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On the genericity of pseudo-Anosov braids I: rigid braids

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Abstract. We prove that, in the l-ball of the Cayley graph of the braid group with $n \geqslant 3$ strands, the proportion of rigid pseudo-Anosov braids is bounded below independently of l by a positive value.

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1. Introduction

A natural question concerning the Nielsen-Thurston classification of braids is: what is the most likely Nielsen-Thurston type of a "long random braid"? Different interpretations can be given to this question, but in this paper we shall use the following setting. We consider the Cayley graph of the braid group B_n (for a fixed number of strands n), with generators the set of simple braids – this is the standard generating set when the braid group is studied as a Garside group. A well-known conjecture since the work of Thurston is as follows.

Conjecture. *The proportion of pseudo-Anosov braids among all elements in the ball of radius l in the Cayley graph converges to* 1 *as l tends to infinity.*

The best known results going into this direction were, to the best of our knowledge, the classical paper of Fathi [9], and the article of Atalan and Korkmaz [1] which deals with the case of three-strand braids. The present paper, together with the article [4], contains a proof of the above conjecture. In this first part, we introduce some essential tools needed for the proof in [4], and already prove a result of independent interest concerning the proportion of *rigid* pseudo-Anosov braids (see Corollary 4.9):

Theorem. *For sufficiently large l, the proportion of rigid pseudo-Anosov braids in the ball of radius l in the Cayley graph of B_n is bounded below by a strictly positive constant which does not depend on l (but might depend on n).*

The proof is in two steps: we shall see that the proportion of so-called *rigid* braids is bounded below independently of l, and among rigid braids the proportion of pseudo-Anosov elements converges to 1.

Another possible interpretation of the original question should be pointed out: the works of Rivin [14], Maher [13], and Sisto [15] deal with braids obtained by a long random walk in the Cayley graph. They prove that in this setting, as well, the probability of obtaining a pseudo-Anosov braid converges to 1 as the length of the walk tends to infinity.

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2. Definitions

Throughout the article, we fix an integer $n \geqslant 3$. All the considered braids will be braids with n strands.

2.1. Garside structure. A general introduction to Garside theory can be found in [7]. The reader can also consult [8]. We shall only recall some facts which are useful for our purposes.

While the group $\mathcal{B}_n$ admits the well-known presentation of groups

$$\mathcal{B}_n = \left\langle \sigma_1, \dots, \sigma_{n-1};\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ and } \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geqslant 2 \right\rangle,$$

the *monoid of positive braids* $\mathcal{B}_n^+$, which is embedded in $\mathcal{B}_n$, is defined by the same presentation, interpreted as a presentation of monoids.

For $i < j \leqslant n$, we denote by Δ_{ij} the element of $\mathcal{B}_n^+$ defined by

$$\Delta_{ij} = (\sigma_i \cdots \sigma_{j-1})(\sigma_i \cdots \sigma_{j-2}) \cdots (\sigma_i\sigma_{i+1})\sigma_i$$

and we denote by $\Delta = \Delta_{1n} \in \mathcal{B}_n^+$.

The pair $(\mathcal{B}_n^+, \Delta)$ defines what we call a Garside structure on $\mathcal{B}_n$. Without giving the complete definition, here are some properties of such a structure. The group $\mathcal{B}_n$ is endowed with a partial order $\preccurlyeq$ defined by

$$x \preccurlyeq y \iff x^{-1}y \in \mathcal{B}_n^+.$$

If $x \preccurlyeq y$, we say that x is a *prefix* of y. Any two elements x, y of $\mathcal{B}_n$ have a unique greatest common prefix.

We also define $\succcurlyeq$ by

$$x \succcurlyeq y \iff xy^{-1} \in \mathcal{B}_n^+.$$

Note that $x \succcurlyeq y$ is not equivalent to $y \preccurlyeq x$.

The elements of the set $\{x \in \mathcal{B}_n, 1 \preccurlyeq x \preccurlyeq \Delta\}$ are called *simple braids*.

Proposition 2.1. *The set of simple braids is in bijection with the set* $\mathfrak{S}_n$ *of permutations of n elements,* ***via*** *the canonical projection from* $\mathcal{B}_n$ *to* $\mathfrak{S}_n$.

Definition 2.2 (left-weighting). Let s_1, s_2 be two simple braids in $\mathcal{B}_n$ (possibly $s_1 = s_2$). We say that s_1 and s_2 are *left-weighted*, or that the pair (s_1, s_2) is left-weighted, if there does not exist any generator σ_i such that $s_1\sigma_i$ and $\sigma_i^{-1}s_2$ are both still simple.

Definition 2.3 (starting set, finishing set). Let $s \in \mathcal{B}_n$ be a simple braid. We call *starting set of* s the set $S(s) = \{i, \sigma_i \preccurlyeq s\}$ and *finishing set of* s the set $F(s) = \{i, s \succcurlyeq \sigma_i\}$.

Remark 2.4. Two simple braids s_1 and s_2 are left-weighted if and only if $S(s_2) \subset F(s_1)$.

Remark 2.5. Let s be a simple braid, and π be the permutation associated to s. Then $i \in S(s)$ if and only if $\pi(i) > \pi(i+1)$, and $i \in F(s)$ if and only if $\pi^{-1}(i) > \pi^{-1}(i+1)$.

Proposition 2.6. *Let* $x \in \mathcal{B}_n$. *There exists a unique decomposition* $x = \Delta^p x_1 \cdots x_r$ *such that* $x_1, \ldots, x_r$ *are simple braids, distinct from* Δ *and* 1, *and such that* x_i *and* x_{i+1} *are left-weighted for all* $i = 1, \ldots, r-1$.

Definition 2.7 (left normal form). In the previous proposition, the writing $x = \Delta^p x_1 \cdots x_r$ is called the *left normal form* of x, p is called the *infimum* of x and is denoted by $\inf x$, $p + r$ is the *supremum* of x and is denoted by $\sup x$, and r is called the *canonical length* of x.

Furthermore, if $r \geqslant 1$, we denote by $\iota(x) = \Delta^p x_1 \Delta^{-p}$ the *initial factor* of x ($\iota(x) = x_1$ if p is even, $\iota(x) = \Delta x_1 \Delta^{-1}$ if p is odd), and $\phi(x) = x_r$ its *final factor*.

A key of the proof will be that every element β of $\mathcal{B}_n$ is represented by a unique normal form word, whose shape determines the distance of β from the neutral element in the Cayley graph (Lemma 4.7). This allows us to replace the counting of elements in a ball in the Cayley graph by the much easier counting of normal form words of bounded length.

Definition 2.8 (rigidity). A braid x of positive canonical length is said to be *rigid* if $\phi(x)$ and $\iota(x)$ are left-weighted.

2.2. Braids and mapping class group of the punctured disk

Definition 2.9 (Mapping class group of the punctured disk). Let D_n be the closed unit disk in $\mathbb{C}$, with n punctures regularly spaced on the real axis. The *mapping class group* of D_n, denoted $\mathrm{Mod}(D_n)$, is the group of homeomorphisms of D_n, modulo the isotopy relation. We also denote $\mathrm{Mod}(D_n, \partial D_n)$ the group of homeomorphisms of D_n fixing pointwise the boundary ∂D_n of D_n, modulo the isotopy relation.

The Artin braid group with n strands is isomorphic to the group $\mathrm{Mod}(D_n, \partial D_n)$.

Recall that the classification theorem of Nielsen and Thurston states that a mapping class $f \in \mathrm{Mod}(D_n)$ is exactly one of the following: periodic, or reducible non-periodic, or pseudo-Anosov. A braid $x \in \mathrm{Mod}(D_n, \partial D_n)$ can be projected on an element of $\mathrm{Mod}(D_n)$. We call *Nielsen-Thurston type of* x the Nielsen-Thurston type of its projection. The definition of periodicity is then transformed as follows: a braid $x \in \mathcal{B}_n$ is periodic if and only if there exist nonzero integers m and l such that $x^m = \Delta^l$, where $\Delta = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2)\sigma_1$. (Geometrically Δ corresponds to the half-twist around the boundary of the disk).

2.3. Round curves and almost round curves. Let us consider a braid as a mapping class in the mapping class group $\mathrm{Mod}(D_n, \partial D_n)$.

Definition 2.10 (curve). We call *closed curve* in D_n the image of the circle $\mathbb{S}^1$ by a continuous map with values in D_n. The curve is said to be *simple* if this map is injective. It is said to be *non degenerated* if it is neither homotopic to a point, nor to the boundary of the disk, and it bounds a least two punctures.

In the following, we simply call curve a homotopy class of non degenerate simple closed curves. We shall use the *right* action of the mapping class group on the set of curves.

Definition 2.11 (round curve). A curve is said to be *round* if it is represented by a circle in D_n.

Definition 2.12 (almost round curve). A curve is said to be *almost round* if it is not round, and is the image by a simple braid of a round curve.

3. Properties of the left-weighting graph

Definition 3.1 (Left-weighting graph). We call *left-weighting graph*, denote by G_{lw}, the following finite oriented graph. The vertices are indexed by the simple braids except 1 and Δ, and there is an edge from the vertex x_1 to the vertex x_2 if and only if the pair (x_1, x_2) is left-weighted.

We call *path* a sequence $(x_1 \to x_2 \to \cdots \to x_l)$ such that there is an edge from the vertex x_i to the vertex x_{i+1}, and the *length* of such a path means the number of edges in the path.

The objective of this section is to study some properties of the graph G_{lw}, especially some asymptotic properties of the number of paths of length l, with l tending to infinity. We introduce the following notations, for all $l \in \mathbb{N}^*$.

- $N(l)$ is the number of paths $(x_1 \to x_2 \to \cdots \to x_{l+1})$ of length l in G_{lw}.
- $N_\circ(l)$ is the number of loops of length $l+1$, with marked base vertex, in G_{lw}. The quantity $N_\circ(l)$ can also be seen as the number of paths of length l, such that there is an edge from the last to the first vertex.
- Let w be a path of length $k \in \mathbb{N}^*$ in G_{lw}. We denote by $N^{(w)}(l)$ the number of paths of length l in G_{lw} that do not pass through w (*ie* that do not contain w as a subpath), and $N_\circ^{(w)}(l)$ the number of loops of length $l+1$ with marked base vertex in G_{lw}, that do not pass through w.

Furthermore, if (u_l) and (v_l) are two sequences of real numbers, we write $u_l = \Theta(v_l)$ if and only if there exist constants $c_1, c_2 > 0$ such that for all large enough l, $c_1 v_l < u_l < c_2 v_l$. We say that u_l *is of the order of* v_l.

We also use the usual notations $u_l \sim v_l$ when u_l *is equivalent to* v_l, that is when for all $\varepsilon > 0$, there exists an integer L such that for all $l > L$, $|u_l - v_l| < \varepsilon |v_l|$, and $u_l = O(v_l)$ when there exists $c_2 > 0$ such that for all large enough l, $u_l < c_2 v_l$.

We will prove some properties of the left-weighting graph by using the notion of adjacency matrix. For more details on graph theory and adjacency matrices, the reader can consult [11]. We recall the following definition and proposition, together with the theorem of Perron-Frobenius.

Definition 3.2 (adjacency matrix). Let G be an oriented finite graph, whose vertices are numbered. We call *adjacency matrix of* G the matrix whose (i, j)-entry contains the number of edges from the vertex i to the vertex j.

Proposition 3.3. *Let G be an oriented finite graph and A its adjacency matrix. Let $l \in \mathbb{N}$. The (i, j)-entry of the matrix A^l contains the number of paths of length l in G linking the vertex i to the vertex j.*

Theorem (Perron-Frobenius). *Let A be a matrix such that there exists $k \in \mathbb{N}^*$, such that all entries of A^k are positive. Then the spectral radius of A is positive, is a simple eigenvalue of A, and is the unique eigenvalue of maximal module.*

Lemma 3.4. *Each pair of vertices in G_{lw} is linked by at least one path of length exactly 5.*

Proof. Let us recall that two simple braids s and t are left-weighted if and only if $S(t) \subset F(s)$. Let s_1 and s_2 be two simple braids distinct from 1 and Δ. There exist i_1 and i_2 in $\{1, \dots, n-1\}$ such that $F(s_1) \supset \{i_1\}$ and $S(s_2) \subset \{1, \dots, n-1\} \backslash \{i_2\}$. We will construct some simple braids x_1, x_2, x_3, x_4 satisfying:

- $S(x_1) = \{i_1\}$,
- $F(x_1) = S(x_2) = \{\lfloor \frac{n}{2} \rfloor\}$,
- $F(x_2) = S(x_3) = \{1, 3, \dots, 2\lfloor \frac{n}{2} \rfloor - 1\}$ (the set of odd numbers between 1 and $n-1$),
- $F(x_3) = S(x_4) = \{1, \dots, n-1\} \backslash \{\lfloor \frac{n}{2} \rfloor\}$,
- $F(x_4) = \{1, \dots, n-1\} \backslash \{i_2\}$.

Thus, $s_1 \to x_1 \to x_2 \to x_3 \to x_4 \to s_2$ will be a path of length 5 in the graph G_{lw}.

Here is how we choose the braids x_1, x_2, x_3, x_4. We set $x_1 = \sigma_{i_1} \cdots \sigma_{\lfloor \frac{n}{2} \rfloor}$. The simple braid x_2 is the braid corresponding to following permutation:

$$\pi_2 = \begin{pmatrix} 1 & 2 & \cdots & \lfloor \frac{n}{2} \rfloor & \lfloor \frac{n}{2} \rfloor + 1 & \lfloor \frac{n}{2} \rfloor + 2 & \cdots & n \\ 2 & 4 & \cdots & 2\lfloor \frac{n}{2} \rfloor & 1 & 3 & \cdots & 2\lceil \frac{n}{2} \rceil - 1 \end{pmatrix}$$

As to the braid x_3, it is equal to $\bar{x}_2 \Delta_{1, \lfloor \frac{n}{2} \rfloor} \Delta_{\lfloor \frac{n}{2} \rfloor + 1, n}$, where $\bar{x}_2$ is the simple braid of permutation π_2^{-1}. Finally, $x_4 = \Delta \sigma_{\lceil \frac{n}{2} \rceil}^{-1} \cdots \sigma_{i_2}^{-1}$ is the left complement of $\sigma_{i_2} \cdots \sigma_{\lceil \frac{n}{2} \rceil}$. The braids x_1 to x_4 are represented for $n = 6$ in Figure 1.

Of course $S(x_1) = \{i_1\}$ and $F(x_1) = \{\lfloor \frac{n}{2} \rfloor\}$. For x_2, the permutation π_2 is increasing on $\{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ and on $\{\lfloor \frac{n}{2} \rfloor + 1, \dots, n\}$, and we have $\pi_2(\lfloor \frac{n}{2} \rfloor + 1) < \pi_2(\lfloor \frac{n}{2} \rfloor)$, so $S(x_2) = \{\lfloor \frac{n}{2} \rfloor\}$. On the other hand, $\pi_2^{-1}(i) > \pi_2^{-1}(i+1)$ if and only if i is odd, hence $F(x_2) = \{1, 3, \dots, 2\lfloor \frac{n}{2} \rfloor - 1\}$. The permutation π_3 associated with x_3 first applies π_2^{-1}, then reverses the order, on the one hand, of the elements from 1 to $\lfloor \frac{n}{2} \rfloor$, and on the other hand, of $\lfloor \frac{n}{2} \rfloor + 1$ to n. It follows that $\pi_3(i) > \pi_3(i+1)$ if and only if i is odd, and that $\pi_3^{-1}(i) > \pi_3^{-1}(i+1)$ for all i except $i = \lfloor \frac{n}{2} \rfloor$. So $S(x_3) = \{1, 3, \dots, 2\lfloor \frac{n}{2} \rfloor - 1\}$ and $F(x_3) = \{1, \dots, n-1\} \backslash \{\lfloor \frac{n}{2} \rfloor\}$. Finally, x_4 is the left complement of $\sigma_{i_2} \cdots \sigma_{\lceil \frac{n}{2} \rceil}$, and thus satisfies $S(x_4) = \{1, \dots, n-1\} \backslash \{n - \lceil \frac{n}{2} \rceil\} = \{1, \dots, n-1\} \backslash \{\lfloor \frac{n}{2} \rfloor\}$ and $F(x_4) = \{1, \dots, n-1\} \backslash \{i_2\}$. □

Lemma 3.5. *The following properties are true.*

(i) *There exists a constant λ such that $N_\circ(l) \sim \lambda^{l+1}$.*

(ii) *We have $N(l) = \Theta(\lambda^l)$. In particular, for large enough l, the proportion $N_\circ(l)/N(l)$ is bounded below, independently of l, by a positive constant.*

(iii) *For all path w, there exists a constant $\mu_{(w)} < \lambda$ such that $N^{(w)}(l) = O(\mu_{(w)}^l)$ and $N_\circ^{(w)}(l) = O(\mu_{(w)}^l)$.*

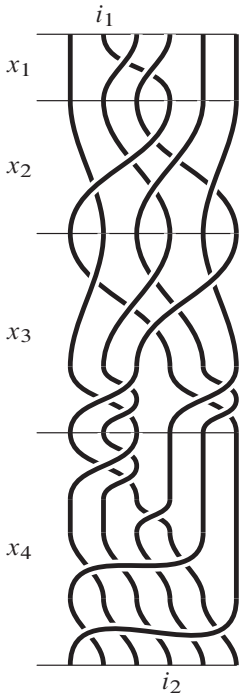

Figure 1. Braids x_1 to x_4

The reader can also consult [6], which contains results and proofs similar to those of this lemma.

Proof of Lemma 3.5. Let A be the adjacency matrix of the graph G_{lw}. According to Proposition 3.3, $N_\circ(l) = \mathrm{tr}(A^{l+1})$ and $N(l) = |A^l|_1$, where $|\cdot|_1$ is the sum of all entries in the matrix.

(i) According to Lemma 3.4, the matrix A^5 has positive entries. So we can apply the Perron-Frobenius theorem to A, and deduce that A has a unique eigenvalue of maximal module. This value is real and positive, and the associated eigenspace has dimension 1. We denote by λ this eigenvalue, and by λ_i ($i = 1, \ldots, n! - 3$) the others (not necessarily distinct and not necessarily real). We have $\mathrm{tr}(A^{l+1}) = \lambda^{l+1} + \lambda_1^{l+1} + \cdots + \lambda_{n!-3}^{l+1}$, hence $N_\circ(l) \sim \lambda^{l+1}$ when l tends to infinity.

(ii) There exists an invertible matrix P such that PAP^{-1} is in Jordan normal form, and we can calculate

$$|PA^lP^{-1}|_1 = \lambda^l + \sum p_i(\lambda_i)$$

where the p_i are some polynomials of degree l. We deduce again the equivalence $|PA^lP^{-1}|_1 \sim \lambda^l$. Furthermore, $|PA^lP^{-1}|_1 = \Theta(|A^l|_1)$, so $N(l) = \Theta(\lambda^l)$.

We deduce that $N_\circ(l)/N(l) = \Theta(1)$, and in particular, that for large enough l, this ratio is bounded below, independently of l, by a positive constant.

(iii) We construct from G_{lw} a graph $G_{lw}^{(k)}$ (where, we recall, k is the length of the path w) as follows: the vertices in $G_{lw}^{(k)}$ are the paths of length $k-1$ in G_{lw}, and two paths $w_1 = (s_1 \to \cdots \to s_k)$ and $w_2 = (t_1 \to \cdots \to t_k)$ are linked by an edge if and only if $s_2 = t_1, s_3 = t_2, \ldots, s_k = t_{k-1}$. Thus, the edges of $G_{lw}^{(k)}$ correspond to the paths of length k in G_{lw}. We denote by A_k the adjacency matrix of $G_{lw}^{(k)}$.

If s and t are two vertices in G_{lw}, a path of length $l \geqslant k$ from s to t in G_{lw} corresponds to a path of length $l-k+1$ from $(s \to s_2 \to \cdots \to s_k)$ to $(t_1 \to \cdots \to t_k \to t)$ in $G_{lw}^{(k)}$ for some $s_2, \ldots, s_{k-1}, t_1, \ldots, t_{k-2}$. This leads to the following consequences. As each pair of vertices in G_{lw} is linked by a path of length 5 (Lemma 3.4), each pair of vertices in $G_{lw}^{(k)}$ is linked by a path of length exactly $k+4$. Furthermore, as the number of paths of length l in G_{lw} is a $\Theta(\lambda^l)$, it is the same for the number of paths of length l in $G_{lw}^{(k)}$. As A_k^{k+4} has positive entries, A_k satisfies the hypothesis of the Perron-Frobenius theorem, and we deduce, as in (ii), that the number of paths of length l in $G_{lw}^{(k)}$ is a $\Theta(\lambda_{(k)}^l)$ where $\lambda_{(k)}$ is the spectral radius of A_k. The two asymptotic estimates obtained ensure that $\lambda_{(k)} = \lambda$.

Moreover, avoiding a path of length k in G_{lw} is equivalent to avoiding an edge in $G_{lw}^{(k)}$. Let $\tilde{G}_{lw}^{(k)}$ be the graph obtained from $G_{lw}^{(k)}$ by removing the edge a_w corresponding to w. We denote by $\tilde{A}_k$ its adjacency matrix, and by $\mu_{(w)}$ the spectral radius of this matrix. Then $\tilde{A}_k$ is a non-negative matrix; thus its spectral radius $\mu_{(w)}$ is a (real non-negative) eigenvalue of $\tilde{A}_k$ [12, Theorem 8.3.1].

Now, the number of paths of length $l-k+1$ in $G_{lw}^{(k)}$ is a $O(\mu_{(w)}^l)$: indeed, as in (ii), there exists an invertible matrix Q such that $|Q\tilde{A}_k^{l-k+1}Q^{-1}|_1$ is a sum of polynomials of degree $l-k+1$ in the eigenvalues of $\tilde{A}_k$. As these eigenvalues are, in module, not greater than the spectral radius $\mu_{(w)}$, we deduce that $|Q\tilde{A}_k^{l-k+1}Q^{-1}|_1 = O(\mu_{(w)}^{l-k+1}) = O(\mu_{(w)}^l)$, and then, that $N^{(w)}(l) = |\tilde{A}_k^{l-k+1}|_1 = O(\mu_{(w)}^l)$.

As for the number of loops of length $l+1$ with marked base point in G_{lw}, their number is not greater than the number of paths of length l, and so we have also $N_\circ^{(w)}(l) = O(\mu_{(w)}^l)$.

It remains to prove that $\mu_{(w)} < \lambda$.

Given two vertices $w_1 = (s_1 \to \cdots \to s_k)$ and $w_2 = (t_1 \to \cdots \to t_k)$ of $G_{lw}^{(k)}$, there always exists a path of length $l_0 = 2k+9$ in $G_{lw}^{(k)}$ from w_1 to w_2 passing through the edge a_w: indeed, it suffices to go with a path of length $k+4$

until the starting vertex of a_w, to go through the edge a_w, and to go again with a path of length $k+4$ until w_2. This means that there are strictly more paths of length l_0 from w_1 to w_2 in $G_{lw}^{(k)}$ than in $\tilde{G}_{lw}^{(k)}$. That is to say, all the entries of the matrix $A_k^{l_0} - \tilde{A}_k^{l_0}$ are positive. Let $\varepsilon > 0$ be such that $A_k^{l_0} - (\tilde{A}_k^{l_0} + \varepsilon I)$ has still only positive entries (where I is the identity matrix). The spectral radius of $A_k^{l_0}$ is λ^{l_0}, the one of $\tilde{A}_k^{l_0} + \varepsilon I$ is $\mu_{(w)}^{l_0} + \varepsilon$. Recall that the spectral radius of a matrix B is the limit of $\|B^k\|^{\frac{1}{k}}$ when k tends to infinity, where $\|\cdot\|$ is any matrix norm. By choosing for $\|\cdot\|$, for example, the infinity-norm, we deduce that, as the entries of $A_k^{l_0}$ are all greater than those of $(\tilde{A}_k^{l_0} + \varepsilon I)$, we have $\lambda^{l_0} \geqslant \mu_{(w)}^{l_0} + \varepsilon$, and thus $\lambda > \mu_{(w)}$. □

Remark 3.6. By similar arguments, we obtain finer results, on the number of paths that do not contain w *in a more localized area of the path*. More precisely, if β is a path of length l, and if a_1, a_2, a_3 are functions of l taking values in $\mathbb{N}$, with $a_1 + a_3$ and a_2 nondecreasing functions that tends to infinity when l tends to infinity, and such that $a_1(l) + a_2(l) + a_3(l) = l$, we can cut the path β into three path β_1, β_2 and β_3 of respective lengths $a_1(l)$, $a_2(l)$ and $a_3(l)$. The number of paths β of length l whose "middle part" β_2 does not contain the path w is a $\Theta(\mu_{(w)}^{a_2(l)} \lambda^{a_1(l)+a_3(l)}) = \Theta(\mu_{(w)}^{a_2(l)} \lambda^{l-a_2(l)})$.

4. Genericity of pseudo-Anosov braids

4.1. Proportion of rigid braids

Proposition 4.1. *Let $l \in \mathbb{N}^*$. Among the braids β such that $\inf \beta = 0$ and $\sup \beta = l$, the proportion of rigid braids is bounded below independently of l by a positive constant.*

Proof. According to the unicity of the left normal form of a braid, the set of all braids β such that $\inf \beta = 0$ and $\sup \beta = l$ is in bijection with the set of paths of length l in the left-weighting graph G_{lw}. The set of *rigid* braids of infimum 0 and supremum l is in bijection with the set of paths of length l, for which there is an edge from the last to the first vertex. Hence, the proposition is a corollary of Lemma 3.5, (ii). □

4.2. Proportion of non pseudo-Anosov braids with infimum 0. The aim of this section is to show that, among the rigid braids of some fixed infimum and canonical length l, the proportion of non pseudo-Anosov braids tends to 0 when l tends to infinity. For this, we can use the following theorem, due to González-Meneses and Wiest [10] (Theorem 5.16):

Theorem 4.2. *Let β be a non-periodic, reducible braid which is rigid. Then there is some positive integer $k \leqslant n$ such that one of the following conditions holds*:

(1) *β^k preserves a round curve, or*

(2) *$\inf(\beta^k)$ and $\sup(\beta^k)$ are even, and either $\Delta^{-\inf(\beta^k)}\beta^k$ or $\beta^{-k}\Delta^{\sup(\beta^k)}$ is a positive braid which preserves an almost round curve whose corresponding interior strands do not cross.*

Let us also state the following theorem of Bernadete, Gutierrez, and Nitecki (Theorem 5.7 in [2]) as given in [3] (Theorem 1). We we recall that $\mathcal{B}_n$ acts on the *right* on the set of curves.

Proposition 4.3. *Let $x \in \mathcal{B}_n$, seen as a mapping class in $\mathrm{Mod}(D_n, \partial D_n)$, with left normal form $x = \Delta^p x_1 \cdots x_r$. Let $\mathcal{C}$ be a round curve in D_n. If $x(\mathcal{C})$ is round, then $\Delta^p x_1 \cdots x_m(\mathcal{C})$ is round for all $m = 1, \ldots, r$.*

Notation 4.4. In what follows we shall use the following two braids, written in normal form as follows:

$$\gamma_1 = \sigma_1\sigma_3\cdots\sigma_{2\lfloor\frac{n}{2}\rfloor-1} \,.\, \sigma_1\sigma_3\cdots\sigma_{2\lfloor\frac{n}{2}\rfloor-1}\sigma_2\sigma_4\cdots\sigma_{2\lceil\frac{n}{2}\rceil-2} \quad \text{(length 2)},$$

$$\gamma_2 = \Delta_{2,n}\sigma_1 \,.\, \sigma_1 \,.\, \sigma_1\sigma_2\cdots\sigma_{n-1} \,.\, \sigma_{n-1} \quad \text{(length 4)}.$$

(See Figures 2 and 3.)

Proposition 4.5. *A rigid braid whose normal form contains both γ_1 and γ_2 as subwords is pseudo-Anosov.*

Proof. Let us study a rigid braid β, denoting $\inf(\beta) = \epsilon$ and the canonical length of β as l.

First, we remark that there is no periodic rigid braid except Δ^ϵ. Indeed, if a braid β is rigid and has canonical length at least 1, then its left normal form is of the shape

$$\beta = \Delta^\epsilon s_1 s_2 \cdots s_l$$

where (s_i, s_{i+1}) $(i = 1, \ldots, l-1)$ and $(s_l, \tau^{-\epsilon}s_1)$ are left-weighted. Therefore, the normal form of a power of this braid is of the shape

$$\beta^k = \Delta^{k\epsilon} s_1^{(1)} s_2^{(1)} \cdots s_l^{(1)} s_1^{(2)} s_2^{(2)} \cdots s_l^{(2)} \cdots\cdots s_1^{(k)} s_2^{(k)} \cdots s_l^{(k)}$$

where $s_i^{(j)} = \tau^{(k-j)\epsilon}(s_i)$, which is never a power of Δ when $l \geqslant 1$.

Let us now deal with the possibility that β might be reducible. According to Theorem 4.2, there are three possible cases.

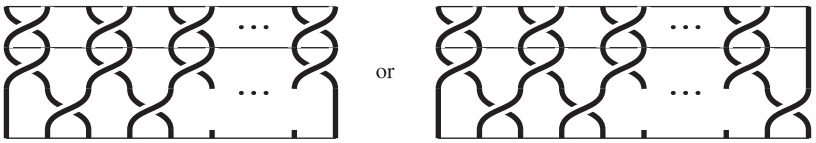

Figure 2. A braid sending no round curve to a round curve

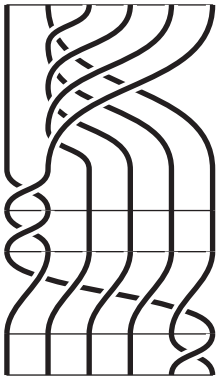

Figure 3. A braid where each pair of strands crosses in some factor, and does not cross in some other factor

The first case correspond to the case (1) of the theorem. A power of β preserves a round curve. The rigidity of β implies that the normal form of a power of β contains the normal form of β (except the initial factors Δ) as a subword. According to Proposition 4.3, we deduce that there exists a round curve whose image by β is still a round curve.

The second case is the case where a k-th power of β is such that $\Delta^{-\inf(\beta^k)}\beta^k = \Delta^{-k\epsilon}\beta^k$ preserves an almost round curve whose interior strands do not cross. If the normal form of β is $\Delta^\epsilon s_1 s_2 \cdots s_l$, then, as before, $\Delta^{-k\epsilon}\beta^k$ has normal form

$$\Delta^{-k\epsilon}\beta^k = s_1^{(1)}s_2^{(1)}\cdots s_l^{(1)}s_1^{(2)}s_2^{(2)}\cdots s_l^{(2)}\cdots\cdots s_1^{(k)}s_2^{(k)}\cdots s_l^{(k)}$$

This word has two strands that never cross, and hence so does the word $s_1 s_2 \cdots s_l$ representing $\Delta^{-\epsilon}\beta$.

Let us look at the third case. This time, it is the braid $\beta^{-k}\Delta^{\sup(\beta^k)} = \beta^{-k}\Delta^{k(l+\epsilon)}$ which has two strands that do not cross. Note that this braid has infimum 0 and supremum $k \cdot l$. Therefore in the braid

$$\Delta^{k\cdot l}\cdot(\beta^{-k}\Delta^{k(l+\epsilon)})^{-1} = \Delta^{-k\epsilon}\beta^k$$

(whose normal form was given in the previous paragraph) there are two strands which cross in every single factor. (This is because two strands in a simple braid s

cross if and only if the corresponding strands in Δs^{-1} do *not* cross.) Hence the same is true for the word $s_1 s_2 \cdots s_l$ representing $\Delta^{-\epsilon}\beta$: it has two strands which cross in every factor.

Now, a braid β whose normal form contains γ_1 cannot send any round curve to a round curve. The reason for this is that no round curve is sent to a round curve by this sequence of two simple braids (see Figure 2), and according to Proposition 4.3, this is also the case for the whole braid β. Similarly, a braid containing γ_2 cannot contain two strands that do not cross at all, or that cross in every single factor (see Figure 3). This completes the proof. □

We now restrict our attention temporarily to the case of braids with infimum 0.

Lemma 4.6. *The number of braids of infimum* 0 *and supremum* l*, which are rigid and pseudo-Anosov, is a* $\Theta(\lambda^l)$*, where* λ *is the constant of Lemma* 3.5*.*

Proof. Let us denote by Ω the set of rigid braids of infimum 0 and supremum l. We also denote by $E_1 \subset \Omega$ the subset of the braids that do not contain, in their normal form, the normal form of γ_1 as a subword. We denote by $E_2 \subset \Omega$ the subset of the braids that do not contain, in their normal form, the normal form of γ_2 as a subword.

According to Lemma 3.5, with the same notations, the cardinality $\#(\Omega)$ is equivalent to λ^{l+1}.

Still from Lemma 3.5, we also have estimates $\#(E_1) = O(\mu_{(\gamma_1)}^l)$ where $\mu_{(\gamma_1)} < \lambda$ and $\#(E_2) = O(\mu_{(\gamma_2)}^l)$ where $\mu_{(\gamma_2)} < \lambda$. Thus the cardinality of the set $E_1 \cup E_2$, whose cardinality is less than $c(\mu_{(\gamma_1)}^l + \mu_{(\gamma_2)}^l)$ for a suitable constant $c > 0$, and this set contains all rigid braids of infimum 0 and supremum l which are non pseudo-Anosov.

As $\mu_{(\gamma_1)} < \lambda$ and $\mu_{(\gamma_2)} < \lambda$, the number of braids of infimum 0 and supremum l which are rigid and pseudo-Anosov, is still of the order of λ^l. □

4.3. Arbitrary infimum. Let us consider the Cayley graph of the braid group, with generators the simple braids. The following lemma, which is an immediate consequence of Lemma 3.1 in [5], gives the possible left normal forms for a braid that is at distance l from the neutral element in this graph.

Lemma 4.7. *A braid* β *is at distance* l *from the neutral element in the Cayley graph if and only if the left normal form of* β *has one of the following shapes*:

(i) $\beta = \Delta^{-l} s_1 \cdots s_k$, $k \in \{0, \ldots, l-1\}$,

(ii) $\beta = \Delta^{-k} s_1 \cdots s_l$, $k \in \{0, \ldots, l\}$,

(iii) $\beta = \Delta^{k} s_1 \cdots s_{l-k}$, $k \in \{1, \ldots, l\}$.

The following theorem is a generalization of the results previously obtained in the particular case of a zero infimum.

Theorem 4.8. *For large enough l, among all braids at distance l from the neutral element in the Cayley graph, the proportion of rigid pseudo-Anosov braids is bounded below by a positive constant.*

Proof. First, let us make a remark: a braid β is pseudo-Anosov if and only if $\Delta^2\beta$ is pseudo-Anosov. The same is true when we replace "pseudo-Anosov" by "rigid." Thus, a braid with left normal form $\Delta^p s_1 \cdots s_r$ with p even is pseudo-Anosov (respectively rigid) if and only if $s_1 \cdots s_r$ is.

According to Lemma 4.6, there exists a constant $c_1 > 0$ such that for all large enough l, the number of rigid pseudo-Anosov braids of the form $s_1 \cdots s_l$ is bounded below by $c_1\lambda^l$. Consequently, the number of rigid pseudo-Anosov braids of the form $\Delta^{-k} s_1 \cdots s_l$ with $k \in \{0, \ldots, l\}$ and k even is bounded below by $c_1 \frac{l}{2} \lambda^l$.

Furthermore, let us bound above the total number of braids at distance l from the neutral element. According to Lemma 3.5, there exists a constant c_2 such that the number of braids with normal forms of the shape $s_1 \cdots s_k$ is bounded above by $c_2\lambda^k$. So:

(i) the number of braids with normal form $\Delta^{-l} s_1 \cdots s_k$ $(0 \leqslant k < l)$ is bounded above by $c_2(1 + \cdots + \lambda^{l-1})$,

(ii) the number of braids with normal form $\Delta^{-k} s_1 \cdots s_l$ $(0 \leqslant k \leqslant l)$ is bounded above by $c_2 l \lambda^l$,

(iii) the number of braids with normal form $\Delta^k s_1 \cdots s_{l-k}$ $(0 < k \leqslant l)$ is bounded above by $c_2(1 + \cdots + \lambda^{l-1})$.

As $c_2(1 + \cdots + \lambda^{l-1}) \sim \frac{c_2}{\lambda-1}\lambda^l$, if we replace c_2 by an even larger constant, we can suppose that, in the cases (i) and (iii), the number of braids is bounded above by $\frac{c_2}{\lambda-1}\lambda^l$. Finally, the proportion of rigid pseudo-Anosov braids among all braids at distance l from the neutral element is bounded below by

$$\frac{c_1 \frac{l}{2} \lambda^l}{\frac{c_2}{\lambda-1}\lambda^l + c_2 l \lambda^l + \frac{c_2}{\lambda-1}\lambda^l} = \frac{c_1}{2c_2} \cdot \frac{1}{1 + \frac{2}{l(\lambda-1)}} \geqslant \frac{c_1}{2c_2} \cdot \frac{1}{1 + \frac{2}{\lambda-1}} > 0,$$

which completes the proof. □

Corollary 4.9. *For large enough l, in the l-ball of the Cayley graph, the proportion of rigid pseudo-Anosov braids is bounded below independently of l by a positive constant.*

Proof. The number of braids in the k-sphere is of the order of $k\lambda^k$, and the l-ball is the union of the k-spheres for $k \leqslant l$. We deduce that the number of braids in the l-ball is of the order of $l\lambda^l$, that is to say, of the order of the number of braids in the l-sphere. So the proportion of rigid pseudo-Anosov braids remains of the order of a constant. □

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