one may define the product $[T_1] \circ [T_2]$ by the equivalence class $[T'_1 \circ T'_2]$ of the product $T'_1 \circ T'_2$.

For an A -adic table $T = \begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ \nu(1) & \nu(2) & \cdots & \nu(m) \end{bmatrix}$, define an A -adic table

$$T^{-1} = \begin{bmatrix} \nu(1) & \nu(2) & \cdots & \nu(m) \\ \mu(1) & \mu(2) & \cdots & \mu(m) \end{bmatrix}.$$

The identity table denoted by I is defined by

$$I = \begin{bmatrix} 1 & 2 & \cdots & N \\ 1 & 2 & \cdots & N \end{bmatrix}$$

where the two rows of I denote the list of the ordered symbols $\{1, 2, \dots, N\} = B_1(X_A)$.

Lemma 5.6. *Keep the above notations.*

- (i) *The equivalence class $[I]$ of I is the unit of the product operations in the equivalence classes of the A -adic tables.*
- (ii) *If $T \approx T'$, then $T^{-1} \approx T'^{-1}$.*

Since $T^{-1} \circ T \approx I$ and $T \circ T^{-1} \approx I$, the class $[T^{-1}]$ of T^{-1} is the inverse of $[T]$ in the equivalence classes of the A -adic tables.

Definition 5.7. Denote by Γ_A^{tab} the group of the equivalence classes of A -adic tables.

Therefore we have

Proposition 5.8. *The correspondence $\tau \in \Gamma_A \rightarrow [T_\tau] \in \Gamma_A^{\text{tab}}$ gives rise to an isomorphism of groups.*

Proof. Let $\tau, \tau' \in \Gamma_A$. By Lemma 5.3, $\tau = \tau'$ if and only if $[T_\tau] = [T_{\tau'}]$. It is direct to see that for $\tau_1, \tau_2 \in \Gamma_A$, the equivalence class $[T_{\tau_1 \circ \tau_2}]$ of an A -adic table $T_{\tau_1 \circ \tau_2}$ representing the composition $\tau_1 \circ \tau_2$ is the product $[T_{\tau_1}] \circ [T_{\tau_2}]$ of the classes $[T_{\tau_1}], [T_{\tau_2}]$. Hence the correspondence $\tau \in \Gamma_A \rightarrow [T_\tau] \in \Gamma_A^{\text{tab}}$ gives rise to an isomorphism of groups. \square

6. Isomorphisms among $\Gamma_A, \Gamma_A^{\text{tab}}$ and Γ_A^{PL}

In the preceding section, we have shown that the two groups $\Gamma_A, \Gamma_A^{\text{tab}}$ are isomorphic. In this section, we will show that these two groups are isomorphic to the group Γ_A^{PL} of A -adic PL functions.

Lemma 6.1. *For an A -adic table $T = \begin{bmatrix} \mu(1) & \mu(2) & \dots & \mu(m) \\ \nu(1) & \nu(2) & \dots & \nu(m) \end{bmatrix}$, there exist an A -adic pattern of rectangles whose rectangle slopes are*

$$\beta^{|\nu(1)|-|\mu(1)|}, \quad \beta^{|\nu(2)|-|\mu(2)|}, \quad \dots, \quad \beta^{|\nu(m)|-|\mu(m)|},$$

and an A -adic PL function f_T having these rectangle slopes such that

$$f_T(I_{\nu(i)}) = I_{\mu(i)}, \quad i = 1, 2, \dots, m. \tag{6.1}$$

Conversely, for an A -adic PL function f with the A -adic pattern of rectangles $I_p \times J_{\sigma(p)}, p = 1, 2, \dots, m$ and a permutation σ on $\{1, \dots, m\}$, there exists an A -adic table $T_f = \begin{bmatrix} \mu(1) & \mu(2) & \dots & \mu(m) \\ \nu(1) & \nu(2) & \dots & \nu(m) \end{bmatrix}$ such that

$$I_p = I_{\nu(p)}, \quad J_{\sigma(p)} = I_{\mu(p)}, \quad p = 1, 2, \dots, m.$$

Proof. We are assuming the ordering such as $\nu(1) < \dots < \nu(m)$. Since X_A is a disjoint union $X_A = \bigsqcup_{j=1}^m U_{\mu(j)}$, there exists a permutation σ_0 on $\{1, 2, \dots, m\}$ such that $\mu(\sigma_0(1)) < \mu(\sigma_0(2)) < \dots < \mu(\sigma_0(m))$. Put

$$x_i = l(\nu(i + 1)), \quad y_i = l(\mu(\sigma_0(i + 1))), \quad i = 0, 1, \dots, m - 1$$

so that $x_0 = y_0 = 0$ and

$$I_p = [x_{p-1}, x_p), \quad J_p = [y_{p-1}, y_p), \quad p = 1, 2, \dots, m$$

where $x_m = y_m = 1$. Define the permutation $\sigma := \sigma_0^{-1}$ on $\{1, 2, \dots, m\}$. We note that $r(\nu(i)) = l(\nu(i + 1)), r(\mu(\sigma_0(i))) = l(\mu(\sigma_0(i + 1)))$ for $i = 1, \dots, m - 1$. Then the rectangles $I_p \times J_{\sigma(p)}, p = 1, 2, \dots, m$ are A -adic rectangles by Lemma 4.4 such that

$$\frac{y_{\sigma(p)} - y_{\sigma(p)-1}}{x_p - x_{p-1}} = \frac{r(\mu(p)) - l(\mu(p))}{r(\nu(p)) - l(\nu(p))}.$$

We then have

$$r(v(p)) - l(v(p)) = \varphi(S_{v(p)}S_{v(p)}^*) = \frac{1}{\beta^{|v(p)|}}\varphi(S_{v(p)}^*S_{v(p)})$$

and similarly

$$r(\mu(p)) - l(\mu(p)) = \frac{1}{\beta^{|\mu(p)|}}\varphi(S_{\mu(p)}^*S_{\mu(p)}).$$

As the condition $\Gamma_*^+(v(p)) = \Gamma_*^+(\mu(p))$ implies $S_{v(p)}^*S_{v(p)} = S_{\mu(p)}^*S_{\mu(p)}$, we have

$$\frac{y_{\sigma(p)} - y_{\sigma(p)-1}}{x_p - x_{p-1}} = \beta^{|v(p)|-|\mu(p)|}, \quad p = 1, 2, \dots, m.$$

By Proposition 4.9, one immediately knows that the associated A -adic PL function denoted by f_T with the above A -adic pattern of rectangles satisfies condition (6.1).

The converse implication is straightforward from Lemma 4.4. □

We may directly construct an A -adic PL function f_T from an A -adic table $T = \begin{bmatrix} \mu(1) & \mu(2) & \dots & \mu(m) \\ v(1) & v(2) & \dots & v(m) \end{bmatrix}$ as follows. Put $x_i = l(v(i + 1))$, $\hat{y}_i = l(\mu(i + 1))$ and $f_T(x_i) = \hat{y}_i$, $i = 0, 1, \dots, m - 1$. Define $f_T(x)$ on $[x_{i-1}, x_i)$ as a linear function with slope $\beta^{|v(i)|-|\mu(i)|} (= \frac{r(\mu(i))-l(\mu(i))}{r(v(i))-l(v(i))} = \frac{\hat{y}_i-\hat{y}_{i-1}}{x_i-x_{i-1}})$ for $i = 1, 2, \dots, m$. It is easy to see that the function f_T is an A -adic PL function. Let us denote by ι the A -adic PL function defined by $\iota(x) = x$, $x \in [0, 1)$. The following lemma is direct.

Lemma 6.2. *For two A -adic tables T_1, T_2 , we have*

- (i) T_1 is equivalent to T_2 if and only if $f_{T_1} = f_{T_2}$ as functions. Hence we may write f_T as $f_{[T]}$,
- (ii) $f_{[T_1] \circ [T_2]} = f_{[T_1]} \circ f_{[T_2]}$,
- (iii) $\iota = f_{[I]}$.

We reach the main result of the paper.

Theorem 6.3. *There exist canonical isomorphisms of discrete groups among the continuous full group Γ_A , the group Γ_A^{tab} of the equivalence classes of A -adic tables, and the group Γ_A^{PL} of A -adic PL functions on $[0, 1)$, that is*

$$\Gamma_A \cong \Gamma_A^{\text{tab}} \cong \Gamma_A^{\text{PL}}.$$

In particular, the continuous full group Γ_A for a topological Markov shift (X_A, σ_A) is realized as the group of all A -adic PL functions on $[0, 1)$.

Proof. By Proposition 5.8, we have an isomorphism from the continuous full group Γ_A to the group Γ_A^{tab} of the equivalence classes of A -adic tables. By Lemma 6.1 and Lemma 6.2, the correspondence $[T] \in \Gamma_A^{\text{tab}} \rightarrow f_T \in \Gamma_A^{\text{PL}}$ yields an isomorphism. □

7. A realization of Γ_A as A -adic PL functions

In this section, we will construct a continuous surjection of the shift space X_A onto the interval $[0, 1]$ which yields a representation of elements of the continuous full group Γ_A to the group Γ_A^{PL} of A -adic PL functions. For $x = (x_i)_{i \in \mathbb{N}} \in X_A$ and $n \in \mathbb{Z}_+$, consider the word $(x_1, \dots, x_n) \in B_n(X_A)$ and set

$$l_n(x) = l(x_1, \dots, x_n), \quad r_n(x) = r(x_1, \dots, x_n).$$

Lemma 7.1. *For $x = (x_i)_{i \in \mathbb{N}} \in X_A$ and $n \in \mathbb{Z}_+$, we have*

- (i) $l_n(x) \leq l_{n+1}(x) \leq r_{n+1}(x) \leq r_n(x)$,
- (ii) $|r_n(x) - l_n(x)| \leq \frac{1}{\beta^n}$.

Proof. (i) For $\mu = (\mu_1, \dots, \mu_n) \in B_n(X_A)$, the condition $\mu \prec (x_1, \dots, x_n)$ implies $\mu j \prec (x_1, \dots, x_n, x_{n+1})$ for all j with $A(\mu_n, j) = 1$ so that

$$\begin{aligned} l_n(x) &= \sum_{\substack{\mu \in B_n(X_A) \\ \mu \prec (x_1, \dots, x_n)}} \varphi(S_\mu S_\mu^*) \\ &= \sum_{j=1}^N A(\mu_n, j) \sum_{\substack{\mu \in B_n(X_A) \\ \mu \prec (x_1, \dots, x_n)}} \varphi(S_{\mu j} S_{\mu j}^*) \\ &\leq \sum_{\substack{v \in B_{n+1}(X_A) \\ v \prec (x_1, \dots, x_n, x_{n+1})}} \varphi(S_v S_v^*) \\ &= l_{n+1}(x). \end{aligned}$$

We note that

$$l_{n+1}(x) = l_n(x) + \sum_{j < x_{n+1}} \varphi(S_{x_1 \dots x_n j} S_{x_1 \dots x_n j}^*) \tag{7.1}$$

so that

$$\begin{aligned} r_{n+1}(x) &= l_{n+1}(x) + \varphi(S_{x_1 \dots x_n x_{n+1}} S_{x_1 \dots x_n x_{n+1}}^*) \\ &= l_n(x) + \sum_{j \leq x_{n+1}} \varphi(S_{x_1 \dots x_n j} S_{x_1 \dots x_n j}^*) \\ &\leq l_n(x) + \sum_{j=1}^N \varphi(S_{x_1 \dots x_n j} S_{x_1 \dots x_n j}^*) \\ &= l_n(x) + \varphi(S_{x_1 \dots x_n} S_{x_1 \dots x_n}^*) \\ &= r_n(x). \end{aligned}$$

(ii) By the equality $r_n(x) = l_n(x) + \varphi(S_{x_1 \dots x_n} S_{x_1 \dots x_n}^*)$ with

$$\varphi(S_{x_1 \dots x_n} S_{x_1 \dots x_n}^*) = \frac{1}{\beta^n} \sum_{j=1}^N A(x_n, j) p_j, \quad \sum_{j=1}^N p_j = 1,$$

we have $|r_n(x) - l_n(x)| \leq \frac{1}{\beta^n}$. \square

Lemma 7.2. For $x = (x_i)_{i \in \mathbb{N}} \in X_A$ and $n \in \mathbb{Z}_+$, we have

- (i) $l_n(x) = l_{n+1}(x)$ if and only if $x_{n+1} = \text{Min}\{j = 1, \dots, N \mid A(x_n, j) = 1\}$,
- (ii) $r_n(x) = r_{n+1}(x)$ if and only if $x_{n+1} = \text{Max}\{j = 1, \dots, N \mid A(x_n, j) = 1\}$.

Proof. (i) By (7.1), one sees that $l_{n+1}(x) = l_n(x)$ if and only if

$$\sum_{j < x_{n+1}} \varphi(S_{x_1 \dots x_n j} S_{x_1 \dots x_n j}^*) = 0.$$

Since the state φ on \mathcal{D}_A is faithful, the latter condition is equivalent to the condition that there does not exist any $j = 1, \dots, N$ such that $j < x_{n+1}$ and $A(x_n, j) = 1$. Hence we have the desired assertion.

(ii) is similar to (i). \square

For a word $\omega = (\omega_1, \dots, \omega_n) \in B_n(X_A)$, let us denote by $\omega_{\min} = (\underline{\omega}_i)_{i \in \mathbb{N}} \in X_A$ (resp. $\omega_{\max} = (\bar{\omega}_i)_{i \in \mathbb{N}} \in X_A$) its minimal (resp. maximal) extension to a right infinite sequence in X_A , which is defined by setting

$$\underline{\omega}_i = \omega_i \quad (\text{resp. } \bar{\omega}_i = \omega_i)$$

for $i = 1, \dots, n$, and

$$\begin{aligned} \underline{\omega}_{n+k} &= \text{Min}\{j = 1, 2, \dots, N \mid A(\underline{\omega}_{n+k-1}, j) = 1\}, \\ (\text{resp. } \bar{\omega}_{n+k} &= \text{Max}\{j = 1, 2, \dots, N \mid A(\bar{\omega}_{n+k-1}, j) = 1\}) \end{aligned}$$

for $k = 1, 2, \dots$. By Lemma 7.2, one has $l(\omega) = l_{n+k}(\omega_{\min})$ and $r(\omega) = r_{n+k}(\omega_{\max})$ for all $k \in \mathbb{N}$. For the two symbols $1, N \in B_1(X_A)$, we may consider the elements $1_{\min}, N_{\max}$ in X_A so that we obtain the following lemma.

Lemma 7.3. $l_n(1_{\min}) = 0, r_n(N_{\max}) = 1$ for all $n \in \mathbb{N}$.

For two sequences $x = (x_n)_{n \in \mathbb{N}}, y = (y_n)_{n \in \mathbb{N}} \in X_A$, we write $x < y$ if $x_1 = y_1, \dots, x_n = y_n, x_{n+1} < y_{n+1}$ for some $n \in \mathbb{Z}_+$. Hence X_A becomes an ordered space such that 1_{\min} (resp. N_{\max}) is minimum (resp. maximum). Recall that for a word $\mu \in B_*(X_A)$, denote by I_μ the interval $[l(\mu), r(\mu)]$, so that $\bar{I}_\mu = [l(\mu), r(\mu)]$.

Proposition 7.4. *There exists an order preserving surjective continuous map $\rho_A: X_A \rightarrow [0, 1]$ such that*

$$\rho_A(1_{\min}) = 0, \quad \rho_A(N_{\max}) = 1, \quad \rho_A(U_\mu) = \bar{I}_\mu, \quad \text{for } \mu \in B_n(X_A).$$

Proof. For $x = (x_i)_{i \in \mathbb{N}} \in X_A$, there exists an element

$$\lim_{n \rightarrow \infty} l_n(x) (= \lim_{n \rightarrow \infty} r_n(x))$$

in $[0, 1]$ which we denote by $\rho_A(x)$. It satisfies the inequalities $l_n(x) \leq \rho_A(x) \leq r_n(x)$ for all $n \in \mathbb{N}$. By the above lemma, we have

$$\rho_A(1_{\min}) = \lim_{n \rightarrow \infty} l_n(1_{\min}) = 0, \quad \rho_A(N_{\max}) = \lim_{n \rightarrow \infty} r_n(N_{\max}) = 1.$$

We will next show that $\rho_A: X_A \rightarrow [0, 1]$ is surjective. For $t \in [0, 1]$, we may assume that $t < 1$ because $\rho_A(N_{\max}) = 1$. For $n \in \mathbb{N}$, by Lemma 4.1 (ii), one may find a word $\mu^{(n)} \in B_n(X_A)$ such that $t \in I_{\mu^{(n)}}$. The first n -symbols of $\mu^{(n+1)}$ coincide with $\mu^{(n)}$ so that the sequence $\{\mu^{(n)}\}_{n \in \mathbb{N}}$ of words defines a right infinite sequence $x_t = (x_n)_{n \in \mathbb{N}}$ of X_A such that $(x_1, \dots, x_n) = \mu^{(n)}$. Since $l(\mu^{(n)}) \leq t \leq r(\mu^{(n)})$ and $|r(\mu^{(n)}) - l(\mu^{(n)})| < \frac{1}{\beta^n}$, one sees that $\rho_A(x_t) = \lim_{n \rightarrow \infty} l(\mu^{(n)}) = t$ so that $\rho_A: X_A \rightarrow [0, 1]$ is surjective.

For $\mu \in B_n(X_A)$ and $x \in U_\mu$, one sees that $l(\mu) = l_n(x) \leq \rho_A(x) \leq r_n(x) = r(\mu)$ so that $\rho_A(x) \in [l(\mu), r(\mu)]$. Hence we have $\rho_A(U_\mu) \subset \bar{I}_\mu$. As $\rho_A(X_A) = [0, 1]$ and $[0, 1) = \bigsqcup_{\mu \in B_n(X_A)} I_\mu$ is a disjoint union for a fixed $n \in \mathbb{N}$, one has $I_\mu \subset \rho_A(U_\mu)$ so that $\rho_A(U_\mu) = \bar{I}_\mu$. This also shows that ρ_A is order preserving. □

We will represent A -adic PL functions on $[0, 1]$ by using the surjection $\rho_A: X_A \rightarrow [0, 1]$. For $\tau \in \Gamma_A$, let $T_\tau = \begin{bmatrix} \mu^{(1)} & \mu^{(2)} & \dots & \mu^{(m)} \\ \nu^{(1)} & \nu^{(2)} & \dots & \nu^{(m)} \end{bmatrix}$ be its reduced representation. Let C_τ be the finite subset of $[0, 1]$ defined by

$$C_\tau = \{l(\nu(i)) \mid i = 2, 3, \dots, m\} (= \{r(\nu(i)) \mid i = 1, 2, \dots, m - 1\}).$$

Then the A -adic PL function f_τ associated with the A -adic table T_τ is continuous and linear on $[0, 1)$ except C_τ . We define a finite subset S_τ of X_A by

$$S_\tau = \{\nu(i)_{\min} \in X_A \mid i = 1, 2, \dots, m\}$$

so that $\rho_A(S_\tau) = C_\tau$.

Proposition 7.5. *For $\tau \in \Gamma$, we have $f_\tau(\rho_A(x)) = \rho_A(\tau(x))$ for all $x \in X_A \setminus S_\tau$.*

Proof. Since X_A is a disjoint union $\bigsqcup_{i=1}^M U_{v(i)}$, for $x \in X_A \setminus S_\tau$ we may take $v(i) = (v(i)_1, \dots, v(i)_{l_i})$ such that $x \in U_{v(i)}$. We write

$$x = (v(i)_1, \dots, v(i)_{l_i}, x_{l_i+1}, x_{l_i+2}, \dots).$$

As $x \notin S_\tau$, the function f_τ is continuous at x . It then follows that

$$\begin{aligned} f_\tau(\rho_A(x)) &= f_\tau(\lim_{n \rightarrow \infty} r(v(i)_1, \dots, v(i)_{l_i}, x_{l_i+1}, \dots, x_{l_i+n})) \\ &= \lim_{n \rightarrow \infty} f_\tau(r(v(i)_1, \dots, v(i)_{l_i}, x_{l_i+1}, \dots, x_{l_i+n})) \\ &= \lim_{n \rightarrow \infty} r(\mu(i)_1, \dots, \mu(i)_{k_i}, x_{l_i+1}, \dots, x_{l_i+n}) \\ &= \rho_A(\tau(x)). \end{aligned} \quad \square$$

We will next define the derivative of $\tau \in \Gamma_A$. For $\tau \in \Gamma_A$, let $l_\tau, k_\tau \in \mathbb{Z}_+$ -valued continuous functions on X_A satisfying (2.5).

Lemma 7.6. *For $\tau \in \Gamma_A$, define $d_\tau: X_A \rightarrow \mathbb{Z}$ by setting*

$$d_\tau(x) = l_\tau(x) - k_\tau(x), \quad x \in X_A.$$

Then d_τ does not depend on the choice of the functions l_τ, k_τ satisfying (2.5).

Proof. Let $l'_\tau, k'_\tau: X_A \rightarrow \mathbb{Z}_+$ be another continuous functions such that

$$\sigma_A^{k'_\tau(x)}(\tau(x)) = \sigma_A^{l'_\tau(x)}(x), \quad x \in X_A. \quad (7.2)$$

For $x = (x_i)_{i \in \mathbb{N}} \in X_A$, the identities (2.5) and (7.2) ensure us that there exist words $(\mu_1(x), \dots, \mu_{k_\tau(x)}(x)) \in B_{k_\tau(x)}(X_A)$ and $(\mu'_1(x), \dots, \mu'_{k'_\tau(x)}(x)) \in B_{k'_\tau(x)}(X_A)$ such that

$$\begin{aligned} \tau(x) &= (\mu_1(x), \dots, \mu_{k_\tau(x)}(x), x_{l_\tau(x)+1}, x_{l_\tau(x)+2}, \dots) \\ &= (\mu'_1(x), \dots, \mu'_{k'_\tau(x)}(x), x_{l'_\tau(x)+1}, x_{l'_\tau(x)+2}, \dots). \end{aligned}$$

For any $n > k_\tau(x), k'_\tau(x)$, by taking the n th coordinates of the above sequences, we see that

$$x_{n-k_\tau(x)+l_\tau(x)} = x_{n-k'_\tau(x)+l'_\tau(x)}.$$

Put $d'_\tau(x) = l'_\tau(x) - k'_\tau(x)$ and $K(x) = \text{Max}\{k_\tau(x), k'_\tau(x)\}$, so that

$$\sigma_A^{K(x)+d_\tau(x)}(x) = \sigma_A^{K(x)+d'_\tau(x)}(x).$$

Suppose that $d_\tau(x) \neq d'_\tau(x)$ for some $x \in X_A$. The above equality implies that x is an eventually periodic point. As the functions K, d_τ, d'_τ are all continuous, all elements of some neighborhood of x are eventually periodic. Since the set of non-eventually periodic points is dense in X_A , we have a contradiction and hence $d_\tau = d'_\tau$. \square

Lemma 7.7. *For $\tau, \tau_1, \tau_2 \in \Gamma_A$, we have*

- (i) $d_{\tau_2 \circ \tau_1} = d_{\tau_1} + d_{\tau_2} \circ \tau_1$.
- (ii) $d_{\tau^{-1}} = -d_{\tau} \circ \tau^{-1}$.

Proof. (i) For $\tau_i \in \Gamma_A$, take continuous functions $k_{\tau_i}, l_{\tau_i}: X_A \rightarrow \mathbb{Z}_+$ such that

$$\sigma_A^{k_{\tau_i}(x)}(\tau_i(x)) = \sigma_A^{l_{\tau_i}(x)}(x), \quad i = 1, 2, x \in X_A$$

so that

$$\sigma_A^{k_{\tau_2}(\tau_1(x))}(\tau_2(\tau_1(x))) = \sigma_A^{l_{\tau_2}(\tau_1(x))}(\tau_1(x)), \quad x \in X_A.$$

It then follows that

$$\begin{aligned} \sigma_A^{k_{\tau_1}(x)}(\sigma_A^{k_{\tau_2}(\tau_1(x))}(\tau_2(\tau_1(x)))) &= \sigma_A^{l_{\tau_2}(\tau_1(x))}(\sigma_A^{k_{\tau_1}(x)}(\tau_1(x))) \\ &= \sigma_A^{l_{\tau_2}(\tau_1(x))}(\sigma_A^{l_{\tau_1}(x)}(x)), \end{aligned}$$

so that

$$\sigma_A^{k_{\tau_1}(x) + k_{\tau_2}(\tau_1(x))}(\tau_2 \circ \tau_1(x)) = \sigma_A^{l_{\tau_1}(x) + l_{\tau_2}(\tau_1(x))}(x).$$

Hence we have

$$\begin{aligned} d_{\tau_2 \circ \tau_1}(x) &= \{l_{\tau_1}(x) + l_{\tau_2}(\tau_1(x))\} - \{k_{\tau_1}(x) + k_{\tau_2}(\tau_1(x))\} \\ &= d_{\tau_1}(x) + d_{\tau_2}(\tau_1(x)). \end{aligned}$$

(ii) By (2.5), we have

$$\sigma_A^{k_{\tau}(\tau^{-1}(x))}(x) = \sigma_A^{l_{\tau}(\tau^{-1}(x))}(\tau^{-1}(x)), \quad x \in X_A$$

so that

$$d_{\tau^{-1}}(x) = k_{\tau}(\tau^{-1}(x)) - l_{\tau}(\tau^{-1}(x)) = -d_{\tau}(\tau^{-1}(x)). \quad \square$$

Definition 7.8. For an element $\tau \in \Gamma_A$, the *derivative* D_{τ} of τ is defined by a real valued continuous function D_{τ} on X_A :

$$D_{\tau}(x) = \beta^{d_{\tau}(x)}, \quad x \in X_A, \tag{7.3}$$

where β is the Perron–Frobenius eigenvalue of the matrix A .

The derivative D_{τ} of τ is regarded as an element of \mathcal{D}_A . Recall that φ stands for the continuous linear functional on \mathcal{D}_A for the unique probability measure on X_A satisfying (3.1). The following proposition shows that D_{τ} satisfies the law of derivatives.

Proposition 7.9. *For $\tau, \tau_1, \tau_2 \in \Gamma_A$, we have*

- (i) $\varphi(D_\tau) = 1$.
- (ii) $D_{\tau_2 \circ \tau_1} = D_{\tau_1} \cdot (D_{\tau_2} \circ \tau_1)$.
- (iii) $D_{\tau^{-1}} = (D_\tau \circ \tau^{-1})^{-1}$.

Proof. (i) Suppose that τ is given by an A -adic table $T = \begin{bmatrix} \mu(1) & \mu(2) & \dots & \mu(m) \\ \nu(1) & \nu(2) & \dots & \nu(m) \end{bmatrix}$ so that $u_\tau = \sum_{i=1}^m S_{\mu(i)} S_{\nu(i)}^*$, $S_{\mu(i)}^* S_{\mu(i)} = S_{\nu(i)}^* S_{\nu(i)}$ and $\sum_{i=1}^m S_{\mu(i)} S_{\mu(i)}^* = \sum_{i=1}^m S_{\nu(i)} S_{\nu(i)}^* = 1$. Recall that the positive operator $\lambda_A: \mathcal{D}_A \rightarrow \mathcal{D}_A$ is defined by $\lambda_A(f) = \sum_{i=1}^N S_i^* f S_i$ for $f \in \mathcal{D}_A$. It then follows that

$$\lambda_A^{|\mu(i)|}(u_\tau S_{\nu(i)} S_{\nu(i)}^* u_\tau^*) = \lambda_A^{|\mu(i)|}(S_{\mu(i)} S_{\mu(i)}^*) = S_{\mu(i)}^* S_{\mu(i)} = S_{\nu(i)}^* S_{\nu(i)}$$

so that

$$\lambda_A^{|\mu(i)|}(u_\tau S_{\nu(i)} S_{\nu(i)}^* u_\tau^*) = \lambda_A^{|\nu(i)|}(S_{\nu(i)} S_{\nu(i)}^*), \quad i = 1, \dots, m.$$

As $\varphi \circ \lambda_A = \beta\varphi$ on \mathcal{D}_A , we have

$$\varphi(u_\tau S_{\nu(i)} S_{\nu(i)}^* u_\tau^*) = \beta^{|\nu(i)| - |\mu(i)|} \varphi(S_{\nu(i)} S_{\nu(i)}^*), \quad i = 1, \dots, m. \tag{7.4}$$

Since $d_\tau(x) = l_\tau(x) - k_\tau(x) = |\nu(i)| - |\mu(i)|$ for $x \in U_{\nu(i)}$, the derivative D_τ is expressed as

$$D_\tau = \sum_{i=1}^m \beta^{|\nu(i)| - |\mu(i)|} S_{\nu(i)} S_{\nu(i)}^*$$

so that by the equality (7.4) one obtains that

$$\varphi(D_\tau) = \sum_{i=1}^m \beta^{|\nu(i)| - |\mu(i)|} \varphi(S_{\nu(i)} S_{\nu(i)}^*) = \sum_{i=1}^m \varphi(u_\tau S_{\nu(i)} S_{\nu(i)}^* u_\tau^*) = \varphi(1) = 1.$$

(ii), (iii) By the previous lemma, we have

$$D_{\tau_2 \circ \tau_1} = \beta^{d_{\tau_2 \circ \tau_1}} = \beta^{d_{\tau_1}} \cdot \beta^{d_{\tau_2} \circ \tau_1} = D_{\tau_1} \cdot D_{\tau_2} \circ \tau_1,$$

$$D_{\tau^{-1}} = \beta^{-d_{\tau \circ \tau^{-1}}} = [D_\tau \circ \tau^{-1}]^{-1}. \quad \square$$

As the function f_τ is linear on the interval $I_{\nu(i)} = [l(\nu(i)), r(\nu(i))]$ with slope $\beta^{|\nu(i)| - |\mu(i)|}$, we may summarize the above discussions in the following theorem.

Theorem 7.10. *There exists an order preserving continuous surjection $\rho_A: X_A \rightarrow [0, 1]$ from the shift space X_A of a one-sided topological Markov shift (X_A, σ_A) to the closed interval $[0, 1]$ such that for any element $\tau \in \Gamma_A$, there exists a finite set $S_\tau \subset X_A$ such that the corresponding A -adic PL function f_τ for τ satisfies the following properties:*

- (i) $f_\tau(\rho_A(x)) = \rho_A(\tau(x))$ for $x \in X_A \setminus S_\tau$,
- (ii) $\frac{df_\tau}{dt}(\rho_A(x)) = D_\tau(x) = \beta^{d_\tau(x)}$ for $x \in X_A \setminus S_\tau$,

where $d_\tau(x) = l_\tau(x) - k_\tau(x)$ for the continuous functions $k_\tau, l_\tau: X_A \rightarrow \mathbb{Z}_+$ satisfying $\sigma_A^{k_\tau(x)}(\tau(x)) = \sigma_A^{l_\tau(x)}(x), x \in X_A$ and β is the Perron–Frobenius eigenvalue of A .

8. Generalizations of other Thompson groups

R. J. Thompson has defined finitely presented infinite subgroups F_2 and T_2 of V_2 which satisfy $F_2 \subset T_2 \subset V_2$. K. S. Brown [1] has extended the subgroups F_2, T_2 of V_2 to the family $F_N \subset T_N \subset V_N$ of finitely presented subgroups F_N, T_N of V_N such that T_N is a group of piecewise linear homeomorphisms $f: [0, 1] \rightarrow [0, 1]$ on the unit circle having finitely many singularities such that all singularities of f are in $\mathbb{Z}[\frac{1}{N}]$, the derivative of f at any non-singular point is N^k for some $k \in \mathbb{Z}$, and F_N is a subgroup of T_N consisting of piecewise linear homeomorphisms $f: [0, 1] \rightarrow [0, 1]$ on the unit interval.

In this section, we generalize the groups F_N, T_N for $1 < N \in \mathbb{N}$ to F_A, T_A for irreducible square matrices A with entries in $\{0, 1\}$ by using the techniques of the preceding sections.

Recall that an element $\tau \in \Gamma_A$ is represented as a cylinder map given by two families $\mu(i), \nu(i), i = 1, \dots, m$ of words satisfying (2.1)–(2.4). We may assume that the words $\nu(i), i = 1, \dots, m$ are ordered such as $\nu(1) \prec \nu(2) \prec \dots \prec \nu(m)$. We define further properties for $\tau \in \Gamma_A$ as follows. $\tau \in \Gamma_A$ is said to be

- (i) *order preserving* if one may take the words $\mu(i), i = 1, \dots, m$ such as

$$\mu(1) \prec \mu(2) \prec \dots \prec \mu(m),$$

- (ii) *cyclic order preserving* if one may take the words $\mu(i), i = 1, \dots, m$ such as

$$\mu(k) \prec \mu(k + 1) \prec \dots \prec \mu(m) \prec \mu(1) \prec \mu(2) \prec \dots \prec \mu(k - 1)$$

for some $k \in \{1, 2, \dots, m\}$.

If τ is order preserving, it is cyclic order preserving. It is easy to see that the set of order preserving cylinder maps forms a subgroup of Γ_A , and the set of cyclic order preserving cylinder maps forms a subgroup of Γ_A . We denote them by F_A and by T_A and call them the order preserving continuous full group and the cyclic order preserving continuous full group, respectively.

In Definition 4.5 (ii), if one may take σ such as

$$\sigma(k) < \sigma(k + 1) < \cdots < \sigma(m) < \sigma(1) < \sigma(2) < \cdots < \sigma(k - 1) \tag{8.1}$$

for some $k \in \{1, \dots, m\}$, the A -adic pattern of rectangles is said to be *A-adic cyclic order preserving pattern of rectangles*. If in particular one may take σ such as $\sigma = \text{id}$, the A -adic pattern of rectangles is said to be *A-adic order preserving pattern of rectangles*.

In Definition 4.6, if one may take σ such as (8.1) for some $k \in \{1, \dots, m\}$, an A -adic PL function f is called a *cyclic order preserving A-adic PL function*. If in particular, one may take $\sigma = \text{id}$, f is called an *order preserving A-adic PL function*.

It is easy to see that the set F_A^{PL} of order preserving A -adic PL functions and the set T_A^{PL} of cyclic order preserving A -adic PL functions form subgroups of the group of the A -adic PL functions. Hence we have subgroups of inclusion relations:

$$F_A^{\text{PL}} \subset T_A^{\text{PL}} \subset \Gamma_A^{\text{PL}}.$$

The following proposition is immediate by definition of order preserving (resp. cyclic order preserving) A -adic PL functions.

Proposition 8.1. *An A-adic order preserving (resp. cyclic order preserving) PL function naturally gives rise to an A-adic order preserving (resp. cyclic order preserving) pattern of rectangles, whose rectangle slopes are the slopes of the A-adic PL function. Conversely, an A-adic order preserving (resp. cyclic order preserving) pattern of rectangles gives rise to an A-adic order preserving (resp. cyclic order preserving) PL function by taking its diagonal lines of the corresponding rectangles.*

In Definition 5.1, let $T = \begin{bmatrix} \mu(1) & \mu(2) & \cdots & \mu(m) \\ v(1) & v(2) & \cdots & v(m) \end{bmatrix}$ be an A -adic table such that $v(1) < v(2) < \cdots < v(m)$. Then T is said to be

- (i) *order preserving* if $\mu(1) < \mu(2) < \cdots < \mu(m)$,
- (ii) *cyclic order preserving* if

$$\mu(k) < \mu(k + 1) < \cdots < \mu(m) < \mu(1) < \mu(2) < \cdots < \mu(k - 1)$$

for some $k \in \{1, 2, \dots, m\}$.

If T is order preserving, it is cyclic order preserving. These two properties of A -adic tables are closed under taking expansions of A -adic tables respectively. We see that the set F_A^{tab} of the equivalence classes of order preserving A -adic tables and the set T_A^{tab} of the equivalence classes of cyclic order preserving A -adic tables form subgroups of Γ_A^{tab} , respectively. Hence we have subgroups of inclusion relations:

$$F_A^{\text{tab}} \subset T_A^{\text{tab}} \subset \Gamma_A^{\text{tab}}.$$

We further see the following:

Lemma 8.2. *For a table T , let f_T be the associated A -adic PL function. Then T is order preserving (resp. cyclic order preserving) if and only if the function $f_{[T]}$ is order preserving (resp. cyclic order preserving).*

We thus have:

Proposition 8.3. *There exist canonical isomorphisms of discrete groups among the order preserving (resp. cyclic order preserving) continuous full group F_A (resp. T_A), the group F_A^{tab} (resp. T_A^{tab}) of the equivalence classes of order preserving (resp. cyclic order preserving) A -adic tables and the group F_A^{PL} (resp. T_A^{PL}) of the order preserving (resp. cyclic order preserving) A -adic PL functions on $[0, 1)$, that is*

$$F_A \cong F_A^{\text{tab}} \cong F_A^{\text{PL}}, \quad T_A \cong T_A^{\text{tab}} \cong T_A^{\text{PL}}.$$

Proof. The isomorphisms in Proposition 5.8 and Theorem 6.3 among Γ_A , Γ_A^{tab} and Γ_A^{PL} preserve the orders of words, so that its restrictions yield desired isomorphisms. □

In [1], K. S. Brown had extended the Higman–Thomson group V_N to infinite families $F_{N,r} \subset T_{N,r} \subset V_{N,r}$ for $N = 2, 3, \dots, r \in \mathbb{N}$ where $V_{N,1} = V_N$ and $F_{N,1} = F_N, T_{N,1} = T_N$. Let A_N be the $N \times N$ matrix whose entries are all 1's. Then our groups $F_{A_N}, T_{A_N}, V_{A_N}$ for the matrix A_N are nothing but the Brown's triple $F_{N,1}, T_{N,1}, V_{N,1}$ for $r = 1$, respectively. Let $A_{N,r}$ be the $r \times r$ block matrix whose entries are $N \times N$ matrices such that

$$\begin{bmatrix} 0 & \dots & \dots & 0 & A_N \\ 1_N & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & 1_N & 0 & 0 \\ 0 & \dots & 0 & 1_N & 0 \end{bmatrix}$$

where 1_N denotes the identity matrix of size N . Since there exists an isomorphism from the Cuntz–Krieger algebra $\mathcal{O}_{A_{N,r}}$ for the matrix $A_{N,r}$ to the tensor product $\mathcal{O}_{A_N} \otimes M_r(\mathbb{C})$ such that $\mathcal{D}_{A_{N,r}} = \mathcal{D}_{A_N} \otimes D_r$, where D_r is the commutative C^* -algebra of the diagonal elements of the $r \times r$ full matrix algebra $M_r(\mathbb{C})$, our groups $F_{A_{N,r}}, T_{A_{N,r}}, V_{A_{N,r}}$ for the matrix $A_{N,r}$ are nothing but the Brown's triple $F_{N,r}, T_{N,r}, V_{N,r}$ (see [18], [19]). Since $\det(\text{id} - A_{N,r}) = 1 - N$, the classification of the Higman–Thompson groups $V_{N,r}$ corresponds to that of the C^* -algebras $\mathcal{O}_N \otimes M_r(\mathbb{C})$ through Theorem 1.1 (see [24, Corollary 6.6], [21]).

In [18], generalization of higher dimensional analogue of Thomson like groups are studied from the view point of étale groupoids.

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