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Anosov structures on Margulis spacetimes

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Abstract. In this paper we describe the stable and unstable leaves for the affine flow on the space of non-wandering spacelike affine lines of a Margulis spacetime and prove contraction properties of the leaves under the flow. We also show that monodromies of Margulis spacetimes are "Anosov representations in non semi-simple Lie groups."

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Contents

1. Introduction

A Margulis spacetime M is a quotient manifold of the three dimensional affine space by a free, non-abelian group acting as affine transformations with discrete linear part. It owes its name to Grigory Margulis, who was the first to use these spaces, in [22] and [23], as examples to answer Milnor's following question in the negative.

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Question 1. Is the fundamental group of a complete, flat, affine manifold virtually polycyclic? (See [25].)

Since then the study of Margulis spacetimes have been carried out extensively by Abels, Margulis, and Soifer [1] and [2], Charette and Drumm [4], Charette, Goldman, and Jones [5], Choi and Goldman [6], Danciger, Guéritaud, and Kassel [7] and [8], Drumm [9] and [10], Drumm and Goldman [11] and [12], Fried and Goldman [13], Goldman [14], Goldman and Labourie [15], Goldman, Labourie, and Margulis [16], Goldman and Margulis [17], Kim [20], and Smilga [26], [27], and [28].

In particular, Fried and Goldman showed in [13] that the fundamental group $\pi_1(\mathsf{M})$ of a Margulis spactime does not contain any translation. Moreover, by combining results of Fried and Goldman in [13] and Mess in [24] we get that a complete flat affine 3-manifold either has a polycyclic fundamental group or is a Margulis spacetime. In this paper we will only consider Margulis spacetimes whose linear part does not contain any parabolics, although by work of Drumm in [10] there exist Margulis spacetimes whose linear part contains parabolics. Fried and Goldman also showed in [13] that a conjugate of the linear part of the affine action of the fundamental group forms a subgroup of $\mathsf{SO}(2,1)$ in $\mathsf{GL}(\mathbb{R}^3)$. Therefore, a Margulis spacetime comes with a parallel Lorentz metric.

The parallelism classes of timelike affine lines of M can be parametrized by a non-compact complete hyperbolic surface Σ. Previous works of Charette, Goldman, and Jones in [5], Goldman and Labourie in [15] and Goldman, Labourie, and Margulis in [16] showed that the dynamics of M is closely related to that of Σ. Jones, Charette, and Goldman showed in [5] that bispiralling affine lines in M exist and they correspond to bispiralling geodesics in Σ. Moreover, Goldman and Labourie showed in [15] that spacelike non-wandering affine lines in M correspond to non-wandering geodesics in Σ. In fact, they constructed an orbit equivalent homeomorphism $\hat{\mathsf{N}}$ between $\mathsf{U}_{\mathrm{rec}}\Sigma$, the space of non-wandering geodesics in Σ and $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$, the space of spacelike non-wandering affine lines in M.

The homeomorphism $\hat{\mathsf{N}}$ gives rise to a metric Anosov structure on $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ with respect to the image of the geodesic flow ϕ on $\mathsf{U}_{\mathrm{rec}}\Sigma$ but in this paper we want to find a metric Anosov structure on $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ with respect to the affine flow Φ on $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$. We note that the image of the flow ϕ under $\hat{\mathsf{N}}$ is not necessarily equal to Φ, it only has the same flow lines. Hence the homeomorphism $\hat{\mathsf{N}}$ does not help us directly in our search. In fact, $\hat{\mathsf{N}} \circ \phi \neq \Phi$ implies that the stable and unstable laminations of the metric Anosov structure on $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ with respect to Φ are not the same as the stable and unstable laminations of the metric Anosov structure on $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ with respect to $\hat{\mathsf{N}} \circ \phi$. However, the central stable and central unstable laminations (Definition 2.5), by contrast, are respected by $\hat{\mathsf{N}}$.

Moreover, we note that, as $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ is compact, the stable and unstable laminations are uniquely determined by the flow and they are independent of the particular distance chosen on $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$, as long as the distances are locally bilipschitz equivalent with each other.

In this paper, we first chalk out some preliminary notions, in order to prepare the grounds to explicitly describe the stable and unstable laminations of $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ with respect to Φ and show that the leaves of the stable lamination contract under the forward flow and the leaves of the unstable lamination contract under the backward flow.

More precisely, let $\mathbb{A}$ be the affine three space and $\mathsf{U}\mathbb{A}$ be the spacelike unit tangent bundle of $\mathbb{A}$. We can alternatively think of $\mathsf{U}\mathbb{A}$ as the space of all tuples (x, ℓ) where x is a point in $\mathbb{A}$ and ℓ is an oriented spacelike affine line containing x. We denote the lift of $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ to $\mathsf{U}\mathbb{A}$ by $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$. Let (x, ℓ) be an element of $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$. We consider the intersection of the plane perpendicular to ℓ at x with respect to the Lorentz metric and the null cone at x. The intersection is the union of two lightlike affine lines. We orient these lightlike affine lines by defining the part lying in the upper light cone to be positive. We denote the two oriented lightlike affine lines by ℓ^- and ℓ^+ such that (ℓ^-, ℓ, ℓ^+) gives the positive orientation on $\mathbb{A}$. Then the stable (respectively unstable) lamination through the projection of $(x, \ell) \in \mathsf{U}_{\mathrm{rec}}\mathbb{A}$ into $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ is the projection of all the elements $(y, \ell') \in \mathsf{U}_{\mathrm{rec}}\mathbb{A}$ into $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ such that $y \in \ell^+$ (respectively ℓ^-) and ℓ' is an oriented spacelike affine line passing through y and lying in the affine plane generated by ℓ and ℓ^+ (respectively ℓ^-) with $(\ell^-, \ell' + (x - y), \ell^+)$ giving the positive orientation on $\mathbb{A}$.

Theorem 1.1. *Let $\underline{\mathcal{L}}^+$ and $\underline{\mathcal{L}}^-$ be two laminations of the metric space $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ as described above. Then Φ contracts $\underline{\mathcal{L}}^+$ exponentially in the forward direction of the flow and contracts $\underline{\mathcal{L}}^-$ exponentially in the backward direction of the flow.*

We can contrast this description of stable (respectively unstable) laminations in $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ to the corresponding laminations in $\mathsf{U}_{\mathrm{rec}}\Sigma$: there, the vectors in a leaf are the vectors perpendicular to a given horocycle, pointing inwards (respectively outwards).

Moreover, in the last section we provide a natural extension of the definition of Anosov representation given in Section 2.0.7 of [21] and define the appropriate notion of an Anosov representation in our context replacing manifolds with metric spaces. Using this definition we furthermore prove the following theorem.

Theorem 1.2. *Let N be the space of all oriented spacelike affine lines in the three dimensional affine Lorentzian space $\mathbb{R}^{2,1}$ and let $\mathcal{L}$ be the orbit foliation of the flow Φ on $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$. Then $(\mathsf{U}_{\mathrm{rec}}\mathsf{M}, \mathcal{L})$ admits a geometric $(\mathsf{N}, \mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3)$ Anosov structure with respect to the pair of foliations $\mathcal{F}^\pm$ on N whose leaves at an oriented spacelike affine line ℓ consist of all possible spacelike affine lines ℓ' with the following properties.*

1. *The affine line ℓ' intersects ℓ,*
2. *Let ℓ' and ℓ both pass through a point x and let $\ell^\pm$ be two oriented lightlike affine lines passing through x as mentioned before. Then $\ell' \in \mathcal{F}^+$ (respectively $\ell' \in \mathcal{F}^-$) if ℓ' lies in the affine plane generated by ℓ and ℓ^+ (respectively ℓ^-) and (ℓ^-, ℓ', ℓ^+) gives the positive orientation on $\mathbb{A}$.*

In other words, monodromies of Margulis spacetimes are "Anosov representations in non semi-simple Lie groups." Here we note that the notion of an Anosov representation of a discrete group into a transformation group G was first introduced by Labourie in [21]. Later, Guichard and Wienhard studied Anosov representations into semisimple Lie groups in more details in [19]. In this paper we consider Anosov representations into the non semi-simple Lie group $\mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3$.

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2. Background

2.1. Affine geometry. An *affine space* is a set $\mathbb{A}$ together with a vector space $\mathbb{V}$ and a faithful and transitive group action of $\mathbb{V}$ on $\mathbb{A}$. We call $\mathbb{V}$ the underlying vector space of $\mathbb{A}$ and refer to its elements as translations. An *affine transformation* between two affine spaces $\mathbb{A}_1$ and $\mathbb{A}_2$ is a map $F\colon \mathbb{A}_1 \to \mathbb{A}_2$ such that there exists a linear map $\mathtt{L}(F)\colon \mathbb{V}_1 \to \mathbb{V}_2$ satisfying the following property: for all x in $\mathbb{A}_1$ and v in $\mathbb{V}_1$,

$$F(x+v) = F(x) + \mathtt{L}(F)v. \tag{2.1}$$

Therefore, by fixing an origin O in $\mathbb{A}$, we can represent an affine transformation F from $\mathbb{A}$ to itself as a combination of a linear transformation and a translation. More precisely,

$$F(O+v) = O + \mathtt{L}(F)v + (F(O) - O). \tag{2.2}$$

We denote $(F(O) - O)$ by $\mathtt{u}(F)$ and the space of affine automorphisms of $\mathbb{A}$ by $\mathsf{Aff}(\mathbb{A})$.

Let $\mathsf{GL}(\mathbb{V})$ be the general linear group of $\mathbb{V}$. We consider the semidirect product $\mathsf{GL}(\mathbb{V}) \ltimes \mathbb{V}$, of the two groups $\mathsf{GL}(\mathbb{V})$ and $\mathbb{V}$, which comes equipped with the following multiplication rule: for g_1, g_2 in $\mathsf{GL}(\mathbb{V})$ and v_1, v_2 in $\mathbb{V}$,

$$(g_1, v_1)(g_2, v_2) := (g_1 g_2, v_1 + g_1 v_2).$$

Using equation (2.2) we obtain that the map

$$F \longmapsto (\mathtt{L}(F), \mathtt{u}(F))$$

defines an isomorphism between $\mathsf{Aff}(\mathbb{A})$ and $\mathsf{GL}(\mathbb{V}) \ltimes \mathbb{V}$.

We denote the tangent bundle of $\mathbb{A}$ by $\mathsf{T}\mathbb{A}$. It is a trivial bundle and is canonically isomorphic to $\mathbb{A} \times \mathbb{V}$ as a bundle. The affine flow $\widetilde{\Phi}$ on $\mathsf{T}\mathbb{A}$ is defined as follows:

$$\begin{aligned} \widetilde{\Phi}_t \colon \mathsf{T}\mathbb{A} &\longrightarrow \mathsf{T}\mathbb{A}, \\ (p, v) &\longmapsto (p + tv, v). \end{aligned}$$

2.2. Hyperboloid model of hyperbolic geometry. Let $(\mathbb{R}^{2,1}, \langle \cdot | \cdot \rangle)$ be the Minkowski spacetime where the quadratic form corresponding to the metric $\langle \cdot | \cdot \rangle$ is given by

$$\mathcal{Q} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.3}$$

Let $\mathsf{SO}(2,1)$ denote the group of linear transformations of $\mathbb{R}^{2,1}$ preserving the metric $\langle \cdot | \cdot \rangle$ and $\mathsf{SO}^0(2,1)$ be the connected component containing the identity of $\mathsf{SO}(2,1)$. The cross product $\boxtimes$ associated with this quadratic form is defined as follows:

$$u \boxtimes v := (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_2 v_1 - u_1 v_2)^{\mathrm{t}},$$

where u, v are denoted by $(u_1, u_2, u_3)^{\mathrm{t}}$ and $(v_1, v_2, v_3)^{\mathrm{t}}$ respectively. The cross product $\boxtimes$ satisfies the following properties: for all u, v, w in $\mathbb{R}^{2,1}$,

$$\begin{cases} u \boxtimes v = -v \boxtimes u, \\ \langle u \mid v \boxtimes w \rangle = \det[u, v, w], \\ \langle u \boxtimes v \mid u \boxtimes v \rangle = \langle u \mid v \rangle^2 - \langle u \mid u \rangle \langle v \mid v \rangle. \end{cases} \tag{2.4}$$

Now for any real number k we define,

$$\mathsf{S}^k := \{ v \in \mathbb{R}^{2,1} \mid \langle v \mid v \rangle = k \}.$$

We note that S^{-1} has two components. We denote the component containing $(0,0,1)^{\mathrm{t}}$ as $\mathbb{H}$. The quadratic form $\mathcal{Q}$ gives rise to a Riemannian metric $d_{\mathbb{H}}$ of constant negative curvature on the submanifold $\mathbb{H}$ of $\mathbb{R}^{2,1}$. The space $\mathbb{H}$ is called the *hyperboloid model of hyperbolic geometry*. Let $\mathsf{U}\mathbb{H}$ denote the unit tangent bundle of $\mathbb{H}$. The map

$$\begin{aligned} \Theta \colon \mathsf{SO}^0(2,1) &\longrightarrow \mathsf{U}\mathbb{H}, \\ g &\longmapsto (g(0,0,1)^{\mathrm{t}}, g(0,1,0)^{\mathrm{t}}), \end{aligned} \tag{2.5}$$

gives an analytic identification between $\mathsf{SO}^0(2,1)$ and $\mathsf{U}\mathbb{H}$. Let $\tilde{\phi}_t$ denote the geodesic flow on $\mathsf{U}\mathbb{H} \cong \mathsf{SO}^0(2,1)$. We note that

$$\tilde{\phi}_t g = g a(t), \tag{2.6}$$

where

$$a(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(t) & \sinh(t) \\ 0 & \sinh(t) & \cosh(t) \end{pmatrix}. \tag{2.7}$$

We also note that $\tilde{\phi}_t$ is the image of the geodesic flow on $\mathsf{PSL}(2,\mathbb{R})$ under the identification of $\mathsf{PSL}(2,\mathbb{R})$ and $\mathsf{SO}^0(2,1)$ given by

$$\Psi\colon \mathsf{PSL}(2,\mathbb{R}) \longrightarrow \mathsf{SO}^0(2,1),$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{pmatrix} ad+bc & ac-bd & ac+bd \\ ab-cd & \dfrac{a^2+d^2-b^2-c^2}{2} & \frac{a^2+b^2-c^2-d^2}{2} \\ ab+cd & \dfrac{a^2+c^2-b^2-d^2}{2} & \frac{a^2+b^2+c^2+d^2}{2} \end{pmatrix}.$$

In particular,

$$a(t) = \Psi\left(\begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}\right).$$

Let $d_{\mathsf{U}\mathbb{H}}$ be a metric on the unit tangent bundle $\mathsf{U}\mathbb{H}$ such that the fibers are orthogonal to the Levi-Civita connection on $\mathsf{U}\mathbb{H}$, the unit balls of $d_{\mathsf{U}\mathbb{H}}$ project onto the unit balls of $d_{\mathbb{H}}$ and each fiber is isometrically preserved by its stabilizer and has a total length of 2π. We note that this last number is somewhat arbitrary. Under the identification Θ the metric $d_{\mathsf{U}\mathbb{H}}$ is invariant under the left action of $\mathsf{SO}^0(2,1)$ and the right action of $\mathsf{SO}(2)$, seen as a maximal compact subgroup of $\mathsf{SO}^0(2,1)$.

Let $g \in \mathsf{SO}^0(2,1) \cong \mathsf{U}\mathbb{H}$. We recall that the *horocycles* $\widetilde{\mathcal{H}}_g^{\pm}$ for the geodesic flow $\tilde{\phi}$ passing through the point g are defined as follows:

$$\widetilde{\mathcal{H}}_g^{+} := \{h \in \mathsf{U}\mathbb{H} \mid \lim_{t\to\infty} d_{\mathsf{U}\mathbb{H}}(\tilde{\phi}_t g, \tilde{\phi}_t h) = 0\}, \tag{2.8}$$

$$\widetilde{\mathcal{H}}_g^{-} := \{h \in \mathsf{U}\mathbb{H} \mid \lim_{t\to-\infty} d_{\mathsf{U}\mathbb{H}}(\tilde{\phi}_t g, \tilde{\phi}_t h) = 0\}.$$

We note that, under the identification Θ the horocycle $\widetilde{\mathcal{H}}_g^{\pm}$ passing through g is given by $gu^{\pm}(t)$, where $u^{\pm}(t)$ are defined as follows:

$$u^{+}(t) := \begin{pmatrix} 1 & -2t & 2t \\ 2t & 1-2t^2 & 2t^2 \\ 2t & -2t^2 & 1+2t^2 \end{pmatrix} = \Psi\left(\begin{bmatrix} 1 & 2t \\ 0 & 1 \end{bmatrix}\right), \tag{2.9}$$

$$u^{-}(t) := \begin{pmatrix} 1 & 2t & 2t \\ -2t & 1-2t^2 & -2t^2 \\ 2t & 2t^2 & 1+2t^2 \end{pmatrix} = \Psi\left(\begin{bmatrix} 1 & 0 \\ 2t & 1 \end{bmatrix}\right). \tag{2.10}$$

Now let ν be defined as follows:

$$\begin{aligned} \nu\colon \mathsf{SO}^0(2,1) &\longrightarrow \mathsf{S}^1, \\ g &\longmapsto g(1,0,0)^{\mathrm{t}}, \end{aligned} \tag{2.11}$$

and also let $\nu^{\pm}$ be defined as follows:

$$\begin{aligned} \nu^{\pm}\colon \mathsf{SO}^0(2,1) &\longrightarrow \mathsf{S}^0, \\ g &\longmapsto g\Big(0, \pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big)^{\mathrm{t}}. \end{aligned} \tag{2.12}$$

The map ν is called the *neutral section* and the maps ν^{+}(respectively ν^{-}) are called the *positive* (respectively *negative*) *limit sections*. We list a few properties of the neutral section and the limit sections as follows:

$$\nu(\tilde{\phi}_t g) = \nu(g), \tag{2.13}$$

$$\nu(hg) = h\nu(g), \tag{2.14}$$

$$\nu^{\pm}(\tilde{\phi}_t g) = e^{\pm t}\nu^{\pm}(g), \tag{2.15}$$

$$\nu^{\pm}(hg) = h\nu^{\pm}(g), \tag{2.16}$$

$$\nu^{+}(gu^{+}(t)) = \nu^{+}(g), \tag{2.17}$$

$$\nu^{-}(gu^{-}(t)) = \nu^{-}(g). \tag{2.18}$$

where $t \in \mathbb{R}$ and $g, h \in \mathsf{SO}^0(2,1)$.

Let Γ be a free, nonabelian subgroup of $\mathsf{SO}^0(2,1)$ with finitely many generators. We consider the left action of Γ on $\mathsf{U}\mathbb{H}$. We notice that the action of Γ being from the left and the action of $a(t)$ (i.e. of the flow $\tilde{\phi}_t$) being from the right, the two actions commute. Furthermore, given a free and proper action of Γ on $\mathsf{U}\mathbb{H}$, one gets an isomorphism between $\Gamma\backslash\mathsf{U}\mathbb{H}$ and $\mathsf{U}\Sigma$, where $\mathsf{U}\Sigma$ is the unit tangent bundle of the surface $\Sigma := \Gamma\backslash\mathbb{H}$. We note that the flow $\tilde{\phi}$ on $\mathsf{U}\mathbb{H}$ gives rise to a flow ϕ on $\mathsf{U}\Sigma$.

Let x_0 be a point in $\mathbb{H}$. Let Γx_0 denote the orbit of x_0 under the action of Γ. We denote the closure of Γx_0 inside the closure of $\mathbb{H}$ in $\mathbb{P}(\mathbb{R}^{2,1})$ by $\overline{\Gamma x_0}$. We define the *limit set* of the group Γ to be the space $\overline{\Gamma x_0} \setminus \Gamma x_0$ and denote it by $\Lambda_\infty\Gamma$. We note that the space $\overline{\Gamma x_0} \setminus \Gamma x_0$ is independent of the particular choice of x_0. We also know that $\Lambda_\infty\Gamma$ is compact.

A point $g \in \mathsf{U}\Sigma$ is called a *wandering point* of the flow ϕ if there exists an ϵ-neighborhood $\mathcal{B}_\epsilon(g) \subset \mathsf{U}\Sigma$ around g and a real number t_0 such that for all $t > t_0$ we have that

$$\mathcal{B}_\epsilon(g) \cap \phi_t\mathcal{B}_\epsilon(g) = \emptyset.$$

Moreover, a point is called *non-wandering* if it is not a wandering point. We note that the geodesic corresponding to a non-wandering point is recurrent in both directions.

Let $\mathsf{U}_{\mathrm{rec}}\Sigma$ be the space of all non-wandering points of the geodesic flow ϕ on $\mathsf{U}\Sigma$. We denote the lift of the space $\mathsf{U}_{\mathrm{rec}}\Sigma$ to $\mathsf{U}\mathbb{H}$ by $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$. Now if the action of Γ on $\mathbb{H}$ is free and proper and moreover Γ contains no parabolics, then the space $\mathsf{U}_{\mathrm{rec}}\Sigma$ is compact. We note that the subspace $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$ can also be given an alternate description as follows:

$$\mathsf{U}_{\mathrm{rec}}\mathbb{H} = \{(x, v) \in \mathsf{U}\mathbb{H} \mid \lim_{t\to\pm\infty} \tilde{\phi}^1_t(x, v) \in \Lambda_\infty\Gamma\},$$

where $\tilde{\phi}_t(x, v) = (\tilde{\phi}^1_t(x, v), \tilde{\phi}^2_t(x, v))$. Furthermore, we note that the space $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$ can be identified with the space

$$(\Lambda_\infty\Gamma \times \Lambda_\infty\Gamma \setminus \{(x, x) \mid x \in \Lambda_\infty\Gamma\}) \times \mathbb{R}.$$

This identification, however, is not canonical as each $\mathbb{R}$-fiber can be shifted independently.

2.3. Metric Anosov property. The definitions in this section, which can also be found in Subsection 3.2 of [3], have been included here for the sake of completeness.

Definition 2.1. Let $\mathcal{X}$ be a topological space. A *lamination* $\mathcal{L}$ of $\mathcal{X}$ is an equivalence relation on $\mathcal{X}$ such that for all x in $\mathcal{X}$ there exist an open neighborhood $\mathcal{U}_x$ of x in $\mathcal{X}$, two topological spaces $\mathcal{U}_1$ and $\mathcal{U}_2$ and a homeomorphism $\mathfrak{f}_x$ from $\mathcal{U}_x$ to $\mathcal{U}_1 \times \mathcal{U}_2$ satisfying the following properties:

1. for all w, z in $\mathcal{U}_x \cap \mathcal{U}_y$ we have $p_2(\mathfrak{f}_x(w)) = p_2(\mathfrak{f}_x(z))$ if and only if $p_2(\mathfrak{f}_y(w)) = p_2(\mathfrak{f}_y(z))$ where p_2 denotes the projection onto the second factor of a Cartesian product;
2. for all w, z in $\mathcal{X}$ we have $w\mathcal{L}z$ if and only if there exists a finite sequence of points $w_1, w_2, \ldots, w_n$ in $\mathcal{X}$ with $w_1 = w$ and $w_n = z$, such that w_{i+1} is in $\mathcal{U}_{w_i}$, where $\mathcal{U}_{w_i}$ is a product neighborhood of w_i and $p_2(\mathfrak{f}_{w_i}(w_i)) = p_2(\mathfrak{f}_{w_i}(w_{i+1}))$ for all i in $\{1, 2, \ldots, n-1\}$.

The homeomorphism $\mathfrak{f}_x$ is called a *chart* and the equivalence classes are called the *leaves*. We denote the leaf containing x by $\mathcal{L}_x$.

A *plaque open set* in the chart corresponding to $\mathfrak{f}_x$ is a set of the form $\mathfrak{f}_x^{-1}(\mathcal{V}_1 \times \{x_2\})$ where $\mathfrak{f}_x(x) = (x_1, x_2)$ and $\mathcal{V}_1$ is an open set in $\mathcal{U}_1$. The *plaque topology* on $\mathcal{L}_x$ is the topology generated by the plaque open sets. A plaque neighborhood of x is a neighborhood for the plaque topology on $\mathcal{L}_x$.

Definition 2.2. A *local product structure* on $\mathcal{X}$ is a pair of two laminations $\mathcal{L}_1$, $\mathcal{L}_2$ satisfying the following property: for all x in $\mathcal{X}$ there exist two plaque neighborhoods $\mathcal{U}_1$, $\mathcal{U}_2$ of x, respectively in $\mathcal{L}_1$, $\mathcal{L}_2$ and a homeomorphism $\mathfrak{f}_x$ from a neighborhood $\mathcal{W}_x$ of x in $\mathcal{X}$ onto $\mathcal{U}_1 \times \mathcal{U}_2$, such that $\mathfrak{f}_x$ defines a chart for both the laminations $\mathcal{L}_1$ and $\mathcal{L}_2$.

Now let us assume that ψ_t is a flow on $\mathcal{X}$. A lamination $\mathcal{L}$ invariant under the flow ψ_t is called *transverse* to the flow, if for all x in $\mathcal{X}$, there exists a plaque neighborhood $\mathcal{U}_x$ of x in $\mathcal{L}_x$, a topological space $\mathcal{V}$, a positive ϵ and a homeomorphism $\mathfrak{f}_x$ from an open neighborhood $\mathcal{W}_x$ of x in $\mathcal{X}$ onto $\mathcal{U}_x \times \mathcal{V} \times (-\epsilon, \epsilon)$ satisfying the following condition:

$$\psi_t(\mathfrak{f}_x^{-1}(u, v, s)) = \mathfrak{f}_x^{-1}(u, v, s + t)$$

for u in $\mathcal{U}_x$, v in $\mathcal{V}$ and for s, t in the interval $(-\epsilon/2, \epsilon/2)$. Let $\mathcal{L}^{\cdot}$ be a lamination which is transverse to the flow ψ_t. We define a new lamination $\mathcal{L}^{\cdot,0}$, called the *central lamination*, starting from $\mathcal{L}^{\cdot}$ as follows. We say y, z in $\mathcal{X}$ belong to the same equivalence class of $\mathcal{L}^{\cdot,0}$ if for some real number t, $\psi_t y$ and z belong to the same equivalence class of $\mathcal{L}^{\cdot}$.

Definition 2.3. Let $(\mathcal{X}, d)$ be a metric space. A lamination $\mathcal{L}$ invariant under a flow ψ_t is said to *contract under the flow* if there exists a positive real number T_0 such that for all x in $\mathcal{X}$, the following holds: there exists an open neighbourhood $\mathcal{W}_x$ of x in $\mathcal{X}$ such that for any two points y, z in $\mathcal{W}_x$ with $\mathcal{L}_y = \mathcal{L}_z$, we have

$$d(\psi_t y, \psi_t z) < \frac{1}{2} d(y, z)$$

for all $t > T_0$.

Remark 2.4. We note that a lamination 'contracts under a flow' if and only if the lamination contracts exponentially under the flow.

Definition 2.5. A flow ψ_t on a compact metric space is called *metric Anosov*, if there exist two laminations $\mathcal{L}^+$ and $\mathcal{L}^-$ of $\mathcal{X}$ such that the following conditions hold:

1. $(\mathcal{L}^+, \mathcal{L}^{-,0})$ defines a local product structure on $\mathcal{X}$,
2. $(\mathcal{L}^-, \mathcal{L}^{+,0})$ defines a local product structure on $\mathcal{X}$,
3. the leaves of $\mathcal{L}^+$ are contracted by the flow,
4. the leaves of $\mathcal{L}^-$ are contracted by the inverse flow.

In such a case we call $\mathcal{L}^+$, $\mathcal{L}^-$, $\mathcal{L}^{+,0}$ and $\mathcal{L}^{-,0}$ respectively the *stable*, *unstable*, *central stable* and *central unstable* laminations.

2.4. Margulis spacetimes and surfaces. A *Margulis spacetime* M is a quotient manifold of the three dimensional affine space $\mathbb{A}$ by a free, non-abelian group Γ which acts freely and properly as affine transformations with discrete linear part. In [22] and [23] Margulis showed the existence of these spaces. Later in [9] Drumm introduced the notion of *crooked planes* and constructed fundamental domains of a certain class of Margulis spacetimes. In his construction the crooked planes give the boundary of appropriate fundamental domains for a certain class of Margulis spacetimes. Recently, in [7] Danciger, Guéritaud, and Kassel showed that for any Margulis spacetime one can find a fundamental domain whose boundaries are given by a union of crooked planes.

If Γ is a subgroup of $\mathsf{GL}(\mathbb{R}^3) \ltimes \mathbb{R}^3$ such that $\mathsf{M} := \Gamma\backslash\mathbb{A}$ is a Margulis spacetime then by a result proved by Fried and Goldman in [13] we get that a conjugate of $\mathsf{L}(\Gamma)$ is a subgroup of $\mathsf{SO}(2,1)$. Moreover, the image is contained in the neutral component $\mathsf{SO}^0(2,1)$ only if the underlying hyperbolic surface is orientable. Therefore, without loss of generality, by restricting Γ to a subgroup of index 2, we can take $\Gamma \subset \mathsf{G} := \mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3$ where Γ is a free non-abelian group with finitely many generators. Now it follows from the definition of a Margulis spacetime that $\mathsf{L}(\Gamma)$ is a discrete subgroup of $\mathsf{SO}^0(2,1)$. In this article I will only consider Margulis spacetimes such that $\mathsf{L}(\Gamma)$ contains no parabolic elements.

Let $\mathsf{M} := \Gamma\backslash\mathbb{A}$ be a Margulis spacetime such that $\mathsf{L}(\Gamma)$ contains no parabolic elements. Then the action of $\mathsf{L}(\Gamma)$ on $\mathbb{H}$ is Schottky i.e. it has a fundamental domain bounded by geodesics not meeting in $\mathbb{H} \cup \partial_\infty\mathbb{H}$. Hence $\Sigma := \mathsf{L}(\Gamma)\backslash\mathbb{H}$ is a non-compact surface with no cusps.

Now let TM be the tangent bundle of M. As $\mathsf{L}(\Gamma) \subset \mathsf{SO}^0(2,1)$ we get that each fiber of TM carries a Lorentzian metric $\langle\cdot|\cdot\rangle$. Let

$$\mathsf{UM} := \{(X, v) \in \mathsf{TM} \mid \langle v \mid v\rangle_X = 1\}.$$

We note that $\mathsf{UM} \cong \Gamma\backslash\mathsf{U}\mathbb{A}$ where $\mathsf{U}\mathbb{A} := \mathbb{A} \times \mathsf{S}^1$. The affine flow $\widetilde{\Phi}$ on $\mathsf{T}\mathbb{A}$ gives rise to a flow Φ on UM.

We recall that a point $(X, v) \in \mathsf{UM}$ is called a *wandering point* of the flow Φ if there exists an ϵ-neighborhood $\mathcal{B}_\epsilon(X, v) \subset \mathsf{UM}$ around (X, v) and a real number t_0 such that for all $t > t_0$ we have that

$$\mathcal{B}_\epsilon(X, v) \cap \Phi_t\mathcal{B}_\epsilon(X, v) = \emptyset.$$

Moreover, a point is called *non-wandering* if it is not a wandering point. We note that the affine line corresponding to a non-wandering point is recurrent in both directions. We denote the space of all non-wandering points of the flow Φ on UM by $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ and the lift of $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ to $\mathsf{U}\mathbb{A}$ by $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$.

In [16] Goldman, Labourie, and Margulis proved the following theorem.

Theorem 2.6 (Goldman, Labourie, and Margulis). *Let Γ be a non-abelian free discrete subgroup of G with finitely many generators giving rise to a Margulis spacetime and suppose the linear part $\mathsf{L}(\Gamma)$ contains no parabolic elements. Then up to replacing the translational part u by $-\mathsf{u}$ there exists a map*

$$N\colon \mathsf{U}_{\mathrm{rec}}\mathbb{H} \longrightarrow \mathbb{A}$$

and a positive Hölder continuous function

$$f\colon \mathsf{U}_{\mathrm{rec}}\mathbb{H} \longrightarrow \mathbb{R}^{>0}$$

such that

1. *for all $\gamma \in \Gamma$ we have $f \circ \mathsf{L}(\gamma) = f$,*
2. *for all $\gamma \in \Gamma$ we have $N \circ \mathsf{L}(\gamma) = \gamma N$, and*
3. *for all $g \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$ and for all $t \in \mathbb{R}$ we have*

$$N(\tilde{\phi}_t g) = N(g) + \left(\int_0^t f(\tilde{\phi}_s g)ds\right)\nu(g).$$

We call N a *neutralised section* and ν, the neutral section, is as defined in (2.11).

Corollary 2.7. *Let $g \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$. Then for all $t \in \mathbb{R}$ there exists a unique $s \in \mathbb{R}$ such that*

$$N(g) + t\nu(g) = N(\tilde{\phi}_s g).$$

Proof. The result follows from Theorem 2.6.(3) and the fact that $f > 0$. ☐

Moreover, using the existence of a neutralised section Goldman and Labourie proved the following theorem in [15].

Theorem 2.8 (Goldman and Labourie). *Let Γ be a non-abelian free discrete subgroup of G with finitely many generators giving rise to a Margulis spacetime and suppose the linear part $\mathsf{L}(\Gamma)$ contains no parabolic elements. Also let $\mathsf{U}_{\mathrm{rec}}\Sigma$ and $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ be defined as above. Now if N is a neutralised section, then there exists an injective map $\hat{\mathtt{N}}$ such that the following diagram commutes*:

$$\begin{array}{ccc} \mathsf{U}_{\mathrm{rec}}\mathbb{H} & \xrightarrow{\mathtt{N}} & \mathsf{U}\mathbb{A} \\ \pi\downarrow & & \downarrow\pi \\ \mathsf{U}_{\mathrm{rec}}\Sigma & \xrightarrow{\hat{\mathtt{N}}} & \mathsf{UM} \end{array}$$

where $\mathtt{N} := (N, \nu)$. Moreover, $\hat{\mathtt{N}}$ is a Hölder homeomorphism onto $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ with $\mathtt{N}(\mathsf{U}_{\mathrm{rec}}\mathbb{H}) = \mathsf{U}_{\mathrm{rec}}\mathbb{A}$ and it is an orbit equivalence (*i.e. $\hat{\mathtt{N}}$ takes full flow lines to full flow lines*).

3. Metric space structure on $\mathsf{U}_{\text{rec}}\mathsf{M}$

Let M be a Margulis spacetime. In this section, we will define a distance function d on $\mathsf{U}_{\text{rec}}\mathsf{M}$ such that $(\mathsf{U}_{\text{rec}}\mathsf{M}, d)$ is a metric space. The restriction of any euclidean metric on $\mathbb{A} \times \mathbb{V}$ to the subspace $\mathsf{U}_{\text{rec}}\mathbb{A}$, defines a distance on $\mathsf{U}_{\text{rec}}\mathbb{A}$. We call this distance the *euclidean distance* on $\mathsf{U}_{\text{rec}}\mathbb{A}$. In this section we will define a distance on the space $\mathsf{U}_{\text{rec}}\mathbb{A}$ such that the distance is locally bilipschitz equivalent to any euclidean distance on $\mathsf{U}_{\text{rec}}\mathbb{A}$ and also is Γ-invariant, so as to get a distance on the quotient space $\mathsf{U}_{\text{rec}}\mathsf{M}$.

The space $\mathsf{U}_{\text{rec}}\mathsf{M}$ is compact as it is homeomorphic to the compact space $\mathsf{U}_{\text{rec}}\Sigma$. Hence we can choose a pre-compact fundamental domain D of $\mathsf{U}_{\text{rec}}\mathsf{M}$ inside $\mathsf{U}_{\text{rec}}\mathbb{A}$. Let $B \subset \mathbb{A}$ be an open euclidean ball containing the closure of D. We choose B in such a way that for all γ in a generating set of Γ we have

$$\gamma B \cap B \neq \emptyset.$$

It follows from the choice of the open ball B that ΓB is path-connected.

Let $U \supset B$ be a larger open ball and let χ' be a smooth function supported on U and positive on B. Hence $\gamma_* \chi' := \chi' \circ \gamma^{-1}$ is a smooth function supported on γU and positive on γB for all $\gamma \in \Gamma$. The action of Γ being proper implies that $\sum_{\gamma \in \Gamma} \gamma_* \chi'$ has only finitely many nonzero terms at any point and hence is well defined and positive on ΓB. Therefore, the function

$$\chi := \frac{\chi'}{\sum_{\gamma \in \Gamma} \gamma_* \chi'} : \Gamma B \longrightarrow \mathbb{R}^{\geqslant 0}$$

is well defined and satisfies the following property:

$$\sum_{\gamma \in \Gamma} \gamma_* \chi = 1.$$

Moreover, we denote the lift of χ from ΓB to $\mathsf{T}(\Gamma B)$ by $\tilde{\chi}$.

Now we fix a euclidean metric g_{euc} on $\mathbb{A} \times \mathbb{V}$. We consider the restriction of g_{euc} on $\mathsf{T}(\Gamma B)$ and we denote the distance corresponding to g_{euc} on $\mathsf{U}_{\text{rec}}\mathbb{A}$ by d_{euc}.

Definition 3.1. We define $\tilde{d}$ to be the distance on $\mathsf{U}_{\text{rec}}\mathbb{A}$ corresponding to the Riemannian metric

$$\mathsf{g} := \sum_{\gamma \in \Gamma} \gamma_* (\tilde{\chi} \mathsf{g}_{\text{euc}})$$

on $\mathsf{T}(\Gamma B)$.

Lemma 3.2. *The distance $\tilde{d}$ is Γ-invariant and is locally bilipschitz equivalent to d_{euc}.*

Proof. Invariance follows from the fact that

$$\eta_* \mathsf{g} = \sum_{\gamma\in\Gamma} (\eta\gamma)_*(\tilde{\chi}\mathsf{g}_{\mathrm{euc}}) = \sum_{\gamma\in\Gamma} \gamma_*(\tilde{\chi}\mathsf{g}_{\mathrm{euc}}) = \mathsf{g}$$

for all $\eta \in \Gamma$.

We note that any two euclidean metrics on $\mathbb{A} \times \mathbb{V}$ are bilipschitz equivalent with each other. Moreover, the cardinality of the following set

$$\Gamma_U := \{\gamma \in \Gamma \mid \gamma U \cap U \neq \emptyset\}$$

is finite. Hence for all γ in Γ_U the metric $\mathsf{g}_{\mathrm{euc}}$ is K-bilipschitz equivalent to $\gamma_*\mathsf{g}_{\mathrm{euc}}$ for some $K \geqslant 1$. Now from the definition of the metric g it follows that at any point on $\mathsf{T}(\eta B)$ the metric $\mathsf{g} = \eta_*\mathsf{g}$ is K-bilipschitz equivalent to the metric $\eta_*\mathsf{g}_{\mathrm{euc}}$. Finally, the fact that any two euclidean metrics on $\mathbb{A}\times\mathbb{V}$ are bilipschitz equivalent with each other implies that $\tilde{d}$ is locally bilipschitz equivalent to d_{euc}. □

Hence $\tilde{d}$ gives rise to a distance d on $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ and the distance $\tilde{d}$ is locally bilipschitz equivalent to any euclidean distance on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$.

4. The lamination and its lift

In this section, we will explicitly describe the two laminations of $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ for the flow $\widetilde{\Phi}$ on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ and show that the laminations are equivariant under the action of the flow and the action of Γ. We will also define the notion of a leaf lift.

Let Z be a point in $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$. We know from Theorem 2.8 that there exists a unique $g \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$ such that $Z = \mathtt{N}(g)$. Recall that $\mathtt{N} = (N, \nu)$ where $\nu\colon \mathsf{U}\mathbb{H} \to \mathbb{V}$ is the neutral section (2.11) and $N\colon \mathsf{U}_{\mathrm{rec}}\mathbb{H} \to \mathbb{A}$ is the neutralised section (Theorem 2.6).

Definition 4.1. The positive and central positive partitions of $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ are respectively given by

$$\mathcal{L}^{+}_{\mathtt{N}(g)} := \widetilde{\mathcal{L}}^{+}_{\mathtt{N}(g)} \cap \mathsf{U}_{\mathrm{rec}}\mathbb{A},$$

$$\mathcal{L}^{+,0}_{\mathtt{N}(g)} := \widetilde{\mathcal{L}}^{+,0}_{\mathtt{N}(g)} \cap \mathsf{U}_{\mathrm{rec}}\mathbb{A},$$

where

$$\widetilde{\mathcal{L}}^{+}_{\mathtt{N}(g)} := \{(N(g) + s_1\nu^+(g), \nu(g) + s_2\nu^+(g)) \mid s_1, s_2 \in \mathbb{R}\},$$

$$\widetilde{\mathcal{L}}^{+,0}_{\mathtt{N}(g)} := \{(N(g) + s_1\nu^+(g) + t\nu(g), \nu(g) + s_2\nu^+(g)) \mid t, s_1, s_2 \in \mathbb{R}\}.$$

Definition 4.2. The negative and central negative partitions of $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ are respectively given by

$$\mathcal{L}^{-}_{\mathtt{N}(g)} := \widetilde{\mathcal{L}}^{-}_{\mathtt{N}(g)} \cap \mathsf{U}_{\mathrm{rec}}\mathbb{A},$$

$$\mathcal{L}^{-,0}_{\mathtt{N}(g)} := \widetilde{\mathcal{L}}^{-,0}_{\mathtt{N}(g)} \cap \mathsf{U}_{\mathrm{rec}}\mathbb{A},$$

where

$$\widetilde{\mathcal{L}}^{-}_{\mathtt{N}(g)} := \{(N(g) + s_1\nu^-(g), \nu(g) + s_2\nu^-(g)) \mid s_1, s_2 \in \mathbb{R}\},$$

$$\widetilde{\mathcal{L}}^{-,0}_{\mathtt{N}(g)} := \{(N(g) + s_1\nu^-(g) + t\nu(g), \nu(g) + s_2\nu^-(g)) \mid t, s_1, s_2 \in \mathbb{R}\}.$$

As we mentioned in the Introduction, we can alternatively think of $\mathsf{U}\mathbb{A}$ as the space of all tuples (x, ℓ) where $x \in \mathbb{A}$ and ℓ is an oriented spacelike affine line containing x. We denote the lift of $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ to $\mathsf{U}\mathbb{A}$ by $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$. Let (x, ℓ) be an element of $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$. We consider the intersection of the plane perpendicular to ℓ at x with respect to the Lorentz metric and the null cone at x. The intersection is the union of two lightlike affine lines. We orient these lightlike affine lines by defining the part lying in the upper light cone to be positive. We denote the two oriented lightlike affine lines by ℓ^- and ℓ^+ such that (ℓ^-, ℓ, ℓ^+) gives the positive orientation on $\mathbb{A}$. Then the lamination $\mathcal{L}^+$ (respectively $\mathcal{L}^-$) through $(x, \ell) \in \mathsf{U}_{\mathrm{rec}}\mathbb{A}$ is the collection of all elements $(y, \ell') \in \mathsf{U}_{\mathrm{rec}}\mathbb{A}$ such that the following two conditions hold:

1. $y \in \ell^+$ (respectively ℓ^-),
2. ℓ' is an oriented spacelike affine line passing through y and lying in the affine plane generated by ℓ and ℓ^+ (respectively ℓ^-) with $(\ell^-, \ell' + (x - y), \ell^+)$ giving the positive orientation on $\mathbb{A}$.

Moreover, the lamination $\mathcal{L}^{+,0}$ (respectively $\mathcal{L}^{-,0}$) through $(x, \ell) \in \mathsf{U}_{\mathrm{rec}}\mathbb{A}$ is the collection of all elements $(y, \ell') \in \mathsf{U}_{\mathrm{rec}}\mathbb{A}$ such that the following two conditions hold:

1. $y \in (z - x) + \ell^+$ (respectively $(z - x) + \ell^-$) for some $z \in \ell$,
2. ℓ' is an oriented spacelike affine line passing through y and lying in the affine plane generated by ℓ and ℓ^+ (respectively ℓ^-) with $(\ell^-, \ell' + (x - y), \ell^+)$ giving the positive orientation on $\mathbb{A}$.

Lemma 4.3. *Let g, h be two points in $\mathsf{U}\mathbb{H}$. Then the following four properties are equivalent*:

1. $h \in \bigcup_{t\in\mathbb{R}} \widetilde{\mathcal{H}}^+_{\tilde{\phi}_t g}$ *where $\widetilde{\mathcal{H}}^+$ is as defined in* (2.8),
2. $\nu(h) - \nu(g) = -\langle \nu(h) \mid \nu^-(g)\rangle \nu^+(g)$,
3. $\nu(h) - \nu(g) \in \mathbb{R}\nu^+(g)$,
4. $\nu^+(h) = c\nu^+(g)$ *where* $c \in \mathbb{R}^{>0}$.

Proof. 1 $\Longrightarrow$ 2. Let h be a point of $\bigcup_{t\in\mathbb{R}}\widetilde{\mathcal{H}}^{+}_{\tilde{\phi}_t g}$. Hence there exist real numbers s,t such that $h = ga(t)u^{+}(s)$. Therefore, we get

$$\begin{aligned}
\nu(h) &= ga(t)u^{+}(s)\begin{pmatrix}1\\0\\0\end{pmatrix}\\
&= ga(t)\begin{pmatrix}1\\2s\\2s\end{pmatrix}\\
&= ga(t)\left(\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\2s\\2s\end{pmatrix}\right)\\
&= \nu(g) + 2sga(t)\begin{pmatrix}0\\1\\1\end{pmatrix}\\
&= \nu(g) + 2se^{t}g\begin{pmatrix}0\\1\\1\end{pmatrix}\\
&= \nu(g) + 2\sqrt{2}\,se^{t}\nu^{+}(g).
\end{aligned} \tag{4.1}$$

Moreover, we notice that

$$\begin{aligned}
\langle \nu(h) \mid \nu^{-}(g)\rangle &= \langle \nu(g) + 2\sqrt{2}\,se^{t}\nu^{+}(g) \mid \nu^{-}(g)\rangle\\
&= 2\sqrt{2}\,se^{t}\langle \nu^{+}(g) \mid \nu^{-}(g)\rangle\\
&= -2\sqrt{2}\,se^{t}.
\end{aligned} \tag{4.2}$$

Hence combining equations (4.1) and (4.2) we get

$$\nu(h) - \nu(g) = -\langle \nu(h) \mid \nu^{-}(g)\rangle \nu^{+}(g).$$

2 $\Longrightarrow$ 3. Property 3 follows directly from Property 2.

3 $\Longrightarrow$ 4. Let g,h be two points in $\mathsf{U}\mathbb{H}$ satisfying

$$\nu(h) = \nu(g) + b\nu^{+}(g)$$

for some $b \in \mathbb{R}$. Using the definition of ν and ν^{+} we observe that the above equation is equivalent to the following equation,

$$h\begin{pmatrix}1\\0\\0\end{pmatrix} = gu^{+}\Big(\frac{b}{2\sqrt{2}}\Big)\begin{pmatrix}1\\0\\0\end{pmatrix}.$$

We know that the only elements of $\mathsf{SO}^0(2,1)$ fixing the vector $(1,0,0)^{\mathrm{t}}$ are of the form $a(t)$ for some real number t. Hence there exists $t \in \mathbb{R}$ such that

$$h = gu^+\Big(\frac{b}{2\sqrt{2}}\Big)a(t).$$

Therefore,

$$\begin{aligned}
\nu^+(h) &= \frac{1}{\sqrt{2}} gu^+\Big(\frac{b}{2\sqrt{2}}\Big)a(t)\begin{pmatrix}0\\1\\1\end{pmatrix}\\
&= \frac{e^t}{\sqrt{2}} gu^+\Big(\frac{b}{2\sqrt{2}}\Big)\begin{pmatrix}0\\1\\1\end{pmatrix}\\
&= \frac{e^t}{\sqrt{2}} g\begin{pmatrix}0\\1\\1\end{pmatrix}\\
&= e^t \nu^+(g).
\end{aligned}$$

Hence we obtain Property 4 by noting that e^t is positive.

$4 \Longrightarrow 1$. Let $\nu^+(h) = c\nu^+(g)$ where $c \in \mathbb{R}^{>0}$. Hence

$$h\begin{pmatrix}0\\1\\1\end{pmatrix} = g\begin{pmatrix}0\\c\\c\end{pmatrix} = ga(\log(c))\begin{pmatrix}0\\1\\1\end{pmatrix}$$

and we get that $a(-\log(c))g^{-1}h$ fixes the vector $(0,1,1)^{\mathrm{t}}$. Therefore, there exists $t \in \mathbb{R}$ such that $a(-\log(c))g^{-1}h = u^+(t)$ i.e.

$$h = ga(\log(c))u^+(t)$$

and we obtain $h \in \bigcup_{t\in\mathbb{R}} \widetilde{\mathcal{H}}^+_{\tilde{\phi}_t g}$. □

Corollary 4.4. *Let g, h be two points in $\mathsf{U}\mathbb{H}$ such that h is in $\bigcup_{t\in\mathbb{R}} \widetilde{\mathcal{H}}^+_{\tilde{\phi}_t g}$. Then*

$$\langle \nu(g) \mid \nu^-(h)\rangle \nu^+(h) = -\langle \nu(h) \mid \nu^-(g)\rangle \nu^+(g).$$

Proof. In Lemma 4.3 Property 4 is symmetric in g and h, hence so must be Property 2: the result follows. □

Definition 4.5. For all g in $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$ we define,

$$\mathcal{H}^{\pm}_g := \widetilde{\mathcal{H}}^{\pm}_g \cap \mathsf{U}_{\mathrm{rec}}\mathbb{H}.$$

Proposition 4.6. *The following equations are true for all g in $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$,*

$$1. \mathcal{L}^{+,0}_{\mathtt{N}(g)} = \left\{\mathtt{N}(h) \,\middle|\, h \in \bigcup_{t\in\mathbb{R}} \mathcal{H}^{+}_{\tilde{\phi}_t g}\right\},$$

$$2. \mathcal{L}^{-,0}_{\mathtt{N}(g)} = \left\{\mathtt{N}(h) \,\middle|\, h \in \bigcup_{t\in\mathbb{R}} \mathcal{H}^{-}_{\tilde{\phi}_t g}\right\}.$$

Proof. We start by defining a function

$$\begin{aligned} F\colon \mathsf{U}_{\mathrm{rec}}\mathbb{H} \times \mathsf{U}_{\mathrm{rec}}\mathbb{H} &\longrightarrow \mathbb{R}, \\ (g,h) &\longmapsto \det[N(g) - N(h), \nu(g), \nu(h)], \end{aligned}$$

where $N\colon \mathsf{U}_{\mathrm{rec}}\mathbb{H} \to \mathbb{A}$ is the neutralised section as mentioned in Theorem 2.6. Using equation (2.13) and Theorem 2.6.(3) we get that

$$F(\tilde{\phi}_s g, \tilde{\phi}_t h) = F(g,h) \tag{4.3}$$

for all $s,t \in \mathbb{R}$. Again using equation (2.14) and Theorem 2.6.(3) we get that the neutralised section N and the neutral section ν are equivariant under, respectively, the affine and linear action of Γ. Hence for all γ in Γ we have

$$\begin{aligned} F(\mathtt{L}(\gamma)g, \mathtt{L}(\gamma)h) &= \det[N(\mathtt{L}(\gamma)g) - N(\mathtt{L}(\gamma)h), \nu(\mathtt{L}(\gamma)g), \nu(\mathtt{L}(\gamma)h)] \\ &= \det[\mathtt{L}(\gamma)(N(g) - N(h)), \mathtt{L}(\gamma)\nu(g), \mathtt{L}(\gamma)\nu(h))] \\ &= \det[\mathtt{L}(\gamma)] \det[N(g) - N(h), \nu(g), \nu(h)] \\ &= \det[N(g) - N(h), \nu(g), \nu(h)] \\ &= F(g,h). \end{aligned} \tag{4.4}$$

Now for a fixed real number c_0 we consider the space

$$\mathfrak{K} := \{(g_1, g_2) \mid d_{\mathsf{U}\mathbb{H}}(g_1, g_2) \leqslant c_0\} \subset \mathsf{U}_{\mathrm{rec}}\mathbb{H} \times \mathsf{U}_{\mathrm{rec}}\mathbb{H}.$$

Compactness of $\mathsf{U}_{\mathrm{rec}}\Sigma$ implies that $\mathfrak{K}_\Gamma$, the projection of $\mathfrak{K}$ in $\Gamma\backslash(\mathsf{U}_{\mathrm{rec}}\mathbb{H} \times \mathsf{U}_{\mathrm{rec}}\mathbb{H})$, is compact, where the Γ action on $\mathsf{U}_{\mathrm{rec}}\mathbb{H} \times \mathsf{U}_{\mathrm{rec}}\mathbb{H}$ is diagonal. Now continuity of F implies that F is uniformly continuous on $\mathfrak{K}_\Gamma$. We note that F vanishes on the diagonal of $\mathsf{U}_{\mathrm{rec}}\mathbb{H} \times \mathsf{U}_{\mathrm{rec}}\mathbb{H}$.

Let g and h be two points in $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$ such that $h \in \bigcup_{t\in\mathbb{R}} \mathcal{H}^{+}_{\tilde{\phi}_t g}$. Then $\nu^+(h)$ and $\nu^+(g)$ are collinear by Lemma 4.3.(4), and there exists $t \in \mathbb{R}$ such that $d_{\mathsf{U}\mathbb{H}}(\tilde{\phi}_{t+s} g, \tilde{\phi}_s h) \to 0$ as $s \to \infty$: hence $F(g,h) = 0$ by uniform continuity. Now the first desired property, namely

$$N(g) - N(h) \in \operatorname{span}(\nu(g), \nu^+(g))$$

can be obtained as follows:

1. if $\nu(g) = \nu(h)$ then $h = \tilde{\phi}_t g$ for some real number t and $N(g) - N(h) \in \operatorname{span}(\nu(g), \nu^+(g))$ by Theorem 2.6.(3);
2. if $\nu(g) \neq \nu(h)$ then $\operatorname{span}(\nu(g), \nu(h)) = \operatorname{span}(\nu(g), \nu^+(g))$ by Lemma 4.3 and $\operatorname{span}(\nu(g), \nu(h))$ contains $N(g) - N(h)$ due to $F(g,h) = 0$.

The second desired property, namely

$$\nu(g) - \nu(h) \in \mathbb{R}\nu^+(g),$$

also follows from Lemma 4.3.

Conversely let $W \in \mathcal{L}^{+,0}_{\mathtt{N}(g)}$. By Theorem 2.8 we know that there exists $h \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$ such that $W = \mathtt{N}(h) = (N(h), \nu(h))$. Now the choice of W implies that there exists some real number s_2 such that

$$\nu(h) = \nu(g) + s_2\nu^+(g).$$

Using lemma 4.3.(3) we get that $h \in \bigcup_{t\in\mathbb{R}} \widetilde{\mathcal{H}}^+_{\tilde{\phi}_t g}$. Therefore, h is in

$$\bigcup_{t\in\mathbb{R}} \mathcal{H}^+_{\tilde{\phi}_t g} = \Big(\mathsf{U}_{\mathrm{rec}}\mathbb{H} \cap \bigcup_{t\in\mathbb{R}} \widetilde{\mathcal{H}}^+_{\tilde{\phi}_t g}\Big)$$

and we have

$$\mathcal{L}^{+,0}_{\mathtt{N}(g)} \subseteq \Big\{\mathtt{N}(h) \;\Big|\; h \in \bigcup_{t\in\mathbb{R}} \mathcal{H}^+_{\tilde{\phi}_t g}\Big\}.$$

Similarly, the other equality follows. □

Let $g, h \in \mathsf{U}\mathbb{H}$. We say

$$g \sim h \;\text{ if and only if }\; h = ga(t)$$

where t is some real number. We notice that

$$\partial_\infty\mathbb{H} \times \partial_\infty\mathbb{H} \smallsetminus \Delta = \mathsf{U}\mathbb{H}/\sim$$

where Δ denotes the diagonal. We recall the definition of the neutral section ν and notice that for any $g \in \mathsf{U}\mathbb{H} \cong \mathsf{SO}^0(2,1)$ and any $t \in \mathbb{R}$ we have

$$\nu(ga(t)) = \nu(g).$$

Hence the map ν from $\mathsf{U}\mathbb{H}$ to the unit quadric S^1 of $\mathbb{R}^{2,1}$ gives rise to a map

$$\nu\colon \partial_\infty\mathbb{H} \times \partial_\infty\mathbb{H} \smallsetminus \Delta \longrightarrow \mathsf{S}^1.$$

Lemma 4.7. *Let $g, h \in \mathsf{U}\mathbb{H}$ and $(g^-, h^+) \in \partial_\infty\mathbb{H} \times \partial_\infty\mathbb{H} \smallsetminus \Delta$ be such that $\nu^-(g) \in g^-$ and $\nu^+(h) \in h^+$. Then*

$$\nu(g^-, h^+) = \frac{\nu^-(g) \boxtimes \nu^+(h)}{\langle \nu^-(g) \mid \nu^+(h)\rangle}.$$

Proof. We know that $\partial_\infty \mathbb{H} \times \partial_\infty \mathbb{H} \setminus \Delta = \mathsf{U}\mathbb{H}/\sim$. Hence there exists $g_1 \in \mathsf{U}\mathbb{H}$ such that $\nu^-(g_1) \in g^-$ and $\nu^+(g_1) \in h^+$. Therefore,

$$\begin{aligned}
\nu(g^-, h^+) &= \nu(g_1) \\
&= g_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= -\frac{1}{2} g_1 \left(\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \boxtimes \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right) \\
&= -\left(\frac{1}{\sqrt{2}} g_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right) \boxtimes \left(\frac{1}{\sqrt{2}} g_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right) \\
&= -\nu^-(g_1) \boxtimes \nu^+(g_1).
\end{aligned}$$

We notice that the image of $\nu^\pm$ is the upper light cone. Hence $\nu^-(g_1), \nu^-(g) \in g^-$ and $\nu^+(g_1), \nu^+(h) \in h^+$ implies that there exist $c_1, c_2 \in \mathbb{R}^{>0}$ such that $\nu^-(g_1) = c_1 \nu^-(g)$ and $\nu^+(g_1) = c_2 \nu^+(h)$. Therefore, using

$$\langle \nu^-(g_1) \mid \nu^+(g_1) \rangle = \frac{1}{2} \langle g_1 (0, -1, 1)^{\mathrm{t}} \mid g_1 (0, 1, 1)^{\mathrm{t}} \rangle = -1,$$

we obtain

$$\nu(g^-, h^+) = -\nu^-(g_1) \boxtimes \nu^+(g_1) = \frac{\nu^-(g) \boxtimes \nu^+(h)}{\langle \nu^-(g) \mid \nu^+(h) \rangle}. \qquad \square$$

Corollary 4.8. *Let $g, h \in \mathsf{U}\mathbb{H}$ and $(g^-, h^+) \in \partial_\infty \mathbb{H} \times \partial_\infty \mathbb{H} \setminus \Delta$ be such that $\nu^-(g) \in g^-$ and $\nu^+(h) \in h^+$. Then*

$$\langle \nu(h) \mid \nu(g^-, h^+) \rangle = 1.$$

Proof. Using Lemma 4.7 and equation (2.4) we get that

$$\langle \nu(h) \mid \nu(g^-, h^+) \rangle = \frac{\langle \nu(h) \mid \nu^-(g) \boxtimes \nu^+(h) \rangle}{\langle \nu^-(g) \mid \nu^+(h) \rangle} = \frac{\langle \nu^-(g) \mid \nu^+(h) \boxtimes \nu(h) \rangle}{\langle \nu^-(g) \mid \nu^+(h) \rangle}.$$

Moreover,

$$\begin{aligned}
\nu^+(h) \boxtimes \nu(h) &= \left(\frac{1}{\sqrt{2}} h \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right) \boxtimes \left(h \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \\
&= \frac{1}{\sqrt{2}} h \left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \boxtimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \\
&= \frac{1}{\sqrt{2}} h \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\
&= \nu^+(h).
\end{aligned}$$

Hence $\langle \nu(h) \mid \nu(g^-, h^+) \rangle = 1$. $\square$

Now let g be a point in $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$. We note that for $g \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$ the points

$$g^{\pm} := \lim_{t\to\pm\infty} \pi(\tilde{\phi}_t g) \in \Lambda_\infty\Gamma$$

where π is the projection from $\mathsf{U}\mathbb{H}$ onto $\mathbb{H}$. We observe that $\partial_\infty\mathbb{H} \setminus \{g^+\}$ is homeomorphic to $\mathbb{R}$. Given any $g \in \mathsf{U}\mathbb{H}$, let $\mathcal{V}_{g^-}$ be a connected open neighborhood of g^- and $\mathcal{V}_{g^+}$ be a connected open neighborhood of g^+ in $\partial_\infty\mathbb{H}$ such that

$$\mathsf{cl}(\mathcal{V}_{g^-}) \cap \mathsf{cl}(\mathcal{V}_{g^+}) = \emptyset.$$

We consider $\mathcal{U}_{g^\pm} := \mathcal{V}_{g^\pm} \cap \Lambda_\infty\Gamma$ and let $\mathcal{U}_g$ be the open subset of $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$ corresponding to the open set $\mathcal{U}_{g^-} \times \mathcal{U}_{g^+} \times \mathbb{R}$. Now we define the following continuous map,

$$\begin{aligned} \mathfrak{N}_g : \mathcal{U}_g &\longrightarrow \mathbb{A}, \\ h &\longmapsto N(h) - \big\langle N(h) - N(g) \mid \nu(g^-, h^+)\big\rangle \nu(h). \end{aligned}$$

Lemma 4.9. *Let $g \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$ and let the map $\mathfrak{N}_g$ be defined as above. Then for all $h \in \mathcal{U}_g$ and $t \in \mathbb{R}$,*

$$\mathfrak{N}_g(ha(t)) = \mathfrak{N}_g(h).$$

Proof. For any real number t we have

$$\begin{aligned} \mathfrak{N}_g(ha(t)) &= N(ha(t)) - \langle N(ha(t)) - N(g) \mid \nu(g^-, h^+)\rangle \nu(ha(t)) \\ &= \mathfrak{N}_g(h) + (1 - \langle \nu(h) \mid \nu(g^-, h^+)\rangle)\left(\int_0^t f(ha(s))ds\right)\nu(h) \end{aligned}$$

where f is as mentioned in Theorem 2.6. Now using Corollary 4.8 we conclude that, for all $h \in \mathcal{U}_g$ and $t \in \mathbb{R}$,

$$\mathfrak{N}_g(ha(t)) = \mathfrak{N}_g(h). \qquad \square$$

We recall that $\partial_\infty\mathbb{H} \times \partial_\infty\mathbb{H} \setminus \Delta = \mathsf{U}\mathbb{H}/\sim$. Therefore, by Lemma 4.9 the map $\mathfrak{N}_g : \mathcal{U}_g \to \mathbb{A}$ gives rise to a map

$$\mathfrak{N}_g : \mathcal{U}_{g^-} \times \mathcal{U}_{g^+} \longrightarrow \mathbb{A}.$$

Lemma 4.10. *Let $g \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$. Then for any $h^\pm \in \mathcal{U}_{g^\pm}$ and $t \in \mathbb{R}$,*

$$(\mathfrak{N}_g(h^-, h^+) + t\nu(h^-, h^+), \nu(h^-, h^+)) \in \mathsf{N}(\mathsf{U}_{\mathrm{rec}}\mathbb{H}) = \mathsf{U}_{\mathrm{rec}}\mathbb{A}.$$

Proof. Let $h \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$ be such that $\nu^\pm(h) \in h^\pm$. We recall that by definition

$$\begin{aligned} \mathfrak{N}_g(h^-, h^+) + t\nu(h^-, h^+) &= \mathfrak{N}_g(h) + t\nu(h) \\ &= N(h) + (t - \langle N(h) - N(g) \mid \nu(g^-, h^+)\rangle)\nu(h). \end{aligned}$$

Hence by Corollary 2.7 there exists $t_1 \in \mathbb{R}$ such that

$$\mathfrak{N}_g(h^-, h^+) + t\nu(h^-, h^+) = N(ha(t_1)).$$

Now we conclude by observing that $\nu(h^-, h^+) = \nu(ha(t_1))$. □

Finally, using Lemma 4.10 we define the following continuous map:

$$\begin{aligned}\Pi_g : \mathcal{U}_{g^-} \times \mathcal{U}_{g^+} \times \mathbb{R} &\longrightarrow \mathsf{U}_{\text{rec}}\mathbb{A},\\ (h^-, h^+, t) &\longmapsto (\mathfrak{N}_g(h^-, h^+) + t\nu(h^-, h^+), \nu(h^-, h^+)).\end{aligned}$$

Furthermore, for a neighborhood $\mathcal{U}_{\mathtt{N}(g)} \subset \mathsf{U}_{\text{rec}}\mathbb{A}$ of a point $\mathtt{N}(g) \in \mathsf{U}_{\text{rec}}\mathbb{A}$ we define another continuous map as folows:

$$\begin{aligned}\amalg_{\mathtt{N}(g)} : \mathcal{U}_{\mathtt{N}(g)} &\longrightarrow (\Lambda_\infty\Gamma \times \Lambda_\infty\Gamma \setminus \Delta) \times \mathbb{R},\\ \mathtt{N}(h) &\longmapsto (h^-, h^+, \langle N(h) - N(g) \mid \nu(g^-, h^+)\rangle),\end{aligned}$$

where $h^\pm := \lim_{t\to\pm\infty} \pi(\tilde{\phi}_t h)$.

Proposition 4.11. *Let $g \in \mathsf{U}_{\text{rec}}\mathbb{H}$ and $\amalg_{\mathtt{N}(g)}$, Π_g be defined as above. Then $\amalg_{\mathtt{N}(g)}$ is a local homeomorphism with its inverse given by Π_g.*

Proof. Using Lemma 4.10 we get that there exists $h \in \mathsf{U}_{\text{rec}}\mathbb{H}$ such that

$$\mathfrak{N}_g(h^-, h^+) + t\nu(h^-, h^+) = N(h) \tag{4.5}$$

Now by the definition of $\mathfrak{N}_g$ we have

$$\mathfrak{N}_g(h^-, h^+) = \mathfrak{N}_g(h) = N(h) - \langle N(h) - N(g) \mid \nu(g^-, h^+)\rangle\nu(h). \tag{4.6}$$

Hence by comparing the two equations (4.5) and (4.6) we get that

$$t = \langle N(h) - N(g) \mid \nu(g^-, h^+)\rangle.$$

Therefore,

$$\begin{aligned}\amalg_{\mathtt{N}(g)} \circ \Pi_g(h^-, h^+, t) &= \amalg_{\mathtt{N}(g)}(\mathfrak{N}_g(h^-, h^+) + t\nu(h^-, h^+), \nu(h^-, h^+))\\ &= \amalg_{\mathtt{N}(g)}(N(h), \nu(h))\\ &= (h^-, h^+, \langle N(h) - N(g) \mid \nu(g^-, h^+)\rangle)\\ &= (h^-, h^+, t).\end{aligned}$$

and

$$\begin{aligned}&\Pi_g \circ \amalg_{\mathtt{N}(g)}(\mathtt{N}(h))\\ &\quad = \Pi_g(h^-, h^+, \langle N(h) - N(g) \mid \nu(g^-, h^+)\rangle)\\ &\quad = (\mathfrak{N}_g(h^-, h^+) + \langle N(h) - N(g) \mid \nu(g^-, h^+)\rangle\nu(h^-, h^+), \nu(h^-, h^+))\\ &\quad = (N(h), \nu(h)) = \mathtt{N}(h).\end{aligned}$$

Finally, we conclude by noting that the two maps $\amalg_{\mathtt{N}(g)}$ and Π_g are continuous. □

Proposition 4.12. *Let $\mathcal{L}^+$ (respectively $\mathcal{L}^-$) be as defined in Definition 4.1. Then $\mathcal{L}^+$ (respectively $\mathcal{L}^-$) is a lamination of* $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$.

Proof. Let $g \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$ and let $\mathcal{U}_g$ be a neighborhood of g in $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$. We will show that the equivalence relation $\mathcal{L}^+$ on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ satisfies Properties 1 and 2 of Definition 2.1 for the local homeomorphisms $\amalg_{\mathtt{N}(g)} = \Pi_g^{-1}$, from $\mathcal{U}_{\mathtt{N}(g)} = \mathtt{N}(\mathcal{U}_g)$ to its image $\mathcal{U}_{g^-} \times (\mathcal{U}_{g^+} \times \mathbb{R}) \subset (\Lambda_\infty\Gamma \times \Lambda_\infty\Gamma \setminus \Delta) \times \mathbb{R}$.

Property 1. Let $h_1, h_2 \in \mathcal{U}_g$ and let p^0 be the projection from $\mathcal{U}_{g^-} \times \mathcal{U}_{g^+} \times \mathbb{R}$ onto $\mathbb{R}$. We notice that to prove Property 1 it is enough to prove that if $h_1^+ = h_2^+$ then

$$p^0 \circ \amalg_{\mathtt{N}(g)}(\mathtt{N}(h_2)) - p^0 \circ \amalg_{\mathtt{N}(g)}(\mathtt{N}(h_1))$$

is independent of g.

Suppose $h_1^+ = h_2^+$. Then $h_2 \in \bigcup_{t\in\mathbb{R}} \widetilde{\mathcal{H}}^+_{\tilde{\phi}_t h_1}$ and by Proposition 4.6 we have

$$N(h_2) = N(h_1) + s_1 \nu^+(h_1) + t_1 \nu(h_1)$$

for some real numbers s_1, t_1. Hence we obtain

$$\begin{aligned} p^0 \circ \amalg_{\mathtt{N}(g)}(\mathtt{N}(h_2)) - p^0 \circ \amalg_{\mathtt{N}(g)}(\mathtt{N}(h_1)) &= \langle N(h_2) - N(h_1) \mid \nu(g^-, h_1^+)\rangle \\ &= \langle s_1 \nu^+(h_1) + t_1 \nu(h_1) \mid \nu(g^-, h_1^+)\rangle \\ &= t_1 \end{aligned}$$

by Corollary 4.8. As desired, t_1 does not depend on g.

Property 2. Let $p^{+,0}$ be the projection from $\mathcal{U}_{g^-} \times \mathcal{U}_{g^+} \times \mathbb{R}$ onto $\mathcal{U}_{g^+} \times \mathbb{R}$ and let $\{\mathtt{N}(h_i)\}_{i\in\{1,2,\ldots,n\}}$ be a sequence of points such that for all $i \in \{1, 2, \ldots, n-1\}$ the following two conditions hold:

1. $\mathtt{N}(h_{i+1}) \in \mathcal{U}_{\mathtt{N}(h_i)}$,
2. $p^{+,0} \circ \amalg_{\mathtt{N}(h_i)}(\mathtt{N}(h_i)) = p^{+,0} \circ \amalg_{\mathtt{N}(h_i)}(\mathtt{N}(h_{i+1}))$.

Hence we have

$$\begin{cases} h_i^+ = h_{i+1}^+, \\ 0 = \langle N(h_i) - N(h_i) \mid \nu(h_i^-, h_i^+)\rangle = \langle N(h_{i+1}) - N(h_i) \mid \nu(h_i^-, h_{i+1}^+)\rangle. \end{cases} \tag{4.7}$$

Moreover, $h_i^+ = h_{i+1}^+$ implies that $h_{i+1} \in \bigcup_{t\in\mathbb{R}} \widetilde{\mathcal{H}}^+_{\tilde{\phi}_t h_i}$. Now using Lemma 4.3.(4) and Proposition 4.6 we get

$$\nu^+(h_{i+1}) = c_i \nu^+(h_i),$$

$$N(h_{i+1}) = N(h_i) + s_i \nu^+(h_i) + t_i \nu(h_i),$$

for some real numbers c_i, s_i and t_i. Hence, by (4.7),

$$\begin{aligned} 0 &= \langle s_i \nu^+(h_i) + t_i \nu(h_i) \mid \nu(h_i^-, h_{i+1}^+)\rangle, \\ &= \langle s_i \nu^+(h_i) + t_i \nu(h_i) \mid \nu(h_i^-, h_i^+)\rangle \\ &= t_i, \end{aligned}$$

and we have $\mathcal{L}^+_{\mathtt{N}(h_i)} = \mathcal{L}^+_{\mathtt{N}(h_{i+1})}$. Therefore, we conclude that

$$\mathcal{L}^+_{\mathtt{N}(h_1)} = \mathcal{L}^+_{\mathtt{N}(h_n)}.$$

Now we show the other direction. Let $h \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$ be such that $\mathtt{N}(h) \in \mathcal{L}^+_{\mathtt{N}(g)}$. Using Proposition 4.6 we get that $h^+ = g^+$. Let $\mathcal{V}_{g^-}$ be a connected bounded open neighborhood of g^- in $\partial_\infty\mathbb{H} \smallsetminus \{g^+\}$ containing the point h^- and let $\mathcal{V}_{g^+}$ be a connected open neighborhood of g^+ in $\partial_\infty\mathbb{H} \smallsetminus \{g^-\}$ such that the intersection $\mathcal{V}_{g^+} \cap \mathcal{V}_{g^-}$ is empty. We denote the sets $\mathcal{V}_{g^\pm} \cap \Lambda_\infty\Gamma$ respectively by $\mathcal{U}_{g^\pm}$ and the open subset of $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ corresponding to the open set $\mathcal{U}_{g^-} \times \mathcal{U}_{g^+} \times \mathbb{R}$ by $\mathcal{U}_{\mathtt{N}(g)}$ i.e.

$$\mathcal{U}_{\mathtt{N}(g)} := \Pi_g(\mathcal{U}_{g^-} \times \mathcal{U}_{g^+} \times \mathbb{R}).$$

Now we consider the chart $\amalg_{\mathtt{N}(g)} : \mathcal{U}_{\mathtt{N}(g)} \to \mathcal{U}_{g^-} \times \mathcal{U}_{g^+} \times \mathbb{R}$ and notice that

$$p^{+,0} \circ \amalg_{\mathtt{N}(g)}(\mathtt{N}(g)) = (g^+, 0).$$

Since $\mathtt{N}(h) \in \mathcal{L}^+_{\mathtt{N}(g)}$, using the definition of $\mathcal{L}^+_{\mathtt{N}(g)}$ we get

$$\langle N(h) - N(g) \mid \nu(g^-, g^+)\rangle = 0.$$

But $g^+ = h^+$, therefore,

$$\langle N(h) - N(g) \mid \nu(g^-, h^+)\rangle = 0$$

and we finally have

$$p^{+,0} \circ \amalg_{\mathtt{N}(g)}(\mathtt{N}(g)) = p^{+,0} \circ \amalg_{\mathtt{N}(g)}(\mathtt{N}(h)).$$

Therefore, we conclude that $\mathcal{L}^+$ defines a lamination with plaque neighborhoods given by the images of the open sets $\mathcal{U}_{g^-}$ for g^- in $\Lambda_\infty\Gamma \smallsetminus \{g^+\}$.

Similarly, $\mathcal{L}^-$ also defines a lamination of $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$. □

Proposition 4.13. *Let $\mathcal{L}^{-,0}$ (respectively $\mathcal{L}^{+,0}$) be as defined in Definition 4.2. Then $\mathcal{L}^{-,0}$ (respectively $\mathcal{L}^{+,0}$) is a lamination of $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$. Moreover, it is the central lamination corresponding to the lamination $\mathcal{L}^-$ (respectively $\mathcal{L}^+$).*

Proof. Let $g \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$ and let $\mathcal{U}_g$ be a neighborhood of g in $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$. We will show that the equivalence relation $\mathcal{L}^{-,0}$ on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ satisfies Properties 1 and 2 of Definition 2.1 for the local homeomorphisms $\amalg_{\mathtt{N}(g)} = \Pi_g^{-1}$, from $\mathcal{U}_{\mathtt{N}(g)} = \mathtt{N}(\mathcal{U}_g)$ to its image

$$\mathcal{U}_{g^-} \times (\mathcal{U}_{g^+} \times \mathbb{R}) \subset (\Lambda_\infty\Gamma \times \Lambda_\infty\Gamma \smallsetminus \Delta) \times \mathbb{R}.$$

1. Let $g_1, g_2 \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$ and let $h_1, h_2 \in \mathcal{U}_{g_1} \cap \mathcal{U}_{g_2}$. Moreover, let p^- be the projection from $\mathcal{U}_{g^-} \times \mathcal{U}_{g^+} \times \mathbb{R}$ onto $\mathcal{U}_{g^-}$. We see that

   $$p^- \circ \amalg_{\mathtt{N}(g_1)}(\mathtt{N}(h_1)) = p^- \circ \amalg_{\mathtt{N}(g_1)}(\mathtt{N}(h_2))$$

   if and only if

   $$p^- \circ \amalg_{\mathtt{N}(g_2)}(\mathtt{N}(h_1)) = p^- \circ \amalg_{\mathtt{N}(g_2)}(\mathtt{N}(h_2)).$$

   Indeed, both left-hand sides are h_1^- and both right-hand sides are h_2^-.

2. Let $\{\mathtt{N}(h_i)\}_{i\in\{1,2,\dots,n\}}$ be a sequence of points such that for all $i \in \{1, 2, \dots, n-1\}$ the following two conditions hold:

   1. $\mathtt{N}(h_{i+1}) \in \mathcal{U}_{\mathtt{N}(h_i)}$,
   2. $p^- \circ \amalg_{\mathtt{N}(h_i)}(\mathtt{N}(h_i)) = p^- \circ \amalg_{\mathtt{N}(h_i)}(\mathtt{N}(h_{i+1}))$.

   Hence for all $i \in \{1, 2, \dots, n-1\}$ we have

   $$h_i^- = h_{i+1}^-.$$

   Now using Proposition 4.6 we get that

   $$\mathcal{L}^{-,0}_{\mathtt{N}(h_i)} = \mathcal{L}^{-,0}_{\mathtt{N}(h_{i+1})}$$

   for all i in $\{1, 2, \dots, n-1\}$. Hence

   $$\mathcal{L}^{-,0}_{\mathtt{N}(h_1)} = \mathcal{L}^{-,0}_{\mathtt{N}(h_n)}.$$

   Now we show the other direction. Let $h \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$ such that $\mathtt{N}(h) \in \mathcal{L}^{-,0}_{\mathtt{N}(g)}$. Using Proposition 4.6 we get that $h^- = g^-$. Let $\mathcal{V}_{g^+}$ be a connected bounded open neighborhood of g^+ in $\partial_\infty\mathbb{H} \smallsetminus \{g^-\}$ containing the point h^+ and let $\mathcal{V}_{g^-}$ be a connected open neighborhood of g^- in $\partial_\infty\mathbb{H} \smallsetminus \{g^+\}$ such that $\mathcal{V}_{g^+} \cap \mathcal{V}_{g^-}$ is empty. We denote the sets $\mathcal{V}_{g^\pm} \cap \Lambda_\infty\Gamma$ respectively by $\mathcal{U}_{g^\pm}$, the open subset of $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$ corresponding to the open set $\mathcal{U}_{g^-} \times \mathcal{U}_{g^+} \times \mathbb{R}$ by $\mathcal{U}_g$ and the open set $\mathtt{N}(\mathcal{U}_g)$ around $\mathtt{N}(g)$ by $\mathcal{U}_{\mathtt{N}(g)}$. Now we consider the chart

   $$\amalg_{\mathtt{N}(g)} \colon \mathcal{U}_{\mathtt{N}(g)} \longrightarrow \mathcal{U}_{g^-} \times \mathcal{U}_{g^+} \times \mathbb{R}$$

   and notice that

   $$p^- \circ \amalg_{\mathtt{N}(g)}(\mathtt{N}(g)) = g^- = h^- = p^- \circ \amalg_{\mathtt{N}(g)}(\mathtt{N}(h)).$$

   Therefore, we conclude that $\mathcal{L}^{-,0}$ defines a lamination with plaque neighborhoods given by the image of the open sets $\mathcal{U}_{g^+} \times \mathbb{R}$ for g^+ in $\Lambda_\infty\Gamma \smallsetminus \{g^+\}$.

Now the fact that $\mathcal{L}^{-,0}$ is the central lamination corresponding to the lamination $\mathcal{L}^-$ follows from Definition 4.2.

Similarly, $\mathcal{L}^{+,0}$ also defines a lamination of $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$. □

Theorem 4.14. *The laminations* $(\mathcal{L}^+, \mathcal{L}^{-,0})$ *and* $(\mathcal{L}^-, \mathcal{L}^{+,0})$ *define a local product structure on* $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$.

Proof. Using Propositions 4.11, 4.12, and 4.13 we get that $(\mathcal{L}^+, \mathcal{L}^{-,0})$ defines a local product structure on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$.

Similarly, $(\mathcal{L}^-, \mathcal{L}^{+,0})$ also defines a local product structure on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$. □

Proposition 4.15. *The laminations* $\mathcal{L}^\pm$ *and* $\mathcal{L}^{\pm,0}$ *are equivariant under the action of* Γ *on* $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$.

Proof. Let $Z \in \mathsf{U}_{\mathrm{rec}}\mathbb{A}$ be such that $Z = \mathtt{N}(g)$ for some $g \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$ and $W \in \mathcal{L}^+_Z$. Therefore, there exist real numbers s_1, s_2 such that

$$W = (N(g) + s_1\nu^+(g), \nu(g) + s_2\nu^+(g)).$$

Now for all $\gamma \in \Gamma$ we have

$$\gamma Z = \gamma \mathtt{N}(g) = \mathtt{N}(\mathtt{L}(\gamma)g)$$

and

$$\begin{aligned} \gamma W &= \gamma(N(g) + s_1\nu^+(g), \nu(g) + s_2\nu^+(g)) \\ &= (\gamma N(g) + s_1\mathtt{L}(\gamma)\nu^+(g), \mathtt{L}(\gamma)\nu(g) + s_2\mathtt{L}(\gamma)\nu^+(g)) \\ &= (N(\mathtt{L}(\gamma)g) + s_1\nu^+(\mathtt{L}(\gamma)g), \nu(\mathtt{L}(\gamma)g) + s_2\nu^+(\mathtt{L}(\gamma)g)). \end{aligned}$$

Hence $\gamma W \in \widetilde{\mathcal{L}}^+_{\gamma Z}$. Moreover, $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ is invariant under the action of Γ. Therefore, $\gamma W \in \mathcal{L}^+_{\gamma Z}$ and we get that for all γ in Γ,

$$\mathcal{L}^+_{\gamma Z} = \gamma \mathcal{L}^+_Z.$$

Similarly, $\mathcal{L}^-$ is also equivariant under the action of Γ on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$.

Now let $W \in \mathcal{L}^{+,0}_Z$. Hence there exist real numbers s_1, s_2, s_3 such that

$$W = (N(g) + s_1\nu^+(g) + s_2\nu(g), \nu(g) + s_3\nu^+(g)).$$

Notice that for all $\gamma \in \Gamma$ we have

$$\begin{aligned} \gamma W &= \gamma(N(g) + s_1\nu^+(g) + s_2\nu(g), \nu(g) + s_3\nu^+(g)) \\ &= (\gamma N(g) + s_1\mathtt{L}(\gamma)\nu^+(g) + s_2\mathtt{L}(\gamma)\nu(g), \mathtt{L}(\gamma)\nu(g) + s_3\mathtt{L}(\gamma)\nu^+(g)) \\ &= (N(\mathtt{L}(\gamma)g) + s_1\nu^+(\mathtt{L}(\gamma)g) + s_2\nu(\mathtt{L}(\gamma)g), \nu(\mathtt{L}(\gamma)g) + s_3\nu^+(\mathtt{L}(\gamma)g)). \end{aligned}$$

Hence $\gamma W \in \widetilde{\mathcal{L}}^{+,0}_{\gamma Z}$. Moreover, $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ is invariant under the action of Γ. Therefore, $\gamma W \in \mathcal{L}^{+,0}_{\gamma Z}$ and we get that for all γ in Γ,

$$\mathcal{L}^{+,0}_{\gamma Z} = \gamma \mathcal{L}^{+,0}_{Z}.$$

Similarly, $\mathcal{L}^{-,0}$ is also equivariant under the action of Γ on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$. □

Proposition 4.16. *The laminations $\mathcal{L}^{\pm}$ and $\mathcal{L}^{\pm,0}$ are equivariant under the flow $\widetilde{\Phi}$ on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$.*

Proof. Let $Z \in \mathsf{U}_{\mathrm{rec}}\mathbb{A}$ be such that $Z = \mathtt{N}(g)$ for some $g \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$ and

$$W \in \mathcal{L}^{+}_{\mathtt{N}(g)} = \mathcal{L}^{+}_{Z}.$$

Hence there exist real numbers s_1, s_2 such that

$$W = (N(g) + s_1\nu^+(g), \nu(g) + s_2\nu^+(g)).$$

Now using Corollary 2.7 we get that for any $t \in \mathbb{R}$ there exists $s \in \mathbb{R}$ such that

$$\widetilde{\Phi}_t \mathtt{N}(g) = \mathtt{N}(ga(s)).$$

We denote $ga(s)$ by g_s and using equation (2.15) we obtain that

$$\begin{aligned}\widetilde{\Phi}_t W &= \widetilde{\Phi}_t(N(g) + s_1\nu^+(g), \nu(g) + s_2\nu^+(g))\\ &= (N(g) + s_1\nu^+(g) + t(\nu(g) + s_2\nu^+(g)), \nu(g) + s_2\nu^+(g))\\ &= (N(g_s) + (s_1 + ts_2)e^{-s}\nu^+(g_s), \nu(g_s) + s_2e^{-s}\nu^+(g_s)).\end{aligned}$$

Therefore, for any real number t we have

$$\widetilde{\Phi}_t W \in \widetilde{\mathcal{L}}^{+}_{\mathtt{N}(ga(s))} = \widetilde{\mathcal{L}}^{+}_{\widetilde{\Phi}_t Z}.$$

Moreover, the invariance of $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ under $\widetilde{\Phi}$ implies that $\widetilde{\Phi}_t W \in \mathcal{L}^{+}_{\widetilde{\Phi}_t Z}$. Hence for any real number t we get that

$$\mathcal{L}^{+}_{\widetilde{\Phi}_t Z} = \widetilde{\Phi}_t \mathcal{L}^{+}_{Z}.$$

Similarly, $\mathcal{L}^{-}$ is also equivariant under the flow $\widetilde{\Phi}$ on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$.

Now let $W \in \widetilde{\mathcal{L}}^{+,0}_{\mathtt{N}(g)} = \mathcal{L}^{+,0}_{Z}$. Hence there exist real numbers s_1, s_2, s_3 such that

$$W = (N(g) + s_1\nu^+(g) + s_2\nu(g), \nu(g) + s_3\nu^+(g)).$$

We denote $ga(s)$ by g_s and using equation (2.15) we obtain that

$$\begin{aligned}\widetilde{\Phi}_t W &= \widetilde{\Phi}_t(N(g) + s_1\nu^+(g) + s_2\nu(g), \nu(g) + s_3\nu^+(g))\\ &= (N(g) + s_1\nu^+(g) + s_2\nu(g) + t(\nu(g) + s_3\nu^+(g)), \nu(g) + s_3\nu^+(g))\\ &= (N(g_s) + (s_1 + ts_3)e^{-s}\nu^+(g_s) + s_2\nu(g_s), \nu(g_s) + s_3e^{-s}\nu^+(g_s)).\end{aligned}$$

Therefore, for any real number t we have

$$\widetilde{\Phi}_t W \in \widetilde{\mathcal{L}}^{+,0}_{\mathtt{N}(ga(s))} = \widetilde{\mathcal{L}}^{+,0}_{\widetilde{\Phi}_t Z}.$$

Moreover, the invariance of $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ under $\widetilde{\Phi}$ implies that $\widetilde{\Phi}_t W \in \mathcal{L}^{+,0}_{\widetilde{\Phi}_t Z}$. Hence for any real number t we get that

$$\mathcal{L}^{+,0}_{\widetilde{\Phi}_t Z} = \widetilde{\Phi}_t \mathcal{L}^{+,0}_{Z}.$$

Similarly, $\mathcal{L}^{-,0}$ is also equivariant under the flow $\widetilde{\Phi}$ on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$. □

Definition 4.17. We respectively denote the projections of $\mathcal{L}^{\pm}, \mathcal{L}^{\pm,0}$ on the space $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ by $\underline{\mathcal{L}}^{\pm}, \underline{\mathcal{L}}^{\pm,0}$.

Now we define the notion of a leaf lift. We will use this notion to estimate the distance $\tilde{d}$ on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ in terms of the norm on the tangent space at any given point. We define the leaf lift as follows.

The *positive leaf lift* is the map

$$\begin{aligned} i^{+}_{\mathtt{N}(g)} \colon \widetilde{\mathcal{L}}^{+}_{\mathtt{N}(g)} &\longrightarrow \mathsf{T}_{\mathtt{N}(g)}\mathsf{U}\mathbb{A}, \\ (N(g) + s_1\nu^{+}(g), \nu(g) + s_2\nu^{+}(g)) &\longmapsto (s_1\nu^{+}(g), s_2\nu^{+}(g)), \end{aligned}$$

where we identify $\mathsf{T}_{\mathtt{N}(g)}\mathsf{U}\mathbb{A}$ with $\mathsf{T}_{N(g)}\mathbb{A} \times \mathsf{T}_{\nu(g)}\mathsf{S}^1$. Similarly, the *negative leaf lift* is the map

$$\begin{aligned} i^{-}_{\mathtt{N}(g)} \colon \widetilde{\mathcal{L}}^{-}_{\mathtt{N}(g)} &\longrightarrow \mathsf{T}_{\mathtt{N}(g)}\mathsf{U}\mathbb{A}, \\ (N(g) + s_1\nu^{-}(g), \nu(g) + s_2\nu^{-}(g)) &\longmapsto (s_1\nu^{-}(g), s_2\nu^{-}(g)). \end{aligned}$$

5. Contraction properties

In this section we will first show that the lamination $\mathcal{L}^{+}$ is a stable lamination and the lamination $\mathcal{L}^{-}$ is an unstable lamination for the affine flow $\widetilde{\Phi}$ on $(\mathsf{U}_{\mathrm{rec}}\mathbb{A}, \tilde{d})$. In fact, we will prove that the leaves of the lamination $\mathcal{L}^{+}$ contract in the forward direction of the affine flow. Similarly, it will follow that the leaves of the lamination $\mathcal{L}^{-}$ contract in the backward direction of the affine flow. Moreover, we will show that $(\mathsf{U}_{\mathrm{rec}}\mathsf{M}, d)$ admits a metric Anosov structure with respect to the affine flow Φ on $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ and its stable (respectively unstable) lamination is given by the projection of the lamination $\mathcal{L}^{+}$ (respectively $\mathcal{L}^{-}$) on $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$.

Now we start with the following construction whose *raison d'être* will be apparent in Proposition 5.2.

Proposition 5.1. *There exists a* Γ*-equivariant map* $Z \mapsto \|\cdot\|_Z$ *from* $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ *into the space of euclidean metrics on* $\mathbb{R}^3 \times \mathbb{R}^3$ *such that for all positive integers* n*, there exists a positive real number* t_n *satisfying the following property*: *if* $t > t_n$*,* $Z \in \mathsf{U}_{\mathrm{rec}}\mathbb{A}$ *and* $W \in \widetilde{\mathcal{L}}^+_Z$ *then*

$$\|i^+_{\widetilde{\Phi}_t Z}(\widetilde{\Phi}_t W) - i^+_{\widetilde{\Phi}_t Z}(\widetilde{\Phi}_t Z)\|_{\widetilde{\Phi}_t Z} \leqslant \frac{1}{2^n}\|i^+_Z(W) - i^+_Z(Z)\|_Z.$$

Proof. Let $g \in \mathsf{U}_{\mathrm{rec}}\mathbb{H}$. We note that the tangent space $\mathsf{T}_{\mathtt{N}(g)}\mathsf{U}\mathbb{A}$ is a five dimensional vector space but it embeds naturally in $\mathsf{T}_{\mathtt{N}(g)}(\mathbb{A} \times \mathbb{V}) \cong \mathbb{R}^3 \times \mathbb{R}^3$ as a hyperplane. Now let $\langle\cdot|\cdot\rangle_{\mathtt{N}(g)}$ be a positive definite bilinear form on the tangent space $\mathsf{T}_{\mathtt{N}(g)}(\mathbb{A} \times \mathbb{V})$ satisfying the following properties:

1. $\langle(\nu^\alpha(g), 0) \mid (\nu^\beta(g), 0)\rangle_{\mathtt{N}(g)} = \langle(0, \nu^\alpha(g)) \mid (0, \nu^\beta(g))\rangle_{\mathtt{N}(g)} = \delta_{\alpha\beta}$,
2. $\langle(\nu^\alpha(g), 0) \mid (0, \nu^\beta(g))\rangle_{\mathtt{N}(g)} = \langle(0, \nu^\alpha(g)) \mid (\nu^\beta(g), 0)\rangle_{\mathtt{N}(g)} = 0$.

where $\delta_{\alpha\beta}$ is the Kronecker delta function with α, β in $\{., +, -\}$. We define the map $\|\cdot\|_{\mathbf{.}}$ as follows:

$$\|X\|_{\mathtt{N}(g)} := \sqrt{\langle X \mid X\rangle_{\mathtt{N}(g)}},$$

where X is in $\mathsf{T}_{\mathtt{N}(g)}(\mathbb{A} \times \mathbb{V})$. Now by equations (2.14), (2.16), and Theorem 2.6 we get that $\|\cdot\|_{\mathbf{.}}$ is Γ-equivariant, i.e.

$$\|\gamma X\|_{\gamma\mathtt{N}(g)} = \|X\|_{\mathtt{N}(g)}.$$

Let $Z := \mathtt{N}(g)$ and $W \in \widetilde{\mathcal{L}}^+_Z$. Hence there exist real numbers s_1 and s_2 such that

$$W = (N(g) + s_1\nu^+(g), \nu(g) + s_2\nu^+(g)).$$

Therefore, we get that

$$\|i^+_Z(W) - i^+_Z(Z)\|_Z = \|(s_1\nu^+(g), s_2\nu^+(g))\|_Z = \sqrt{s_1^2 + s_2^2}. \tag{5.1}$$

Moreover, for any $t \in \mathbb{R}$, by Corollary 2.7 there exists $t_1 \in \mathbb{R}$ such that

$$\widetilde{\Phi}_t Z = \widetilde{\Phi}_t\mathtt{N}(g) = \mathtt{N}(ga(t_1)). \tag{5.2}$$

Therefore, we get that

$$\begin{aligned}
&\|i^+_{\widetilde{\Phi}_t Z}(\widetilde{\Phi}_t W) - i^+_{\widetilde{\Phi}_t Z}(\widetilde{\Phi}_t Z)\|_{\widetilde{\Phi}_t Z} \\
&\quad = \|((s_1 + ts_2)\nu^+(g), s_2\nu^+(g))\|_{\mathtt{N}(ga(t_1))} \\
&\quad = e^{-t_1}\|(\nu^+(ga(t_1)), 0)\|_{\mathtt{N}(ga(t_1))}\sqrt{(s_1 + ts_2)^2 + s_2^2} \\
&\quad \leqslant e^{-t_1}\sqrt{2}(1 + |t|)\sqrt{s_1^2 + s_2^2}.
\end{aligned} \tag{5.3}$$

Furthermore, by equation (5.2) and Theorem 2.6.(3) we have

$$t = \int_0^{t_1} f(ga(s))ds.$$

Now compactness of $\mathsf{U}_{\mathrm{rec}}\Sigma$ implies that f is bounded on $\mathsf{U}_{\mathrm{rec}}\mathbb{H}$ by some constant $k \in \mathbb{R}^{>0}$. Moreover, if $t \in \mathbb{R}^{>0}$ then by positivity of f we have $t_1 \in \mathbb{R}^{>0}$ and

$$t \leqslant \int_0^{t_1} k ds = kt_1.$$

Let c be a constant which is bigger than $\max\{1, 2k\}$. Then for $t \in \mathbb{R}^{>0}$ we get that

$$(1 + |t|)e^{-t_1} = (1 + t)e^{-t_1} \leqslant ce^{-\frac{t}{2k}}. \tag{5.4}$$

Now by combining equation (5.1), inequalities (5.3) and (5.4) we obtain

$$\|i^+_{\widetilde{\Phi}_t Z}(\widetilde{\Phi}_t W) - i^+_{\widetilde{\Phi}_t Z}(\widetilde{\Phi}_t Z)\|_{\widetilde{\Phi}_t Z} \leqslant \sqrt{2}ce^{-\frac{t}{2k}}\|i^+_Z(W) - i^+_Z(Z)\|_Z$$

when t is positive. Hence for any positive integer n, there exists $t_n \in \mathbb{R}$ such that if $t > t_n$, $Z \in \mathsf{U}_{\mathrm{rec}}\mathbb{A}$ and $W \in \mathcal{L}^+_Z$ then

$$\|i^+_{\widetilde{\Phi}_t Z}(\widetilde{\Phi}_t W) - i^+_{\widetilde{\Phi}_t Z}(\widetilde{\Phi}_t Z)\|_{\widetilde{\Phi}_t Z} \leqslant \frac{1}{2^n}\|i^+_Z(W) - i^+_Z(Z)\|_Z. \qquad \square$$

Proposition 5.2. *Let $\tilde{d}$ be a Γ-invariant distance on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ which is locally bilipschitz equivalent to a euclidean distance on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ and let $\|\cdot\|_{\cdot}$ be the Γ-equivariant map from $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ to the space of euclidean metrics on $\mathbb{R}^3 \times \mathbb{R}^3$ as constructed in the proof of Proposition 5.1. Then there exist positive constants K and α such that for any $Z \in \mathsf{U}_{\mathrm{rec}}\mathbb{A}$ and for any $W \in \mathcal{L}^+_Z$, the following statements are true*:

1. *if $\tilde{d}(W, Z) \leqslant \alpha$, then $\|i^+_Z(Z) - i^+_Z(W)\|_Z \leqslant K\tilde{d}(W, Z)$;*
2. *if $\|i^+_Z(Z) - i^+_Z(W)\|_Z \leqslant \alpha$, then $\tilde{d}(W, Z) \leqslant K\|i^+_Z(Z) - i^+_Z(W)\|_Z$.*

Proof. Since Γ acts cocompactly on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$, $\tilde{d}$ is Γ-invariant and $\|\cdot\|_{\cdot}$ is Γ-equivariant, it suffices to prove the above assertion for Z in a compact subset D of $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$, where D is the closure of a suitably chosen fundamental domain.

We can define a euclidean distance d_Z on $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$, uniquely using the euclidean metric $\|\cdot\|_Z$ on $\mathbb{R}^3\times\mathbb{R}^3$, by taking the embedding of $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ in $\mathbb{A}\times\mathbb{V}$. We notice that for any Z in $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ and for any W in $\mathcal{L}^+_Z$, $d_Z(W, Z)$ is equal to $\|i^+_Z(W)-i^+_Z(Z)\|_Z$.

Now, any two euclidean distances are bilipschitz equivalent with each other and by our hypothesis, $\tilde{d}$ is locally bilipschitz equivalent to a euclidean distance. Therefore, in particular, $\tilde{d}$ is locally bilipschitz equivalent with d_Z for Z in D, that is, there exist constants K_Z depending on Z, and open sets U_Z around Z, such that the distances d_Z and $\tilde{d}$ are K_Z bilipschitz equivalent with each other on U_Z.

Let $C_{(X,Y)}$ for any X and Y in D, be a constant such that the distances d_X and d_Y are $C_{(X,Y)}$ bilipschitz equivalent with each other. It follows from the construction of the norm $\|\cdot\|_{\cdot}$, as done in Proposition 5.1, that we can choose the constants $C_{(X,Y)}$ in such a way that $C_{(X,Y)}$ vary continuously in (X,Y). As D is compact it follows that $C_{(X,Y)}$ is bounded above by some constant C. Hence, for all X and Y in D, d_X and d_Y are C bilipschitz equivalent with each other.

Now we consider the open cover of D by the open sets U_Z. As D is compact, there exist points $Z_1, Z_2, \ldots, Z_n$ in D such that $U_{Z_1}, U_{Z_2}, \ldots, U_{Z_n}$ covers D. There exists a real number $\beta > 0$, called a *Lebesgue number* for this covering with respect to the distance $\tilde{d}$, such that for any Z in D, the open $\tilde{d}$-ball of radius β around Z, denoted by $B_{\tilde{d}}(Z,\beta)$, lies inside U_{Z_j} for some $j \in \{1,2,\ldots,n\}$. Also, let K_0 be the maximum of $K_{Z_1}, K_{Z_2}, \ldots, K_{Z_n}$. Hence $\tilde{d}$ and d_{Z_j} are K_0 bilipschitz equivalent with each other on $B_{\tilde{d}}(Z,\beta)$. As d_Z and d_{Z_j} are C bilipschitz equivalent with each other, it follows that $\tilde{d}$ and d_Z are CK_0 bilipschitz equivalent with each other on $B_{\tilde{d}}(Z,\beta)$. Moreover, we note that the constants β, C, K_0 and hence also CK_0, do not depend on Z. Therefore, $\tilde{d}$ and d_Z are CK_0 bilipschitz equivalent with each other on $B_{\tilde{d}}(Z,\beta)$, for all Z in D.

As any two distances d_X and d_Y, for all X, Y in D are C bilipschitz equivalent with each other, without loss of generality we can choose a point X in D and consider the distance d_X. We note that the set $\{B_{\tilde{d}}(Z,\beta) \colon Z \in D\}$ is an open cover of D. Let β_1 be a Lebesgue number for this cover for the metric space (D, d_X). Therefore, the open ball $B_{d_X}(Y_1,\beta_1)$ for any Y_1 in D, lies inside an open ball $B_{\tilde{d}}(Y_2,\beta)$ for some point Y_2 in D. Now, as $\tilde{d}$ and d_Z are CK_0 bilipschitz equivalent with each other on the ball $B_{\tilde{d}}(Z,\beta)$ for all Z in D, it follows that $\tilde{d}$ and d_X are CK_0 bilipschitz equivalent with each other on the ball $B_{d_X}(Y_2,\beta_1)$. As Y_2 was chosen arbitrarily we have that $\tilde{d}$ and d_X are CK_0 bilipschitz equivalent with each other on the ball $B_{d_X}(Y,\beta_1)$, for all Y in D.

Moreover, we know that d_X and d_Z are C bilipschitz equivalent with each other. Therefore, we get that $\tilde{d}$ and d_Z are CK_0 bilipschitz equivalent with each other on the ball $B_{d_Z}(Y,\frac{\beta_1}{C})$, for all Y in D. In particular, we get that, $\tilde{d}$ and d_Z are CK_0 bilipschitz equivalent with each other on the ball $B_{d_Z}(Z,\frac{\beta_1}{C})$. Finally, we set α to be $\min\{\frac{\beta_1}{C},\beta\}$ and K to be CK_0 to get that for any Z in $\mathsf{U}_{\mathrm{rec}}\mathbb{A}$ and W in $\mathcal{L}^+_Z$,

1. if $\tilde{d}(W,Z) \leqslant \alpha$, then $\|i^+_Z(Z) - i^+_Z(W)\|_Z \leqslant K\tilde{d}(W,Z)$,
2. if $\|i^+_Z(Z) - i^+_Z(W)\|_Z \leqslant \alpha$, then $\tilde{d}(W,Z) \leqslant K\|i^+_Z(Z) - i^+_Z(W)\|_Z$. □

Theorem 5.3. *Let $\mathcal{L}^{\pm}$ be two laminations on $\mathsf{U}_{\text{rec}}\mathbb{A}$ as defined in Definitions 4.1 and 4.2 and let $\tilde{d}$ be the Γ-invariant distance, as defined in Definition 3.1. Under these assumptions, for the distance $\tilde{d}$ on $\mathsf{U}_{\text{rec}}\mathbb{A}$ we have that*

1. *$\mathcal{L}^{+}$ is contracted in the forward direction of the affine flow,*
2. *$\mathcal{L}^{-}$ is contracted in the backward direction of the affine flow.*

Proof. Let $\|\cdot\|_{.}$ be the Γ-equivariant map from $\mathsf{U}_{\text{rec}}\mathbb{A}$ to the space of euclidean metrics on $\mathbb{R}^3 \times \mathbb{R}^3$ as constructed in the proof of Proposition 5.1 and let K and α be as in the Proposition 5.2 for the distance $\tilde{d}$. We choose a positive integer n such that

$$\frac{K}{2^n} < 1\,,\ \frac{K^2}{2^n} \leqslant \frac{1}{2}.$$

Let t_n be the constant as in Proposition 5.1 for our chosen n. Also let Z be in $\mathsf{U}_{\text{rec}}\mathbb{A}$ and W be in $\mathcal{L}^{+}_{Z}$, so that $\tilde{d}(W,Z) \leqslant \alpha$. Then using Proposition 5.2 we get

$$\|i^{+}_{Z}(W) - i^{+}_{Z}(Z)\|_{Z} \leqslant K\tilde{d}(W,Z). \tag{5.5}$$

Furthermore, using Proposition 5.1 we get that for all $t > t_n$,

$$\|i^{+}_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t W) - i^{+}_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t Z)\|_{\tilde{\Phi}_t Z} \leqslant \frac{1}{2^n}\|i^{+}_{Z}(W) - i^{+}_{Z}(Z)\|_{Z}. \tag{5.6}$$

Hence it follows that

$$\|i^{+}_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t W) - i^{+}_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t Z)\|_{\tilde{\Phi}_t Z} \leqslant \frac{K\alpha}{2^n} \leqslant \alpha.$$

Now again using Proposition 5.2 we get

$$\tilde{d}(\tilde{\Phi}_t W, \tilde{\Phi}_t Z) \leqslant K\|i^{+}_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t W) - i^{+}_{\tilde{\Phi}_t Z}(\tilde{\Phi}_t Z)\|_{\tilde{\Phi}_t Z}. \tag{5.7}$$

Hence combining the inequalities (5.5), (5.6), and (5.7) we obtain

$$\tilde{d}(\tilde{\Phi}_t W, \tilde{\Phi}_t Z) \leqslant \frac{K^2}{2^n}\tilde{d}(W,Z) \leqslant \frac{1}{2}\,\tilde{d}(W,Z), \tag{5.8}$$

for all $t > t_n$. Therefore, $\mathcal{L}^{+}$ is contracted in the forward direction of the affine flow.

Similarly, $\mathcal{L}^{-}$ is contracted in the backward direction of the affine flow. □

Finally, we consider what happens in the quotient, i.e. $\mathsf{U}_{\text{rec}}\mathsf{M}$. Let $Z \in \mathsf{U}_{\text{rec}}\mathbb{A}$ and ϵ be a positive real number. Then we define,

$$\mathcal{L}^{\pm}_{\epsilon}(Z) := \mathcal{L}^{\pm}_{Z} \cap B_{\tilde{d}}(Z, \epsilon),$$

and

$$\mathcal{K}_{\epsilon}(Z) := \Pi_Z(\mathcal{L}^{+}_{\epsilon}(Z) \times \mathcal{L}^{-}_{\epsilon}(Z) \times (-\epsilon, \epsilon)) \subset \mathsf{U}_{\text{rec}}\mathbb{A}$$

where Π_Z is the local product structure at Z defined by the stable and the unstable leaves.

We know that there exists a positive real number ϵ_0 such that for any non identity element γ in Γ and for $Z \in \mathsf{U}_{\text{rec}}\mathbb{A}$ we have,

$$\gamma(\mathcal{K}_{\epsilon_0}(Z)) \cap \mathcal{K}_{\epsilon_0}(Z) = \emptyset.$$

Proof of Theorem 1.1. Let us fix α as in Proposition 5.2 and let ϵ_1 be from the open interval $\left(0, \min\left\{\alpha, \frac{\epsilon_0}{2}\right\}\right)$. Now let z be any point of $\mathsf{U}_{\text{rec}}\mathsf{M}$ and let Z be a point in $\mathsf{U}_{\text{rec}}\mathbb{A}$ in the preimage of z. Our choice of ϵ_1 gives us that the inequality (5.8) holds for the affine flow on $\mathsf{U}_{\text{rec}}\mathbb{A}$ for the points in the chart $\mathcal{K}_{\epsilon_1}(Z)$. Hence the inequality (5.8) also holds for the affine flow on $\mathsf{U}_{\text{rec}}\mathsf{M}$ for points in the chart which are in the projection of $\mathcal{K}_{\epsilon_1}(Z)$.

Therefore, $\underline{\mathcal{L}}^{+}$, the projection of $\mathcal{L}^{+}$ in $\mathsf{U}_{\text{rec}}\mathsf{M}$, is contracted in the forward direction of the affine flow on $\mathsf{U}_{\text{rec}}\mathsf{M}$.

Similarly, $\underline{\mathcal{L}}^{-}$, the projection of $\mathcal{L}^{-}$ in $\mathsf{U}_{\text{rec}}\mathsf{M}$, is contracted in the backward direction of the affine flow on $\mathsf{U}_{\text{rec}}\mathsf{M}$. □

6. Anosov representations

In this section we define the notion of an Anosov representation in the context of the non-semisimple Lie group $\mathsf{G} := \mathsf{SO}^0(2,1) \ltimes \mathbb{R}^3$.

6.1. Pseudo-Parabolic subgroups. Let $\mathbb{X}$ be the space of all affine null planes. We observe that G acts transitively on $\mathbb{X}$. Hence for all $P \in \mathbb{X}$ we have

$$\mathbb{X} = \mathsf{G}.P \cong \mathsf{G}/\mathsf{Stab}_{\mathsf{G}}(P).$$

Definition 6.1. If $P \in \mathbb{X}$ then we define

$$\mathsf{P}_P := \mathsf{Stab}_{\mathsf{G}}(P).$$

We call P_P a *pseudo-parabolic* subgroup of G.

Let $\mathsf{V}(P)$ denote the vector space underlying a null plane P, let $v_0 := (1,0,0)^{\mathsf{t}}$ and $v_0^{\pm} := (0,\pm 1,1)^{\mathsf{t}}$ and let $\mathcal{C}$ be the upper half of the isotropic cone $\mathsf{S}^0 \setminus \{0\}$. Now we consider the space

$$\mathcal{N} := \{(P_1,P_2) \mid P_1,P_2 \in \mathbb{X}, \mathsf{V}(P_1) \neq \mathsf{V}(P_2)\}$$

and define the following map onto the unit quadric S^1 of $\mathbb{R}^{2,1}$:

$$\begin{aligned} v\colon \mathcal{N} &\longrightarrow \mathsf{S}^1, \\ (P_1,P_2) &\longmapsto v(P_1,P_2), \end{aligned}$$

where $v(P_1,P_2) \in \mathsf{V}(P_1) \cap \mathsf{V}(P_2) \cap \mathsf{S}^1$ is such that if $v_1 \in \mathsf{V}(Q_1) \cap \mathcal{C}$ and $v_2 \in \mathsf{V}(Q_2) \cap \mathcal{C}$ then $(v_1, v(Q_1,Q_2), v_2)$ gives the same orientation as (v_0^-, v_0, v_0^+). We observe that

$$v(P_1,P_2) = -v(P_2,P_1).$$

Proposition 6.2. *The space $\mathcal{N}$ is the unique open G orbit in $\mathbb{X}\times\mathbb{X}$ for the diagonal action of G on $\mathbb{X}\times\mathbb{X}$.*

Proof. Let (P_1,P_2) and (Q_1,Q_2) be two arbitrary points in $\mathcal{N}$. We consider the vector $v(P_1,P_2) \in \mathsf{S}^1$ corresponding to the point (P_1,P_2) and the vector $v(Q_1,Q_2) \in \mathsf{S}^1$ corresponding to the point (Q_1,Q_2). Now as $\mathsf{SO}^0(2,1)$ acts transitively on S^1 we get that there exist $g \in \mathsf{SO}^0(2,1)$ such that

$$v(Q_1,Q_2) = g v(P_1,P_2).$$

We choose $X(Q_1,Q_2) \in Q_1 \cap Q_2$ and $X(P_1,P_2) \in P_1 \cap P_2$ and observe that

$$(e, X(Q_1,Q_2) - O) \circ (g,0) \circ (e, X(P_1,P_2) - O)^{-1} P_1 = Q_1,$$

$$(e, X(Q_1,Q_2) - O) \circ (g,0) \circ (e, X(P_1,P_2) - O)^{-1} P_2 = Q_2,$$

where e is the identity element in $\mathsf{SO}^0(2,1)$. Therefore, $\mathcal{N}$ is an open G orbit in $\mathbb{X}\times\mathbb{X}$. Now as $\mathcal{N}$ is dense in $\mathbb{X}\times\mathbb{X}$ and $\mathbb{X}\times\mathbb{X}$ is connected, the result follows. □

Let N be the space of oriented spacelike affine lines. We think of N as the space $\mathsf{U}\mathbb{A}/\sim$ where $(X,v) \sim (X_1,v_1)$ if and only if $(X_1,v_1) = \widetilde{\Phi}_t(X,v)$ for some $t \in \mathbb{R}$. We denote the equivalence class of (X,v) by $[(X,v)]$. Now we consider the following map:

$$\begin{aligned} \iota'\colon \mathcal{N} &\longrightarrow \mathsf{N}, \\ (P_1,P_2) &\longmapsto [(X(P_1,P_2), v(P_1,P_2))] \end{aligned}$$

where $X(P_1,P_2)$ is any point in $P_1 \cap P_2$. We observe that ι' gives a G-equivariant map.

Let us denote the plane passing through X with underlying vector space generated by the vectors w_1 and w_2 by P_{X,w_1,w_2}. Now we consider another map

$$\begin{aligned} \imath\colon \mathsf{U}\mathbb{A} &\longrightarrow \mathcal{N}, \\ (X, v) &\longmapsto (P_{X,v,v^-}, P_{X,v,v^+}) \end{aligned}$$

where $v^\pm \in \mathcal{C}$ such that $\langle v^\pm \mid v\rangle = 0$ and (v^-, v, v^+) gives the same orientation as (v_0^-, v_0, v_0^+). We observe that $\imath$ is a G-equivariant map. Now as $P_{X+tv,v,v^+} = P_{X,v,v^+}$ and $P_{X+tv,v,v^-} = P_{X,v,v^-}$ we get that the map $\imath$ gives rise to a map, which we again denote by $\imath$,

$$\imath\colon \mathsf{N} \longrightarrow \mathcal{N}.$$

Moreover, we observe that $\imath \circ \imath' = \mathsf{Id}$ and $\imath' \circ \imath = \mathsf{Id}$.

6.2. Geometric Anosov structure. Geometric Anosov structures were first intoduced by Labourie in [21]. In this subsection we give an appropriate definition of the geometric Anosov property and show that $(\mathsf{U}_{\mathrm{rec}}\mathsf{M}, \mathcal{L})$ admits a geometric Anosov structure.

Let $(P^-, P^+) \in \mathcal{N}$ be such that $P^+ := P_{O,v_0,v_0^+}$ and $P^- := P_{O,v_0,v_0^-}$. We denote $\mathsf{Stab}_{\mathsf{G}}(P^\pm)$ respectively by $\mathsf{P}^\pm$. We note that the pair $\mathbb{X}^\pm := \mathsf{G}/\mathsf{P}^\pm$ gives a pair of continuous foliations on the space N whose tangential distributions $\mathsf{E}^\pm$ satisfy

$$\mathsf{TN} = \mathsf{E}^- \oplus \mathsf{E}^+.$$

Definition 6.3. We say that a vector bundle E over a compact topological space whose total space is equipped with a flow $\{\varphi_t\}_{t\in\mathbb{R}}$ of bundle automorphisms is *contracted* by the flow as $t \to \infty$ if for any metric $\|\cdot\|$ on E, there exist positive constants t_0, A and c such that for all $t > t_0$ and for all v in E we have

$$\|\varphi_t(v)\| \leqslant A e^{-ct}\|v\|.$$

Definition 6.4. Let $\mathcal{L}$ denote the orbit foliation of $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ under the flow Φ. We say that $(\mathsf{U}_{\mathrm{rec}}\mathsf{M}, \mathcal{L})$ admits a *geometric* (N, G)-*Anosov structure* if there exists a map

$$F\colon \widetilde{\mathsf{U}_{\mathrm{rec}}\mathsf{M}} \longrightarrow \mathsf{N}$$

such that the following conditions hold:

1. for all $\gamma \in \Gamma$ we have $F \circ \gamma = \gamma \circ F$;
2. for all $t \in \mathbb{R}$ we have $F \circ \widetilde{\Phi}_t = F$;
3. by the flow invariance, the bundles $F^\pm := F^*\mathsf{E}^\pm$ are equipped with a parallel transport along the orbits of $\widetilde{\Phi}$. The bundle F^+ (respectively F^-) gets contracted by the lift of the flow $\widetilde{\Phi}_t$ as $t \to -\infty$ (respectively $t \to \infty$).

Proof of Theorem 1.2. Let us define a map F as follows:

$$F\colon \widetilde{\mathsf{U}_{\mathrm{rec}}\mathsf{M}} \longrightarrow \mathsf{N}, \quad (X, v) \longmapsto [(X, v)].$$

We note that the map F is clearly Γ-equivariant and is also invariant under the flow $\widetilde{\Phi}$. Now we observe that

$$\mathsf{T}_{\iota([(X,v)])}\mathsf{G}/\mathsf{P}^- \cong \mathbb{R}(0, v^+) \oplus \mathbb{R}(v^+, 0)$$

and

$$\mathsf{T}_{\iota([(X,v)])}\mathsf{G}/\mathsf{P}^+ \cong \mathbb{R}(0, v^-) \oplus \mathbb{R}(v^-, 0),$$

where $v^+, v^- \in \mathcal{C}$ such that $\langle v^\pm \mid v\rangle = 0$ and (v^-, v, v^+) gives the same orientation as (v_0^-, v_0, v_0^+).

Now using Proposition 5.1 we notice that F^+ gets contracted by the lift of the flow $\widetilde{\Phi}_t$ as $t \to -\infty$ and F^- gets contracted by the lift of the flow $\widetilde{\Phi}_t$ as $t \to \infty$. Moreover, as $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$ is compact we get that the convergence is independent of the particular distance choosen on $\mathsf{U}_{\mathrm{rec}}\mathsf{M}$, as long as the distances are locally bilipschitz equivalent to each other. □

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