A sharp threshold for collapse
of the random triangular group

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Abstract. The random triangular group $\Gamma(n, p)$ is the group given by a random group presentation with n generators in which every relator of length three is present independently with probability p. We show that in the evolution of $\Gamma(n, p)$ the property of collapsing to the trivial group admits a very sharp threshold.

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1. Introduction

Let $P = \langle S \mid R \rangle$ denote a group presentation, where S is the set of generators and R is the set of relators. A group generated by a presentation P is called a *triangular group* if R consists of cyclically reduced words of length three over the alphabet $S \cup S^{-1}$, that is if R consists of words of the form abc such that $a \neq b^{-1}$, $b \neq c^{-1}$ and $c \neq a^{-1}$. Here we consider the random triangular group $\Gamma(n, p)$ defined as a group given by a random triangular group presentation with n generators and such that each cyclically reduced word of length three over the alphabet $S \cup S^{-1}$ is present in R independently with probability $p = p(n)$.

We study the asymptotic properties of the random triangular group when the number of generators n goes to infinity. Thus, for a group property $\mathcal{P}$ and a function $p(n)$, we say that $\Gamma(n, p(n))$ has $\mathcal{P}$ *asymptotically almost surely* (a.a.s.), if the probability that $\Gamma(n, p(n))$ has this property tends to 1 as $n \to \infty$.

The notion of the random triangular group was introduced by Żuk [11]. In particular, he showed that for every constant $\epsilon > 0$, if $p \leq n^{-3/2-\epsilon}$, then a.a.s. $\Gamma(n, p)$ is an infinite, hyperbolic group, while for $p \geq n^{-3/2+\epsilon}$, a.a.s. $\Gamma(n, p)$ collapses to the trivial group (his result is stated for a somewhat different, yet equivalent,

model of random triangular group). Antoniuk, Łuczak and Świątkowski [2] improved this result from one side and showed that there exists a constant $C > 0$ such that for $p \geq Cn^{-3/2}$ a.a.s. $\Gamma(n, p)$ collapses to the trivial group. They also asked if there exists a constant $c > 0$ such that for $p < cn^{-3/2}$ a.a.s. $\Gamma(n, p)$ is infinite.

Note that the property that a group is trivial is monotone, i.e. if $\langle S \mid R \rangle$ is trivial then for any $R' \supseteq R$ the group $\langle S \mid R' \rangle$ is trivial as well. Hence, by a well known argument of Bollobás and Thomason [3], there exists a 'coarse' threshold function for collapsibility i.e. there exists a function $\theta(n)$ such that if $p(n)/\theta(n) \to 0$, then a.a.s. $\Gamma(n, p)$ is non-trivial, whereas for $p(n)/\theta(n) \to \infty$ a.a.s. $\Gamma(n, p)$ collapses to the trivial group. However, the result of Antoniuk, Łuczak and Świątkowski [2] and their conjecture we have just mentioned suggest that $\Gamma(n, p)$ collapses more rapidly, i.e. that the collapsibility has a 'sharp' threshold. Our main result states that this is indeed the case.

Theorem 1. *Let $h(n, p)$ denote the probability that $\Gamma(n, p)$ is trivial. There exists a function $\tilde{c}(n)$ such that for any $\epsilon > 0$,*

$$\lim_{n \to \infty} h(n, (1 - \epsilon)\tilde{c}(n)n^{-3/2}) = 0 \quad \text{and} \quad \lim_{n \to \infty} h(n, (1 + \epsilon)\tilde{c}(n)n^{-3/2}) = 1.$$

Unfortunately, the argument we use does not give any information on the asymptotic behaviour of $\tilde{c}(n)$. Nonetheless we strengthen the conjecture from [2] and predict that $\tilde{c}(n)$ tends to a limit.

Conjecture. *There exists a constant $c > 0$ such that for every constant $\epsilon > 0$ the following holds*:

(i) *if $p \leq (c - \epsilon)n^{-3/2}$, then a.a.s. $\Gamma(n, p)$ is infinite and hyperbolic;*

(ii) *if $p \geq (c + \epsilon)n^{-3/2}$, then a.a.s. $\Gamma(n, p)$ is trivial.*

As we have already remarked it was shown in [2] that

$$\limsup \tilde{c}(n) < \infty.$$

Although we cannot verify the conjecture and prove that

$$\liminf \tilde{c}(n) > 0,$$

we show however that $\tilde{c}(n)$ cannot tend to 0 too quickly.

Theorem 2. *Let ω and p be functions of n such that $\omega(n) \to \infty$ as $n \to \infty$, and*

$$p(n) = n^{-3/2 - \omega/\log^{1/3} n}.$$

Then a.a.s. $\Gamma(n, p)$ is infinite, torsion-free, and hyperbolic.

The structure of the paper is the following. In the next section we prove Theorem 1. The argument is based on a result of the second author [6] which, up to our knowledge, has never been used to show that properties of random groups have sharp thresholds. It states, roughly, that if a property does not admit a sharp threshold then it is 'local', i.e. its probability can be significantly changed by a local modification of the random structure (see Lemma 3 below). We show that it is not the case with the collapsibility. In particular, we show that adding to R a few more specially selected relators affects the probability of collapsing less than a tiny increase of the probability p, which in turn corresponds to adding to R a small number of random relators. Hence, a local modification of the random structure cannot have large impact on the probability of the property in question.

Then we prove Theorem 2. We follow closely the argument of Ollivier who in [8] showed that the assertion holds for some function $p(n) = n^{-3/2+o(1)}$. This result was initially stated by Gromov [5] however it seems that Ollivier was the first one who gave a complete proof of this statement. We basically rewrite Ollivier's argument (who, following Żuk, used a slightly different model of the random triangular group) to replace $o(1)$ in the power by some explicit function.

2. Proof of Theorem 1

As mentioned in the introduction, the tool we use in order to prove the sharpness of the threshold, as expressed in Theorem 1, is a result from Friedgut [6]. In [6] the author gives a general necessary condition for a property to have a coarse threshold, namely that it can be well approximated by a local property. Although the main theorem in that paper refers to graphs, the proof extends to hypergraph-like settings where the number of isomorphism types of bounded size is bounded. This includes random hypergraphs, random SAT Boolean formulae, and also the model of random groups that we are addressing in the current paper. A different, but very similar tool that can be used here is Bourgain's theorem that appears in the appendix of [6], which has a weaker conclusion, but does not assume the symmetry of the property in question, such as we have in our current problem. To make things simpler we will use the "working-mathematicians-version" of these theorems, as described in Friedgut [7]. We present below the lemma we will use, stated in terms of the problem at hand, but first let us introduce some notation. For each value of n we denote by S the set S_n of generators, and assume that $S_n \subset S_{n+1}$, so that any fixed relator is meaningful for all sufficiently large values of n. Next, let $\Gamma(n, p)$ be given by a presentation $P = \langle S \mid R \rangle$, where R is random, and let R^* be a set of relators. We use the notation

$$h(n, p|R^*) := \Pr[\langle S \mid R \cup R^* \rangle \text{ is trivial}].$$

We will use this notation both for $R^* = R_{\text{fixed}} = \{r_1, \ldots, r_k\}$, a fixed set of cyclically reduced relators of length three, and for $R^* = R_{\epsilon p}$, a random set of relators chosen from S_n with probability ϵp (in which case the probability is over both the choice of R and of R^*). The following is an adaptation of Theorems 2.2, 2.3, and 2.4 from [7] to the current setting.

Lemma 3. *Assume that there exists a function $p = p(n)$, and constants $0 < \alpha$, $\epsilon < 1$, such that there exist infinitely many values of n for which it holds that*

$$\alpha < h(n, p) < h(n, (1 + \epsilon)p) < 1 - \alpha.$$

Then there exists a fixed (finite, independent of n but possibly dependent on α and ϵ) set $R_{\text{fixed}} = \{r_1, \ldots, r_k\}$ of cyclically reduced relators of length three, and a constant $\delta > 0$ such that for all such n

(i) $h(n, p|R_{\text{fixed}}) > h(n, p) + 2\delta$ *and*

(ii) $h(n, p|R_{\epsilon p}) < h(n, p) + \delta$.

We will now see how this lemma, together with the fact that $\Gamma(n, p)$ collapses when $p = n^{-3/2+o(1)}$ (see either Olliver [8], or Antoniuk, Łuczak and Świątkowski [2] and Theorem 2) implies Theorem 1.

Proof of Theorem 1. Assume, by way of contradiction, that Theorem 1 does not hold. Then the assumptions of Lemma 3 are met. Indeed, let $p_c = p_c(n)$ be defined so that $h(n, p_c) = 1/2$. Then specifically Theorem 1 does not hold with the choice of $\tilde{c}(n)n^{-3/2} = p_c$. Hence there exists an $\epsilon_0 > 0$ and a positive constant $\alpha < 1/2$ such that for infinitely many values of n either one has $h(n, (1 + \epsilon_0)p_c) < 1 - \alpha$, or $h(n, (1 - \epsilon_0)p_c) > \alpha$. In the first case $p = p_c, \alpha$, $\epsilon = \epsilon_0$ meet the assumptions of Lemma 3, in the latter one can take $p = (1-\epsilon_0)p_c$, α, and $\epsilon = \epsilon_0/(1 - \epsilon_0)$.

Now, let R_{fixed} be the set of relators guaranteed by Lemma 3, and let $Z := \{z_1, z_2, \ldots, z_\ell\}$ be the set of all generators involved in R_{fixed} and all of their inverses. Let R_{strong} be the following relation: $z_1 = z_2 = \cdots = z_\ell = e$, where e is the identity. Clearly

$$h(n, p|R_{\text{strong}}) \geq h(n, p|R_{\text{fixed}}).$$

Next, consider the graph $G = (V, E)$, where $V = (S \cup S^{-1}) \setminus Z$, and E consists of all pairs xy such that there is a relator in R which involves x, y and an element of Z (implying $x = y^{-1}$, since all elements in Z are set by R_{strong} to be equal to the identity). The probability that a given pair xy forms an edge is less than $6\ell p$, and these events are independent, so G can be coupled with the Erdős-Rényi random graph $G(2n - \ell, q)$ with $q = O(n^{-3/2+o(1)})$. Elementary first moment estimates imply that a.a.s. G has fewer than $n^{0.6}$ non-trivial components each of

them consisting of at most two edges. Indeed, the expected number of connected subgraphs with exactly 3 edges is bounded by

$$(2n)^3 q^3 + 16(2n)^4 q^3 = O(n^{-0.5}),$$

hence by Markov's inequality a.a.s. there are no such subgraphs and isolated edges and paths of length two are the only non-trivial components. Moreover, the expected number of edges in G can be bounded by

$$(2n)^2 q = O(n^{0.5}),$$

and again by Markov's inequality a.a.s. there are at most $n^{0.6}$ of them. Let R' be the set of relators in R that are disjoint from Z. Slightly abusing the notation we will also use E to denote the set of relators $\{xy: \{x, y\} \in E\}$. We have

$$h(n, p|R_{\text{strong}}) = \Pr[\langle S \setminus Z \mid R' \cup E \rangle \text{ is trivial}] + o(n^{-0.4}), \qquad (1)$$

where the $o(n^{-0.4})$ accounts for the case where there exists in R a relator involving two elements of Z. Note that both E and R' are random.

Now let us consider the effect of $R_{\epsilon p}$. First let us choose arbitrarily a set M, $|M| = m = \lfloor n^{1.9} \rfloor$, of pairs of generators $\{a, b\}, a, b \in S \cup S^{-1}$. Define a graph $G' = (V', E')$ with $V' = (S \cup S^{-1}) \setminus Z$ and E' consisting of all pairs xy such that $R_{\epsilon p}$ includes two relators of the form abx and aby^{-1}, where $\{a, b\} \in M$. Note that the existence of such two relators clearly implies that $x = y^{-1}$. Let X denote the number of paths of length two in $G' = (V', E')$. It is easy to see that for the expectation of X we have

$$\mathbb{E}X \geq 0.5(2n)^3 m^2 (\epsilon p)^4 = 4n^3 n^{3.8} n^{4(-3/2 + o(1))} = 4n^{0.8 - o(1)} \geq 4n^{0.75}.$$

It is also easy to check that the standard deviation of X is also of order $O(n^3 m^2 p^4)$, so from Chebyshev's inequality we infer that a.a.s. the number of such paths is larger than $3n^{0.75}$. On the other hand, let Y be the number of pairs of paths which share at least one vertex. The expectation of Y is dominated by the number of pairs which share one edge and is bounded from above by

$$\mathbb{E}Y \leq (2n)^4 m^3 (\epsilon p)^6 = 16n^4 n^{5.7} n^{6(-3/2 + o(1))} = 16n^{0.7 + o(1)}.$$

Thus, from Markov's inequality, a.a.s. the number of such pairs is of order smaller than $n^{0.75}$. Consequently, a.a.s. $G' = (V', E')$ contains at least $n^{0.75} \gg n^{0.6}$ disjoint paths of length two.

Now

$$h(n, p|R_{\epsilon p}) = \Pr[\langle S \setminus Z \mid R' \cup E' \rangle \text{ is trivial}] - o(n^{-0.4}), \qquad (2)$$

where the term $o(n^{-0.4})$ accounts for the fact that even if the group generated by $S \setminus Z$ collapses there are the generators in Z to account for. However, if all

generators in $S \setminus Z$ are set to be equal to the identity it suffices that for each element $z \in Z$ there will be in R a relator involving z and two elements of $S \setminus Z$. The probability of this event is at least as large as $1 - o(n^{-0.4})$ as n tends to infinity.

We have shown that G' contains at least $n^{0.1}$ edge-disjoint subgraphs isomorphic to the graph spanned by the edges of G, i.e. there is a coupling which shows that $E' \supseteq E$. Thus, the equations (1) and (2) contradict the items 1 and 2 in the conclusion of Lemma 3. Consequently, the hypothesis of the lemma cannot hold, and the property in question must have a sharp threshold.   □

3. Proof of Theorem 2

In order to show Theorem 2 we need to introduce a number of somewhat technical definitions. Let $P = \langle S \mid R \rangle$ be a group presentation. A *van Kampen diagram* with respect to the presentation P is a finite planar 2-cell complex $\mathcal{D}$ given with an embedding $\mathcal{D} \subseteq \mathbb{R}^2$ and satisfying the following conditions.

- $\mathcal{D}$ is connected and simply connected.
- For each edge e and one of its orientations we assign a generator $s \in S$. If we change the orientation of e to the opposite one, we replace the generator s by s^{-1}.
- Each 2-cell c is assigned a relator $r \in R$, the number of edges on the boundary of c is equal to the length of the relator r.
- For each 2-cell c there is a vertex v such that the word read from v in some direction of the boundary of the cell is the relator $r \in R$ assigned to c.

For a van Kampen diagram $\mathcal{D}$ the *size* of the diagram, denoted by $|\mathcal{D}|$, is the number of faces (2-cells) of $\mathcal{D}$. The *boundary* of $\mathcal{D}$, denoted by $\partial\mathcal{D}$, is the boundary of the complement of $\mathcal{D}$ in $\mathbb{R}^2$ and $|\partial\mathcal{D}|$ denotes its size, that is the number of edges in $\partial\mathcal{D}$. The *boundary word* is any word read from some vertex in $\partial\mathcal{D}$ in one of the directions around the boundary. In particular, the length of this word is precisely $|\partial\mathcal{D}|$.

A van Kampen diagram is said to be *reduced* if there is no pair of cells c and c' sharing at least one edge e, which are assigned the same relator r, and are such that if we read the word r on the boundaries of c and c' the edge e has the same orientation and corresponds to the same letter in the relator with respect to the starting point. A van Kampen diagram is said to be *minimal* if it is reduced and no other reduced van Kampen diagram with smaller number of faces has the same boundary word.

Let Γ be the group given by a presentation $P = \langle S \mid R \rangle$. In order to verify whether Γ is hyperbolic it is enough to consider minimal reduced van Kampen diagrams with respect to the presentation P and to show that they fulfill a certain geometric condition. In particular, it is known that a group generated by

a presentation $P = \langle S \mid R \rangle$ is hyperbolic if and only if there exists a coefficient $\delta > 0$ such that every minimal reduced van Kampen diagram $\mathcal{D}$ with respect to the presentation P satisfies the linear isoperimetric inequality $|\mathcal{D}| \leq \delta|\partial\mathcal{D}|$ (cf. [1]).

However, verifying that every reduced van Kampen diagram satisfies a certain isoperimetric inequality may turn out fairly hard since it requires showing that this inequality holds for all of them. At this point, the so called *local to global principle for hyperbolic geometry* (or *Cartan–Hadamard–Gromov–Papasoglu theorem*) (cf.[9]) comes to an aid. This principle states that it is enough to verify the isoperimetric inequality for a finite family of van Kampen diagrams.

Theorem 4 (Cartan–Hadamard–Gromov–Papasoglu). *Let $P = \langle S \mid R \rangle$ be a triangular group presentation. Assume that for some integer $K > 0$ every minimal reduced van Kampen diagram $\mathcal{D}$ with respect to P and of size $K^2/2 \leq |\mathcal{D}| \leq 240K^2$ satisfies the inequality*

$$|\mathcal{D}| \leq \frac{K}{200}|\partial\mathcal{D}|.$$

Then for every minimal reduced van Kampen diagram with respect to P the following isoperimetric inequality is true

$$|\mathcal{D}| \leq K^2|\partial\mathcal{D}|.$$

Following Ollivier [8], in order to simplify the verification of the isoperimetric condition for van Kampen diagrams, we introduce a k-labeled *decorated abstract van Kampen diagram* (davKd). For simplicity, we do it only for groups with triangular presentations. A k-labeled davKd is a finite planar 2-cell complex $\hat{\mathcal{D}}$ given with an embedding $\hat{\mathcal{D}} \subseteq \mathbb{R}^2$ and satisfying the following conditions:

- $\hat{\mathcal{D}}$ is connected and simply connected,
- each 2-cell c is a triangle with a label i from $\{1, \ldots, k\}$, with a marked vertex on its boundary and an orientation at this vertex,
- for all $i \in \{1, \ldots, k\}$ and for any 2-cell c labeled by i, starting from the marked vertex and going around according to prescribed orientation at this vertex, the edges of c get abstract labels i_1, i_2, i_3.

Now, let $P = \langle S \mid R \rangle$ be a triangular presentation and consider a one-to-one map $\phi\colon \{1, \ldots, k\} \to R$ which assigns relators to faces of $\hat{\mathcal{D}}$. Let $\phi_1(i), \phi_2(i), \phi_3(i)$ be the generators appearing on the first, second and third position of the relator $\phi(i)$ respectively. Then ϕ induces a map $\tilde{\phi}\colon \{i_r\}_{1 \leq i \leq k, 1 \leq r \leq 3} \to S$ which assigns to each abstract label i_r a generator from S, namely $\tilde{\phi}(i_r) = \phi_r(i)$. The map ϕ is called a *fulfillment map*, if additionally whenever there is an edge with two abstract labels i_r, j_s, then $\tilde{\phi}(i_r) = \tilde{\phi}(j_s)$. We say that a given davKd $\hat{\mathcal{D}}$ is *fulfillable*

with respect to the presentation $P = \langle S \mid R \rangle$ if there exists a fulfillment map $\phi\colon\{1,\ldots,k\} \to R$ for $\widehat{\mathcal{D}}$.

A davKd is said to be *reduced* if there is no pair of faces sharing at least one edge, which are assigned the same label i and have opposite orientations, and such that the common edge gets the same abstract label i_r from both faces. A davKd is said to be *minimal* if there is no other davKd with smaller number of faces and having the same boundary word (with respect to the abstract labels of edges).

Our aim is to show that for a function $f = f(n) = \omega/\log^{1/3} n$, where $\omega = \omega(n) \to \infty$ and $p = n^{-3/2-f}$, a.a.s. all minimal reduced van Kampen diagrams $\mathcal{D}$ with respect to the random presentation in $\Gamma(n, p)$ satisfy the isoperimetric inequality with a coefficient $\delta = \delta(n) = (200/f)^2$. But to do it, it is enough to verify this inequality for all minimal reduced k-labeled davKd's, so we show that the following statement holds.

Lemma 5. *Let $\omega = \omega(n) \to \infty$, $\omega < \log\log n$, $f = f(n) = \omega/\log^{1/3} n$, and $p = p(n) = n^{-3/2-f}$. Then a.a.s. for each minimal reduced k-labeled davKd $\widehat{\mathcal{D}}$, fulfillable with respect to $\Gamma(n, p)$, we have*

$$|\widehat{\mathcal{D}}| \leq (200/f)^2 |\partial\widehat{\mathcal{D}}|.$$

In particular, a.a.s. each minimal reduced van Kampen diagram $\mathcal{D}$ with respect to $\Gamma(n, p)$ satisfies the linear isoperimetric inequality

$$|\mathcal{D}| \leq (200/f)^2 |\partial\mathcal{D}|.$$

Proof. Let $f = f(n) = \omega/\log^{1/3} n$. From Theorem 4 it is enough to show that a.a.s. each given davKd $\widehat{\mathcal{D}}$ of size at most $|\widehat{\mathcal{D}}| \leq 240(200/f)^2$ satisfies the linear isoperimetric inequality with the coefficient $1/f$. We do it in two steps. First, we show that for any davKd $\widehat{\mathcal{D}}$ with size bounded by $|\widehat{\mathcal{D}}| \leq 240(200/f)^2$ one of the following two possibilities holds:

(i) $\widehat{\mathcal{D}}$ satisfies the isoperimetric inequality with the coefficient $1/f$;

(ii) the probability that $\widehat{\mathcal{D}}$ is fulfillable by $\Gamma(n, p)$ is bounded from above by $n^{-f/2}$.

Using this dichotomy, we then show that the probability that there is a bounded size fulfillable davKd $\widehat{\mathcal{D}}$ not satisfying the isoperimetric inequality in question goes to 0 with $n \to \infty$. Hence, a.a.s. all sufficiently small fulfillable davKd's satisfy this inequality.

Let $\widehat{\mathcal{D}}$ be a davKd with $m = |\widehat{\mathcal{D}}|$ faces having k distinct labels and with l_1 internal edges and $l_2 = |\partial\widehat{\mathcal{D}}|$ boundary edges. If each face is assigned a different label, i.e. each cell of $\widehat{\mathcal{D}}$ corresponds to a different relator, the probability that $\widehat{\mathcal{D}}$ is fulfillable is bounded above by $n^{l_1+l_2} p^m$. This is in fact a rather easy case and showing that for all diagrams with different labels and fulfillable in $\Gamma(n, p)$ a.a.s.

an isoperimetric inequality holds with a coefficient $1/f$ is rather straightforward. Indeed, assume that for a given $\widehat{\mathcal{D}}$ the isoperimetric inequality does not hold, that is $fm = f|\widehat{\mathcal{D}}| > |\partial\widehat{\mathcal{D}}| = l_2$. Notice also that $3m = 2l_1 + l_2 \geq 2l_1 + 1$ as there is at least one edge in the boundary of $\widehat{\mathcal{D}}$. Then the probability that $\widehat{\mathcal{D}}$ is fulfillable is bounded by $n^{l_1+l_2}p^m = n^{l_1+l_2}n^{(-3/2-f)m} \leq n^{-1/2}$. Moreover, as we will show later, the number of different davKd's $\widehat{\mathcal{D}}$ with $|\widehat{\mathcal{D}}| \leq 240(200/f)^2$ is of order much smaller than $n^{1/2}$, hence a.a.s. there are no such diagrams which are fulfillable and at the same time do not satisfy the isoperimetric inequality.

The main challenge is to deal with diagrams where some of the labels may appear more than once. On one hand, this reduces the number of distinct relators used to fulfill the diagram. On the other hand, this also imposes some restrictions on the generators used in this assignment. To control the influence of these two factors we follow an approach of Ollivier from [8]. To this end let m_i denote the number of faces labeled with i. Without loss of generality we may assume that $m_1 \geq m_2 \geq \cdots \geq m_k$. We want to count the probability that $\widehat{\mathcal{D}}$ is fulfillable with respect to the random presentation given by $\Gamma(n, p)$. We introduce an auxiliary graph $G = G(\widehat{\mathcal{D}})$ which captures all the constraints resulting from the structure of the davKd. The vertices of the graph G are the abstract labels $\{i_r\}_{1\leq i\leq k, 1\leq r\leq 3}$ and two vertices i_r, j_s are adjacent if there is an edge in $\widehat{\mathcal{D}}$ carrying labels i_r and j_s. We also define a family of induced subgraphs $G_1 \subset G_2 \subset \cdots \subset G_k$ of G, where G_l is a subgraph of G induced by vertices $\{i_r\}_{1\leq i\leq l, 1\leq r\leq 3}$. Let us remark that the main reason why this approach to davKd is so convenient is the fact that G contains no loops if and only if davKd is reduced.

Now, the number of connected components in the graph G is the total number of distinct generators which can appear in relators used in the fulfillment map ϕ for $\widehat{\mathcal{D}}$. In some sense, this gives us the number of degrees of freedom we have while choosing relators for the fulfillment map. Indeed, if two vertices are adjacent in G, then the corresponding abstract labels in $\widehat{\mathcal{D}}$ are mapped by $\tilde{\phi}$ to the same generator. Therefore, if we denote the number of connected components in the graph G by C, then we obtain the estimate

$$\Pr(\widehat{\mathcal{D}} \text{ is fulfillable}) \leq n^C p^k.$$

A similar argument works for the graphs G_l, which correspond to a partial assignment, namely we assign relators to faces bearing labels $1, \ldots, l$. Let C_l denote the number of connected components in G_l. Then

$$\Pr(\widehat{\mathcal{D}} \text{ is fulfillable}) \leq n^{C_l} p^l = n^{C_l - l(3/2+f)},$$

therefore putting

$$d_l = C_l - l(3/2 + f)$$

we get the estimate

$$\Pr(\widehat{\mathcal{D}} \text{ is fulfillable}) \leq n^{\min d_l}.$$

Thus, if for some l we have $d_l < -f/2$, then

$$\Pr(\widehat{\mathcal{D}} \text{ is fulfillable}) \le n^{-f/2}.$$

On the other hand, we claim that in the case of $\min d_l \ge -f/2$, the diagram $\widehat{\mathcal{D}}$ satisfies the isoperimetric inequality with the coefficient $1/f$. Indeed, as was observed by Ollivier [8] (see p. 613) one gets that

$$|\partial\widehat{\mathcal{D}}| \ge 3|\widehat{\mathcal{D}}|(1-2d) + 2\sum_{l=1}^{k} d_l(m_l - m_{l+1}),$$

where the parameter d is the density of the random triangular group, which in our notation is equal to $1/2 - f/3$. Thus

$$|\partial\widehat{\mathcal{D}}| \ge 2f|\widehat{\mathcal{D}}| + 2\sum_{l=1}^{k} d_l(m_l - m_{l+1}).$$

Next, observe that $m_l - m_{l+1} \ge 0$ for every l and $\sum m_l = |\widehat{\mathcal{D}}|$. Hence, if $\min d_l \ge -f/2$, then

$$|\partial\widehat{\mathcal{D}}| \ge 2f|\widehat{\mathcal{D}}| - f\sum_{l=1}^{k}(m_l - m_{l+1}) \ge f|\widehat{\mathcal{D}}|,$$

and we arrive at the desired isoperimetric inequality

$$|\widehat{\mathcal{D}}| \le \frac{1}{f}|\partial\widehat{\mathcal{D}}|. \tag{3}$$

   To complete our argument we use the local to global principle. In our case the coefficient K from Theorem 4 is equal to $200/f$. We need to show that the probability that there exists a diagram of size at most $240(200/f)^2$ violating the isoperimetric inequality (3) tends to 0. If this is the case, then the random presentation in the $\Gamma(n, p)$ model a.a.s. meets the assumptions of the local to global principle, hence a.a.s. each diagram satisfies the isoperimetric inequality with the coefficient $(200/f)^2$.

   First, we need to count the number of all possible davKd's with precisely m faces. To do this we take the number of all possible triangulations of a polygon which consist of exactly m triangles, and then for each triangle we choose the orientation in 2 ways, the starting point in 3 ways and the label of this face in m ways.

   A triangulation of a polygon with m triangles has at most $m + 2$ vertices. Thus, the number of such triangulations is bounded from above by the number of distinct triangulations $t(N)$ of a 2-dimensional sphere with N vertices, where $N \le m+3$,

which in turn we bound from above by α^m for some absolute constant $\alpha > 0$ (see Tutte [10]). Hence, the total number of davKd's with exactly m faces can be bounded by $\alpha^m \cdot 6^m \cdot m^m / m! \leq \beta^m$, where $\beta > 0$ is an appropriate constant. Therefore, the probability that a fulfillable davKd of size at most $240(200/f)^2$ violates the isoperimetric inequality (3) is at most

$$\sum_{m \leq 240(200/f)^2} \beta^m n^{-f/2} \leq \beta^{\frac{\gamma}{f^2}} n^{-f/2},$$

for some constant $\gamma > 0$. It is easy to verify that the right hand side of this inequality tends to 0 as $n \to \infty$ provided $f = f(n) = \omega / \log^{1/3} n$. Hence, a.a.s. for every davKd $\widehat{\mathcal{D}}$ with $|\widehat{\mathcal{D}}| \leq 240(200/f)^2$ fulfillable in $\Gamma(n, p)$ the isoperimetric inequality holds with a coefficient $1/f$ and so the assertion follows from Theorem 4. □

Proof of Theorem 2. As the group properties in question are monotone decreasing, it is enough to consider $p(n) = n^{-3/2 - \omega / \log^{1/3} n}$ with $\omega < \log \log n$. Observe first that a.a.s. $\Gamma(n, p)$ is aspherical, i.e. there exists no reduced spherical van Kampen diagram with respect to the random presentation $\Gamma(n, p)$. Indeed, such a spherical reduced van Kampen diagram has zero boundary, which violates the isoperimetric inequality proved in Lemma 5. Since $\Gamma(n, p)$ is aspherical, it is torsion-free (see, for instance, Brown [4], p. 187). Consequently, a.a.s. $\Gamma(n, p)$ is an infinite, hyperbolic group. □

Let us conclude with a remark that in order to show the conjecture we have to prove Theorem 2 with $\tau(n) = O(1/ \log n)$ instead of $\tau(n) = \omega / (\log n)^{1/3}$. Such an improvement seems to require a new approach and, perhaps, a stronger version of Theorem 4.

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