Small doubling in ordered groups:
generators and structure

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Abstract. We prove several new results on the structure of the subgroup generated by a small doubling subset of an ordered group, abelian or not. We obtain precise results generalizing Freiman's $3k - 3$ and $3k - 2$ theorems in the integers and several further generalizations.

1. Introduction

Let G denote an arbitrary group (multiplicatively written). If S is a subset of G, we define its *square* S^2 by

$$S^2 = \{x_1 x_2 \mid x_1, x_2 \in S\}.$$

In the abelian context, G will usually be additively written and we shall rather speak of the *double* of S.

In this paper, we are concerned with the following general problem. Suppose that we are given two real numbers $\alpha \geq 1$ and β. While in general, the size of S^2 should typically be quadratic in the one of S, we would like to identify those finite sets S having the property that

$$|S^2| \leq \alpha|S| + \beta. \tag{1}$$

Typically, we may need to add the natural assumption that $|S|$ is not too small. It is clear that the difficulty in describing S in this problem increases with α, which is related with the *doubling constant* of S defined as $|S^2|/|S|$. When $|S|$ is large, solving (1) is tantamount to asking for a complete description of sets with a bounded doubling constant.

Problems of this kind are called *inverse problems* in additive number theory. During the last two decades, they became the most central issue in a fast growing area, known as additive combinatorics. Inverse problems of small doubling type have been first investigated by G.A. Freiman very precisely in the additive group of the integers (see [6], [7], [8], and [9]) and by many other authors in general abelian groups, starting with M. Kneser [21] (see, for example, [17], [2], [22], [28], and [16]). More recently, small doubling problems in non-necessarily abelian groups have been also studied, see [15], [29], and [4] for recent surveys on these problems and [24] and [34] for two important books on the subject.

There are two main types of questions one may ask. First, find the general type of structure that S can have and how this type of structure behaves when α increases. In this case, very powerful general results have been obtained (leading to a qualitatively complete structure theorem thanks, notably, to the concepts of nilprogressions and approximate groups), but they are not very precise quantitatively. Second, for a given (in general quite small) range of values for α, find the precise (and possibly complete) description of those sets S which satisfy (1). The archetypal results in this area are Freiman's $3k - 4$, $3k - 3$ or $3k - 2$ theorems in the integers (see [6], [7], and [8]). See also for instance [22], [5], [18], [19], [31] or [32] for other results of this type. In this paper, we investigate problems of the second type.

Here, for a given non-necessarily abelian group G, we would like to understand precisely what happens in the case when $\alpha = 3$. We restrict ourselves to the already quite intricate case of *ordered groups* started in papers [10], [12], [13], [20], and [14].

We recall that if G is a group and $\leq$ is a total order relation defined on the set G, we say that $(G, \leq)$ is *an ordered group* if for all $a, b, x, y \in G$, the inequality $a \leq b$ implies that $xay \leq xby$. A group G is *orderable* if there exists an order $\leq$ on G such that $(G, \leq)$ is an ordered group.

Obviously the group of integers with the usual ordering is an ordered group. More generally, the class of orderable groups contains all nilpotent torsion-free groups (see, for example, [23] or [25]). We will come back to nilpotent groups later in this paper.

In [10], some of us proved that if S is a finite subset of an orderable group satisfying $|S| \geq 3$ and $|S^2| \leq 3|S| - 3$, then $\langle S \rangle$, the subgroup generated by S, is abelian. Moreover, if $|S| \geq 3$ and $|S^2| \leq 3|S| - 4$, then S is a subset of a short commutative *geometric progression*, that is, a set of the form

$$P_l(u, t) = \{u, ut, \ldots, ut^{l-1}\} \subseteq \langle u, t \rangle$$

for some *commuting* elements u and t of G and l an integer (t will be called the *ratio* of the progression and l its *length*). In particular, $\langle S \rangle$ is abelian and at most 2-*generated* (by which we mean that it is generated by a set having at most 2 elements). Finally, a group was constructed such that for any integer $k \geq 3$ there exists a subset S of cardinality k such that $|S^2| = 3|S| - 2$ and $\langle S \rangle$ is non-abelian.

In this paper, we continue to study the structure of finite subsets S of an orderable group when an inequality of the form (1) with $\alpha = 3$ holds and improve drastically our preceding results, both by extending their range of application and by making them more precise. We particularly concentrate on the size of a generating set and the structure of $\langle S \rangle$, as will be explained in the following section.

2. New results and the plan of this paper

The present paper is organized as follows.

In Section 3, we consider the case when $\langle S \rangle$ is abelian. This can be seen as a continuation of [10] (see Theorem 1.3 there). We first obtain a $(3k - 3)$-type theorem generalizing Freiman's classical theorem in the integers.

Theorem 1 (Ordered $3k - 3$ Theorem). *Let G be an orderable group and S be a finite subset of G satisfying $|S^2| \leq 3|S| - 3$. Then $\langle S \rangle$ is abelian and at most 3-generated.*

Moreover, if $|S| \geq 11$, then one of the following two possibilities occurs:

 (i) *S is a subset of a geometric progression of length at most $2|S| - 1$;*

 (ii) *S is the union of two geometric progressions with the same ratio, such that the sum of their lengths is equal to $|S|$.*

In general, this result cannot be improved. However, it will follow from our proof that a slightly more detailed result (namely, describing those sets S with $|S| \leq 10$ involved in this statement) can be easily achieved if one employs more carefully the tools used in our proof. It will be shortly explained how to do this at the end of our proof. For the sake of simplicity and avoiding technicalities, here we prefer a cleaner statement.

We can go a step further and formulate also a precise generalized $(3k - 2)$-type result.

Theorem 2 (Abelian ordered $3k - 2$ Theorem). *Let G be an orderable group and S be a finite subset of G such that $|S^2| = 3|S| - 2$ and $\langle S \rangle$ is abelian. Then either $|S| = 4$ or $\langle S \rangle$ is at most 3-generated.*

Moreover, if $|S| \geq 12$, then one of the following two possibilities occurs:

 (i) *S is a subset of a geometric progression of length at most $2|S| + 1$;*

 (ii) *S is contained in the union of two geometric progressions with the same ratio, such that the sum of their lengths is equal to $|S| + 1$.*

Again, this result is in general best possible and, again, at the price of increasing the number of sporadic cases and doing a more careful examination of our proofs, one could replace the assumption $|S| \geq 12$ in Theorem 2 by a more precise list of possible situations. For the sake of clarity, we prefer such a neat statement.

We then investigate the general abelian case. We obtain the following result.

Theorem 3. *Let c be an integer, $c \geq 2$. Let G be an orderable group and S be a finite subset of G such that $\langle S \rangle$ is abelian. If*

$$|S^2| < (c+1)|S| - \frac{c(c+1)}{2},$$

then $\langle S \rangle$ is at most c-generated.

We obtain for instance the following corollary, corresponding to the case $c = 3$ above.

Corollary 1. *Let G be an ordered group and S be a finite subset of G such that $\langle S \rangle$ is abelian. If $|S^2| \leq 4|S| - 7$, then $\langle S \rangle$ is at most 3-generated.*

If needed, we could even make this result more precise and give the precise structure of S by using the results of [30], [31] and [32].

In Section 4, we go back to the general case of non-necessarily abelian groups and study the maximal number of generators of $\langle S \rangle$. When merged with some of our preceding results, we obtain the following general result.

Theorem 4. *Let G be an ordered group and S be a finite subset of G. Then the following statements hold.*

(i) $|S^2| \geq 2|S| - 1$.

(ii) *If $2|S| - 1 \leq |S^2| \leq 3|S| - 4$, then $\langle S \rangle$ is abelian and at most 2-generated.*

(iii) *If $|S^2| = 3|S| - 3$, then $\langle S \rangle$ is abelian and at most 3-generated.*

(iv) *Let $|S^2| = 3|S| - 3 + b$ for some integer $b \geq 1$. Then either $|S| = 4$, $b = 1$, $\langle S \rangle$ is abelian and at most $(b+3)$-generated or $\langle S \rangle$ is at most $(b+2)$-generated.*

In Sections 5, 6, 7 and 8, we look for a complete description of $\langle S \rangle$, if S is a finite subset of cardinality ≥ 4 of an orderable group and $|S^2| = 3|S| - 2$. The flavour of these results is more group-theoretical. Here we briefly recall for completeness sake a few standard notations that we shall need. We refer to [27] for all the group-theoretic complementary notation that the reader may need for reading this paper.

If a and b are elements of a group G, their *commutator* is denoted by

$$[a, b] = a^{-1} b^{-1} ab$$

and the *derived subgroup* of G, denoted by G', is simply the subgroup of G generated by its commutators. More generally, if H and K are two subgroups of G, we denote by $[H, K]$ the subgroup generated by the commutators of the form $[h, k]$ for h in H and k in K. With this notation

$$G' = [G, G].$$

We shall also use the classical notation

$$a^b = b^{-1} ab.$$

If $X \subseteq G$, we use the notation $C_G(X)$ for the *centralizer* of X in G defined as the subgroup of elements of G commuting with all the elements of X and

$$Z(G) = C_G(G)$$

is the *center* of G. If H is a subgroup of G, then the *normalizer* of H in G is by definition the subgroup

$$N_G(H) = \{g \in G \mid g^{-1} Hg = H\},$$

and it is the largest subgroup of G containing H in which H is normal. Recall finally that, if n is a positive integer, a *soluble group of length at most n* is a group which has an *abelian series* of length n that is, a finite chain

$$\{1\} = G_0 \le G_1 \le \cdots \le G_n = G \tag{2}$$

of subgroups, such that G_i is a normal subgroup of G_{i+1} and G_{i+1}/G_i is abelian for any index $0 \le i \le n - 1$; and a *nilpotent group of class at most n* is a group which has a *central series* of length n of the form (2) such that G_{i+1}/G_i is contained in the center of G/G_i, for any index $0 \le i \le n - 1$. Soluble groups of length at most 2 are also called *metabelian*; they are exactly the groups G such that G' is abelian. Nilpotent groups of class at most 2 are exactly the groups G such that the derived subgroup G' is contained in the center $Z(G)$.

In order to state our $(3k-2)$-type result, we introduce a definition. In this paper, we shall say that an ordered group G is *young* if one of the following occurs:

 (i) $G = \langle a, b \rangle$ with $[[a, b], a] = [[a, b], b] = 1$,

 (ii) $G = \langle a \rangle \times \langle b, c \rangle$ with either $c^b = c^2$ or $(c^2)^b = c$ and $c \ne 1$,

 (iii) $G = \langle a, b \rangle$ with $a^b = a^2$ and $a \ne 1$,

 (iv) $G = \langle a, b \rangle$ with $a^{b^2} = aa^b$, $[a, a^b] = 1$ and $a \ne 1$.

More precisely, we shall speak of an ordered young group of type (j) if the group satisfies the definition (j) in the list above (j being i, ii, iii or iv). Notice that an ordered young group of type (iii) is a quotient of $B(1, 2)$, the Baumslag-Solitar group [1].

Notice also that nilpotent ordered young groups are of type (i). This follows from the fact that in nilpotent groups the derived subgroup is contained in the set of all non-generators (see for example [26], Lemma 2.22). Consequently, nilpotent ordered young groups are of class at most 2.

Moreover, we claim the following.

Lemma 1. *An ordered young group is metabelian and a nilpotent ordered young group is of nilpotency class at most 2.*

Proof. This is obvious in case (i), since in this case G is nilpotent of class at most 2.

So suppose that either (ii) or (iii) or (iv) holds. Write $H = < c^{b^i} | i \in \mathbb{Z} >$ in case (ii) and $H = < a^{b^i} | i \in \mathbb{Z} >$ in cases (iii) and (iv). Then H is normal in G and G/H is abelian. Thus G' is contained in H, and since the converse is also true, it follows that $G' = H$. By induction on n it is easy to prove that for every $n \in \mathbb{N}$, $[c, c^{b^n}] = 1$ in case (ii) and $[a, a^{b^n}] = 1$ in cases (iii) and (iv). This result also implies that $[c^{b^{-n}}, c] = 1$ in case (ii) and $[a^{b^{-n}}, a] = 1$ in cases (iii) and (iv). Thus $c \in Z(H)$ in case (ii) and $a \in Z(H)$ in the other cases, which implies that $H \leqslant Z(H)$. Hence H is abelian and G is metabelian, as claimed. $\qquad\square$

Our main result in this paper is the following theorem.

Theorem 5. *Let G be an ordered group and let S be a finite subset of G. If $|S| \geq 4$ and $|S^2| = 3|S| - 2$, then $\langle S \rangle$ is either abelian or young.*

Theorem 1, Theorem 5 and Lemma 1 yield the following two corollaries.

Corollary 2. *Let G be an ordered group and let S be a finite subset of G. If $|S| \geq 4$ and $|S^2| \leq 3|S| - 2$, then $\langle S \rangle$ is metabelian.*

Corollary 3. *Let G be an ordered group and let S be a finite subset of G such that $\langle S \rangle$ is nilpotent. If $|S| \geq 4$ and $|S^2| \leq 3|S| - 2$, then $\langle S \rangle$ is of nilpotency class at most 2.*

Our preceding results show that if $|S| \geq 3$ and $|S^2| \leq 3|S| - 3$, then $\langle S \rangle$ is abelian, but, for any integer $k \geq 3$, there exist an ordered group G and a subset S of G of size k with $|S^2| = 3|S| - 2$ and $\langle S \rangle$ non-abelian [10]. Moreover, we have proved in Corollary 2 that if $|S| \geq 4$ and $|S^2| \leq 3|S| - 2$, then $\langle S \rangle$ is metabelian. It is now natural to ask whether there exist an ordered group G and a positive integer b, such that for any integer k there is a subset S of G of order k with $|S^2| = 3|S| - 2 + b$ and $\langle S \rangle$ non-metabelian or more generally non-soluble. We give a negative answer to this question in Section 9 by proving the following result.

Theorem 6. *Let G be an ordered group and s be any positive integer. If S is a subset of G of cardinality $\geq 2^{s+2}$ such that $|S^2| = 3|S| - 2 + s$, then $\langle S \rangle$ is metabelian. Moreover, if G is nilpotent, then $\langle S \rangle$ is nilpotent of class at most 2.*

Our final result, Theorem 7, is a constructive result.

Theorem 7. *There exists a non-soluble ordered group G such that for any integer $k \geq 3$, there exists a subset S of G of cardinality k such that $|S^2| = 4|S| - 5$ and $\langle S \rangle = G$. In particular, $\langle S \rangle$ is non-soluble.*

Notice that the results of the present paper will be used, as a cornerstone, to derive the main result of [11] (Theorem 1), that is a complete description of the structure of S if S is a finite subset of an orderable group G with $|S^2| = 3|S| - 2$ and $\langle S \rangle$ is non-abelian.

3. The abelian case: proofs of Theorems 1, 2 and 3

Let S be a finite subset of an ordered group G and suppose that $\langle S \rangle$ is abelian. Since $\langle S \rangle$ is finitely generated and ordered (thus torsion-free), it is isomorphic to some $(\mathbb{Z}^m, +)$, for $m = m(S)$, an integer. In other words, the additive group $\langle S \rangle$ is an *m-generated group*. We may thus make our reasoning in this setting, which is simpler and was already studied. Notice that in this section we shall always use the additive notation. In particular, we use the term 'difference' instead of 'ratio' in the description of an arithmetic progression.

The notion of *Freiman dimension* of the set S will be needed here [8]. Recall first that S is *Freiman isomorphic* to a set $A \subseteq \mathbb{Z}^d$ if there exists a bijective mapping $F: S \to A$ such that the equations $g_1 + g_2 = g_3 + g_4$ and $F(g_1) + F(g_2) = F(g_3) + F(g_4)$ are equivalent for all $g_1, g_2, g_3, g_4 \in S$ (i.e. sums of two elements of S coincide if and only if their images have the same property).

The *Freiman dimension* of S is then defined as the largest integer $d = d(S)$ such that S is Freiman-isomorphic to a subset A of $\mathbb{Z}^d$ not contained in an affine hyperplane of $\mathbb{Z}^d$ (i.e. A is not contained in a subset of $\mathbb{Z}^d$ of the form $L = a + W$, where $a \in \mathbb{Z}^d$ and $W = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_{d-1}$).

We recall immediately the basic inequalities linking $m(S), d(S)$ and $|S|$ for a subset S of some ordered abelian group.

Lemma 2. *Let S be a finite subset of an ordered group G. Assume that $\langle S \rangle$ is abelian. Then*

$$m(S) \leq d(S) + 1 \leq |S|.$$

Both inequalities are tight as shown by the example

$$S = \{(0, 1), (1, 1)\} \subseteq \mathbb{Z}^2.$$

Indeed, $d(S) = 1$, while $\langle S \rangle = \mathbb{Z}^2$ and $m(S) = 2 = |S|$.

Proof. By our assumptions, the additive group $\langle S \rangle$ is isomorphic to $(\mathbb{Z}^m, +)$, for $m = m(S)$, an integer. Without loss of generality we may assume that

$$S \subseteq \mathbb{Z}^m \quad \text{and} \quad \langle S \rangle = \mathbb{Z}^m.$$

Let r be the smallest dimension of a subspace $W = \mathbb{Z}w_1 \oplus \cdots \oplus \mathbb{Z}w_r \subseteq \mathbb{Z}^m$ such that

$$L = a + W = a + \mathbb{Z}w_1 \oplus \cdots \oplus \mathbb{Z}w_r$$

contains S for some $a \in \mathbb{Z}^m$. Then

$$r \leq m = m(S) \leq r + 1,$$

which implies that either $r = m - 1$ or $r = m$.

If $r = m - 1$, then there is a point $a \in \mathbb{Z}^m$ and a subspace

$$W = \mathbb{Z}w_1 \oplus \cdots \oplus \mathbb{Z}w_{m-1} \subseteq \mathbb{Z}^m$$

such that

$$S \subseteq a + W.$$

The set $S^* = S - a = \{s - a \mid s \in S\}$ is Freiman isomorphic to S by the trivial correspondence: $s \leftrightarrow s - a$ for all $s \in S$ and it is contained in $W = \mathbb{Z}w_1 \oplus \cdots \oplus \mathbb{Z}w_{m-1}$. Let

$$T: W \longrightarrow \mathbb{Z}^{m-1}$$

be the unique linear isomorphism defined by

$$T(w_1) = e_1 = (1, 0, \ldots, 0), \ldots, T(w_{m-1}) = e_{m-1} = (0, \ldots, 0, 1).$$

Define $A = T(S^*)$. Using the hypothesis $r = m - 1$, it follows that the set A is not contained in an affine hyperplane of $\mathbb{Z}^{m-1}$. Therefore $d(A) \geq m - 1$. The sets A, S^* and S are Freiman isomorphic to each other and thus

$$d(S) = d(S^*) = d(A) \geq m - 1.$$

If $r = m$, then the set S is not contained in an affine hyperplane of $\mathbb{Z}^m$. Thus $d(S) \geq m > m - 1$. Hence $m \leq d(S) + 1$ in all cases.

We prove now the inequality $d + 1 \leq |S|$. Let $|S| = k$. The set S is Freiman-isomorphic to a subset $A = \{a_0, a_1, \ldots, a_{k-1}\} \subseteq \mathbb{Z}^d$ not contained in an affine hyperplane of $\mathbb{Z}^d$. Without loss of generality, we may assume that $a_0 = 0$ (indeed, we may replace A by the translate $A - a_0$ and use $d(A) = d(A - a_0)$ and $|A| = |A - a_0|$). The set A is contained in the subspace

$$W = \mathbb{Z}a_0 + \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_{k-1} = \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_{k-1}.$$

If $k \leq d$, then $m(W) \leq k - 1 \leq d - 1$ and this contradicts our hypothesis that A is not contained in an affine hyperplane of $\mathbb{Z}^d$. Hence $k = |A| = |S| \geq d + 1$ as required. The proof of the lemma is complete. □

For the proof of the Theorems we shall use Freiman's theorem (Lemma 1.14 of [8]), stating that for any finite subset S of a torsion free group with $\langle S \rangle$ abelian and with Freiman dimension d, the following lower bound for the cardinality of S^2 holds:

$$|S^2| \geq (d + 1)|S| - \frac{d(d + 1)}{2}. \tag{3}$$

This inequality was actually proved in Lemma 1.14 of [8] for sets with affine dimension d. However, if S has Freiman dimension d, then S is Freiman isomorphic to some $S^* \subseteq \mathbb{Z}^d$ which has affine dimension d and satisfies the equations $|S^*| = |S|$ and $|(S^*)^2| = |S^2|$. Therefore

$$|S^2| = |(S^*)^2| \geq (d + 1)|S^*| - \frac{(d + 1)d}{2} = (d + 1)|S| - \frac{(d + 1)d}{2}.$$

The proof of the Theorems can now be obtained easily. Notice that by Theorem 1.3 in [10], also the set S of Theorem 1 generates an abelian group.

Proof of Theorems 1 and 2, the common part. Let $c = 2$ or 3. By our assumptions and (3) we obtain

$$3|S| - c \geq |S^2| \geq (d + 1)|S| - \frac{d(d + 1)}{2},$$

where $d = d(S)$. This yields

$$(d - 2)|S| \leq \frac{d(d + 1)}{2} - c. \tag{4}$$

Then, using Lemma 2, we obtain

$$(d - 2)(d + 1) \leq \frac{d(d + 1)}{2} - c$$

and therefore

$$(d - 1)(d - 2) \leq 2(3 - c). \tag{5}$$

$\square$

Now, we have to continue separately the study of the cases $c = 2$ and 3.

Proof of Theorem 1, concluded. Here $c = 3$, and as mentioned above, $\langle S \rangle$ is abelian. Thus (5) implies that either $d = 1$ or $d = 2$. Hence by Lemma 2 $m(S) \leq 3$ and $\langle S \rangle$ is at most 3-generated, as required.

Suppose now that $|S| \geq 11$. Since either $d = 1$ or $d = 2$, we are back to the case of the 1- or the 2-dimensional $3k - 3$ theorem in the integers.

If $d = 1$, such sets are described in Theorem 1.11 in [8] (or see [18]): they are subsets of an arithmetic progression of length $2|S| - 1$ or the union of two arithmetic progressions with the same difference (a remaining case corresponds to bounded cardinality). If $d = 2$, such sets are described by Theorem 1.17 in [8] (or see Theorem B in [31]): they are the union of two arithmetic progressions with the same difference (there are also remaining cases of bounded cardinality).

This is enough to prove Theorem 1. If one wants to be more precise, it is enough to use the same Theorems 1.11 and 1.17 of [8] quoted above, but in their precise forms. ☐

Proof of Theorem 2, concluded. Here $c = 2$, thus the bound (5) implies that $d = 1, 2$ or 3. If $d = 3$, then by (4), we obtain that $|S| \leq 4$ and if $d = 1, 2$, then $\langle S \rangle$ is at most 3-generated by Lemma 2. Hence either $|S| = 4$ or $\langle S \rangle$ is at most 3-generated, as required.

If $d = 1$, we are back to the $3k - 2$ theorem in the integers. Hence it follows by Theorem 1.13 of [8] (or see the original publication [7]; we notice that in the original statement of this theorem, there is a missing sporadic case of size 11 which makes it necessary to have 12 here instead of 11) that if $|S| \geq 12$, then S must be either a subset of a short geometric progression or a geometric progression minus its second element, together with an isolated point, or, finally, a geometric progression together with another geometric progression of length 3 with the same ratio and with the middle term missing. The last possibility is missing in Theorem 1.13, as printed in [8]. Thus S satisfies either (i) or (ii).

If $d = 2$, then the set is of Freiman dimension 2 and its structure is described in Theorem 1.17 of [8] (or see Theorem B in [31]). This case leads to possibility (ii) in the statement of Theorem 2. Again, if one wishes to deal with small cardinalities for S, one simply has to use more precise versions of Freiman's theorems. ☐

The proof of Theorem 3 follows the same lines.

Proof of Theorem 3. As above, if $d = d(S)$, we must have

$$(c + 1)|S| - \frac{c(c + 1)}{2} > |S^2| \geq (d + 1)|S| - \frac{d(d + 1)}{2}.$$

This implies by factorization that

$$(d - c)|S| < \frac{(c - d)(c + d + 1)}{2}$$

from which it follows that one cannot have $d \geq c$. Thus $d \leq c - 1$ and the conclusion follows by Lemma 2. ☐

4. On the generators of $\langle S \rangle$: proof of Theorem 4

We now come back to the general case of non-necessarily abelian groups and start with a lemma.

Lemma 3. *Let G be an ordered group. Let T be a non-empty finite subset of G and let t denote its maximal element* $\max T$. *Let* $x \in G$ *be an element satisfying* $x > t$. *If* $yx, xy \in T^2$ *for each* $y \in T \setminus \{t\}$, *then* $T \subseteq \langle t, x \rangle$.

Proof. Write $l = |T|$ and

$$T = \{t_l, t_{l-1}, \ldots, t_1\}, \quad t_l < t_{l-1} < \cdots < t_1 = t.$$

We prove by induction on j ($1 \le j \le l$) that $t_j \in \langle t, x \rangle$.

If $j = 1$, we have $t_1 = t \in \langle t, x \rangle$.

Assume that we have proved that $t_1, \ldots, t_j \in \langle t, x \rangle$ for some j satisfying $1 \le j \le l - 1$. Without loss of generality, we may assume that $xt_{j+1} \le t_{j+1}x$.

By assumption, we have $xt_{j+1}, t_{j+1}x \in T^2$. It follows that $t_{j+1}x = t_u t_v$ for some $t_u, t_v \in T$. But $x > t \ge t_u, t_v$ and $t_{j+1}x = t_u t_v \ge xt_{j+1}$, so the ordering implies that $t_u, t_v > t_{j+1}$. Therefore $u, v \le j$, and the induction hypothesis implies that $t_u, t_v \in \langle t, x \rangle$. Consequently $t_{j+1} = t_u t_v x^{-1} \in \langle t, x \rangle$ and the inductive step is completed. Hence $T \subseteq \langle t, x \rangle$, as required. $\square$

We pass now to the proof of Theorem 4.

Proof of Theorem 4. For parts (i), and (ii), see [10], Theorem 1.1 and Corollary 1.4, respectively. Assertion (iii) follows by our Theorem 1.

Our aim now is to prove part (iv). Suppose that $|S| = k$ and $S = \{x_1 < x_2 < \cdots < x_k\}$. We argue by induction on k. If $|S| = k \le b + 2$, then the result is trivial. So suppose that $k \ge b + 3$ and that part (iv) of Theorem 4 is true up to $k - 1$.

Assume, first, that $\langle S \rangle$ is abelian. If $b = 1$, then $|S^2| = 3k - 2$ and by Theorem 2 either $|S| = 4$ or $\langle S \rangle$ is at most 3-generated, as required since $b+2 = 3$. So assume that $b \ge 2$. Since $k \ge b + 3 \ge 5$, we have $b \le k - 3$ and

$$|S^2| \le 4k - 6 < 5k - 10.$$

But $5k - 10 = (4+1)k - 4(4+1)/2$, so by Theorem 3 $\langle S \rangle$ is at most 4-generated, as required since $4 \le b + 2$. This concludes the proof of the abelian case.

So assume that $\langle S \rangle$ is non-abelian and let $T = \{x_1, \ldots, x_{k-1}\}$. If $x_i x_k, x_k x_i \in T^2$ for all $i < k - 1$, then it follows by Lemma 3 that $T \le \langle x_{k-1}, x_k \rangle$ and hence $\langle S \rangle = \langle x_{k-1}, x_k \rangle$ is at most 2-generated, as required since $2 < b + 2$.

Hence we may assume that $x_i x_k$ (or $x_k x_i$) $\notin T^2$ for some $i < k - 1$. Then, by the ordering in G, we have $x_i x_k, x_{k-1} x_k, x_k^2 \notin T^2$, which implies that

$$|T^2| \leq |S^2| - 3 = 3k - 3 + b - 3 = 3(k-1) - 3 + b.$$

Applying induction, we may conclude that either $\langle T \rangle$ is abelian and $|T| = 4$, or $\langle T \rangle$ is at most $(b+2)$-generated.

If $x_k \in \langle T \rangle$, then $\langle T \rangle = \langle S \rangle$ and $\langle T \rangle$ is non-abelian. Hence $\langle T \rangle$ is at most $(b+2)$-generated and so is $\langle S \rangle$, as required.

So assume that $x_k \notin \langle T \rangle$. Then $x_k x_i, x_i x_k \notin T^2$ for all $i \leq k-1$ and $x_k^2 \notin T^2$ by the ordering in G. Thus

$$S^2 = T^2 \dot{\cup} (x_k T \cup T x_k) \dot{\cup} \{x_k^2\}$$

and since $|x_k T \cup T x_k| \geq k - 1$, it follows that

$$|T^2| \leq |S^2| - k \leq 3k - 3 + b - (b+3) = 3(k-1) - 3.$$

Hence, by Theorem 1.3 in [10], $\langle T \rangle$ is abelian and since $\langle S \rangle$ is non-abelian, it follows that $x_k \notin C_G(T)$. Consequently, by Corollary 1.4 in [10], $|x_k T \cup T x_k| \geq k$, which implies that

$$|T^2| \leq |S^2| - (k+1) \leq 3k - 3 + b - (b+4) = 3(k-1) - 4.$$

Hence, by Proposition 3.1 in [10], $\langle T \rangle$ is at most 2-generated, so $\langle S \rangle$ is at most 3-generated, as required since $3 \leq b + 2$.

The proof of Theorem 4 is now complete. $\qquad\square$

5. The structure of $\langle S \rangle$ if $|S^2| = 3|S| - 2$: cardinality 3

In Theorem 5 we assume that $|S| \geq 4$. However, for the inductive proof of that theorem, we need some special results concerning the case when $|S| = 3$. These results are proved in the next two propositions.

Proposition 1. *Let G be an ordered group. Let x_1, x_2, x_3 be elements of G, such that $x_1 < x_2 < x_3$ and let $S = \{x_1, x_2, x_3\}$. Assume that $\langle S \rangle$ is non-abelian and either $x_1 x_2 = x_2 x_1$ or $x_2 x_3 = x_3 x_2$.*

Then $|S^2| = 7$ if and only if one of the following holds:

(i) $S \cap Z(\langle S \rangle) \neq \emptyset$, *or*

(ii) S *is of the form* $\{a, a^b, b\}$, *where* $a a^b = a^b a$.

Proof. Suppose that $|S^2| = 7$. Assume first that $x_1 x_2 = x_2 x_1$. The other case $(x_2 x_3 = x_3 x_2)$ follows by reversing the ordering in G.

If either $x_2 x_3 = x_3 x_2$ or $x_3 x_1 = x_1 x_3$, then either $x_2 \in Z(\langle S \rangle)$ or $x_1 \in Z(\langle S \rangle)$, respectively, and (i) holds.

So suppose that $x_2 x_3 \neq x_3 x_2$ and $x_3 x_1 \neq x_1 x_3$. It follows that $x_3 \notin \langle x_1, x_2 \rangle$. If $x_1 x_3 < x_3 x_1$, then $x_1 x_3 \neq x_2 x_3$, $x_1 x_3 \neq x_3 x_2$, $x_1 x_3 \neq x_3^2$ and

$$S^2 = \{x_1^2, x_1 x_2, x_2^2, x_2 x_3, x_3 x_2, x_3^2, x_1 x_3\}.$$

Hence $x_3 x_1 = x_2 x_3$ and $x_1 = x_2^{x_3}$. Thus (ii) follows by taking $a = x_2$ and $b = x_3$, since then $x_1 = a^b$. Similar arguments yield also (ii) if $x_1 x_3 > x_3 x_1$.

Conversely, if either (i) or (ii) holds, then it is easy to verify that $|S^2| = 7$. $\square$

Proposition 2. *Let G be an ordered group. Let x_1, x_2, x_3 be elements of G, such that $x_1 < x_2 < x_3$ and let $S = \{x_1, x_2, x_3\}$. Assume that both $x_1 x_2 \neq x_2 x_1$ and $x_2 x_3 \neq x_3 x_2$ (in particular $\langle S \rangle$ is non abelian) and $|S^2| = 7$. Then S is of one of the following forms*:

(a) *either $\{x, xc, xc^x\}$ or $\{x^{-1}, x^{-1}c, x^{-1}c^x\}$ for some $c \in G'$ satisfying $c > 1$, with $c^{x^2} = cc^x$ and $cc^x = c^x c$;*

(b) *either $\{x, xc, xcc^x\}$ or $\{x^{-1}, x^{-1}c, x^{-1}cc^x\}$ for some $c \in G'$ satisfying $c > 1$, with $c^{x^2} = cc^x$ and $cc^x = c^x c$;*

(c) *$\{x, xc, xc^2\}$ for some $c \in G'$ satisfying $c > 1$, with either $c^x = c^2$ or $(c^2)^x = c$.*

Moreover, in cases (a) and (b), one has

$$\langle S \rangle = \langle a, b \rangle \quad \text{with} \quad a^{b^2} = aa^b \text{ and } aa^b = a^b a$$

and $\langle S \rangle$ is young of type (iv), while in case (c)

$$\langle S \rangle = \langle a, b \rangle, \quad \text{with } a^b = a^2$$

and $\langle S \rangle$ is young of type (iii).

Proof. Write $T = \{x_1, x_2\}$, then $S^2 = T^2 \dot{\cup} \{x_2 x_3, x_3 x_2, x_3^2\}$.

First suppose that $x_1 x_3 \leq x_3 x_1$. Then $x_1 x_3 \notin \{x_2 x_3, x_3 x_2, x_3^2\}$, and we have either $x_1 x_3 = x_2^2$ or $x_1 x_3 = x_2 x_1$. We distinguish between three cases.

CASE 1: $x_1 x_3 = x_2^2$. In this case $x_1 x_3 < x_3 x_1$, since otherwise $x_1 \in C_G(x_3)$, $[x_1, x_2^2] = 1$ and then $[x_1, x_2] = 1$, a contradiction. Hence

$$x_3 x_1 = x_2 x_3. \tag{6}$$

Write $x_1 = x$. Then $x_2 = xc$ for some $c \in \langle S \rangle'$, with $c > 1$. Moreover, since $x_1 x_3 = x_2^2$, we obtain $x_3 = cxc = xc^x c$. Hence, by (6), $cxcx = xccxc$, so $c^x cx = c^2 xc$ and $c^{x^2} c^x = (c^x)^2 c$. Thus

$$[c, c^x]^x = [c^x, c^{x^2}] = [c^x, (c^x)^2 c (c^x)^{-1}] = [c^x, c]^{(c^x)^{-1}},$$

yielding $[c, c^x]^{xc^x} = [c^x, c] = [c, c^x]^{-1}$. Since G is an ordered group, it follows that $[c, c^x] = 1$. Thus $c^{x^2} = cc^x$ and (b) holds.

CASE 2: $x_1 x_3 = x_2 x_1$ and $x_3 x_1 \in T^2$. In this case $x_3 = x_2^{x_1}$ and $x_3 x_1 \in \{x_1 x_2, x_2^2\}$. If $x_3 x_1 = x_1 x_2$, then $x_2 = x_3^{x_1} = x_2^{x_1^2}$, so $[x_1^2, x_2] = 1$ and hence $[x_1, x_2] = 1$, a contradiction. So we may suppose that

$$x_3 x_1 = x_2^2. \tag{7}$$

Hence $(x_3 x_1)^{-1} x_1 x_3 = x_2^{-2} x_2 x_1 = x_2^{-1} x_1$ and $x_2 = x_1 c$ for some $c \in \langle S \rangle'$ with $c > 1$. Moreover, $x_3 = x_2^{x_1} = x_1 c^{x_1}$. Write $x_1 = x$; then $x_2 = xc$ and $x_3 = xc^x$. Moreover $xc^x x = xcxc$, by (7), hence $c^{x^2} = c^x c$, and arguing as before $[c, c^x] = 1$ and (a) holds.

CASE 3: $x_1 x_3 = x_2 x_1$ and $x_3 x_1 \notin T^2$. In this case

$$x_1 x_3 = x_2 x_1, \quad \text{and} \quad x_3 x_1 = x_2 x_3.$$

Write $x_1 = x$. Then $x_2 = xc$ for some $c \in \langle S \rangle'$ with $c > 1$, $x_3 = xc^x$, and $xc^x x = xcxc^x$. Then $c^{x^2} = c^x c^x$, hence $c^x = c^2$ and (c) holds.

We argue similarly if $x_3 x_1 \leq x_1 x_3$. In this case $x_3 x_1 \notin \{x_2 x_3, x_3 x_2, x_3^2\}$. If $x_3 x_1 = x_2^2$, then as before $x_1 x_3 \neq x_3 x_1$ and $x_1 x_3 = x_3 x_2$. Setting $x_1 = x^{-1}$, $x_2 = x^{-1} c$ for some $c \in \langle S \rangle'$ with $c > 1$, we obtain that $x_3 = x^{-1} c c^x$ and we see that $cc^x = c^x c$ and (b) holds. If $x_3 x_1 = x_1 x_2$ and $x_1 x_3 = x_3 x_2$, then with $x_1 = x^{-1}$ we obtain that (c) holds. If $x_3 x_1 = x_1 x_2$ and $x_1 x_3 = x_2^2$, then with $x_1 = x^{-1}$, $x_2 = x^{-1} c$ for some $c \in \langle S \rangle'$ with $c > 1$, we obtain $x_3 = x^{-1} c^x$ and we see that (a) holds.   □

6. The structure of $\langle S \rangle$ if $|S^2| = 3|S| - 2$: two general lemmas

In this section we present two general lemmas which will be useful in the inductive process entering the proof of Theorem 5, as well as in the study of the special case $|S| = 4$ (in the next section).

Lemma 4. *Let G be an ordered group. Let S be a finite subset of G with at least two elements. Let either $m = \max S$ or $m = \min S$ and $T = S \setminus \{m\}$. Then either $\langle S \rangle$ is a 2-generated abelian group, or $|T^2| \leq |S^2| - 3$.*

Proof. Write $k = |S| \geq 2$ and let $S = \{x_1, x_2, \ldots, x_k\}$, with $x_1 < x_2 < \cdots < x_k$. Moreover, suppose that $T = \{x_1, \ldots, x_{k-1}\}$.

Obviously $x_k^2, x_{k-1}x_k, x_k x_{k-1} \notin T^2$, because of the ordering. If $x_k x_{k-1} \neq x_{k-1}x_k$, then the result holds immediately. Therefore suppose that $x_k x_{k-1} = x_{k-1}x_k$.

If $x_i x_k, x_k x_i \in T^2$ for each $i < k-1$, then, by Lemma 3, $T \subseteq \langle x_{k-1}, x_k \rangle$. Thus $\langle S \rangle = \langle x_{k-1}, x_k \rangle$ and hence $\langle S \rangle$ is 2-generated and abelian, as required.

If there exists $j < k-1$ such that either $x_j x_k \notin T^2$ or $x_k x_j \notin T^2$, then $|T^2| \leq |S^2| - 3$, since $x_j x_k$ and $x_k x_j$ are both less than $x_{k-1}x_k = x_k x_{k-1}$, as required.

If $T = \{x_2, \ldots, x_k\}$, then the result follows from the previous arguments by reversing the ordering in G. $\qquad\square$

Now, we study the case when $S = T \dot{\cup} \{y\}$, where $\langle T \rangle$ is abelian.

Proposition 3. *Let G be an ordered group and let T be a finite subset of G such that $|T| \geq 3$ and $\langle T \rangle$ is abelian. Let $y \in G \setminus T$. Define $S = T \dot{\cup} \{y\}$ and assume that $|S^2| = 3|S| - 2$. Then either $\langle S \rangle$ is abelian, or there are elements $a, c \in G$ such that S is of the form*

$$S = \{a, ac, \ldots, ac^{k-2}, y\}$$

where $k = |S|$ and one of the following holds:

 (a) $[a, y] = c$ *or* $[y, a] = c$, $[c, y] = [c, a] = 1$;
 (b) $[a, y] = c$ *or* $[a, y] = 1$, $[c, a] = 1$, $(c^2)^y = c$, $|S| = 4$;
 (c) $[a, y] = 1$ *or* $[y, a] = c^2$, $[c, a] = 1$, $c^y = c^2$, $|S| = 4$.

In particular, $\langle S \rangle$ is either abelian or young of type (i) or (ii). If it is young of type (ii), then $|S| = 4$.

Proof. If $y \in C_G(T)$, then $\langle S \rangle$ is abelian, as required. So we may assume that $y \notin C_G(T)$. Then $|yT \cup Ty| \geq k$ by Proposition 2.4 of [10].

We have

$$y^2 \notin T^2$$

since otherwise, for any $t \in T$, $[y^2, t] = 1$ and hence $[y, t] = 1$, thus $y \in C_G(T)$, which is not the case.

Similarly,

$$(yT \cup Ty) \cap T^2 = \emptyset.$$

Hence

$$S^2 = T^2 \dot{\cup} (yT \cup Ty) \dot{\cup} \{y^2\},$$

and therefore

$$|T^2| \leqslant |S^2| - k - 1 = 3k - 2 - k - 1 = 2k - 3 = 2|T| - 1.$$

But by Theorem 1.1 of [10], it is also true that $|T^2| \geq 2|T| - 1$, hence $|T^2| = 2|T| - 1$, and by Corollary 1.4 of [10], T is of the form

$$T = \{a, ac, ac^2, \ldots, ac^{k-2}\},$$

where $ac = ca$, and we may assume that $c > 1$. Moreover $|yT \cup Ty| = k$. We consider now three cases.

CASE 1: $ya < ay$. In this case

$$yT \cup Ty = \{ya, ay, acy, ac^2y, \ldots, ac^{k-2}y\},$$

with $ya < ay < acy < ac^2y < \cdots < ac^{k-2}y$. Now consider the elements $yac, yac^2 \in yT \cup Ty$. Since $yac < yac^2 < yac^3 < \cdots < yac^{k-2}$, yac is less than $k - 2 - 1$ elements of $yT \cup Ty$ and yac^2 is less than $k - 4$ elements of $yT \cup Ty$. Thus either $yac = ay$ or $yac = acy$.

If $yac = acy$, then the only possibility is that $yac^2 = ac^2y$. But then $1 = [y, ac] = [y, ac^2]$, yielding $[y, c] = 1 = [y, a]$, a contradiction.

Now suppose that $yac = ay$, which implies $[a, y] = c$. In this case either $yac^2 = acy$ or $yac^2 = ac^2y$.

If $yac^2 = acy$, then $ayc = acy$ and $[c, y] = 1$. Thus the derived group $\langle S \rangle' = \langle c \rangle \leq Z(\langle S \rangle)$, $\langle S \rangle$ is of class 2, and (a) holds.

If $yac^2 = ac^2y$, then $ayc = ac^2y$ and $c = (c^2)^y$. Moreover, in this case $k = 4$. Indeed, if $k > 4$, then $yac^3 \in yT \cup Ty$ and the only possibility is $yac^3 = ac^3y$. But then $yac^3 = ayc^2 = ac^3y$ and $c^2 = (c^3)^y = (c^2)^yc^y = cc^y$, yielding $cy = yc$. Thus $yac^2 = ayc^2$, in contradiction to our assumption that $[a, y] = c$ and $c > 1$. Therefore $k = 4$ and (b) holds.

CASE 2: $ay < ya$. We argue similarly and obtain that either $[y, a] = c$, $[c, y] = 1$ and (a) holds, or $[y, a] = c^2$, $c^y = c^2$, $|S| = 4$ and (c) holds.

CASE 3: $ay = ya$. We have $yac \neq acy$, since otherwise $[y, a] = 1 = [y, c]$ and $y \in C_G(T)$, a contradiction.

Assume first $yac < acy$. Then

$$yT \cup Ty = \{ya = ay, yac, acy, ac^2y, \ldots, ac^{k-2}y\},$$

with $ya = ay < yac < acy < ac^2y < \cdots < ac^{k-2}y$. Consider the element yac^2. Then $yac^2 \in yT \cup Ty$ and $yac < yac^2 < yac^3 < \cdots < yac^{k-2}$, so, as before, we may conclude that either $yac^2 = acy$ or $yac^2 = ac^2y$.

If $yac^2 = acy$, then $ya = ay$ implies that $yc^2 = cy$. Thus $c^y = c^2$. Moreover, in this case $|S| = 4$, since otherwise $yac^3 \in T^2$ and either $yac^3 = ac^2y$, $yc^3 = c^2y$ yielding the contradiction $c^3 = (c^2)^y = c^4$, or $yac^3 = c^3ay$ yielding the contradiction $cy = yc$. Therefore (c) holds.

If $yac^2 = ac^2 y$, then $ayc^2 = ac^2 y$ and $[c^2, y] = 1$. But then $[c, y] = 1$, again a contradiction.

If $acy < yac$, then arguing similarly we obtain $(c^2)^y = c$, $|S| = 4$ and (b) holds.

We now prove the 'in particular' part of our statement.

If (a) holds, then $\langle S \rangle$ is young of type (i).

If (b) holds and $[a, y] = 1$, then $\langle S \rangle$ is young of type (ii).

If (b) holds and $[a, y] = c$, then $[ac^2, y] = cc^{-1} = 1$, $(c^2)^y = c$, $[ac^2, c] = 1$ and again $\langle S \rangle$ is young of type (ii).

If (c) holds, and $[a, y] = 1$, we have $\langle S \rangle = \langle a \rangle \times \langle c, y \rangle$, with $c^y = c^2$, and again $\langle S \rangle$ is young of type (ii).

Finally, if (c) holds and $[y, a] = c^2$, then $[y, c] = c^{-1}$, thus $[y, ac^2] = 1$ and again $\langle S \rangle = \langle ac^2 \rangle \times \langle c, y \rangle$ in a young groups of type (ii). $\qquad\square$

7. The structure of $\langle S \rangle$ if $|S^2| = 3|S| - 2$: cardinality 4

In this section, we study the special case of Theorem 5 when S has cardinality 4 and prove several lemmas.

Lemma 5. *Let G be an ordered group. Let x_1, x_2, x_3, x_4 be elements of G, such that $x_1 < x_2 < x_3 < x_4$ and let $S = \{x_1, x_2, x_3, x_4\}$. Suppose that $|S^2| = 10 = 3|S| - 2$. If $x_3 \in Z(\langle x_2, x_3, x_4 \rangle)$, then either $\langle x_2, x_3, x_4 \rangle$ or $\langle x_1, x_2, x_3 \rangle$ is abelian.*

Proof. We have $x_2 x_3 = x_3 x_2$ and $x_3 x_4 = x_4 x_3$.

If $x_2 x_4 = x_4 x_2$, then the result holds.

We may therefore restrict ourselves to the case where $x_2 x_4 \neq x_4 x_2$, and assume that $x_2 x_4 < x_4 x_2$ (we argue similarly in the symmetric case).

Let $T = \{x_1, x_2, x_3\}$. We first notice that

$$x_4 x_2 \notin T^2.$$

Indeed, if $x_4 x_2 \in T^2$, then the only possibility is $x_4 x_2 = x_3^2$. But then $x_2 x_4 = x_4 x_2$, a contradiction.

Obviously $x_2 x_4 \notin \{x_4 x_2, x_3 x_4, x_4^2\}$. If also $x_2 x_4 \notin T^2$, then

$$|T^2| \leq 10 - 4 = 6 = 3|T| - 3,$$

and by Theorem 1.3 of [10] T is abelian, as required.

Thus we may assume that $x_2x_4 \in T^2$ and it follows that

$$x_2x_4 \in \{x_3x_1, x_3x_2, x_3^2\}.$$

If $x_2x_4 = x_3x_2 = x_2x_3$, we obtain the contradiction $x_4 = x_3$. If $x_2x_4 = x_3x_1$, then $x_1 \in \langle x_2, x_3, x_4 \rangle$, which implies that $x_1x_3 = x_3x_1$. But then $x_2x_4 = x_1x_3$, in contradiction to the ordering in S. Finally, if $x_2x_4 = x_3^2$, then $x_2x_4 = x_4x_2$, again a contradiction. $\square$

Lemma 6. *Let G be an ordered group. Let x_1, x_2, x_3, x_4 be elements of G such that $x_1 < x_2 < x_3 < x_4$ and let $S = \{x_1, x_2, x_3, x_4\}$. Suppose that $|S^2| = 10 = 3|S| - 2$. If $x_2x_3 = x_3x_2$, then either $\langle x_2, x_3, x_4 \rangle$ or $\langle x_1, x_2, x_3 \rangle$ is abelian.*

Proof. If $x_3x_4 = x_4x_3$, then the result follows from Lemma 5. So we may assume that

$$x_3x_4 \neq x_4x_3.$$

Write $T = \{x_1, x_2, x_3\}$. If $|T^2| < 7$, then Theorem 1.3 of [10] implies that $\{x_1, x_2, x_3\}$ is abelian, as required. Hence we may assume that $|T^2| = 7$. Then

$$S^2 = T^2 \dot{\cup} \{x_3x_4, x_4x_3, x_4^2\}.$$

We assume now that

$$x_2x_4 \leq x_4x_2,$$

the symmetric case being similar.

We first notice that

$$x_4x_2 \notin T^2,$$

since otherwise $x_4x_2 \in T^2$ and the only possibility is $x_4x_2 = x_3^2$. But then $x_3x_4 = x_4x_3$, a contradiction. Hence

$$x_4x_2 = x_3x_4$$

and in particular $x_2x_4 \neq x_4x_2$. Moreover, $x_2x_4 \neq x_4x_3$ since $x_2x_4 < x_4x_2 < x_4x_3$. Hence $x_2x_4 \in T^2$ and $x_2x_4 \in \{x_3x_1, x_3x_2, x_3^2\}$.

If $x_2x_4 = x_3x_2 = x_2x_3$, then $x_3 = x_4$, a contradiction.
If $x_2x_4 = x_3^2$, then $x_2x_4 = x_4x_2$, a contradiction.
Hence the only possibility is

$$x_2x_4 = x_3x_1, \tag{8}$$

which implies that

$$x_1x_4 < x_2x_4 = x_3x_1 < x_4x_1. \tag{9}$$

If $x_4x_1 \notin T^2$, then $x_4x_1 = x_3x_4 = x_4x_2$, a contradiction. Thus $x_4x_1 \in T^2$ and we have

$$x_4x_1 \in \{x_1x_2, x_2^2, x_2x_3 = x_3x_2, x_1x_3, x_3^2\}.$$

Because of (9), the only possibility is

$$x_4x_1 = x_3^2. \tag{10}$$

Now, from (8) we get $x_1x_4^{-1} = x_3^{-1}x_2 \in C_G(x_3)$, and by (10) also $x_4x_1 = x_3^2 \in C_G(x_3)$. Thus $x_1^2 \in C_G(x_3)$ and $x_1x_3 = x_3x_1 = x_2x_4$, a contradiction. □

Lemma 7. *Let G be an ordered group. Let x_1, x_2, x_3, x_4 be elements of G, such that $x_1 < x_2 < x_3 < x_4$ and let $S = \{x_1, x_2, x_3, x_4\}$. Suppose that $|S^2| = 10 = 3|S| - 2$.*

If $x_1x_2 = x_2x_1$ and $x_3x_4 = x_4x_3$, then $\langle S \rangle$ is either abelian or young of type (i), (ii), or (iii).

Proof. Assume that $\langle S \rangle$ is non-abelian and let us prove that it is young of type (i), (ii), or (iii). Let $T = \{x_1, x_2, x_3\}$.

By Lemma 4, $|T^2| \leq 7$. If $\langle T \rangle$ is abelian, then by Proposition 3 $\langle S \rangle$ is young of type (i) or (ii), as required. So assume that $\langle T \rangle$ is non-abelian. Then Theorem 1.3 of [10] implies that $|T^2| = 7$ and Proposition 1 applies.

Suppose that $x_1 \in Z(\langle T \rangle)$. Since $|T^2| = 7 = |S^2| - 3$, it is impossible that $x_1x_4, x_2x_4, x_3x_4, x_4^2 \notin T^2$. Hence $x_4 \in \langle T \rangle$, which implies that $x_1x_4 = x_4x_1$. Thus $\langle x_1, x_3, x_4 \rangle$ is abelian and, by Proposition 3, $\langle S \rangle$ is young of type (i) or (ii), as required.

If x_2 or $x_3 \in Z(\langle x_1, x_2, x_3 \rangle)$, then $x_2x_3 = x_3x_2$ and the result follows by Lemma 6 and Proposition 3.

So we may assume that $Z(\langle x_1, x_2, x_3 \rangle) \cap \{x_1, x_2, x_3\} = \emptyset$. Similarly, we may assume that $\langle x_2, x_3, x_4 \rangle$ is non-abelian, $|\{x_2, x_3, x_4\}^2| = 7$ and $Z(\langle x_2, x_3, x_4 \rangle) \cap \{x_2, x_3, x_4\} = \emptyset$. Hence Proposition 1 implies that

$$\{x_1, x_2, x_3\} = \{a, a^b, b\} \quad \text{and} \quad \{x_2, x_3, x_4\} = \{c, c^d, d\},$$

for some $a, b, c, d \in G$ satisfying $aa^b = a^ba$, $cc^d = c^dc$. Therefore

$$[[a, b], a] = [[a^b, b], a^b] = 1 \quad \text{and} \quad [[c, d], c] = [[c^d, d], c^d] = 1.$$

Since $[x_1, x_3] \neq 1$ and $[x_1, x_2] = 1$, we have $x_3 = b$, $x_2 \in \{a, a^b\}$ and thus $[[x_2, x_3], x_2] = 1$. Similarly, since $[x_2, x_4] \neq 1$ and $[x_3, x_4] = 1$, we have $x_2 = d$, $x_3 \in \{c, c^d\}$ and thus $[[x_3, x_2], x_3] = 1$. It follows that $\langle x_2, x_3 \rangle$ is nilpotent of class 2. Moreover, $a, b \in \langle x_2, x_3 \rangle$ and $c, d \in \langle x_2, x_3 \rangle$, so $\langle S \rangle = \langle x_2, x_3 \rangle$ is 2-generated and nilpotent of class 2. Therefore $\langle S \rangle$ is young of type (i), as required. □

Lemma 8. *Let G be an ordered group. Let x_1, x_2, x_3, x_4 be elements of G, such that $x_1 < x_2 < x_3 < x_4$ and let $S = \{x_1, x_2, x_3, x_4\}$. Suppose that $|S^2| = 10 = 3|S| - 2$ and $\langle S \rangle$ is non-abelian.*

If $x_1x_2 = x_2x_1$, then $\langle S \rangle$ is young.

Proof. If $x_3x_4 = x_4x_3$, then the result follows from Lemma 7. Thus we may suppose that $x_3x_4 \neq x_4x_3$.

If $x_2x_3 = x_3x_2$, then the result follows by Lemma 6 and Proposition 3. Thus we may also assume that $x_2x_3 \neq x_3x_2$. Moreover, Lemma 4 and Theorem 1.3 in [10] imply that $|\{x_2, x_3, x_4\}^2| = 7$. Therefore $\langle x_2, x_3, x_4 \rangle$ satisfies the hypotheses of Proposition 2 and hence it is young.

Since $|\{x_2, x_3, x_4\}^2| = 7$ and $x_1^2 \notin \{x_2, x_3, x_4\}^2$, it follows that

$$x_1x_2, x_1x_3, x_1x_4 \notin \{x_2, x_3, x_4\}^2$$

is impossible. Hence $x_1 \in \langle x_2, x_3, x_4 \rangle$ and $\langle S \rangle = \langle x_2, x_3, x_4 \rangle$ is young, as required. $\square$

8. The structure of $\langle S \rangle$ if $|S^2| = 3|S| - 2$: proof of Theorem 5

Now we can prove Theorem 5.

Proof. Write $S = \{x_1, x_2, \ldots, x_{k-1}, x_k\}$, $x_1 < x_2 < \cdots < x_k$, and define

$$T = \{x_1, x_2, \ldots, x_{k-1}\},$$

and

$$V = \{x_2, \ldots, x_{k-1}, x_k\}.$$

We argue by induction on k.

We start with the basic case $k = 4$. By Lemma 4, either $\langle S \rangle$ is abelian, as required, or $|T^2| \leq 7$. Similarly, by considering the order opposite to $<$, we may suppose that $|V^2| \leq 7$.

If either T or V is abelian, then, by Proposition 3, $\langle S \rangle$ is either abelian or young of type (i) or (ii), as required. So, from now on, we may assume that T and V are non-abelian and $|T| = |V| = 7$ by Theorem 1.3 in [10].

If $x_1x_2 = x_2x_1$, then by Lemma 8, $\langle S \rangle$ is a young group, as required. So we may suppose that $x_1x_2 \neq x_2x_1$ and, by Lemma 6, also $x_2x_3 \neq x_3x_2$. Thus $\langle T \rangle$ satisfies the hypotheses of Proposition 2 and hence $\langle T \rangle$ is young. Moreover, one of the elements $x_1x_4, x_2x_4, x_3x_4 \in T^2$, so $\langle S \rangle = \langle T \rangle$ is also young.

Now we move to the inductive step. So suppose that $k > 4$ and that the result is true for $k - 1$. Moreover, suppose that $\langle S \rangle$ is non-abelian. Then

$$|T^2| \leq 3k - 2 - 3 = 3(k - 1) - 2$$

by Lemma 4. We may assume that the equality holds, since otherwise $\langle T \rangle$ is abelian by Theorem 1.3 in [10] and the result follows from Proposition 3. Hence, by induction, $\langle T \rangle$ has the required structure.

Now consider the k elements $x_1 x_k, x_2 x_k, \ldots, x_{k-1} x_k, x_k^2$. If one of them is in T^2, then $x_k \in \langle T \rangle$, hence $\langle S \rangle = \langle T \rangle$ has the required structure. On the other hand, if $x_1 x_k, x_2 x_k, \ldots, x_k^2 \notin T^2$, then $S^2 \supseteq T^2 \dot\cup \{x_1 x_k, \ldots, x_k^2\}$ and, in view of $k > 4$,

$$|T^2| \leq |S^2| - k = 2k - 2 \leq 3(k - 1) - 3 = 3|T| - 3.$$

Therefore T is abelian by Theorem 1.3 in [10] and the result follows from Proposition 3. $\qquad\square$

9. On the structure of $\langle S \rangle$ if $|S^2| \leq 3|S| - 2 + s$ and $|S|$ is large

We start with the following lemma.

Lemma 9. *Let G be an ordered group. Suppose that $a, b, c \in G$ such that $[a, c] = 1$ and let T be a subset of G satisfying $T \subseteq \{a, ac, ac^2, \ldots, ac^h\}$ for some positive integer h. Moreover, let $b \in G \setminus T$ such that $|Tb \cap bT| \geq 2$. Then $\langle a, b, c \rangle$ is metabelian. Moreover, if G is nilpotent, then $\langle a, b, c \rangle$ is nilpotent of class at most 2.*

Proof. Since $|Tb \cap bT| \geq 2$, there exist $r \neq l$, $s, t \in \mathbb{Z}$ such that $(ac^r)^b = ac^s$, $(ac^l)^b = ac^t$. Suppose, without loss of generality, that $r < l$. Then

$$(ac^r c^{l-r})^b = ac^s (c^{l-r})^b = ac^t,$$

which implies that $(c^{l-r})^b = c^{t-s}$. Write $l - r = m$, $t - s = n$. Then $(c^m)^b = c^n$, so $[(c^m)^b, c] = 1$ and $[c^b, c] = 1$.

Now we claim that the subgroup $C = \langle c^{b^j} \mid j \in \mathbb{Z} \rangle$ is abelian.

Obviously it suffices to prove that $[c, c^{b^j}] = 1$ for any integer j. From $(c^m)^b = c^n$ we get easily by induction that $(c^{m^i})^{b^i} = c^{n^i}$ for any positive integer i.

Indeed, suppose that $(c^{m^{i-1}})^{b^{i-1}} = c^{n^{i-1}}$. Then we have

$$(c^{m^i})^{b^i} = (((c^{m^{i-1}})^{b^{i-1}})^m)^b = ((c^{n^{i-1}})^m)^b = ((c^m)^b)^{n^{i-1}} = c^{n^i}.$$

Therefore $[(c^{b^i})^{m^i}, c] = 1$ for each positive integer i, and hence $[c^{b^i}, c] = 1$. This result also implies that $[c, c^{b^{-i}}] = 1$ for each positive integer i. Thus $[c, c^{b^j}] = 1$ for every integer j and the claim follows.

Obviously $b \in N_G(C)$. We claim that also $a \in C_G(C)$. We need only to show that if v is an integer, then $a^{b^v} \in C_G(c)$.

We show first that $a^b \in C_G(c)$. Indeed, since $a \in C_G(c)$ and $(ac^r)^b = ac^s$, it follows that

$$a^b(c^b)^r = ac^s = (ac^s)^c = (a^b(c^b)^r)^c = (a^b)^c(c^b)^r,$$

which implies that $(a^b)^c = a^b$, as required.

Suppose, by induction, that $a^{b^v} \in C_G(c)$ for some positive integer v. Since $(ac^r)^{b^{v+1}} = ((ac^r)^b)^{b^v} = (ac^s)^{b^v}$, we have

$$a^{b^{v+1}}(c^r)^{b^{v+1}} = (ac^s)^{b^v} = ((ac^s)^{b^v})^c = (a^{b^{v+1}}(c^r)^{b^{v+1}})^c = (a^{b^{v+1}})^c(c^r)^{b^{v+1}}.$$

Hence also $a^{b^{v+1}} \in C_G(c)$. It follows that $a^{b^v} \in C_G(c)$ for each positive integer v.

Similarly, from $(ac^s)^{b^{-1}} = ac^r$, we get that $a^{b^{-1}} \in C_G(c)$ and by induction $a^{b^{-v}} \in C_G(c)$ for each positive integer v.

Hence $a \in C_G(C) \subseteq N_G(C)$. Thus C is normal in $\langle a, b, c \rangle$, and obviously $\langle a, b, c \rangle / C$ is abelian. Hence $\langle a, b, c \rangle$ is metabelian.

If G is a torsion-free nilpotent group, then $(c^m)^b = c^n$ implies that $c^b = c$. In fact, we can argue by induction on the nilpotency class d of G. The result is obvious if $d = 1$, that is if G is abelian. By induction we have $c^b Z(G) = c Z(G)$, since $G/Z(G)$ is torsion-free of class $d - 1$ (see for example 5.2.19 of [27]). Thus $c^b = cz$ for some $z \in Z(G)$ and we have $c^n = (c^b)^m = (cz)^m = c^m z^m$, which implies that $c^{n-m} \in Z(G)$. If $n - m \neq 0$, then $c \in Z(G)$, and obviously $c^b = c$ in this case. If $m = n$, then $[c^m, b] = 1$ and $[c, b] = 1$, since G is an ordered group. Hence $C \subseteq Z(\langle a, b, c \rangle)$ and $\langle a, b, c \rangle$ is nilpotent of class at most 2, as required. $\square$

Lemma 9 has the following useful Corollary.

Corollary 4. *Let G be an ordered group and let S be a finite subset of G of finite size > 3. Let $m = \max S$ and $T = S \setminus \{m\}$. Suppose that $|S^2| = 3|S| + b$ for some $b \leq |S| - 6$. If $m \notin \langle T \rangle$ and $m \notin C_G(T)$, then $\langle S \rangle$ is metabelian and if G is nilpotent, then $\langle S \rangle$ is nilpotent of class at most 2.*

Proof. Set $|S| = k$ and $m = x_k$. Since $x_k \notin \langle T \rangle$ and $x_k = \max S$, we have the partition

$$S^2 = T^2 \dot\cup (x_k T \cup T x_k) \dot\cup \{x_k^2\}.$$

Moreover $|x_k T \cup T x_k| \geq k$ by Proposition 2.4 of [10]. Hence

$$3k + b = |T^2| + 1 + |x_k T \cup T x_k| \geq |T^2| + 1 + k,$$

and

$$|T^2| \leq 2k + b - 1 \leq 2k + k - 6 - 1 = 3(k-1) - 4.$$

Thus, by Corollary 1.4 of [10], there exist elements a, c in G, with $ac = ca$, such that $T \subseteq \{a, ac, \ldots, ac^h\}$ for some positive integer h.

Furthermore, $|T^2| \geq 2|T| - 1 = 2k - 3$ by Theorem 1.1 in [10]. Therefore

$$|x_k T \cup T x_k| = |S^2| - |T^2| - 1 \leq (4k - 6) - (2k - 3) - 1 = 2k - 4$$

and

$$|x_k T \cap T x_k| = 2|T| - |x_k T \cup T x_k| \geq 2(k - 1) - (2k - 4) = 2.$$

Thus Lemma 9 applies. Hence $\langle S \rangle \subseteq \langle a, c, x_k \rangle$ is metabelian and and if G is nilpotent, then $\langle S \rangle$ is nilpotent of class at most 2, as required. $\square$

Finally, in order to prove Theorem 6, we shall use the following easy consequence of Lemma 4.

Lemma 10. *Let G be an ordered group and let $x_1 < \cdots < x_k$ be elements of G. Let $S = \{x_1, \ldots, x_k\}$, with $k \geq 2$ and suppose that $|S^2| \leq 3|S| + b$ for some integer b. Then for any integer i, $1 \leq i < k$, either $\langle x_1, \ldots, x_{i+1} \rangle$ is abelian, or $|\{x_1, \ldots, x_i\}^2| \leq 3i + b$.*

Proof. Notice that if $b \leq -3$, then $\langle S \rangle$ is abelian by Theorem 1.3 in [10]. Thefore we may assume that $b \geq -2$.

Our proof is by induction from $i = k - 1$ down to $k = 1$. If $i = k - 1$, then $\langle x_1, \ldots, x_{i+1} \rangle = \langle S \rangle$. Hence, by Lemma 4, either $\langle S \rangle$ is abelian, or

$$|\{x_1, \ldots, x_i\}^2| \leq |S^2| - 3 \leq 3k - 3 + b = 3(k - 1) + b,$$

as required.

So assume that $1 \leq j < k - 1$ and that the Lemma holds for $i = j + 1$. Moreover, suppose that $\langle x_1, \ldots, x_{j+1} \rangle$ is non-abelian. Then $\langle x_1, \ldots, x_{j+2} \rangle$ is nonabelian and by our assumptions $|\{x_1, \ldots, x_{j+1}\}^2| \leq 3(j + 1) + b$. It follows then by Lemma 4 that

$$|\{x_1, \ldots, x_j\}^2| \leq |\{x_1, \ldots, x_{j+1}\}^2| - 3 \leq 3j + b,$$

as required. Therefore the Lemma holds for $1 \leq i < k$, as required. $\square$

We can now prove Theorem 6.

Proof of Theorem 6. Write $S = \{x_1, \ldots, x_k\}$, with $x_1 < \cdots < x_k$, $T = \{x_1, \ldots, x_t\}$, $V = \{x_{t+1}, \ldots, x_k\}$, with $|T| = |V| = k/2$ if k is even, and $|T| = (k - 1)/2$, $|V| = (k + 1)/2$, if k is odd. Then $t = |T| \geq 2^{s+1}$ and $v = |V| \geq 2^{s+1}$.

We claim that either $\langle T \rangle$ or $\langle V \rangle$ is metabelian (and nilpotent of class at most 2 if G is nilpotent). We shall prove this claim by induction on s.

The claim is obvious if either $\langle T \rangle$ or $\langle V \rangle$ is abelian. So suppose that both are non-abelian. Then, by Lemma 10, $|T^2| \leq 3t - 2 + s$ and also $|V^2| \leq 3v - 2 + s$, by considering the ordering opposite to $<$.

Moreover, because of the ordering, $T^2 \cap V^2 = \emptyset$ and $x_t x_{t+1}, x_{t+1} x_t \notin T^2 \cup V^2$. If $x_t x_{t+1} = x_{t+1} x_t$, then by Lemma 3 there exists $x_j < x_t$ such that either $x_j x_{t+1}$ or $x_{t+1} x_j$ is not in T^2, since $\langle T \rangle$ is not abelian. Obviously also $x_j x_{t+1}, x_{t+1} x_j \notin V^2$. Therefore, in any case $S^2 \setminus (T^2 \cup V^2) \geq 2$.

Now, if

$$|T^2| \geq 3t - 2 + \frac{s+1}{2}$$

and

$$|V^2| \geq 3v - 2 + \frac{s+1}{2},$$

then

$$|S^2| \geq 3k - 4 + s + 1 + 2 = 3k + s - 1,$$

a contradiction.

If $s = 1$, then this contradiction implies that either $|T^2| \leq 3t - 2$ or $|V^2| \leq 3v - 2$ and by Corollaries 2 and 3 either $\langle T \rangle$ or $\langle V \rangle$ is metabelian (and nilpotent of class at most 2 if G is nilpotent), as claimed.

So suppose that $s \geq 2$, and that our claim holds for all smaller values of s. By the above contradiction, we must have either

$$|T^2| \leq 3t - 2 + \frac{s}{2}$$

or

$$|V^2| \leq 3v - 2 + \frac{s}{2}.$$

Moreover

$$t \geq 2^{s+1} \geq 2^{\frac{s}{2}+2}$$

and similarly $v \geq 2^{\frac{s}{2}+2}$, since $s \geq 2$. Hence, by induction, either $\langle T \rangle$ or $\langle V \rangle$ is metabelian (and nilpotent of class at most 2 if G is nilpotent), as claimed. The proof of the claim is complete.

Suppose, without loss of generality, that $\langle T \rangle$ is metabelian (and nilpotent of class at most 2, if G is nilpotent). Let $X = \{x_1, \ldots, x_j\} \supseteq T$ be a subset of S maximal under the condition that $\langle X \rangle$ is metabelian (and nilpotent of class at most 2, if G is nilpotent). If $X = S$, then we have the result.

So suppose that X is a proper subset of S. Under this assumption we shall reach a contradiction, thus concluding the proof of the Theorem. Because of the maximality of X, $x_{j+1} \notin \langle X \rangle$ and $x_{j+1} \notin C_G(X)$. Hence, if we write $W = \{x_1, \ldots, x_{j+1}\}$ and $w = |W|$, then $|W^2| \leq 3w - 2 + s$, by Lemma 10. Moreover $s - 2 \leq w - 6$, since $w \geq t + 1 \geq 2^{s+1} + 1 \geq s + 4$. Therefore Corollary 4 applies, and $\langle W \rangle$ is metabelian (and nilpotent of class at most 2, if G is nilpotent), a contradiction. $\qquad \square$

Finally we can prove our final statement.

Proof of Theorem 7. Let $G = \langle a \rangle \times \langle b, c \rangle$, where $\langle a \rangle$ is an infinite cyclic group and $\langle b, c \rangle$ is a free group of rank 2. Then G is a direct product of two orderable groups and therefore it is an orderable group. Let k be an integer ≥ 3 and define $S = \{a, ac, \ldots, ac^{k-2}, b\}$. Write $T = \{a, ac, \ldots, ac^{k-2}\}$. Then $b \notin C_G(T)$, so in particular $b \notin \langle T \rangle$. Hence $S^2 = T^2 \dot\cup (bT \cup Tb) \dot\cup \{b^2\}$. We also have $ab = ba$ and $bac^i = abc^i \neq ac^j b$ for any $i \neq j$, since $\langle b, c \rangle$ is free. Hence $|bT \cap Tb| = 1$, which implies that

$$|bT \cup Tb| = k - 1 + k - 1 - 1 = 2k - 3.$$

Since $S^2 = T^2 \dot\cup (bT \cup Tb) \dot\cup \{b^2\}$, it follows that

$$|S^2| = 2(k-1) - 1 + 2k - 3 + 1 = 4k - 5.$$

Obviously $\langle S \rangle = \langle a, b, c \rangle = G$. In particular, $\langle S \rangle$ is not soluble. □

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