

Linear programming and the intersection of free subgroups in free products of groups

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Abstract. We study the intersection of finitely generated factor-free subgroups of free products of groups by utilizing the method of linear programming. For example, we prove that if H_1 is a finitely generated factor-free noncyclic subgroup of the free product $G_1 * G_2$ of two finite groups G_1, G_2 , then the WN-coefficient $\sigma(H_1)$ of H_1 is rational and can be computed in exponential time in the size of H_1 . This coefficient $\sigma(H_1)$ is the minimal positive real number such that, for every finitely generated factor-free subgroup H_2 of $G_1 * G_2$, it is true that $\bar{r}(H_1, H_2) \leq \sigma(H_1)\bar{r}(H_1)\bar{r}(H_2)$, where $\bar{r}(H) = \max(r(H) - 1, 0)$ is the reduced rank of H , $r(H)$ is the rank of H , and $\bar{r}(H_1, H_2)$ is the reduced rank of the generalized intersection of H_1 and H_2 . In the case of the free product $G_1 * G_2$ of two finite groups G_1, G_2 , it is also proved that there exists a factor-free subgroup $H_2^* = H_2^*(H_1)$ such that $\bar{r}(H_1, H_2^*) = \sigma(H_1)\bar{r}(H_1)\bar{r}(H_2^*)$, H_2^* has at most doubly exponential size in the size of H_1 , and H_2^* can be constructed in exponential time in the size of H_1 .

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1. Introduction

Let G_α , $\alpha \in I$, be some nontrivial groups and let $\mathcal{F} = \prod_{\alpha \in I}^* G_\alpha$ denote the free product of these groups. According to the classic Kurosh subgroup theorem [20] and [21], every subgroup H of \mathcal{F} is a free product $F(H) * \prod_{\alpha, \gamma}^* t_{\alpha, \gamma} H_{\alpha, \gamma} t_{\alpha, \gamma}^{-1}$, where $H_{\alpha, \gamma}$ is a subgroup of G_α , $t_{\alpha, \gamma} \in \mathcal{F}$, and $F(H)$ is a free subgroup of \mathcal{F} such that, for every $s \in \mathcal{F}$ and $\gamma \in I$, it is true that $F(H) \cap {}_s G_\gamma s^{-1} = \{1\}$. We say that H is a *factor-free* subgroup of \mathcal{F} if $H = F(H)$ in the above form of H , i.e., for every $s \in \mathcal{F}$ and $\gamma \in I$, we have $H \cap {}_s G_\gamma s^{-1} = \{1\}$. Let $r(F)$ denote the rank of a (finitely generated) free group F . Since a factor-free subgroup H of \mathcal{F} is free, the reduced rank $\bar{r}(H) := \max(r(H) - 1, 0)$ of H , where $r(H)$ is the rank of H , is well defined.

Let $q^* = q^*(G_\alpha, \alpha \in I)$ denote the minimum of orders > 2 of finite subgroups of groups G_α , $\alpha \in I$, and let $q^* := \infty$ if there are no such subgroups. It is clear that either q^* is an odd prime or $q^* \in \{4, \infty\}$. If $q^* = \infty$, define $\frac{q^*}{q^*-2} := 1$. Dicks and the author [6] proved that if H_1 and H_2 are finitely generated factor-free subgroups of \mathcal{F} , then

$$\bar{r}(H_1 \cap H_2) \leq 2 \frac{q^*}{q^* - 2} \bar{r}(H_1) \bar{r}(H_2). \quad (1)$$

Dicks and the author [6] conjectured that if groups G_α , $\alpha \in I$, contain no involutions, then the coefficient 2 could be left out and

$$\bar{r}(H_1 \cap H_2) \leq \frac{q^*}{q^* - 2} \bar{r}(H_1) \bar{r}(H_2). \quad (2)$$

This conjecture can be regarded as a far reaching generalization of the Hanna Neumann conjecture [23] on rank of the intersection of subgroups in free groups. Recall that the Hanna Neumann conjecture [23] claims that if H_1, H_2 are finitely generated subgroups of a free group, then $\bar{r}(H_1 \cap H_2) \leq \bar{r}(H_1) \bar{r}(H_2)$. For more discussion, partial results and proofs of this conjecture the reader is referred to [4], [5], [8], [18], [22], [24], [27], and [28].

The conjecture (2) is established by Dicks and the author [7] in the case when \mathcal{F} is the free product of two groups of order 3 in which case $q^* = 3$ and (2) turns into

$$\bar{r}(H_1 \cap H_2) \leq \frac{q^*}{q^* - 2} \bar{r}(H_1) \bar{r}(H_2) = 3 \bar{r}(H_1) \bar{r}(H_2).$$

Another special case in which the conjecture (2) is known to be true is the case when \mathcal{F} is the free product of infinite cyclic groups, i.e., \mathcal{F} is a free group, as follows from Friedman's [8] and Mineyev's [22] proofs of the Hanna Neumann conjecture, see also Dicks's proof [5]. In this case $q^* = \infty$ and the inequality (2) turns into

$$\bar{r}(H_1 \cap H_2) \leq \bar{r}(H_1) \bar{r}(H_2). \quad (3)$$

More generally, the inequality (3) also holds in the case when \mathcal{F} is the free product of right orderable groups as follows from results of Antolín, Martino, and Schwabrow [1], see also [18]. We mention that it follows from results of [6] that the conjectured inequality (2) is sharp and may not be improved.

In an attempt to improve on the bound (1) in a special case, Dicks and the author [7] showed that

$$\bar{r}(H_1 \cap H_2) \leq \left(2 - \frac{(4 + 2\sqrt{3})p}{(2p - 3 + \sqrt{3})^2}\right) \cdot \frac{p}{p - 2} \bar{r}(H_1)\bar{r}(H_2) \tag{4}$$

for finitely generated factor-free subgroups H_1, H_2 of the free product $C_p * C_p$ of two cyclic groups of prime order $p > 2$.

Note that for $p = 3$ the inequality (4) yields the conjectured inequality (2). However, for prime $p \geq 5$, the problem whether the inequality (2) holds for the free product of two cyclic groups of order p remains open and seems to be the most basic and appealing case of the conjecture (2) for groups with torsion. In this connection, we remark that the ideas of articles [1], [5], [8], and [22] do not look to be applicable to the case of free products with torsion and shed no light on the conjecture (2) for free products of groups with torsion, especially, for free products of finite groups.

In this article, however, we will not attempt to prove or improve on any upper bounds. Instead, we will look at generalized intersections of finitely generated factor-free subgroups in free products of groups from a disparate standpoint and prove results of quite different flavor by utilizing the method of linear programming.

First we recall a stronger version of the conjecture (2) that generalizes the strengthened Hanna Neumann conjecture which was put forward by Walter Neumann [24] for subgroups of free groups. Let H_1 and H_2 be finitely generated factor-free subgroups of an arbitrary free product $\mathcal{F} = \prod_{\alpha \in I}^* G_\alpha$ of groups G_α , $\alpha \in I$, let the number $\frac{q^*}{q^* - 2}$ be defined for \mathcal{F} as above, and let $S(H_1, H_2)$ denote a set of representatives of those double cosets $H_1 t H_2$ of \mathcal{F} , $t \in \mathcal{F}$, that have the property $H_1 \cap t H_2 t^{-1} \neq \{1\}$. Then the strengthened version of the conjecture (2) claims that

$$\bar{r}(H_1, H_2) := \sum_{s \in S(H_1, H_2)} \bar{r}(H_1 \cap s H_2 s^{-1}) \leq \frac{q^*}{q^* - 2} \bar{r}(H_1)\bar{r}(H_2), \tag{5}$$

where $\bar{r}(H_1, H_2)$ is the reduced rank of the generalized intersection of H_1 and H_2 consisting of subgroups $H_1 \cap s H_2 s^{-1}$, $s \in S(H_1, H_2)$.

Let $\mathcal{K}_{\text{ff}}(\mathcal{F})$ denote the set of all finitely generated noncyclic factor-free subgroups of the free product \mathcal{F} . Pick a subgroup $H_1 \in \mathcal{K}_{\text{ff}}(\mathcal{F})$. We will say that a

real number $\sigma(H_1) > 0$ is the *Walter Neumann coefficient* for H_1 , or, briefly, the WN-coefficient for H_1 , if, for every $H_2 \in \mathcal{K}_{\text{ff}}(\mathcal{F})$, we have

$$\bar{r}(H_1, H_2) \leq \sigma(H_1)\bar{r}(H_1)\bar{r}(H_2) \tag{6}$$

and $\sigma(H_1)$ is minimal with this property. Clearly,

$$\sigma(H_1) = \sup_{H_2} \left\{ \frac{\bar{r}(H_1, H_2)}{\bar{r}(H_1)\bar{r}(H_2)} \right\}$$

over all subgroups $H_2 \in \mathcal{K}_{\text{ff}}(\mathcal{F})$.

For every integer $d \geq 3$, we also define the number

$$\sigma_d(H_1) := \sup_{H_2} \left\{ \frac{\bar{r}(H_1, H_2)}{\bar{r}(H_1)\bar{r}(H_2)} \right\} \tag{7}$$

over all subgroups $H_2 \in \mathcal{K}_{\text{ff}}(\mathcal{F}, d)$, where $\mathcal{K}_{\text{ff}}(\mathcal{F}, d)$ is a subset of $\mathcal{K}_{\text{ff}}(\mathcal{F})$ consisting of those subgroups whose irreducible core graphs have all of its vertices of degree at most d , see Section 2 for definitions. This number $\sigma_d(H_1)$ is called the WN_d -coefficient for H_1 . Since $\mathcal{K}_{\text{ff}}(\mathcal{F}, d) \subseteq \mathcal{K}_{\text{ff}}(\mathcal{F}, d + 1)$, it follows from the definitions that $\sigma_d(H_1) \leq \sigma_{d+1}(H_1) \leq \sigma(H_1)$ and $\sup_d \{\sigma_d(H_1)\} = \sigma(H_1)$.

For example, it follows from results of [6] and [7] mentioned above that if $\mathcal{F} = C_p * C_p$ is the free product of two cyclic groups of prime order $p > 2$ and $H_1 \in \mathcal{K}_{\text{ff}}(\mathcal{F})$, then

$$\frac{p}{p-2} \leq \sigma_d(H_1) \leq \sigma(H_1) \leq \left(2 - \frac{(4 + 2\sqrt{3})p}{(2p - 3 + \sqrt{3})^2} \right) \cdot \frac{p}{p-2}.$$

The main technical result of this article is the following.

Theorem 1.1. *Suppose that $\mathcal{F} = G_1 * G_2$ is the free product of two nontrivial groups G_1, G_2 and H_1 is a finitely generated factor-free noncyclic subgroup of \mathcal{F} . Then the following are true.*

- (a) *For every integer $d \geq 3$, there exists a linear programming problem (LP-problem)*

$$\mathcal{P}(H_1, d) = \max\{c(d)x(d) \mid A(d)x(d) \leq b(d)\} \tag{8}$$

with integer coefficients whose solution is equal to $-\sigma_d(H_1)\bar{r}(H_1)$.

- (b) *There is a finitely generated factor-free subgroup H_2^* of \mathcal{F} , $H_2^* = H_2^*(H_1)$, such that H_2^* corresponds to a vertex solution of the dual problem*

$$\mathcal{P}^*(H_1, d) = \min\{b(d)^\top y(d) \mid A(d)^\top y(d) = c(d)^\top, y(d) \geq 0\}$$

of the primal LP-problem (8) of part (a) and

$$\bar{r}(H_1, H_2^*) = \sigma_d(H_1)\bar{r}(H_1)\bar{r}(H_2^*).$$

In particular, the WN_d -coefficient $\sigma_d(H_1)$ of H_1 is rational.

Furthermore, if $\Psi(H_1)$ and $\Psi(H_2^*)$ denote irreducible core graphs representing subgroups H_1 and H_2^* , resp., and $|E\Psi|$ is the number of oriented edges in the graph Ψ , then

$$|E\Psi(H_2^*)| < 2^{2^{|E\Psi(H_1)|/4 + \log_2 \log_2(4d)}}.$$

- (c) There exists a linear semi-infinite programming problem (LSIP-problem) $\mathcal{P}(H_1) = \sup\{cx \mid Ax \leq b\}$ with finitely many variables in x and with countably many constraints in the system $Ax \leq b$ whose dual problem

$$\mathcal{P}^*(H_1) = \inf\{b^\top y \mid A^\top y = c^\top, y \geq 0\}$$

has a solution equal to $-\sigma(H_1)\bar{r}(H_1)$.

- (d) Let the word problem for both groups G_1, G_2 be solvable and let an irreducible core graph $\Psi(H_1)$ of H_1 be given. Then the LP-problem (8) of part (a) can be algorithmically written down and the WN_d -coefficient $\sigma_d(H_1)$ for H_1 can be computed. In addition, an irreducible core graph $\Psi(H_2^*)$ of the subgroup H_2^* of part (b) can be algorithmically constructed.
- (e) Let both groups G_1 and G_2 be finite, let $d_m := \max(|G_1|, |G_2|) \geq 3$, and let an irreducible core graph $\Psi(H_1)$ of H_1 be given. Then the LP-problem (8) of part (a) for $d = d_m$ coincides with the LSIP-problem $\mathcal{P}(H_1)$ of part (c) and the WN -coefficient $\sigma(H_1)$ for H_1 is rational and computable.

It is worthwhile to mention that the correspondence between subgroups $H_2 \in \mathcal{X}_{\text{ff}}(\mathcal{F}, d)$ and vectors of the feasible polyhedron $\{y(d) \mid A(d)^\top y(d) = c(d)^\top, y(d) \geq 0\}$ of the dual problem $\mathcal{P}^*(H_1, d)$, mentioned in part (b) of Theorem 1.1, plays an important role in proofs and is reminiscent of the correspondence between (almost) normal surfaces in 3-dimensional manifolds and their (resp. almost) normal vectors in the Haken theory of normal surfaces and its generalizations, see [10], [11], [12], [16], and [19]. In particular, the idea of a vertex solution works equally well both in the context of almost normal surfaces [16], see also [11] and [19], and in the context of factor-free subgroups, providing in either situation both the connectedness of the underlying object associated with a vertex solution and an upper bound on the size of the underlying object.

Relying on the linear programming approach of Theorem 1.1, in the following Theorem 1.2, we look at the computational complexity of the problem to compute the WN -coefficient $\sigma(H_1)$ for a factor-free subgroup H_1 of the free product of two finite groups and at other relevant questions.

Theorem 1.2. *Suppose that $\mathcal{F} = G_1 * G_2$ is the free product of two nontrivial finite groups G_1, G_2 and H_1 is a subgroup of \mathcal{F} given by a finite generating set \mathcal{S} of words over the alphabet $G_1 \cup G_2$. Then the following are true.*

- (a) *In deterministic polynomial time in the size of \mathcal{S} , one can detect whether H_1 is factor-free and noncyclic and, if so, one can construct an irreducible graph $\Psi_o(H_1)$ of H_1 .*
- (b) *If H_1 is factor-free and noncyclic, then, in deterministic exponential time in the size of \mathcal{S} , one can write down and solve an LP-problem $\mathcal{P} = \max\{cx \mid Ax \leq b\}$ whose solution is equal to $-\sigma(H_1)\bar{r}(H_1)$. In particular, the WN-coefficient $\sigma(H_1)$ of H_1 is computable in exponential time in the size of \mathcal{S} .*
- (c) *If H_1 is factor-free and noncyclic, then there exists a finitely generated factor-free subgroup $H_2^* = H_2^*(H_1)$ of \mathcal{F} such that*

$$\bar{r}(H_1, H_2^*) = \sigma(H_1)\bar{r}(H_1)\bar{r}(H_2^*)$$

and the size of an irreducible core graph $\Psi(H_2^)$ of H_2^* is at most doubly exponential in the size of $\Psi(H_1)$. Specifically,*

$$|E\Psi(H_2^*)| < 2^{2^{|E\Psi(H_1)|/4 + \log_2 \log_2(4d_m)}},$$

where $\Psi(H_1)$ is an irreducible core graph of H_1 , $|E\Psi|$ denotes the number of oriented edges of the graph Ψ , and $d_m := \max(|G_1|, |G_2|)$.

In addition, an irreducible core graph $\Psi(H_2^)$ of H_2^* can be constructed in deterministic exponential time in the size of \mathcal{S} or $\Psi(H_1)$.*

It is of interest to observe that our construction of the graph $\Psi(H_2^*)$ is somewhat succinct (cf. the definition of succinct representations of graphs in [25]) in the sense that, despite the fact that the size of $\Psi(H_2^*)$ could be doubly exponential, we are able to give a description of $\Psi(H_2^*)$ in exponential time. In particular, vertices of $\Psi(H_2^*)$ are represented by exponentially long bit strings and edges of $\Psi(H_2^*)$ are drawn in blocks. As a result, we can find out in exponential time whether two given vertices of $\Psi(H_2^*)$ are connected by an edge.

The situation with free products of more than two factors is more difficult to study and we will make additional efforts to obtain the following results.

Theorem 1.3. *Suppose that $\mathcal{F} = \prod_{\alpha \in I}^* G_\alpha$ is the free product of nontrivial groups G_α , $\alpha \in I$, and H_1 is a finitely generated factor-free noncyclic subgroup of \mathcal{F} . Then there are two disjoint finite subsets I_1, I_2 of the index set I such that if $\hat{G}_1 := \prod_{\alpha \in I_1}^* G_\alpha$, $\hat{G}_2 := \prod_{\alpha \in I_2}^* G_\alpha$, and $\hat{\mathcal{F}} := \hat{G}_1 * \hat{G}_2$, then there exists a finitely generated factor-free subgroup \hat{H}_1 of $\hat{\mathcal{F}}$ with the following properties.*

- (a) $\bar{r}(\hat{H}_1) = \bar{r}(H_1)$, $\sigma_d(\hat{H}_1) \geq \sigma_d(H_1)$ for every $d \geq 3$, and $\sigma(\hat{H}_1) \geq \sigma(H_1)$. In particular, if the conjecture (5) fails for H_1 then the conjecture (5) also fails for \hat{H}_1 .
- (b) If the word problem for every group G_α , where $\alpha \in I_1 \cup I_2$, is solvable and a finite irreducible graph of H_1 is given, then the LP-problem $\mathcal{P}(\hat{H}_1, d)$ for \hat{H}_1 of part (a) of Theorem 1.1 can be algorithmically written down and the WN_d -coefficient $\sigma_d(\hat{H}_1)$ for \hat{H}_1 can be computed.
- (c) Let every group G_α , where $\alpha \in I_1 \cup I_2$, be finite, let H_1 be given either by a finite irreducible graph or by a finite generating set, and let

$$d_M := \max\{|I_1 \cup I_2|, \max\{|G_\alpha| \mid \alpha \in I_1 \cup I_2\}\}.$$

Then $\sigma_{d_M}(\hat{H}_1) \geq \sigma(H_1)$ and there is an algorithm that decides whether the conjecture (5) holds for H_1 .

We remark that the proofs of Theorems 1.2–1.3 provide a practical deterministic algorithm (with exponential running time, though) to compute the WN -coefficient $\sigma(H_1)$ for a finitely generated factor-free subgroup H_1 of the free product of two finite groups and to determine whether a certain finitely generated factor-free subgroup of a free product of finite groups satisfies the conjecture (5). It would be of interest to implement this algorithm and experiment with it.

The article is structured as follows. In Section 2, we define basic notions and recall geometric ideas that are used to study finitely generated factor-free subgroups and their intersections in a free product \mathcal{F} . In particular, we define a finite labeled graph $\Psi(H)$ associated with such a subgroup H of \mathcal{F} . In Section 3, we consider the free product $\mathcal{F} = G_1 * G_2$ of two nontrivial groups G_1, G_2 and introduce certain linear inequalities associated with the groups G_1, G_2 and with the graph $\Psi(H_1)$ of H_1 , where H_1 is a finitely generated factor-free noncyclic subgroup of \mathcal{F} . Informally, these inequalities are used for construction of cores of potential fiber product graphs $\Psi(H_1) \times \Psi(H_2)$, where H_2 is another finitely generated factor-free subgroup of \mathcal{F} , and for subsequent translation to linear programming problems. More formally, these inequalities enable us to define an LP-problem $\max\{c(d)x(d) \mid A(d)x(d) \leq b(d)\}$, corresponding to $\Psi(H_1)$ and to an integer $d \geq 3$, and to define an LSIP-problem $\sup\{cx \mid Ax \leq b\}$, corresponding to $\Psi(H_1)$. We also consider and make use of the dual problems of the primal problems

$$\max\{c(d)x(d) \mid A(d)x(d) \leq b(d)\}, \quad \sup\{cx \mid Ax \leq b\}.$$

Basic results and terminology of linear programming are discussed in Section 4. These LP-, LSIP-problems, and their dual problems are investigated in Sections 3 and 4. In Section 5, we look at the case of free products of more than two groups and prove a few more technical lemmas. Proofs of Theorems 1.1–1.3 are given in Section 6.

2. Preliminaries

Let G_α , $\alpha \in I$, be nontrivial groups, let $\mathcal{F} = \prod_{\alpha \in I}^* G_\alpha$ be their free product, and let H be a finitely generated factor-free subgroup of \mathcal{F} , $H \neq \{1\}$. Consider the alphabet $\mathcal{A} = \bigcup_{\alpha \in I} G_\alpha$, where $G_\alpha \cap G_{\alpha'} = \{1\}$ if $\alpha \neq \alpha'$.

Analogously to the graph-theoretic approach of articles [6], [7], [13], [14], [15], [17], and [18], we first define a labeled \mathcal{A} -graph $\Psi(H)$ which geometrically represents H in a manner similar to the way Stallings graphs represent subgroups of a free group, see [27].

If Γ is a graph, $V\Gamma$ denotes the vertex set of Γ and $E\Gamma$ denotes the set of oriented edges of Γ . For $e \in E\Gamma$ let e_- , e_+ denote the initial, terminal, resp., vertices of e and let e^{-1} be the edge with the opposite orientation, where $e^{-1} \neq e$ for every $e \in E\Gamma$, $(e^{-1})_- = e_+$, $(e^{-1})_+ = e_-$.

A *path* $p = e_1 \dots e_k$ in Γ is a sequence of edges e_1, \dots, e_k such that $(e_i)_+ = (e_{i+1})_-$, $i = 1, \dots, k-1$. Define $p_- := (e_1)_-$, $p_+ := (e_k)_+$, and $|p| := k$, where $|p|$ is the *length* of p . We allow the possibility that $p = \{p_-\} = \{p_+\}$ and $|p| = 0$. A path p is *closed* if $p_- = p_+$. A path p is called *reduced* if p contains no subpaths of the form ee^{-1} , $e \in E\Gamma$. A closed path $p = e_1 \dots e_k$ is *cyclically reduced* if $|p| > 0$ and both p and the cyclic permutation $e_2 \dots e_k e_1$ of p are reduced paths. The *core* of a graph Γ , denoted $\text{core}(\Gamma)$, is the minimal subgraph of Γ that contains every edge e which can be included into a cyclically reduced path in Γ .

Let Ψ be a graph whose vertex set $V\Psi$ consists of two disjoint nonempty parts $V_P\Psi, V_S\Psi$, so $V\Psi = V_P\Psi \cup V_S\Psi$. Vertices in $V_P\Psi$ are called *primary* and vertices in $V_S\Psi$ are called *secondary*. Every edge $e \in E\Psi$ connects primary and secondary vertices, hence, Ψ is a bipartite graph.

Ψ is called a *labeled \mathcal{A} -graph*, or briefly *\mathcal{A} -graph*, if Ψ is equipped with functions

$$\varphi: E\Psi \longrightarrow \mathcal{A}, \quad \theta: V_S\Psi \longrightarrow I$$

such that, for every edge $e \in E\Psi$, it is true that

$$\varphi(e) \in \mathcal{A} = \bigcup_{\alpha \in I} G_\alpha, \quad \varphi(e^{-1}) = \varphi(e)^{-1},$$

and, if $e_+ \in V_S\Psi$, then $\varphi(e) \in G_\alpha$, where $\alpha = \theta(e_+)$.

If $e_+ \in V_S\Psi$, define

$$\theta(e) := \theta(e_+), \quad \theta(e^{-1}) := \theta(e_+)$$

and call $\theta(e_+)$, $\theta(e)$ the *type* of a vertex $e_+ \in V_S\Psi$ and of an edge $e \in E\Psi$. Thus, for every $e \in E\Psi$, we have defined an element $\varphi(e) \in \mathcal{A}$, called the *label* of e , and an element $\theta(e) \in I$, called the *type* of e .

The reader familiar with van Kampen diagrams over a free product of groups, as defined in [21], will recognize that our labeling function $\varphi: E\Psi \rightarrow \mathcal{A}$ is defined in the way analogous to labeling functions on van Kampen diagrams over free products of groups. Recall that van Kampen diagrams are planar 2-complexes whereas graphs are 1-complexes, however, apart from this, the ideas of cancellations and edge folding work equally well for both diagrams and graphs.

An \mathcal{A} -graph Ψ is called *irreducible* if the following properties (P1)–(P3) hold true.

- (P1) If $e, f \in E\Psi$, $e_- = f_- \in V_P\Psi$, and $e_+ \neq f_+$, then $\theta(e) \neq \theta(f)$.
- (P2) If $e, f \in E\Psi$, $e \neq f$, and $e_+ = f_+ \in V_S\Psi$, then $\varphi(e) \neq \varphi(f)$ in $G_{\theta(e)}$.
- (P3) Ψ has no multiple edges, $\deg_{\Psi} v > 0$ for every $v \in V\Psi$, and there is at most one vertex of degree 1 in Ψ which, if exists, is primary.

Suppose Ψ is a connected finite irreducible \mathcal{A} -graph and a primary vertex $o \in V_P\Psi$ is distinguished so that $\deg_{\Psi} o = 1$ if Ψ happens to have a vertex of degree 1. Then o is called the *base* vertex of $\Psi = \Psi_o$.

As usual, elements of the free product $\mathcal{F} = \prod_{\alpha \in I}^* G_{\alpha}$ are regarded as words over the alphabet $\mathcal{A} = \bigcup_{\alpha \in I} G_{\alpha}$, where $G_{\alpha} \cap G_{\alpha'} = \{1\}$ if $\alpha \neq \alpha'$. A *syllable* of a word W over \mathcal{A} is a maximal nonempty subword of W all of whose letters belong to the same factor G_{α} . The *syllable length* $\|W\|$ of W is the number of syllables of W , while the *length* $|W|$ of W is the number of all letters in W . For example, if $a_1, a_2 \in G_{\alpha}$, then $|a_1 a_2| = 3$, $\|a_1 a_2\| = 1$, and $|1| = \|1\| = 1$.

A nonempty word W over \mathcal{A} is called *reduced* if every syllable of W consists of a single letter. Clearly, $|W| = \|W\|$ if W is reduced. Note that an arbitrary nontrivial element of the free product \mathcal{F} can be uniquely written as a reduced word. A word W is called *cyclically reduced* if W^2 is reduced. We write $U \stackrel{0}{=} W$ if words U, W are equal as elements of \mathcal{F} . The literal (or letter-by-letter) equality of words U, W is denoted $U \equiv W$.

If $p = e_1 \dots e_k$ is a path in an \mathcal{A} -graph Ψ and e_1, \dots, e_k are edges of Ψ , then the *label* $\varphi(p)$ of p is the word $\varphi(p) := \varphi(e_1) \dots \varphi(e_k)$.

The significance of irreducible \mathcal{A} -graphs for geometric interpretation of factor-free subgroups H of \mathcal{F} is given in the following lemma.

Lemma 2.1. *Suppose H is a finitely generated factor-free subgroup of the free product $\mathcal{F} = \prod_{\alpha \in I}^* G_{\alpha}$, $H \neq \{1\}$. Then there exists a finite connected irreducible \mathcal{A} -graph $\Psi = \Psi_o(H)$, with a base vertex o , such that a reduced word W over the alphabet \mathcal{A} belongs to H if and only if there is a reduced path p in $\Psi_o(H)$ such that $p_- = p_+ = o$, $\varphi(p) \stackrel{0}{=} W$ in \mathcal{F} , and $|p| = 2|W|$.*

In addition, assume that all factors G_α , $\alpha \in I$, are finite and V_1, \dots, V_k are words over \mathcal{A} . Then there is a deterministic algorithm which, in polynomial time depending on the sum $|V_1| + \dots + |V_k|$, decides whether the subgroup $H_V = \langle V_1, \dots, V_k \rangle$, generated by V_1, \dots, V_k , is factor-free and, if so, constructs an irreducible \mathcal{A} -graph $\Psi_o(H_V)$ for H_V .

Proof. The proof is based on Stallings’s folding techniques and is somewhat analogous to the proof of van Kampen lemma for diagrams over free products of groups, see [21] (in fact, it is simpler because foldings need not preserve the property of being planar for diagrams).

Let $H_V = \langle V_1, \dots, V_k \rangle$ be a subgroup of \mathcal{F} , generated by some words V_1, \dots, V_k over \mathcal{A} . Without loss of generality we can assume that V_1, \dots, V_k are reduced words. Consider a graph $\tilde{\Psi}$ which consists of k closed paths p_1, \dots, p_k such that they have a single common vertex $o = (p_i)_-$, and $|p_i| = 2|V_i|$, $i = 1, \dots, k$. We distinguish o as the base vertex of $\tilde{\Psi}$ and call o primary, the vertices adjacent to o are called secondary vertices and so on. Denote $V \equiv a_{i,1} \dots a_{i,\ell_i}$, where $a_{i,j} \in \mathcal{A}$ are letters, $i = 1, \dots, k$, and let $p_i = e_{i,1} f_{i,1} \dots e_{i,\ell_i} f_{i,\ell_i}$, where $e_{i,j}, f_{i,j}$ are edges of the path p_i . The labeling functions φ, θ on the path p_i are defined so that if $a_{i,j} \in G_{\alpha(i,j)}$, then

$$\begin{aligned} \theta(e_{i,j}) &:= \alpha(i, j), & \theta(f_{i,j}) &:= \alpha(i, j), \\ \varphi(e_{i,j}) &:= a_{i,j} b_{i,j}^{-1}, & \varphi(f_{i,j}) &:= b_{i,j}, \end{aligned}$$

where $b_{i,j}$ is an element of the group $G_{\alpha(i,j)}$.

Clearly, $\varphi(p_i) \stackrel{0}{=} V_i$ in \mathcal{F} for all $i = 1, \dots, k$.

It is also clear that $\tilde{\Psi} = \tilde{\Psi}_o$ is a finite connected \mathcal{A} -graph with the base vertex o that has the following property.

(Q) A word $W \in \mathcal{F}$ belongs to H if and only if there is a path p in $\tilde{\Psi}_o$ such that $p_- = p_+ = o$ and $\varphi(p) \stackrel{0}{=} W$.

However, $\tilde{\Psi}_o$ need not be irreducible and we will do foldings of edges in $\tilde{\Psi}_o$ which preserve property (Q) and which are aimed to achieve properties (P1) and (P2).

Assume that property (P1) fails for edges e, f with $e_- = f_- \in V_P \tilde{\Psi}_o$ so that $e_+ \neq f_+$ and $\theta(e) = \theta(f)$. Let us redefine the labels of all edges e' with $e'_+ = e_+$ so that $\varphi(e')\varphi(e)^{-1}$ does not change and $\varphi(e) = \varphi(f)$ in $G_{\theta(e)}$. This can be done by multiplication of φ -labels on the right by $\varphi(e)^{-1}\varphi(f)$. Since $\varphi(e) = \varphi(f)$ and $\theta(e) = \theta(f)$, we may now identify the edges e, f and vertices e_+, f_+ . Observe that this folding preserves property (Q) ((P2) might fail) and decreases the total edge number $|E\tilde{\Psi}_o|$. This operation changes the labels of edges and can be done in time polynomial in $|V_1| + \dots + |V_k|$ if all factors G_α , $\alpha \in I$, are finite. Note that

if G_α were not finite, then there would be a problem with increasing space needed to store φ -labels of edges and subsequent computations with larger labels.

If property (P2) fails for edges e, f and $\varphi(e) = \varphi(f)$ in $G_{\theta(e)}$, then we identify the edges e, f . Note property (Q) still holds ((P1) might fail) and the number $|E\tilde{\Psi}_o|$ decreases.

Suppose property (P3) fails and there are two distinct edges e, f in $\tilde{\Psi}_o$ such that $e_- = f_-$ and $e_+ = f_+ \in V_S\tilde{\Psi}_o$. If $\varphi(e) \neq \varphi(f)$ in $G_{\theta(e)}$, then a conjugate of $\varphi(e)\varphi(f)^{-1} \in G_{\theta(e)}$ is in H_V , hence we conclude that H_V is not factor-free. So we may assume that $\varphi(e) = \varphi(f)$ in $G_{\theta(e)}$. Then we identify the edges e, f , thus preserving property (Q) and decreasing the number $|E\tilde{\Psi}_o|$. If property (P3) fails so that there is a vertex v of degree 1, different from o , then we delete v along with the incident edge. Clearly, property (Q) still holds and the number $|E\tilde{\Psi}_o|$ decreases.

Thus, by induction on $|E\tilde{\Psi}_o|$ in polynomially many (relative to $\sum_{i=1}^k |V_i|$) steps as described above, we either establish that the subgroup H_V is not factor-free or construct an irreducible \mathcal{A} -graph Ψ_o with property (Q).

It follows from the definitions and from property (Q) of the graph Ψ_o that H_V is factor-free (see also Lemma 2.2). Other stated properties of Ψ_o are straightforward.

Finally, we observe that if all factors $G_\alpha, \alpha \in I$, are finite, then the space required to store the φ -label of an edge of intermediate graphs is constant and multiplication (or inversion) of φ -labels would require time bounded by a constant. Therefore, the above procedure implies the existence of a polynomial algorithm for finding out whether a subgroup $H_V = \langle V_1, \dots, V_k \rangle$ of \mathcal{F} is factor-free and for construction of a finite irreducible \mathcal{A} -graph Ψ_o for H_V . \square

The following lemma further elaborates on the correspondence between finitely generated factor-free subgroups of the free product \mathcal{F} and finite irreducible \mathcal{A} -graphs.

Lemma 2.2. *Let Ψ_o be a finite connected irreducible \mathcal{A} -graph with the base vertex o and let $H = H(\Psi_o)$ be a subgroup of the free product \mathcal{F} that consists of all words $\varphi(p)$, where p is a path in Ψ_o such that $p_- = p_+ = o$. Then H is a factor-free subgroup of \mathcal{F} and $\bar{\chi}(H) = -\chi(\Psi_o)$, where*

$$\chi(\Psi_o) = |V\Psi_o| - \frac{1}{2}|E\Psi_o|$$

is the Euler characteristic of Ψ_o .

Proof. This follows from the facts that the fundamental group $\pi_1(\Psi_o, o)$ of Ψ_o at o is free of rank $-\chi(\Psi_o) + 1$ and that the homomorphism $\pi_1(\Psi_o, o) \rightarrow \mathcal{F}$, given by $p \rightarrow \varphi(p)$, where p is a path with $p_- = p_+ = o$, has the trivial kernel in view of properties (P1) and (P2). \square

Suppose H is a nontrivial finitely generated factor-free subgroup of a free product $\mathcal{F} = \prod_{\alpha \in I}^* G_\alpha$, and $\Psi_o = \Psi_o(H)$ is a finite irreducible \mathcal{A} -graph for H as in Lemma 2.1. We say that $\Psi_o(H)$ is an *irreducible graph* of H .

Let $\Psi(H) := \text{core}(\Psi_o(H))$ denote the core of an irreducible graph $\Psi_o(H)$ of H . Clearly, $\Psi(H)$ has no vertices of degree ≤ 1 and $\Psi(H)$ is also an irreducible \mathcal{A} -graph. We say that $\Psi(H)$ is an *irreducible core graph* of H .

It is easy to see that an irreducible graph $\Psi_o(H)$ of H can be obtained back from an irreducible core graph $\Psi(H)$ of H by attaching a suitable path p to $\Psi(H)$ so that p starts at a primary vertex o , ends in $p_+ \in V_P \Psi(H)$, and then by doing foldings of edges as in the proof of Lemma 2.1, see Figure 2.1.

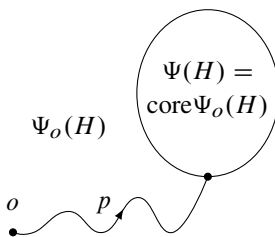


Figure 2.1

Now suppose H_1, H_2 are nontrivial finitely generated factor-free subgroups of \mathcal{F} . Consider a set $S(H_1, H_2)$ of representatives of those double cosets $H_1 t H_2$ of \mathcal{F} , $t \in \mathcal{F}$, that have the property $H_1 \cap t H_2 t^{-1} \neq \{1\}$. For every $s \in S(H_1, H_2)$, define the subgroup $K_s := H_1 \cap s H_2 s^{-1}$. Similarly to articles [13], [14], [15], [17], and [18] and analogously to the case of free groups, see [4] and [24], we now construct a finite irreducible \mathcal{A} -graph $\Psi(H_1, H_2)$, also denoted $\text{core}(\Psi(H_1) \times \Psi(H_2))$, whose connected components are irreducible core graphs $\Psi(K_s)$, $s \in S(H_1, H_2)$.

First we define an \mathcal{A} -graph $\Psi'_o(H_1, H_2)$. The set of primary vertices of $\Psi'_o(H_1, H_2)$ is $V_P \Psi'_o(H_1, H_2) := V_P \Psi_{o_1}(H_1) \times V_P \Psi_{o_2}(H_2)$. Let

$$\tau_i: V_P \Psi'_o(H_1, H_2) \longrightarrow V_P \Psi_{o_i}(H_i)$$

denote the projection map, $\tau_i((v_1, v_2)) = v_i, i = 1, 2$.

The set of secondary vertices $V_S \Psi'_o(H_1, H_2)$ of $\Psi'_o(H_1, H_2)$ consists of equivalence classes $[u]_\alpha$, where $u \in V_P \Psi'_o(H_1, H_2)$, $\alpha \in I$, with respect to the minimal equivalence relation generated by the following relation $\overset{\alpha}{\sim}$ on the set $V_P \Psi'_o(H_1, H_2)$. Define $v \overset{\alpha}{\sim} w$ if and only if there are edges $e_i, f_i \in E \Psi_{o_i}(H_i)$ such that

$$(e_i)_- = \tau_i(v), (f_i)_- = \tau_i(w), (e_i)_+ = (f_i)_+$$

for each $i = 1, 2$, the edges e_i, f_i have type α , and $\varphi(e_1)\varphi(f_1)^{-1} = \varphi(e_2)\varphi(f_2)^{-1}$ in G_α . It is easy to see that the relation $\overset{\sim}{\sim}$ is symmetric and transitive on pairs and triples of distinct elements (but it could lack the reflexive property).

The edges in $\Psi'_o(H_1, H_2)$ are defined so that the vertices

$$u \in V_P\Psi'_o(H_1, H_2) \quad \text{and} \quad [v]_\alpha \in V_S\Psi'_o(H_1, H_2)$$

are connected by an edge if and only if $u \in [v]_\alpha$.

The type $\theta([v]_\alpha)$ of a vertex $[v]_\alpha \in V_S\Psi'_o(H_1, H_2)$ is α and if

$$e \in E\Psi'_o(H_1, H_2), \quad e_- = u, \quad e_+ = [v]_\alpha,$$

then $\varphi(e) := \varphi(e_1)$, where $e_1 \in E\Psi_{o_1}(H_1)$ is an edge of type α with $(e_1)_- = \tau_1(u)$, when such an e_1 exists, and $\varphi(e_1) := g_\alpha$, where $g_\alpha \in G_\alpha, g_\alpha \neq 1$, otherwise.

It follows from the definitions and properties (P1) and (P2) of $\Psi_{o_i}(H_i), i = 1, 2$, that $\Psi'_o(H_1, H_2)$ is an \mathcal{A} -graph with properties (P1) and (P2). Hence, taking the core of $\Psi'_o(H_1, H_2)$, we obtain a finite irreducible \mathcal{A} -graph which we denote by $\Psi(H_1, H_2)$ or by $\text{core}(\Psi(H_1) \times \Psi(H_2))$.

It is not difficult to see that, when taking the connected component

$$\Psi'_o(H_1, H_2, o)$$

of $\Psi'_o(H_1, H_2)$ that contains the vertex $o = (o_1, o_2)$ and inductively removing from $\Psi'_o(H_1, H_2, o)$ the vertices of degree 1 different from o , we will obtain an irreducible \mathcal{A} -graph $\Psi_o(H_1 \cap H_2)$ with the base vertex o that corresponds to the intersection $H_1 \cap H_2$ as in Lemma 2.1.

It follows from the definitions and property (P1) for $\Psi(H_i), i = 1, 2$, that, for every edge $e \in E\Psi(H_1, H_2)$ with $e_- \in V_P\Psi(H_1, H_2)$, there are unique edges $e_i \in E\Psi(H_i)$ such that $\tau_i(e_-) = (e_i)_-, i = 1, 2$. Hence, by setting $\tau_i(e) = e_i, \tau_i(e_+) = (e_i)_+, i = 1, 2$, we extend τ_i to the graph map

$$\tau_i: \Psi(H_1, H_2) \longrightarrow \Psi(H_i), \quad i = 1, 2. \tag{9}$$

It follows from the definitions that τ_i is locally injective and τ_i preserves syllables of the word $\varphi(p)$ for every path p with primary vertices p_-, p_+ .

Lemma 2.3. *Suppose H_1, H_2 are finitely generated factor-free subgroups of the free product \mathcal{F} and $S(H_1, H_2) \neq \emptyset$. Then the connected components of the graph $\Psi(H_1, H_2)$ are core graphs $\Psi(H_1 \cap sH_2s^{-1})$ of subgroups $H_1 \cap sH_2s^{-1}, s \in S(H_1, H_2)$. In particular,*

$$\bar{r}(H_1, H_2) := \sum_{s \in S(H_1, H_2)} \bar{r}(H_1 \cap sH_2s^{-1}) = -\chi(\Psi(H_1, H_2)).$$

Proof. This is straightforward, details can be found in [18]. □

3. The system of linear inequalities SLI[Y₁]

In this section, we let $\mathcal{F}_2 = G_1 * G_2$ be the free product of two nontrivial groups G_1, G_2 , let $\mathcal{A} := G_1 \cup G_2$ be the alphabet, $G_1 \cap G_2 = \{1\}$, and let Y_1 be a finite connected irreducible \mathcal{A} -graph such that $\text{core}(Y_1) = Y_1$ and $\bar{r}(Y_1) := -\chi(Y_1) > 0$. In particular, Y_1 has no vertices of degree 1 and Y_1 contains a vertex of degree > 2 .

Let $S_2(G_\alpha)$, where $\alpha = 1, 2$, denote the set of all finite subsets of G_α of cardinality ≥ 2 and let $S_1(V_P Y_1)$ denote the set of all nonempty subsets of $V_P Y_1$. For a set $T \in S_2(G_\alpha)$, consider a function

$$\Omega_T: T \longrightarrow S_1(V_P Y_1).$$

We also consider a relation \sim_{Ω_T} on the set of all pairs (a, u) , where $a \in T$ and $u \in \Omega_T(a)$, defined as follows. Two pairs $(a, u), (b, v)$ are related by \sim_{Ω_T} , written $(a, u) \sim_{\Omega_T} (b, v)$, if and only if the following holds. Either $(a, u) = (b, v)$ or, otherwise, there exist edges $e, f \in EY_1$ with the properties that $e_- = u, f_- = v$, the secondary vertex $e_+ = f_+$ has type α , and $\varphi(e)\varphi(f)^{-1} = ab^{-1}$ in G_α , see an example depicted in Figure 3.1. It is easy to see that the relation \sim_{Ω_T} is an equivalence one.

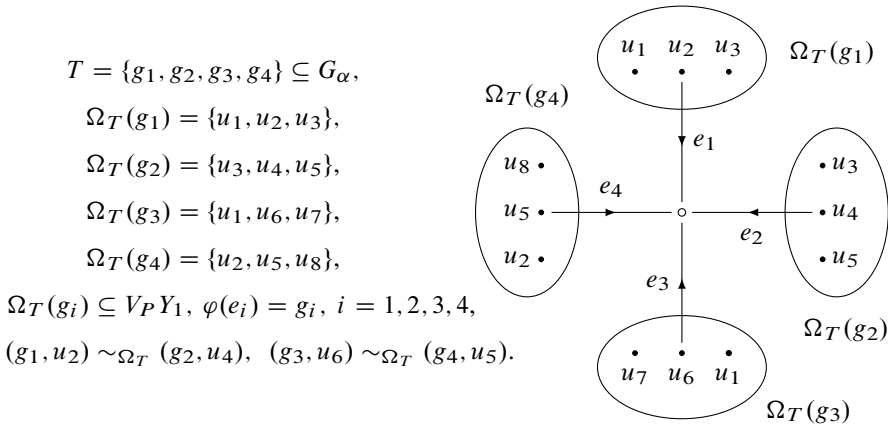


Figure 3.1

Let $[(a, u)]_{\sim_{\Omega_T}}$ denote the equivalence class of (a, u) and let

$$|[(a, u)]_{\sim_{\Omega_T}}|$$

denote the cardinality of $[(a, u)]_{\sim_{\Omega_T}}$. It follows from the definitions that

$$1 \leq |[(a, u)]_{\sim_{\Omega_T}}| \leq |T|. \tag{10}$$

We will say that the equivalence class $[(a, u)]_{\sim_{\Omega_T}}$ is associated with a secondary vertex $w \in V_S Y_1$ of type α if $w = e_+$, where $e \in E Y_1$ and $e_- = u$. It is easy to see that the definition of the secondary vertex w is independent of the primary vertex u in $[(a, u)]_{\sim_{\Omega_T}}$.

A function $\Omega_T: T \rightarrow S_1(V_P Y_1)$, $T \in S_2(G_\alpha)$, is called α -admissible if

$$|[[(a, u)]_{\sim_{\Omega_T}}| \geq 2$$

for every equivalence class $[(a, u)]_{\sim_{\Omega_T}}$, where $a \in T$, $u \in \Omega_T(a)$. The set of all α -admissible functions is denoted $\Omega(Y_1, \alpha)$, $\alpha = 1, 2$.

Let $\Omega_T \in \Omega(Y_1, \alpha)$ be an α -admissible function, $T \in S_2(G_\alpha)$, and let

$$N_\alpha(\Omega_T) := \sum (|[[(a, u)]_{\sim_{\Omega_T}}| - 2) \tag{11}$$

denote the sum of cardinalities minus two over all equivalence classes $[(a, u)]_{\sim_{\Omega_T}}$, where $a \in T$ and $u \in \Omega_T(a)$, of the equivalence relation \sim_{Ω_T} .

Let r be the number of all equivalence classes $[(a, u)]_{\sim_{\Omega_T}}$ of the equivalence relation \sim_{Ω_T} , where $a \in T$ and $u \in \Omega_T(a)$. If $r \geq |V_P Y_1|$, then it follows from (11) and definitions that

$$N_\alpha(\Omega_T) = \sum (|[[(a, u)]_{\sim_{\Omega_T}}| - 2) \leq |T| \cdot |V_P Y_1| - 2r \leq (|T| - 2)|V_P Y_1|.$$

On the other hand, if $r \leq |V_P Y_1|$, then it follows from (10) and (11) that

$$N_\alpha(\Omega_T) = \sum (|[[(a, u)]_{\sim_{\Omega_T}}| - 2) \leq (|T| - 2)r \leq (|T| - 2)|V_P Y_1|.$$

Thus, in any case, it is shown that

$$N_\alpha(\Omega_T) \leq (|T| - 2)|V_P Y_1|. \tag{12}$$

For every set $A \in S_1(V_P Y_1)$, we consider a variable x_A . We also introduce a special variable x_s . Now we will define a system of linear inequalities in these variables.

For every α -admissible function Ω_T , where $T \in S_2(G_\alpha)$ and $\alpha = 1, 2$, we denote $T = \{b_1, \dots, b_k\}$ and we set $A_i := \Omega_T(b_i)$, $i = 1, \dots, k$.

If $\alpha = 1$, then the inequality, corresponding to the α -admissible function Ω_T , is defined as follows.

$$-x_{A_1} - \dots - x_{A_k} - (k - 2)x_s \leq -N_1(\Omega_T). \tag{13}$$

If $\alpha = 2$, then the inequality corresponding to the α -admissible function Ω_T is defined as follows.

$$x_{A_1} + \dots + x_{A_k} - (k - 2)x_s \leq -N_2(\Omega_T). \tag{14}$$

Let

$$\text{SLI}[Y_1] \tag{15}$$

denote the system of linear inequalities (13)–(14) over all α -admissible functions Ω_T , where $\Omega_T \in \Omega(Y_1, \alpha)$ and $\alpha = 1, 2$. Since the set $S_2(G_\alpha)$ is in general infinite (unless G_α is finite) and the set $S_1(V_P Y_1)$ is finite (because Y_1 is finite), it follows that $\text{SLI}[Y_1]$ is an infinite system of linear inequalities with integer coefficients over a finite set of variables $x_A, A \in S_1(V_P Y_1), x_s$.

Let $d \geq 3$ be an integer and let

$$\text{SLI}_d[Y_1] \tag{16}$$

denote the subsystem of the system (15) whose linear inequalities (13)–(14) are defined for all α -admissible functions Ω_T , where $\Omega_T \in \Omega(Y_1, \alpha)$ and $\alpha = 1, 2$, such that $|T| \leq d$.

If q is an inequality of $\text{SLI}_d[Y_1]$ then the coefficient of x_s in the left hand side of q is the integer $-k + 2$, where

$$2 \leq k = k(q) = |T| \leq d,$$

and the right hand side of q is the integer $-N_\alpha(\Omega_T)$, where

$$0 \leq N_\alpha(\Omega_T) \leq (d - 2)|V_P Y_1|,$$

as follows from inequality (12). Also, the number of subsets $A \subseteq V_P Y_1$ that index variables $\pm x_A$ in the left hand side of q is finite and the total number of occurrences of such variables $\pm x_A$ in q is $k = |T| \leq d$. Therefore, $\text{SLI}_d[Y_1]$ is a finite system of linear inequalities and

$$\text{SLI}[Y_1] = \bigcup_{d=3}^{\infty} \text{SLI}_d[Y_1].$$

Consider the following property of a graph Y_2 (which need not be connected).

- (B) Y_2 is a finite irreducible \mathcal{A} -graph, the map $\tau_2: \text{core}(Y_1 \times Y_2) \rightarrow Y_2$ is surjective, $\text{core}(Y_2) = Y_2$, and $\bar{r}(Y_2) := -\chi(Y_2) > 0$.

For example, Y_1 has property (B).

If Γ is a finite graph, let $\text{deg } \Gamma$ denote the maximum of degrees of vertices of Γ . Recall that the degree of a vertex $v \in V\Gamma$ is the number of edges $e \in E\Gamma$ such that $e_+ = v$. For later references, we introduce one more property of a graph Y_2 .

- (Bd) Y_2 has property (B) and $\text{deg } Y_2 \leq d$, where $d \geq 3$ is an integer.

Suppose Y_2 is a graph with property (B). For a secondary vertex $u \in V_S Y_2$ of type α , we consider all edges e_1, \dots, e_ℓ , where $\deg u = \ell$, such that

$$u = (e_1)_+ = \dots = (e_\ell)_+$$

and denote $v_j := (e_j)_-, j = 1, \dots, \ell$. Define

$$T_u := \{\varphi(e_1), \dots, \varphi(e_\ell)\}.$$

Clearly, $T_u \subseteq S_2(G_\alpha)$. For every $j = 1, \dots, \ell$, let $\tau_2^{-1}(v_j)$ denote the full preimage of the vertex v_j in $\text{core}(Y_1 \times Y_2)$. Define the sets

$$A_j(u) := \tau_1 \tau_2^{-1}(v_j) \subseteq V_P Y_1 \tag{17}$$

for $j = 1, \dots, \ell$ and consider the function

$$\Omega_{T_u}: T_u \longrightarrow S_1(V_P Y_1) \tag{18}$$

so that $\Omega_{T_u}(\varphi(e_j)) := A_j(u)$.

It is easy to check that Ω_{T_u} is α -admissible. Since every α -admissible function $\Omega \in \Omega(Y_1, \alpha)$ gives rise to an inequality (13) if $\alpha = 1$ or to an inequality (14) if $\alpha = 2$ and every secondary vertex $u \in V_S Y_2$ of type α defines, as indicated above, an α -admissible function Ω_{T_u} , it follows that every $u \in V_S Y_2$ is mapped to a certain inequality of the system $\text{SLI}[Y_1]$, denoted $\text{inq}_S(u)$. Thus we obtain a function

$$\text{inq}_S: V_S Y_2 \longrightarrow \text{SLI}[Y_1] \tag{19}$$

defined from the set $V_S Y_2$ of secondary vertices of a finite irreducible \mathcal{A} -graph Y_2 with property (B) to the set of inequalities of $\text{SLI}[Y_1]$.

If q is an inequality of the system $\text{SLI}[Y_1]$, denoted $q \in \text{SLI}[Y_1]$, we let q^L denote the left hand side of q , let q^R denote the integer in the right hand side of q and let $k(q) \geq 2$ denote the parameter k for q , see the definition of inequalities (13) and (14).

Lemma 3.1. *Suppose Y_2 is a finite irreducible \mathcal{A} -graph such that the map $\tau_2: \text{core}(Y_1 \times Y_2) \rightarrow Y_2$ is surjective, $\text{core}(Y_2) = Y_2$ and $\deg Y_2 \leq d$. Then $\text{inq}_S(V_S Y_2) \subseteq \text{SLI}_d[Y_1]$. Furthermore,*

$$\sum_{u \in V_S Y_2} \text{inq}_S(u)^L = -2\bar{r}(Y_2)x_s,$$

$$\sum_{u \in V_S Y_2} \text{inq}_S(u)^R = -2\bar{r}(\text{core}(Y_1 \times Y_2)).$$

Proof. The inclusion $\text{inq}_S(V_S Y_2) \subseteq \text{SLI}_d[Y_1]$ is evident from the definitions.

Suppose $v \in V_P Y_2$ and let e_1, e_2 be the edges such that $(e_1)_- = (e_2)_- = v$ and $u_\alpha := (e_\alpha)_+, \alpha = 1, 2$, is a secondary vertex of type α in Y_2 . Clearly, $\varphi(e_\alpha) \in G_\alpha$ for $\alpha = 1, 2$. Denote $A_v := \tau_1 \tau_2^{-1}(v)$. It follows from the definitions that $A_v \in S_1(V_P Y_1)$ and that the variables $-x_{A_v}, x_{A_v}$ occur in $\text{inq}_S(u_1)^L, \text{inq}_S(u_2)^L$, resp., and will cancel out in the sum $\text{inq}_S(u_1)^L + \text{inq}_S(u_2)^L$. It is easy to see that all occurrences of variables $\pm x_A, A \in S_1(V_P Y_1)$, in the formal sum

$$\sum_{u \in V_S Y_2} \text{inq}_S(u)^L, \tag{20}$$

before any cancellations are made, can be paired down by using primary vertices of Y_2 as indicated above. Since every secondary vertex u of Y_2 contributes $-(\text{deg } u - 2)$ to the coefficient of x_s in the sum (25) and

$$\sum_{u \in V_S Y_2} (\text{deg } u - 2) = 2\bar{r}(Y_2), \tag{21}$$

it follows that the first equality of Lemma 3.1 is true. The second equality follows from the analogous to (21) formula

$$\sum_{u \in V_S(\text{core}(Y_1 \times Y_2))} (\text{deg } u - 2) = 2\bar{r}(\text{core}(Y_1 \times Y_2)),$$

and from the definition (11) of numbers $N_\alpha(\Omega_{T_u}), u \in V_S Y_2$. Here the function Ω_{T_u} is defined for u as in (18). \square

Let A be a finite set. A *combination with repetitions* B of A , which we denote

$$B = [[b_1, \dots, b_\ell]] \subseteq A,$$

is a finite unordered collection of multiple copies of elements of A . Hence, $b_i \in A$ and $b_i = b_j$ is possible when $i \neq j$. If $B = [[b_1, \dots, b_\ell]]$ is a combination with repetitions then the cardinality $|B|$ of B is $|B| := \ell$.

Observe that a finite irreducible \mathcal{A} -graph Y_2 with property (Bd) can be used to construct a combination with repetitions, denoted

$$\text{inq}_d(V_S Y_2),$$

of the system $\text{SLI}_d[Y_1]$, whose elements are individual inequalities of $\text{SLI}_d[Y_1]$ so that every inequality $q = \text{inq}_S(u)$ of $\text{SLI}_d[Y_1]$, see (19), occurs in $\text{inq}_d(V_S Y_2)$ as many times as the number of preimages of q in $V_S Y_2$ under inq_S . Note that, in general, $\text{inq}_d(V_S Y_2) \neq \text{inq}_S(V_S Y_2)$ because $\text{inq}_S(V_S Y_2)$ is a subset of $\text{SLI}_d[Y_1]$ while $\text{inq}_d(V_S Y_2)$ is a combination with repetitions of $\text{SLI}_d[Y_1]$.

It follows from Lemma 3.1 that if $\text{inq}_d(V_S Y_2) = [[q_1, \dots, q_\ell]]$ is a combination of $\text{SLI}_d[Y_1]$, then

$$\sum_{q \in \text{inq}_d(V_S Y_2)} q^L := \sum_{j=1}^m q_j^L = -2\bar{r}(Y_2)x_s.$$

In the opposite direction, we will prove the following.

Lemma 3.2. *Suppose Q is a nonempty combination with repetitions of $\text{SLI}_d[Y_1]$ and*

$$\sum_{q \in Q} q^L = -C(Q)x_s, \tag{22}$$

where $C(Q) > 0$ is an integer. Then there exists a finite irreducible \mathcal{A} -graph $Y_{2,Q}$ with property (Bd) such that, letting $\tilde{Q} = \text{inq}_d(V_S Y_{2,Q})$, one has $|\tilde{Q}| = |Q|$ and

$$\sum_{q \in Q} q^L = \sum_{q \in \tilde{Q}} q^L = -2\bar{r}(Y_{2,Q})x_s, \tag{23}$$

$$\sum_{q \in Q} q^R \geq \sum_{q \in \tilde{Q}} q^R = -2\bar{r}(\text{core}(Y_1 \times Y_{2,Q})). \tag{24}$$

Proof. We will construct an \mathcal{A} -graph $Y_{2,Q}$ whose secondary vertices u_j are in bijective correspondence

$$u_j \mapsto q_j,$$

where $j = 1, \dots, |Q|$, with elements of the combination

$$Q = [[q_1, \dots, q_{|Q|}]] \sqsubseteq \text{SLI}_d[Y_1]$$

so that the secondary vertices of type $\alpha = 1$ in $Y_{2,Q}$ correspond to the inequalities of type (13) in Q , and the secondary vertices of type $\alpha = 2$ in $Y_{2,Q}$ correspond to the inequalities of type (14) in Q .

To fix the notation, we let the inequality q_j of Q be defined by means of an α_j -admissible function

$$\Omega_{T_j}: T_j \longrightarrow S_1(V_P Y_1),$$

where $T_j \in S_2(G_{\alpha_j})$ and $T_j = \{b_{1,j}, \dots, b_{k_j,j}\}$, $2 \leq k_j \leq d$, $b_{i,j} \in G_{\alpha_j}$. Here $k_j = k(q_j)$ denotes the parameter k for q_j , see (13) and (14).

Consider a secondary vertex u_j of type α_j and k_j edges $e_{1,j}, \dots, e_{k_j,j}$ whose terminal vertex is u_j and whose φ -labels are

$$\varphi(e_{1,j}) = b_{1,j}, \dots, \varphi(e_{k_j,j}) = b_{k_j,j}.$$

This is the local structure of the graph $Y_{2,Q}$ around its secondary vertices.

Now we will identify in pairs the initial vertices of the edges $e_{1,j}, \dots, e_{k_j,j}$, $j = 1, \dots, |Q|$, which will form the set of primary vertices $V_P Y_{2,Q}$ of the graph $Y_{2,Q}$. In the notation introduced above, it follows from the definitions (13) and (14) that a typical term $\pm x_A$ of q_j^L has the form $(-1)^{\alpha_j} x_A$, where $A = \Omega_{T_j}(b_{i,j})$ for some $i = 1, \dots, k_j$.

It follows from the equality (22) that there is an involution ι on the set of all terms $\pm x_A$ of the formal sum

$$\sum_{j=1}^{|Q|} q_j^L \tag{25}$$

such that ι takes every term $\pm x_A$ of $q_{j_1}^L$ to a term $\mp x_A$ of $q_{j_2}^L$, $j_1 \neq j_2$, and $\iota^2 = \text{id}$. Therefore, if

$$(-1)^{\alpha_{j_1}} x_{\Omega_{T_{j_1}}(b_{i_1,j_1})} \quad \text{and} \quad (-1)^{\alpha_{j_2}} x_{\Omega_{T_{j_2}}(b_{i_2,j_2})}$$

are two terms of the formal sum (25) which are ι -images of each other, then

$$\{\alpha_{j_1}, \alpha_{j_2}\} = \{1, 2\} \quad \text{and} \quad \Omega_{T_{j_1}}(b_{i_1,j_1}) = \Omega_{T_{j_2}}(b_{i_2,j_2}).$$

We identify the initial vertices of the edges e_{i_1,j_1}, e_{i_2,j_2} so that the vertex

$$(e_{i_1,j_1})_- = (e_{i_2,j_2})_-$$

becomes a primary vertex of $Y_{2,Q}$. We do this identification of the initial vertices of all pairs of edges, corresponding as described above to all pairs of terms $\pm x_A, \mp x_A$ in (25) that are ι -images of each other. As a result, we obtain an \mathcal{A} -graph $Y_{2,Q}$. It is clear from the definitions that $Y_{2,Q}$ is a finite irreducible \mathcal{A} -graph such that the degree of any secondary vertex u_j of $Y_{2,Q}$ is k_j such that

$$2 \leq k_j = |T_j| \leq d,$$

and the degree of every primary vertex of $Y_{2,Q}$ is 2.

Looking at the coefficients of $-x_s$ in (25), we can see from (22), (13), and (14) that

$$C(Q) = \sum_{j=1}^{|Q|} (k_j - 2) > 0.$$

Hence, the graph $Y_{2,Q}$ has a vertex of degree at least 3.

Therefore, $Y_{2,Q}$ is a finite irreducible \mathcal{A} -graph such that $\text{core}(Y_{2,Q}) = Y_{2,Q}$ and $\bar{r}(Y_{2,Q}) > 0$. Note that $Y_{2,Q}$ is not uniquely determined by Q (because there are many choices to define the involution ι , i.e., to do cancellations in the left hand side of (22)).

Consider the graph $\text{core}(Y_1 \times Y_{2,\mathcal{Q}})$ and the associated graph maps

$$\alpha_1: \text{core}(Y_1 \times Y_{2,\mathcal{Q}}) \longrightarrow Y_1, \quad \alpha_2: \text{core}(Y_1 \times Y_{2,\mathcal{Q}}) \longrightarrow Y_{2,\mathcal{Q}}.$$

It follows from the definitions, in particular, from the α -admissibility of functions $\Omega_{T_1}, \dots, \Omega_{T_{|\mathcal{Q}|}}$, that α_2 is surjective. Hence, $Y_{2,\mathcal{Q}}$ has property (Bd).

Let the sets $A_1(u_j), \dots, A_{k_j}(u_j)$ be defined for a secondary vertex u_j of $Y_{2,\mathcal{Q}}$ as in (17) so that $A_i(u_j)$ is defined by means of the primary vertex $(e_{i,j})_-$, where $i = 1, \dots, k_j$. It is not difficult to see from the definitions that

$$\Omega_{T_j}(b_{1,j}) \subseteq A_1(u_j), \dots, \Omega_{T_j}(b_{k_j,j}) \subseteq A_{k_j}(u_j).$$

This observation means that if $\tilde{\mathcal{Q}} := \text{inq}_d(V_S Y_{2,\mathcal{Q}})$ then $|\tilde{\mathcal{Q}}| = |\mathcal{Q}|$ and $Y_{2,\tilde{\mathcal{Q}}} = Y_{2,\mathcal{Q}}$ for a suitable involution $\tilde{\iota} = \tilde{\iota}(\tilde{\mathcal{Q}})$.

Hence, if q_j has the form (13), where $k_j = k(q_j)$ as before, then we have

$$\text{inq}_S(u_j)^L = -x_{A_1(u_j)} - \dots - x_{A_{k_j}(u_j)} - (k_j - 2)x_S,$$

where $\Omega_{T_j}(b_{i,j}) \subseteq A_i(u_j)$ for $i = 1, \dots, k_j$, and

$$N_1(\Omega_{T_j}) \leq N_1(\tilde{\Omega}_{T_j}) = -\text{inq}_S(u_j)^R,$$

here $\tilde{\Omega}_{T_j}$ is the function,

$$\tilde{\Omega}_{T_j}: \{b_{1,j}, \dots, b_{k_j,j}\} \longrightarrow S_1(V_p Y_1),$$

defined by $\tilde{\Omega}_{T_j}(b_{i,j}) := A_i(u_j)$ for $i = 1, \dots, k_j$.

Analogously, if q_j has the form (14), where $k_j = k(q_j)$, then we have

$$\text{inq}_S(u_j)^L = x_{A_1(u_j)} + \dots + x_{A_{k_j}(u_j)} - (k_j - 2)x_S,$$

where $\Omega_{T_j}(b_{i,j}) \subseteq A_i(u_j)$ for $i = 1, \dots, k_j$, and

$$N_1(\Omega_{T_j}) \leq N_1(\tilde{\Omega}_{T_j}) = -\text{inq}_S(u_j)^R,$$

here $\tilde{\Omega}_{T_j}$ is the function,

$$\tilde{\Omega}_{T_j}: \{b_{1,j}, \dots, b_{k_j,j}\} \longrightarrow S_1(V_p Y_1),$$

defined by $\tilde{\Omega}_{T_j}(b_{i,j}) := A_i(u_j)$ for $i = 1, \dots, k_j$.

Therefore,

$$\sum_{q \in \text{inq}_d(V_S Y_{2,\mathcal{Q}})} q^R = \sum_{q \in \tilde{\mathcal{Q}}} q^R \leq \sum_{q \in \mathcal{Q}} q^R.$$

Now both (23)–(24) follow from Lemma 3.1. □

We summarize Lemmas 3.1–3.2 in the following.

Lemma 3.3. *The function*

$$\text{inq}_d: Y_2 \mapsto \text{inq}_d(V_S Y_2) = Q$$

from the set of finite irreducible A -graphs Y_2 with property (Bd) to the set of combinations Q with repetitions of the system $\text{SLI}_d[Y_1]$ with the property $\sum_{q \in Q} q^L = -C(Q)x_s$, where $C(Q) > 0$ is an integer, is such that

$$\sum_{q \in \text{inq}_d(V_S Y_2)} q^L = -2\bar{r}(Y_2)x_s \quad \text{and} \quad \sum_{q \in \text{inq}_d(V_S Y_2)} q^R = -2\bar{r}(\text{core}(Y_1 \times Y_2)).$$

In addition, for every Q in the codomain of the function inq_d , there exists a graph $Y_{2,Q}$ in the domain of inq_d such that, letting $\tilde{Q} = \text{inq}_d(V_S Y_{2,Q})$, one has $|\tilde{Q}| = |Q|$ and

$$\sum_{q \in Q} q^L = \sum_{q \in \tilde{Q}} q^L = -2\bar{r}(Y_{2,Q})x_s,$$

$$\sum_{q \in Q} q^R \geq \sum_{q \in \tilde{Q}} q^R = -2\bar{r}(\text{core}(Y_1 \times Y_{2,Q})).$$

Proof. This is straightforward from Lemmas 3.1 and 3.2 and their proofs. □

4. Utilizing linear and linear semi-infinite programming

First we briefly review relevant results from the theory of linear programming (LP) over the field \mathbb{Q} of rational numbers. Following the notation of Schrijver's monograph [26], let $A \in \mathbb{Q}^{m' \times n'}$ be an $m' \times n'$ -matrix, let $b \in \mathbb{Q}^{m' \times 1} = \mathbb{Q}^{m'}$ be a column vector, let $c \in \mathbb{Q}^{1 \times n'}$ be a row vector, $c = (c_1, \dots, c_{n'})$, and let x be a column vector consisting of variables $x_1, \dots, x_{n'}$, so $x = (x_1, \dots, x_{n'})^\top$, where M^\top means the transpose of a matrix M . The inequality $x \geq 0$ means that $x_i \geq 0$ for every i .

A typical LP-problem asks about the maximal value of the objective linear function

$$cx = c_1x_1 + \dots + c_{n'}x_{n'}$$

over all $x \in \mathbb{Q}^{n'}$ subject to a finite system of linear inequalities $Ax \leq b$. This value (and often the LP-problem itself) is denoted

$$\max\{cx \mid Ax \leq b\}.$$

We write $\max\{cx \mid Ax \leq b\} = -\infty$ if the set $\{cx \mid Ax \leq b\}$ is empty. We write $\max\{cx \mid Ax \leq b\} = +\infty$ if the set $\{cx \mid Ax \leq b\}$ is unbounded from above and say that $\max\{cx \mid Ax \leq b\}$ is finite if the set $\{cx \mid Ax \leq b\}$ is nonempty and bounded from above. The notation and terminology for an LP-problem

$$\min\{cx \mid Ax \leq b\} = -\max\{-cx \mid Ax \leq b\}$$

is analogous with $-\infty$ and $+\infty$ interchanged.

If $\max\{cx \mid Ax \leq b\}$ is an LP-problem as defined above, then the problem

$$\min\{b^\top y \mid A^\top y = c^\top, y \geq 0\},$$

where $y = (y_1, \dots, y_m)^\top$, is called the *dual* problem of the *primal* LP-problem $\max\{cx \mid Ax \leq b\}$.

The (weak) duality theorem of linear programming can be stated as follows, see [26, Section 7.4].

Theorem A. *A Let $\max\{cx \mid Ax \leq b\}$ be an LP-problem and let $\min\{b^\top y \mid A^\top y = c^\top, y \geq 0\}$ be its dual LP-problem. Then for every $x \in \mathbb{Q}^{n'}$ such that $Ax \leq b$ and for every $y \in \mathbb{Q}^{m'}$ such that $A^\top y = c^\top, y \geq 0$, one has $cx = y^\top Ax \leq b^\top y$ and*

$$\max\{cx \mid Ax \leq b\} = \min\{b^\top y \mid A^\top y = c^\top, y \geq 0\} \tag{26}$$

provided both polyhedra $\{x \mid Ax \leq b\}$ and $\{y \mid A^\top y = c^\top, y \geq 0\}$ are not empty. In addition, the minimum, whenever it is finite, is attained at a vector \hat{y} which is a vertex of the polyhedron $\{y \mid A^\top y = c^\top, y \geq 0\}$.

Since the system of inequalities $\text{SLI}[Y_1]$, as defined in Section 3, is infinite in general, we also recall basic terminology and results regarding duality in linear semi-infinite programming (LSIP), see [2], [3], and [9]. Consider a generalized LP-problem $\max\{cx \mid Ax \leq b\}$ that has countably many linear inequalities in the system $Ax \leq b$ while the number of variables in x is still finite. Hence, in this setting, A is a matrix with countably many rows and n' columns, or $A \in \mathbb{Q}^{\infty \times n'}$ is an $\infty \times n'$ -matrix, $b \in \mathbb{Q}^{\infty \times 1} = \mathbb{Q}^\infty$, or b is an infinite column vector, $c \in \mathbb{Q}^{1 \times n'}$ is a row vector, and $x = (x_1, \dots, x_{n'})^\top$.

A typical LSIP-problem over \mathbb{Q} asks about the supremum of the objective linear functional cx over all $x \in \mathbb{Q}^{n'}$ subject to $Ax \leq b$. This number and the problem itself is denoted $\sup\{cx \mid Ax \leq b\}$. As above, we write $\sup\{cx \mid Ax \leq b\} = -\infty$ if the set $\{cx \mid Ax \leq b\}$ is empty, $\sup\{cx \mid Ax \leq b\} = +\infty$ if the set $\{cx \mid Ax \leq b\}$ is not bounded from above and say that $\sup\{cx \mid Ax \leq b\}$ is finite if the set $\{cx \mid Ax \leq b\}$ is nonempty and bounded from above. The notation and the terminology for an LSIP-problem $\inf\{cx \mid Ax \leq b\} = -\sup\{-cx \mid Ax \leq b\}$ is analogous with $-\infty$ and $+\infty$ interchanged. Let A_i denote the

submatrix of A of size $i \times n'$ whose first i rows are those of A and b_i is the starting subcolumn of b of length i . Then $\max\{cx \mid A_i x \leq b_i\}$ is an LP-problem which is called the i -approximate of the LSIP-problem $\sup\{cx \mid Ax \leq b\}$.

Let $M_i = \max\{cx \mid A_i x \leq b_i\}$ denote the optimal value of the i -approximate LP-problem $\max\{cx \mid A_i x \leq b_i\}$ and M is the number $\sup\{cx \mid Ax \leq b\}$. Clearly, for every i , $M_i \geq M_{i+1} \geq M$. Note that in general $\lim_{i \rightarrow \infty} M_i \neq M$, see [2], [3].

Similarly to [2], [3], [9], we say that if $\sup\{cx \mid Ax \leq b\}$ is an LSIP-problem as above, then the problem

$$\inf\{b^\top y \mid A^\top y = c^\top, y \geq 0\},$$

where $y = (y_1, y_2, \dots)^\top$ is an infinite vector whose set of nonzero components is finite, is called the *dual problem* of $\sup\{cx \mid Ax \leq b\}$.

For later references, we state the analogue of Theorem A for linear semi-infinite programming which, in fact, is an easy corollary of Theorem A.

Theorem B. *Suppose that $\sup\{cx \mid Ax \leq b\}$ is an LSIP-problem whose set $\{cx \mid Ax \leq b\}$ is nonempty and bounded from above and whose dual problem is $\inf\{b^\top y \mid A^\top y = c^\top, y \geq 0\}$. Then*

$$\sup\{cx \mid Ax \leq b\} \leq \inf\{b^\top y \mid A^\top y = c^\top, y \geq 0\} \quad (27)$$

and the equality holds if and only if $\sup\{cx \mid Ax \leq b\}$ is equal to $\lim_{i \rightarrow \infty} M_i$, where $M_i := \max\{cx \mid A_i x \leq b_i\}$ is the optimal solution of the i -approximate LP-problem $\max\{cx \mid A_i x \leq b_i\}$ of the primal LSIP-problem $\sup\{cx \mid Ax \leq b\}$.

In the situation when the inequality (27) is strict, the difference

$$\inf\{b^\top y \mid A^\top y = c^\top, y \geq 0\} - \sup\{cx \mid Ax \leq b\} > 0$$

is called the *duality gap* of the LSIP-problem $\sup\{cx \mid Ax \leq b\}$.

We now consider the problem of maximizing the objective linear function

$$cx := -x_s$$

over all rational vectors x , $x \in \mathbb{Q}^{n'}$ for a suitable n' , subject to the system of linear inequalities $\text{SLI}[Y_1]$, see (15), as an LSIP-problem $\sup\{cx \mid Ax \leq b\}$.

We also consider a subsequence of $m_{\text{inq},d}$ -approximate LP-problems

$$\max\{cx \mid A_{m_{\text{inq},d}} x \leq b_{m_{\text{inq},d}}\}$$

of the LSIP-problem $\sup\{cx \mid Ax \leq b\}$ whose systems $A_{m_{\text{inq},d}} x \leq b_{m_{\text{inq},d}}$ of inequalities are finite subsystems $\text{SLI}_d[Y_1]$ of $\text{SLI}[Y_1]$, where $d = 3, 4, \dots$, as defined in (16).

It is straightforward to verify that the dual problem

$$\inf\{b^\top y \mid A^\top y = c^\top, y \geq 0\}$$

of this LSIP-problem $\sup\{cx \mid Ax \leq b\}$ can be equivalently stated as follows.

$$\sum_{j=1}^{\infty} y_j q_j^R \longrightarrow \inf \quad \text{subject to } y \geq 0, \sum_{j=1}^{\infty} y_j q_j^L = -x_s, \quad (28)$$

where almost all $y_j, j = 1, 2, \dots$, are zeros. We rewrite (28) in the form

$$\inf \left\{ \sum_{j=1}^{\infty} y_j q_j^R \mid y \geq 0, \sum_{j=1}^{\infty} y_j q_j^L = -x_s \right\}. \quad (29)$$

Analogously, the dual problem of the $m_{\text{inq},d}$ -approximate LP-problem

$$\max\{cx \mid A_{m_{\text{inq},d}} x \leq b_{m_{\text{inq},d}}\}$$

can be stated in the form

$$\sum_{j=1}^{m_{\text{inq},d}} y_j q_j^R \longrightarrow \min \quad \text{subject to } y \geq 0, \sum_{j=1}^{m_{\text{inq},d}} y_j q_j^L = -x_s$$

which we write as follows:

$$\min \left\{ \sum_{j=1}^{m_{\text{inq},d}} y_j q_j^R \mid y \geq 0, \sum_{j=1}^{m_{\text{inq},d}} y_j q_j^L = -x_s \right\}. \quad (30)$$

In Lemma 3.3, we established the existence of a function

$$\text{inq}_d: Y_2 \longmapsto \text{inq}_d(Y_2)$$

from the set of finite irreducible \mathcal{A} -graphs Y_2 with property (Bd) to a certain set of combinations with repetitions of $\text{SLI}_d[Y_1]$. Now we will relate these combinations with repetitions of $\text{SLI}_d[Y_1]$ to solutions of the dual LP-problem (30).

Consider a combination with repetitions Q of $\text{SLI}_d[Y_1]$ that has the property that

$$\sum_{q \in Q} q^L = -C(Q)x_s, \quad (31)$$

where $C(Q) > 0$ is an integer. As above in (30), let the inequalities of $\text{SLI}_d[Y_1]$ be indexed and let

$$\text{SLI}_d[Y_1] = \{q_1, \dots, q_{m_{\text{inq},d}}\}.$$

Let $\eta_j(Q)$ denote the number of times that q_j occurs in Q , and let κ_j be the coefficient of x_s in q_j . Then it follows from the definitions and (31) that

$$\sum_{q \in Q} q^L = \sum_{j=1}^{m_{\text{in},d}} \kappa_j \eta_j(Q) x_s = -C(Q) x_s. \tag{32}$$

Consider the map

$$\text{sol}_d: Q \mapsto y_Q = (y_{Q,1}, \dots, y_{Q,m_{\text{in},d}})^\top, \tag{33}$$

where $y_{Q,j} := \frac{\eta_j(Q)}{C(Q)}$ for $j = 1, \dots, m_{\text{in},d}$. It follows from the definitions that y_Q is a rational vector, $y_Q \geq 0$ and, by (32), y_Q satisfies the condition

$$\sum_{j=1}^{m_{\text{in},d}} y_{Q,j} q_j^L = -x_s.$$

Hence, y_Q is a vector in the feasible polyhedron

$$\left\{ y \mid y \geq 0, \sum_{j=1}^{m_{\text{in},d}} y_j q_j^L = -x_s \right\} \tag{34}$$

of the dual LP-problem (30).

Note that, in place of (33), we could also write

$$\text{sol}_d: Q \mapsto C(Q)^{-1} \eta(Q)^\top, \tag{35}$$

where $\eta(Q) = (\eta_1(Q), \dots, \eta_{m_{\text{in},d}}(Q))$, as $y_Q = C(Q)^{-1} \eta(Q)^\top$.

Conversely, let $z = (z_1, \dots, z_{m_{\text{in},d}})^\top$ be a vector of the feasible polyhedron (34) of the dual LP-problem (30). Let $C > 0$ be a common multiple of positive denominators of the rational numbers $z_1, \dots, z_{m_{\text{in},d}}$. Consider a combination with repetitions $Q(z)$ of $\text{SLI}_d[Y_1]$ such that every q_j of $\text{SLI}_d[Y_1]$ occurs in $Q(z)$ exactly $C z_j = n_j$ many times. Then it follows from the definitions that

$$\sum_{q \in Q(z)} q^L = \sum_{j=1}^{m_{\text{in},d}} n_j q_j^L = \sum_{j=1}^{m_{\text{in},d}} C z_j q_j^L = C \sum_{j=1}^{m_{\text{in},d}} z_j q_j^L = -C x_s. \tag{36}$$

Now we can see from

$$\frac{\eta_j(Q(z))}{C} = \frac{C z_j}{C} = z_j, \tag{37}$$

where $j = 1, \dots, m_{\text{in},d}$, that the vector $y_{Q(z)} = \text{sol}_d(Q(z))$, defined by (33) for $Q(z)$, is equal to z .

Lemma 4.1. *The map*

$$\text{sol}_d: Q \mapsto y_Q$$

defined by (33) is a surjective function from the set of combinations Q with repetitions of $\text{SLI}_d[Y_1]$ that satisfy the equation $\sum_{q \in Q} q^L = -C(Q)x_s$, where $C(Q) > 0$ is an integer, to the feasible polyhedron (34) of the dual LP-problem (30). Furthermore, the composition of the maps inq_d and sol_d ,

$$\text{sol}_d \circ \text{inq}_d: Y_2 \mapsto \text{sol}_d(\text{inq}_d(Y_2)) = y_{Y_2},$$

is a function from the set of graphs with property (Bd) to the feasible polyhedron (34) of the dual LP-problem (30). Under this map, the value of the objective function $\sum_{j=1}^{m_{\text{inq},d}} y_{Y_2,j} q_j^R$ of the dual LP-problem (30) at y_{Y_2} satisfies the equality

$$\sum_{j=1}^{m_{\text{inq},d}} y_{Y_2,j} q_j^R = -\frac{\bar{r}(\text{core}(Y_1 \times Y_2))}{\bar{r}(Y_2)}. \tag{38}$$

In addition, for every z in the polyhedron (34), there is a vector \tilde{z} in (34) such that $\tilde{z} = \text{sol}_d(\text{inq}_d(Y_2))$ for some graph Y_2 with property (Bd) and

$$\sum_{j=1}^{m_{\text{inq},d}} \tilde{z}_j q_j^R \leq \sum_{j=1}^{m_{\text{inq},d}} z_j q_j^R.$$

Proof. As was observed above, see (36) and (37), sol_d is a surjective function.

Consider a finite irreducible \mathcal{A} -graph Y_2 with property (Bd) and define

$$Q := \text{inq}_d(Y_2), \quad y_{Y_2} := \text{sol}_d(Q).$$

It follows from Lemma 3.3 that

$$\sum_{q \in Q} q^L = -2\bar{r}(Y_2)x_s \quad \text{and} \quad \sum_{q \in Q} q^R = -2\bar{r}(\text{core}(Y_1 \times Y_2)). \tag{39}$$

It follows from (32) and (39) that $C(Q) = 2\bar{r}(Y_2)$. Hence, using the definition (33) and equalities (39), we obtain

$$\sum_{j=1}^{m_{\text{inq},d}} y_{Y_2,j} q_j^R = \frac{\sum_{q \in Q} q^R}{C(Q)} = -\frac{\bar{r}(\text{core}(Y_1 \times Y_2))}{\bar{r}(Y_2)},$$

as required in (38).

To prove the additional statement, consider a vector z in the polyhedron (34).

Since sol_d is surjective, there is a combination with repetitions Q such that $\text{sol}_d(Q) = z$. By Lemma 3.3 for this Q , there is a graph $Y_{2,Q}$ such that if $\text{inq}_d(V_S Y_{2,Q}) = \tilde{Q}$ then $|\tilde{Q}| = |Q|$ and

$$\sum_{q \in Q} q^L = \sum_{q \in \tilde{Q}} q^L = -2\bar{r}(Y_{2,Q})x_s = -C(Q)x_s = -C(\tilde{Q})x_s, \tag{40}$$

$$\sum_{q \in Q} q^R \geq \sum_{q \in \tilde{Q}} q^R = -2\bar{r}(\text{core}(Y_1 \times Y_{2,Q})). \tag{41}$$

Let $\tilde{z} := \text{sol}_d(\tilde{Q})$. Then, in view of (40)–(41), we obtain

$$\sum_{j=1}^{m_{\text{inq},d}} \tilde{z}_j q_j^R = \frac{\sum_{q \in \tilde{Q}} q^R}{C(\tilde{Q})} \leq \frac{\sum_{q \in Q} q^R}{C(Q)} = \sum_{j=1}^{m_{\text{inq},d}} z_j q_j^R,$$

as required. □

We will say that a real number $\sigma(Y_1) \geq 0$ is the *WN-coefficient* for Y_1 if

$$\bar{r}(\text{core}(Y_1 \times Y_2)) \leq \sigma(Y_1)\bar{r}(Y_1)\bar{r}(Y_2)$$

for every finite irreducible \mathcal{A} -graph Y_2 with property (B) and $\sigma(Y_1)$ is minimal with this property.

We also consider the WN_d -coefficient $\sigma_d(Y_1)$, where $d \geq 3$ is an integer, for Y_1 defined so that

$$\bar{r}(\text{core}(Y_1 \times Y_2)) \leq \sigma_d(Y_1)\bar{r}(Y_1)\bar{r}(Y_2)$$

for every finite irreducible \mathcal{A} -graph Y_2 with property (Bd) and $\sigma_d(Y_1)$ is minimal with this property.

It is clear from the definitions that

$$\sigma_d(Y_1) \leq \sigma_{d+1}(Y_1) \leq \sigma(Y_1)$$

for every $d = 3, 4, \dots$ and

$$\sup_d \{\sigma_d(Y_1)\} = \sigma(Y_1). \tag{42}$$

Observe that

$$\sigma(Y_1) = \sup_{Y_2} \left\{ \frac{\bar{r}(\text{core}(Y_1 \times Y_2))}{\bar{r}(Y_1)\bar{r}(Y_2)} \right\}$$

over all finite irreducible \mathcal{A} -graphs Y_2 with property (B). Similarly,

$$\sigma_d(Y_1) = \sup_{Y'_2} \left\{ \frac{\bar{r}(\text{core}(Y_1 \times Y'_2))}{\bar{r}(Y_1)\bar{r}(Y'_2)} \right\} \tag{43}$$

over all finite irreducible \mathcal{A} -graphs Y'_2 with property (Bd).

Lemma 4.2. *Both optima*

$$\max\{-x_s \mid \text{SLI}_d[Y_1]\} \quad \text{and} \quad \min\left\{\sum_{j=1}^{m_{\text{inq},d}} y_j q_j^R \mid y \geq 0, \sum_{j=1}^{m_{\text{inq},d}} y_j q_j^L = -x_s\right\}$$

are finite and satisfy the following inequalities and equalities

$$\begin{aligned} -2\frac{q^*}{q^*-2}\bar{r}(Y_1) &\leq \sup\{-x_s \mid \text{SLI}[Y_1]\} \\ &\leq \max\{-x_s \mid \text{SLI}_d[Y_1]\} \\ &= \min\left\{\sum_{j=1}^{m_{\text{inq},d}} y_j q_j^R \mid y \geq 0, \sum_{j=1}^{m_{\text{inq},d}} y_j q_j^L = -x_s\right\} \\ &= -\sigma_d(Y_1)\bar{r}(Y_1). \end{aligned} \tag{44}$$

Furthermore, the minimum is attained at a vector $\tilde{y}_V = \tilde{y}_V(d)$ of the feasible polyhedron (34) of the dual LP-problem (30) such that there is a graph Y_{2,Q_V} that has property (Bd), $\tilde{y}_V = \text{sol}_d(\text{inq}_d(Y_{2,Q_V}))$ and the following hold

$$\begin{aligned} \inf\left\{\sum_{j=1}^{\infty} y_j q_j^R \mid y \geq 0, \sum_{j=1}^{\infty} y_j q_j^L = -x_s\right\} \\ = -\sigma(Y_1)\bar{r}(Y_1) \\ \leq \min\left\{\sum_{j=1}^{m_{\text{inq},d}} y_j q_j^R \mid y \geq 0, \sum_{j=1}^{m_{\text{inq},d}} y_j q_j^L = -x_s\right\} \\ = -\sigma_d(Y_1)\bar{r}(Y_1). \end{aligned} \tag{45}$$

In particular, $\sigma_d(Y_1) \leq \sigma(Y_1) \leq 2\frac{q^*}{q^*-2}$.

Proof. Recall that every primary vertex of Y_1 has degree 2 and $d \geq 3$. Hence, if the graph Y_1 contains a vertex u of degree $> d$, then u is secondary and we may take some edges out of Y_1 to get a subgraph \hat{Y}_1 of Y_1 such that $|E\hat{Y}_1| < |EY_1|$, $\bar{r}(\hat{Y}_1) > 0$ and $\text{core}(\hat{Y}_1) = \hat{Y}_1$. It is clear that the natural projection

$$\tau_2: \text{core}(Y_1 \times \hat{Y}_1) \longrightarrow \hat{Y}_1$$

is surjective. Hence, either the graph \hat{Y}_1 has property (Bd) or, otherwise, \hat{Y}_1 has a vertex of degree greater than d . Iterating this argument, we can prove that Y_1 contains a subgraph $Y_{1,d}$ with property (Bd).

Setting $Y_2 := Y_{1,d}$, we obtain, by Lemma 4.1, a solution $\hat{y} = \text{sol}_d(\text{in}_d(Y_2))$ to the equalities and inequalities that define the feasible polyhedron (34) of (30). Hence, both sets

$$\left\{ y \mid y \geq 0, \sum_{j=1}^{m_{\text{in},d}} y_j q_j^L = -x_s \right\}, \quad \left\{ y \mid y \geq 0, \sum_{j=1}^{\infty} y_j q_j^L = -x_s \right\}$$

are not empty.

To see that the sets $\{x \mid \text{SLI}[Y_1]\}$, $\{x \mid \text{SLI}_d[Y_1]\}$ are not empty either, we will show that the vector \hat{x} , whose components are $\hat{x}_A := 0$ for every nonempty $A \subseteq V_P Y_1$ and $\hat{x}_s := 2 \frac{q^*}{q^* - 2} \bar{r}(Y_1)$, is a solution both to $\text{SLI}_d[Y_1]$ and to $\text{SLI}[Y_1]$. To do this, we will check that every inequality of $\text{SLI}[Y_1]$ is satisfied with these values of variables, that is,

$$-(k - 2) \cdot \frac{2q^*}{q^* - 2} \bar{r}(Y_1) \leq -N_\alpha(\Omega_T) \tag{46}$$

for every α -admissible function

$$\Omega_T: T \longrightarrow S_1(V_P Y_1),$$

where $T \in S_2(G_\alpha)$ and $|T| = k$.

Let $T = \{a_1, \dots, a_k\}$, $k \geq 2$, $a_i \in G_\alpha$, and $\Omega_T(a_i) = A_i$, $i = 1, \dots, k$.

Consider a secondary vertex u of Y_1 , suppose $\text{deg } u = \ell$ and let e_1, \dots, e_ℓ be all edges of Y_1 such that $u = (e_1)_+ = \dots = (e_\ell)_+$. Denote

$$B := \{\varphi(e_1), \dots, \varphi(e_\ell)\}.$$

It is not difficult to see from the definition (11) of the number $N_\alpha(\Omega_T)$ that the contribution to the sum $N_\alpha(\Omega_T)$, made by those equivalence classes that are associated with the vertex $u \in V_S Y_1$, does not exceed

$$\sum_{g \in G_\alpha} \max(|T \cap Bg| - 2, 0).$$

Hence, it follows from the definition of the number $\frac{q^*}{q^* - 2}$, see (1), and from the results of Dicks and the author [6, Corollary 3.5] that

$$\begin{aligned} \sum_{g \in G_\alpha} \max(|T \cap Bg| - 2, 0) &\leq \frac{q^*}{q^* - 2} (|T| - 2)(|B| - 2) \\ &= \frac{q^*}{q^* - 2} (k - 2)(\ell - 2). \end{aligned} \tag{47}$$

Therefore, summing up inequalities (47) over all $u \in V_S Y_1$, we obtain

$$\begin{aligned} N_\alpha(\Omega) &\leq \frac{q^*}{q^* - 2}(k - 2) \cdot 2\bar{r}_\alpha(Y_1) \\ &\leq \frac{q^*}{q^* - 2}(k - 2) \cdot 2\bar{r}(Y_1), \end{aligned}$$

where $2\bar{r}_\alpha(Y_1)$ is the sum $\sum_u (\deg u - 2)$ over all secondary vertices $u \in V_S Y_1$ of type α . This proves (46) and also shows that

$$-\frac{2q^*}{q^* - 2}\bar{r}(Y_1) \leq \sup\{-x_s \mid \text{SLI}[Y_1]\} \tag{48}$$

because \hat{x} with $\hat{x}_s = \frac{2q^*}{q^* - 2}\bar{r}(Y_1)$ is a solution to $\text{SLI}[Y_1]$.

Therefore, both sets $\{x \mid \text{SLI}[Y_1]\}$ and $\{x \mid \text{SLI}_d[Y_1]\}$ are not empty as required.

According to Theorem A, the maximum and minimum in (44) are finite and equal. The first inequality in (44) is shown in (48) and the second one follows from the definitions.

It follows from the definition (43) and Lemma 4.1 that the supremum

$$\sup_{Y_2} \left\{ \frac{\bar{r}(\text{core}(Y_1 \times Y_2))}{\bar{r}(Y_2)} \right\} = \sigma_d(Y_1)\bar{r}(Y_1) = -\inf_{Y_2} \left\{ -\frac{\bar{r}(\text{core}(Y_1 \times Y_2))}{\bar{r}(Y_2)} \right\}$$

over all graphs Y_2 with property (Bd) is equal to

$$\begin{aligned} \sigma_d(Y_1)\bar{r}(Y_1) &= -\inf \left\{ \sum_{j=1}^{m_{\text{inq},d}} y_j q_j^R \mid y \geq 0, \sum_{j=1}^{m_{\text{inq},d}} y_j q_j^L = -x_s \right\} \\ &= -\min \left\{ \sum_{j=1}^{m_{\text{inq},d}} y_j q_j^R \mid y \geq 0, \sum_{j=1}^{m_{\text{inq},d}} y_j q_j^L = -x_s \right\}, \end{aligned}$$

as stated in the last equality of (44).

The inequalities and equalities of (44) are now proven.

By Theorem A, the minimum in (45) of the LP-problem (30) is attained at a vertex $y_V = y_V(d)$ of the feasible polyhedron (34).

It follows from Lemma 4.1 that, for the vertex y_V , there exists a vector \tilde{y}_V in the polyhedron (34) such that

$$\sum_{j=1}^{m_{\text{inq},d}} \tilde{y}_{V,j} q_j^R \leq \sum_{j=1}^{m_{\text{inq},d}} y_{V,j} q_j^R.$$

and $\tilde{y}_V = \text{sol}_d(\text{inq}_d(Y_{2,Q_V}))$ for some graph Y_{2,Q_V} with property (Bd). Hence, the minimum in (45) is also attained at \tilde{y}_V .

In view of the last equality of (44) and (42), we obtain

$$\begin{aligned}
 & \inf_d \left\{ \min \left\{ \sum_{j=1}^{m_{\text{inq},d}} y_j q_j^R \mid y \geq 0, \sum_{j=1}^{m_{\text{inq},d}} y_j q_j^L = -x_s \right\} \right\} \\
 &= \inf_d \{-\sigma_d(Y_1)\bar{r}(Y_1)\} \\
 &= -\sigma(Y_1)\bar{r}(Y_1) \tag{49} \\
 &\leq \min \left\{ \sum_{j=1}^{m_{\text{inq},d}} y_j q_j^R \mid y \geq 0, \sum_{j=1}^{m_{\text{inq},d}} y_j q_j^L = -x_s \right\} \\
 &= -\sigma_d(Y_1)\bar{r}(Y_1).
 \end{aligned}$$

On the other hand, it is clear that

$$\begin{aligned}
 & \inf \left\{ \sum_{j=1}^{\infty} y_j q_j^R \mid y \geq 0, \sum_{j=1}^{\infty} y_j q_j^L = -x_s \right\} \\
 &= \inf_d \left\{ \min \left\{ \sum_{j=1}^{m_{\text{inq},d}} y_j q_j^R \mid y \geq 0, \sum_{j=1}^{m_{\text{inq},d}} y_j q_j^L = -x_s \right\} \right\}. \tag{50}
 \end{aligned}$$

Now the equalities and inequalities (45) follow from (49) and (50)

The inequalities $\sigma_d(Y_1) \leq \sigma(Y_1) \leq 2 \frac{q^*}{q^*-2}$ follow from (44) and (42). □

Lemma 4.3. *There exists a finite irreducible \mathcal{A} -graph $Y_{2,Q_V} = Y_{2,Q_V}(Y_1)$ with property (Bd) such that*

$$\bar{r}(\text{core}(Y_1 \times Y_{2,Q_V})) = \sigma_d(Y_1)\bar{r}(Y_1)\bar{r}(Y_{2,Q_V}),$$

Y_{2,Q_V} is connected, and

$$|EY_{2,Q_V}| < 2^{2|EY_1|/4 + \log_2 \log_2(4d)}.$$

Proof. According to Lemma 4.2 and to Theorem A, we may assume that the minimum of the dual LP-problem (30) is attained at a vertex y_V of the feasible polyhedron (34) of (30).

It is convenient to switch back to the general LP and LSIP notation as was introduced in the beginning of this section. In particular, let $A_{m_{\text{inq},d}} x \leq b_{m_{\text{inq},d}}$ be the matrix form of the system (16). Since y_V is a vertex solution of the LP-problem (30) and (30) is stated in the form

$$\min\{b_{m_{\text{inq},d}}^\top y \mid A_{m_{\text{inq},d}}^\top y = c^\top, y \geq 0\},$$

it follows that the vertex solution y_V will satisfy $m_{\text{in},d}$ equalities among

$$A_{m_{\text{in},d}}^\top y = c^\top, \quad y_j = 0, \quad j = 1, \dots, m_{\text{in},d},$$

whose left hand side parts are linearly independent (as formal linear combinations in variables $y_1, \dots, y_{m_{\text{in},d}}$). We call these $m_{\text{in},d}$ equalities *distinguished*.

The foregoing observation implies that there are r , $r \leq m_{\text{in},d}$, distinguished equalities in the system $A_{m_{\text{in},d}}^\top y = c^\top$ such that the submatrix $A_{m_{\text{in},d},r}^\top$ of $A_{m_{\text{in},d}}^\top$, consisting of the rows of $A_{m_{\text{in},d}}^\top$ that correspond to the r distinguished equalities, has the following property. The rank of $A_{m_{\text{in},d},r}^\top$ is r and deletion of the columns of $A_{m_{\text{in},d},r}^\top$, that correspond to the variables y_j that in turn correspond to the distinguished equalities $y_j = 0$, produces an $r \times r$ matrix $A_{m_{\text{in},d},r \times r}^\top$ with $\det A_{m_{\text{in},d},r \times r}^\top \neq 0$. Reordering the equalities in the system $A_{m_{\text{in},d}}^\top y = c^\top$ and variables y_j if necessary, we may assume that $A_{m_{\text{in},d},r}^\top$ consists of the first r rows of $A_{m_{\text{in},d}}^\top$ and $A_{m_{\text{in},d},r \times r}^\top$ is an upper left submatrix of $A_{m_{\text{in},d}}^\top$.

Let

$$\bar{y}_V = (y_{V,1}, \dots, y_{V,r})$$

be the truncated version of y_V consisting of the first r components. It follows from the definitions that \bar{y}_V contains all nonzero components of y_V and

$$A_{m_{\text{in},d},r \times r}^\top \bar{y}_V = \bar{c}^\top = (c_1, \dots, c_r)^\top.$$

Since $\sum_{j=1}^{m_{\text{in},d}} y_{V,j} q_j^L = -x_s$, it follows that $c_i = 0$ if c_i corresponds to a variable x_B and $c_i = -1$ if c_i corresponds to the variable x_s . Since $y_V \neq 0$ following from the definition of the LP-problem (30), we conclude that $\bar{c}^\top \neq 0$, i.e., one of c_i is -1 and all other entries in \bar{c}^\top are equal to 0.

Note that every row of $A_{m_{\text{in},d},r \times r}$ contains at most $d + 1$ nonzero entries such that one is $-(k - 2)$, where $2 \leq k \leq d$ (this is the coefficient of x_s that could be zero), and the other nonzero entries have the same sign and their sum is at least $-d$ and at most d , see the definitions (13) and (14). Hence, the standard Euclidian norm of any row of $A_{m_{\text{in},d},r \times r}$ is at most

$$(d^2 + (d - 2)^2)^{1/2} < 2d$$

as $d \geq 3$. Hence, by the Hadamard's inequality, we have that

$$|\det A_{m_{\text{in},d},r \times r}| < (2d)^r. \tag{51}$$

Invoking the Cramer's rule, we further obtain that

$$y_{V,j} = \frac{\det A_{m_{\text{in},d},r \times r,j}^\top(\bar{c}^\top)}{\det A_{m_{\text{in},d},r \times r}^\top}, \tag{52}$$

where $A_{m_{\text{inq},d},r \times r,j}^\top(\bar{c}^\top)$ is the matrix obtained from $A_{m_{\text{inq},d},r \times r}^\top$ by replacing the j th column with \bar{c}^\top , $j = 1, \dots, r$. Since \bar{c}^\top has a unique nonzero entry which is -1 , we have from the Hadamard's inequality, similarly to (51), that

$$|\det A_{m_{\text{inq},d},r \times r,j}(\bar{c}^\top)| < (2d)^{r-1}. \tag{53}$$

In view of (51)–(53), we can see that there is a common denominator $C > 0$ for the rational numbers $y_{V,1}, \dots, y_{V,r}$ that satisfies $C < (2d)^r$ and that the nonnegative integers $Cy_{V,1}, \dots, Cy_{V,r}$ are less than $(2d)^{r-1}$.

Hence, it follows from the definition of the function sol_d , see also Lemma 4.1, that if Q_V is a combination such that $y_V = \text{sol}_d(Q_V)$ and $|Q_V|$ is minimal with this property, i.e., the entries of $\eta(Q_V)$ are coprime, then

$$|Q_V| < r(2d)^{r-1}. \tag{54}$$

Recall that the cardinality $|Q|$ of a combination with repetitions Q is defined so that every $q \in Q$ is counted as many times as it occurs in Q .

We now construct a graph Y_{2,Q_V} from Q_V as described in the proof of Lemma 3.2. Recall that if $\text{inq}_d(V_S Y_{2,Q_V}) = \tilde{Q}_V$ then \tilde{Q}_V could be different from Q_V but $|\tilde{Q}_V| = |Q_V|$ and Y_{2,Q_V} could also be constructed by means of \tilde{Q}_V .

It follows from the definitions and Lemmas 4.1 and 4.2 that if

$$\tilde{y}_V := \text{sol}_d(\text{inq}_d(V_S Y_{2,Q_V}))$$

then the minimum of the dual LP-problem (30) is also attained at \tilde{y}_V and this minimum is equal to $-\sigma_d(Y_1)\bar{r}(Y_1)$. Hence,

$$\bar{r}(\text{core}(Y_1 \times Y_{2,Q_V})) = \sigma_d(Y_1)\bar{r}(Y_1)\bar{r}(Y_{2,Q_V}).$$

Since $|V_S Y_{2,Q_V}| = |Q_V|$ and the degree of every secondary vertex of Y_{2,Q_V} is at most d , it follows from (54) that

$$|EY_{2,Q_V}| \leq 2d|V_S Y_{2,Q_V}| = 2d|Q_V| < r(2d)^r. \tag{55}$$

Note that r does not exceed the total number n_{inq} of variables of $\text{SLI}[Y_1]$. Since every primary vertex of Y_1 has degree 2 and edges of Y_1 are oriented, we have $|EY_1| = 4|V_P Y_1|$. Since each variable x_B of $\text{SLI}[Y_1]$, different from x_s , is indexed with a nonempty set $B \subseteq V_P Y_1$, it follows that

$$r \leq n_{\text{inq}} \leq (2^{|V_P Y_1|} - 1) + 1 = (2^{|EY_1|/4} - 1) + 1 = 2^{|EY_1|/4}. \tag{56}$$

Finally, we obtain from (55) and (56) that

$$\begin{aligned}
 |EY_{2,Q_V}| &< r(2d)^r \\
 &\leq 2^{|EY_1|/4} \cdot (2d)^{2^{|EY_1|/4}} \\
 &= 2^{|EY_1|/4} \cdot 2^{(\log_2(2d)) \cdot 2^{|EY_1|/4}} \\
 &< 2^{(\log_2(2d)+1) \cdot 2^{|EY_1|/4}} \\
 &= 2^{2^{|EY_1|/4 + \log_2 \log_2(4d)}},
 \end{aligned}
 \tag{57}$$

as desired.

It remains to show that the graph Y_{2,Q_V} is connected.

Arguing on the contrary, assume that the graph Y_{2,Q_V} is the disjoint union of its two subgraphs Y_3 and Y_4 . First we assume that

$$\bar{r}(Y_3) > 0 \quad \text{and} \quad \bar{r}(Y_4) > 0.
 \tag{58}$$

Clearly, Y_3 and Y_4 are graphs with property (Bd). Recall that the secondary vertices of the graph Y_{2,Q_V} bijectively correspond to the inequalities of the combination Q_V , see the proof of Lemma 3.2. In particular, we can consider the combinations Q_3 and Q_4 , whose inequalities bijectively correspond to the secondary vertices of Y_3 and Y_4 , resp. It is clear that Q_V is the union of the combinations Q_3 and Q_4 and

$$\eta(Q_V) = \eta(Q_3) + \eta(Q_4).
 \tag{59}$$

We specify that by the union $B_1 \sqcup B_2$ of two combinations B_1, B_2 we mean the combination whose elements are all elements of both B_1 and B_2 , in particular, $|B_1 \sqcup B_2| = |B_1| + |B_2|$.

Furthermore, the graphs Y_3 and Y_4 could be constructed from Q_3 and Q_4 , resp., in the same manner as Y_{2,Q_V} was constructed from Q_V . In particular, the combinations Q_3 and Q_4 belong to the domain of the function sol_d .

Invoking Lemma 4.1, denote $y_V(j) := \text{sol}_d(Q_j)$, $j = 3, 4$. We also denote

$$\sum_{q \in Q_V} q^L = -C(Q_V)x_s, \quad \sum_{q \in Q_j} q^L = -C(Q_j)x_s,$$

where $j = 3, 4$.

Since $Q_V = Q_3 \sqcup Q_4$, it follows that $C(Q_V) = C(Q_3) + C(Q_4)$. According to the definition (33) of the function sol_d , we have

$$y_{V,i} = \frac{\eta_i(Q_V)}{C(Q_V)}, \quad y_{V,i}(j) = \frac{\eta_i(Q_j)}{C(Q_j)}
 \tag{60}$$

for all suitable i, j . Hence, in view of (59), for every $i = 1, \dots, m_{\text{inq},d}$, we obtain

$$\begin{aligned}
 y_{V,i} &= \frac{\eta_i(Q_V)}{C(Q_V)} \\
 &= \frac{\eta_i(Q_3) + \eta_i(Q_4)}{C(Q_V)} \\
 &= \frac{C(Q_3)}{C(Q_V)} \cdot \frac{\eta_i(Q_3)}{C(Q_3)} + \frac{C(Q_4)}{C(Q_V)} \cdot \frac{\eta_i(Q_4)}{C(Q_4)} \\
 &= \lambda_3 y_{V,i}(3) + \lambda_4 y_{V,i}(4),
 \end{aligned} \tag{61}$$

where $\lambda_3 = \frac{C(Q_3)}{C(Q_V)}$ and $\lambda_4 = \frac{C(Q_4)}{C(Q_V)}$ are positive rational numbers that satisfy $\lambda_3 + \lambda_4 = 1$.

The equalities (61) imply that

$$y_V = \lambda_3 y_V(3) + \lambda_4 y_V(4). \tag{62}$$

Since y_V is a vertex of the polyhedron (34), $y_V(3)$ and $y_V(4)$ are vectors in (34), and $0 < \lambda_3, \lambda_4 < 1, \lambda_3 + \lambda_4 = 1$, it follows from (62) that

$$y_V(3) = y_V(4) = y_V.$$

Hence, in view of (60), the tuples $\eta(Q_V), \eta(Q_3), \eta(Q_4)$ that have integer entries are rational multiples of each other. Referring to (59), we conclude that the entries of $\eta(Q_V)$ are not coprime, contrary to the definition of the combination Q_V . This contradiction completes the case (58).

We now assume that the graph Y_{2,Q_V} is the disjoint union of its two subgraphs Y_3 and Y_4 such that

$$\bar{r}(Y_3) > 0 \quad \text{and} \quad \bar{r}(Y_4) = 0. \tag{63}$$

Let $2Q_V$ denote the combination such that $\eta(2Q_V) = 2\eta(Q_V)$, i.e., to get $2Q_V$ from Q_V we double the number of occurrences of each inequality in Q_V . Using this combination $2Q_V$, we can construct, as in the proof of Lemma 3.2, a graph $Y_{2,2Q_V}$ which consists of two disjoint copies of Y_{2,Q_V} , denoted \bar{Y}_{2,Q_V} and \hat{Y}_{2,Q_V} . Since $Y_{2,Q_V} = Y_3 \cup Y_4$, we can represent the graph $Y_{2,2Q_V}$ in the form

$$Y_{2,2Q_V} = Y_5 \cup Y_6,$$

where $Y_5 := \bar{Y}_3 \cup \bar{Y}_4 \cup \hat{Y}_4$ and $Y_6 := \hat{Y}_3$

Clearly, $\bar{r}(Y_5) > 0, \bar{r}(Y_6) > 0$, and both Y_5, Y_6 have property (Bd). As above, we remark that the secondary vertices of $Y_{2,2Q_V}$ are in bijective correspondence with the inequalities of $2Q_V$. Hence, the combination $2Q_V$ is the union of the

combinations Q_5 and Q_6 that consist of those inequalities that correspond to the secondary vertices of Y_5 and Y_6 , resp., and that can be used to construct the graphs Y_5 and Y_6 in the same manner as Y_{2,Q_V} was constructed from Q_V .

As above, we can write

$$\eta(2Q_V) = \eta(Q_5) + \eta(Q_6). \tag{64}$$

Note that the combinations Q_5 and Q_6 belong to the domain of the function sol_d . Using Lemma 4.1, denote $y_V(j) := \text{sol}_d(Q_j)$, $j = 5, 6$. As above, denote

$$\sum_{q \in 2Q_V} q^L = -C(2Q_V)x_s, \quad \sum_{q \in Q_j} q^L = -C(Q_j)x_s,$$

where $j = 5, 6$.

Since $2Q_V = Q_5 \sqcup Q_6$, it follows that $C(2Q_V) = C(Q_5) + C(Q_6)$. According to the definition (33) of the function sol_d , we have

$$y_{V,i} = \frac{\eta_i(Q_V)}{C(Q_V)} = \frac{\eta_i(2Q_V)}{C(2Q_V)}, \quad y_{V,i}(j) = \frac{\eta_i(Q_j)}{C(Q_j)} \tag{65}$$

for all suitable i, j . Hence, in view of (64), for every $i = 1, \dots, m_{\text{inq},d}$, we obtain

$$\begin{aligned} y_{V,i} &= \frac{\eta_i(2Q_V)}{C(2Q_V)} \\ &= \frac{\eta_i(Q_5) + \eta_i(Q_6)}{C(2Q_V)} \\ &= \frac{C(Q_5)}{C(2Q_V)} \cdot \frac{\eta_i(Q_5)}{C(Q_5)} + \frac{C(Q_6)}{C(2Q_V)} \cdot \frac{\eta_i(Q_6)}{C(Q_6)} \\ &= \lambda_5 y_{V,i}(5) + \lambda_6 y_{V,i}(6), \end{aligned} \tag{66}$$

where $\lambda_5 = \frac{C(Q_5)}{C(2Q_V)}$ and $\lambda_6 = \frac{C(Q_6)}{C(2Q_V)}$ are positive rational numbers that satisfy $\lambda_5 + \lambda_6 = 1$.

The equalities (66) imply that

$$y_V = \lambda_5 y_V(5) + \lambda_6 y_V(6). \tag{67}$$

Since y_V is a vertex of the polyhedron (34), $y_V(5)$ and $y_V(6)$ are vectors in the polyhedron (34), and $0 < \lambda_5, \lambda_6 < 1$, $\lambda_5 + \lambda_6 = 1$, it follows from (67) that

$$y_V(5) = y_V(6) = y_V.$$

Hence, in view of (65), the tuples $\eta(2Q_V)$, $\eta(Q_5)$, $\eta(Q_6)$ that have integer entries are rational multiples of each other. Referring to (64) and keeping in mind that the entries of $\eta(Q_V)$ are coprime, we conclude that

$$\eta(Q_V) = \eta(Q_5) = \eta(Q_6), \tag{68}$$

i.e., $Q_V = Q_5 = Q_6$. However, $Y_6 = \hat{Y}_3$ and \hat{Y}_3 is a subgraph of \hat{Y}_{2,Q_V} that consists of several connected components of \hat{Y}_{2,Q_V} and $\hat{Y}_3 \neq \hat{Y}_{2,Q_V}$. Hence, $Q_5 \neq Q_V$. This contradiction to (68) completes the second case (63). Thus the graph Y_{2,Q_V} is connected. The proof of Lemma 4.3 is complete. \square

5. More lemmas

We now let $\mathcal{F} = \prod_{\alpha \in I}^* G_\alpha$ be an arbitrary free product of nontrivial groups G_α , $\alpha \in I$, and $|I| > 1$. Let H be a finitely generated factor-free subgroup of \mathcal{F} . As in Section 2, let $\Psi_o(H)$ denote an irreducible \mathcal{A} -graph of H , where $\mathcal{A} = \bigcup_{\alpha \in I} G_\alpha$, with the base vertex o and let $\Psi(H)$ denote the core of $\Psi_o(H)$.

Let $I(H)$ denote a subset of the index set I such that $\alpha \in I(H)$ if and only if there is a secondary vertex $u \in V_S \Psi(H)$ of type α . Since H is finitely generated, it follows that the set $I(H)$ is finite.

Let us fix a finitely generated factor-free subgroup H_1 of \mathcal{F} with positive reduced rank $\bar{r}(H_1) = -\chi(\Psi(H_1)) > 0$.

We say that a finitely generated factor-free subgroup H_2 of \mathcal{F} has *property (B)* (relative to H_1) if the core graph $\Psi(H_2)$ of H_2 has the original property (B) in which the graphs Y_1 and Y_2 are replaced with core graphs $\Psi(H_1)$ and $\Psi(H_2)$, resp., i.e., $\bar{r}(H_2) = -\chi(\Psi(H_2)) > 0$ and the map

$$\tau_2: \text{core}(\Psi(H_1) \times \Psi(H_2)) \longrightarrow \Psi(H_2)$$

is surjective.

Let $d \geq 3$ be an integer. Analogously, we say that a finitely generated factor-free subgroup H_2 of \mathcal{F} has *property (Bd)* (relative to H_1) if the core graph $\Psi(H_2)$ of H_2 has the original property (Bd) in which the graphs Y_1 and Y_2 are replaced with core graphs $\Psi(H_1)$ and $\Psi(H_2)$, resp., i.e.,

$$\bar{r}(H_2) = -\chi(\Psi(H_2)) > 0, \quad \text{deg } \Psi(H_2) \leq d$$

and the map $\tau_2: \text{core}(\Psi(H_1) \times \Psi(H_2)) \rightarrow \Psi(H_2)$ is surjective.

Recall that if Γ is a finite graph then $\text{deg } \Gamma$ is the maximum degree of a vertex of Γ .

Lemma 5.1. *Suppose H_2 is a finitely generated factor-free subgroup of \mathcal{F} such that $\text{deg } \Psi(H_2) \leq d$, where $d \geq 3$ is an integer or $d = \infty$, $\bar{r}(H_2) = -\chi(\Psi(H_2)) > 0$, and the map*

$$\tau_2: \text{core}(\Psi(H_1) \times \Psi(H_2)) \longrightarrow \Psi(H_2)$$

is not surjective. Then there exists a finitely generated factor-free subgroup H_4 of \mathcal{F} with property (Bd) if $d < \infty$ or with property (B) if $d = \infty$ such that

$$\frac{\bar{r}(H_1, H_4)}{\bar{r}(H_4)} > \frac{\bar{r}(H_1, H_2)}{\bar{r}(H_2)}. \tag{69}$$

Proof. Recall that $\bar{r}(H_1, H_2) = \bar{r}(\text{core}(\Psi(H_1) \times \Psi(H_2)))$ and $\bar{r}(H_i) = \bar{r}(\Psi(H_i))$, $i = 1, 2$. If $\bar{r}(\text{core}(\Psi(H_1) \times \Psi(H_2))) = 0$, then we may take $H_4 = H_1$ and the inequality (69) holds. Assume that $\bar{r}(\text{core}(\Psi(H_1) \times \Psi(H_2))) > 0$ and that the map

$$\tau_2: \text{core}(\Psi(H_1) \times \Psi(H_2)) \longrightarrow \Psi(H_2)$$

is not surjective. Consider the subgraph $\Gamma := \tau_2(\text{core}(\Psi(H_1) \times \Psi(H_2)))$ of $\Psi(H_2)$. It follows from the definitions and assumptions that $\bar{r}(\Gamma) < \bar{r}(\Psi(H_2))$ and

$$\bar{r}(\text{core}(\Psi(H_1) \times \Gamma)) = \bar{r}(\text{core}(\Psi(H_1) \times \Psi(H_2))) > 0,$$

whence $\bar{r}(\Gamma) > 0$. It is also clear that $\text{core}(\Gamma) = \Gamma$. Therefore,

$$\frac{\bar{r}(\text{core}(\Psi(H_1) \times \Gamma))}{\bar{r}(\Gamma)} > \frac{\bar{r}(\text{core}(\Psi(H_1) \times \Psi(H_2)))}{\bar{r}(\Psi(H_2))}. \tag{70}$$

Let $\Gamma_1, \dots, \Gamma_k$ be connected components of the graph Γ . Since

$$\bar{r}(\text{core}(\Psi(H_1) \times \Gamma)) > 0,$$

it follows that $\bar{r}(\Gamma) > 0$. Note that the graph

$$\text{core}(\Psi(H_1) \times \Gamma)$$

consists of disjoint graphs $\text{core}(\Psi(H_1) \times \Gamma_j)$, $j = 1, \dots, k$. In particular,

$$\bar{r}(\Gamma) = \sum_{j=1}^k \bar{r}(\Gamma_j), \quad \bar{r}(\text{core}(\Psi(H_1) \times \Gamma)) = \sum_{j=1}^k \bar{r}(\text{core}(\Psi(H_1) \times \Gamma_j)),$$

hence,

$$\frac{\bar{r}(\text{core}(\Psi(H_1) \times \Gamma))}{\bar{r}(\Gamma)} = \frac{\sum_{j=1}^k \bar{r}(\text{core}(\Psi(H_1) \times \Gamma_j))}{\sum_{j=1}^k \bar{r}(\Gamma_j)}. \tag{71}$$

Note that if $\bar{r}(\Gamma_j) = 0$ then $\bar{r}(\text{core}(\Psi(H_1) \times \Gamma_j)) = 0$.

Let Γ_{j^*} be chosen so that $\bar{r}(\Gamma_{j^*}) > 0$ and the ratio

$$\frac{\bar{r}(\text{core}(\Psi(H_1) \times \Gamma_{j^*}))}{\bar{r}(\Gamma_{j^*})}$$

is maximal over those graphs Γ_j with $\bar{r}(\Gamma_j) > 0$. It follows from

$$\bar{r}(\Gamma) = \sum_{j=1}^k \bar{r}(\Gamma_j) > 0$$

that such j^* does exist. It is not difficult to see that

$$\frac{\sum_{j=1}^k \bar{r}(\text{core}(\Psi(H_1) \times \Gamma_j))}{\sum_{j=1}^k \bar{r}(\Gamma_j)} \leq \frac{\bar{r}(\text{core}(\Psi(H_1) \times \Gamma_{j^*}))}{\bar{r}(\Gamma_{j^*})}.$$

This, together with (70) and (71), implies that

$$\frac{\bar{r}(\text{core}(\Psi(H_1) \times \Gamma_{j^*}))}{\bar{r}(\Gamma_{j^*})} \geq \frac{\bar{r}(\text{core}(\Psi(H_1) \times \Gamma))}{\bar{r}(\Gamma)} > \frac{\bar{r}(\text{core}(\Psi(H_1) \times \Psi(H_2)))}{\bar{r}(\Psi(H_2))}.$$

Hence, picking an arbitrary primary vertex $v \in V_P \Gamma_{j^*}$ in Γ_{j^*} as a base vertex, and letting $H_4 := H(\Gamma_{j^*, v})$, as in Lemma 2.2, we obtain a subgroup H_4 with the desired inequality (69). □

Lemma 5.2. *The supremum*

$$\sup_{H_3} \left\{ \frac{\bar{r}(H_1, H_3)}{\bar{r}(H_3)} \right\}$$

over all finitely generated factor-free subgroups H_3 of \mathcal{F} such that $\bar{r}(H_3) > 0$ and $\text{deg } \Psi(H_3) \leq d$, where $d \geq 3$ is an integer or $d = \infty$, is equal to $\sup_{H_2} \left\{ \frac{\bar{r}(H_1, H_2)}{\bar{r}(H_2)} \right\}$ over all finitely generated factor-free subgroups H_2 of \mathcal{F} that possess property (Bd) when $d < \infty$ or property (B) when $d = \infty$, and satisfy the condition $I(H_2) \subseteq I(H_1)$. In particular, we have

$$\begin{aligned} \sigma_d(H_1)\bar{r}(H_1) &= \sigma_d(\Psi(H_1))\bar{r}(\Psi(H_1)), \\ \sigma(H_1)\bar{r}(H_1) &= \sigma(\Psi(H_1))\bar{r}(\Psi(H_1)). \end{aligned}$$

Proof. The first claim follows from Lemma 5.1 and the observation that if the map

$$\tau_2: \text{core}(\Psi(H_1) \times \Psi(H_2)) \longrightarrow \Psi(H_2)$$

is surjective then $I(H_2) \subseteq I(H_1)$. The equalities follow from the first claim, the definitions of the numbers $\sigma_d(H_1)$, $\sigma(H_1)$, $\sigma_d(\Psi(H_1))$, $\sigma(\Psi(H_1))$, and Lemma 5.1. □

In view of Lemma 5.2, when investigating the supremum

$$\sup_{H_3} \left\{ \frac{\bar{r}(H_1, H_3)}{\bar{r}(H_3)} \right\}$$

over all finitely generated factor-free subgroups H_3 of \mathcal{F} with $\bar{r}(H_3) > 0$ and $\deg \Psi(H_3) \leq d$, we may assume that the index set I is finite, i.e., $I = I(H_1)$, say, $I = \{1, \dots, m\}$, and so $\mathcal{F} = G_1 * G_2 * \dots * G_m$.

Furthermore, in order to be able to make use of results of Sections 3 and 4, we consider \mathcal{F} as the following free product

$$\mathcal{F}_2(1) = G_1 * G(2, m)$$

of two groups G_1 and $G(2, m) := G_2 * \dots * G_m$. Let $g_\alpha \in G_\alpha$ be some nontrivial element of G_α , $\alpha \in I = \{1, \dots, m\}$. For every $a_\alpha \in G_\alpha$, consider the map

$$a_\alpha \mapsto (g_{\alpha+1} \dots g_m g_1 \dots g_\alpha)^{-1} a_\alpha g_{\alpha+1} \dots g_m g_1 \dots g_\alpha, \tag{72}$$

where $g_{\alpha+1} \dots g_m g_1 \dots g_\alpha$ is a cyclic permutation of the word $g_1 g_2 \dots g_m$.

Recall that a subgroup K of a group G is called *antinormal* if, for every $g \in G$, $gKg^{-1} \cap K \neq \{1\}$ implies $g \in K$.

Lemma 5.3. *Let $|I| = m \geq 3$ and let H_1 be a finitely generated factor-free subgroup of \mathcal{F} . Then the map (72) extends to monomorphisms*

$$\mu: \mathcal{F} \longrightarrow \mathcal{F}, \quad \mu_2: \mathcal{F} \longrightarrow \mathcal{F}_2(1)$$

that have the following properties.

- (a) A word $U \in \mathcal{F}$ with $|U| > 1$ is cyclically reduced if and only if $\mu(U)$ is cyclically reduced.
- (b) The subgroups $\mu_2(\mathcal{F})$ and $\mu(\mathcal{F})$ are antinormal in $\mathcal{F}_2(1)$ and \mathcal{F} , resp.
- (c) $\mu_2(H_1)$ is a factor-free subgroup of $\mathcal{F}_2(1)$ and $\mu(H_1)$ is factor-free in \mathcal{F} . Furthermore, $\deg \Psi(H_1) = \deg \Psi(\mu_2(H_1))$.
- (d) If K_1 and K_2 are finitely generated factor-free subgroups of \mathcal{F} , then

$$\bar{r}(K_1, K_2) = \bar{r}(\mu_2(K_1), \mu_2(K_2)).$$

- (e) The supremum

$$\sup_{H_2} \left\{ \frac{\bar{r}(H_1, H_2)}{\bar{r}(H_2)} \right\}$$

over all finitely generated factor-free subgroups H_2 of \mathcal{F} such that $\bar{r}(H_2) > 0$ and $\deg \Psi(H_2) \leq d$, where $d \geq 3$ is an integer, does not exceed the supremum

$$\sup_{K_2} \left\{ \frac{\bar{r}(\mu_2(H_1), K_2)}{\bar{r}(K_2)} \right\}$$

over all finitely generated factor-free subgroups K_2 of $\mathcal{F}_2(1)$ with property (Bd) relative to $\mu_2(H_1)$. In particular,

$$\sigma_d(H_1) \leq \sigma_d(\mu_2(H_1)) \quad \text{and} \quad \sigma(H_1) \leq \sigma(\mu_2(H_1)).$$

Proof. It is clear that the map (72) extends to homomorphisms

$$\mu: \mathcal{F} \longrightarrow \mathcal{F}, \quad \mu_2: \mathcal{F} \longrightarrow \mathcal{F}_2(1).$$

Note that if $a_1 \in G_{\alpha_1}$ and $a_2 \in G_{\alpha_2}$ are nontrivial elements and $\alpha_1 \neq \alpha_2$, then $\mu(a_1)\mu(a_2)$ is a cyclically reduced word. This remark implies that the kernels of the maps μ, μ_2 are trivial, whence μ, μ_2 are monomorphisms.

(a) It follows from the foregoing remark that a word $U \in \mathcal{F}$ with $|U| > 1$ is cyclically reduced if and only if $\mu(U)$ is cyclically reduced.

(b) Let $U_1, U_2 \in \mathcal{F}$ be reduced words and $W\mu(U_1)W^{-1} = \mu(U_2)$ in \mathcal{F} . Using induction on $|U_1| + |U_2|$, we will prove that $W \in \mu(\mathcal{F})$.

Suppose U_1 is not cyclically reduced and

$$U_1 \equiv a_1 U_3 a_2,$$

where $a_1, a_2 \in G_\alpha \setminus \{1\}$ are letters of U_1 . Then we can replace U_1 with $U'_1 := U_3 a_3$, where $a_3 \in G_\alpha, a_3 = a_2 a_1$ in G_α if $a_3 \neq 1$ or with $U'_1 := U_3$ if $a_3 = 1$, and we replace W with $W' := W\mu_2(a_1)$. This way we obtain an equality

$$W'\mu(U'_1)(W')^{-1} \stackrel{0}{=} \mu(U_2)$$

in \mathcal{F} in which $|U'_1| + |U_2| < |U_1| + |U_2|$. Hence, it follows from the induction hypothesis that $W' \in \mu(\mathcal{F})$, as required. If U_2 is not cyclically reduced, then, analogously to what we did above for U_1 , we can decrease the sum $|U_1| + |U_2|$ and use the induction hypothesis.

Thus we may assume that both words U_1, U_2 are cyclically reduced. By part (a), the words $\mu(U_1), \mu(U_2)$ are also cyclically reduced. Observe that if

$$WV_1W^{-1} \stackrel{0}{=} V_2$$

in \mathcal{F} , where V_1, V_2 are cyclically reduced and W is reduced, then V_2 is a cyclic permutation of V_1 . More specifically, there is a factorization

$$V_1 \equiv V_{11}V_{12}$$

and an integer k such that if $k \geq 0$ then $W \equiv V_{12}V_1^k$ and if $k \leq 0$ then $W \equiv V_{11}^{-1}V_1^k$. In either case, $V_2 \equiv V_{12}V_{11}$. Applying this observation to the equality

$$W\mu(U_1)W^{-1} \stackrel{0}{=} \mu(U_2)$$

in \mathcal{F} , we can see from (72), when $m \geq 3$, that a cyclic permutation of $\mu(U_1)$ equal to $\mu(U_2)$ must have the form $\mu(\bar{U}_1)$, where \bar{U}_1 is a cyclic permutation of U_1 . For similar reasons, $W \equiv \mu(V)$ for some $V \in \mathcal{F}$ and part (b) is proven for the subgroup $\mu(\mathcal{F})$. It now follows that $\mu_2(\mathcal{F})$ is also antinormal in $\mathcal{F}_2(1)$.

(c) Arguing on the contrary, suppose H is a factor-free subgroup of \mathcal{F} and one of $\mu(H)$, $\mu_2(H)$ is not factor-free in \mathcal{F} , $\mathcal{F}_2(1)$, resp. Then it follows from the definitions that $\mu_2(H)$ is not factor-free in $\mathcal{F}_2(1)$. Hence, there is a reduced word U such that U is not conjugate in \mathcal{F} to a word of length ≤ 1 and

$$\mu(U) \stackrel{0}{=} WVW^{-1} \tag{73}$$

in \mathcal{F} , where W is either empty or reduced and V is either a letter of $G_1 \setminus \{1\}$ or V is a reduced word with no letters of G_1 . Thus, V is reduced and either $V \in G_1$ or $V \in G(2, m)$.

Assume that the word U in (73) is not cyclically reduced. Then

$$U \equiv a_1U_1a_2,$$

where $a_1, a_2 \in G_\alpha \setminus \{1\}$ are letters of U . If $a_1a_2 = a_3$ in G_α and $a_3 \in G_\alpha \setminus \{1\}$, then the word $U' \equiv U_1a_3$, similarly to U , is not conjugate to \mathcal{F} to a word of length ≤ 1 and $\mu(U')$, being conjugate to $\mu(U)$ in \mathcal{F} , has a representation of the form (73), so U can be replaced with U' . If $a_1a_2 = 1$ in G_α , then the word U_1 can be taken as U . Hence, by induction on $|U|$, we may assume that U is cyclically reduced.

If the word WVW^{-1} in (73) is not reduced, then there are words W', V' such that

$$\mu(U) \stackrel{0}{=} W'V'(W')^{-1},$$

W', V' have the foregoing properties of W, V , resp., and

$$2|W'| + |V'| < 2|W| + |V|.$$

Indeed, if, say $W \equiv W_1a_1$ and $V \equiv a_2V_1$, where $a_1, a_2 \in G_\alpha \setminus \{1\}$, then we set $W' := W_1$ and V' is a reduced word equal in \mathcal{F} to $a_1a_2V_1a_1^{-1}$. Note that W', V' have the foregoing properties of W, V , resp., and

$$|W'| = |W| - 1, \quad |V'| \leq |V| + 1,$$

whence $2|W'| + |V'| < 2|W| + |V|$. Thus, by induction on $2|W| + |V|$, we may assume that the word WVW^{-1} in (73) is reduced.

Since U is cyclically reduced and $|U| > 1$, it follows from part (a) that $\mu(U)$ is cyclically reduced. Hence, the word W is empty and $\mu(U) \equiv V$, where V is a single letter of $G_1 \setminus \{1\}$ or V has no letters of G_1 . However, neither situation is possible by the definition (72). This contradiction completes the proof of the first statement of part (c).

Now we will prove the equality

$$\deg \Psi(H_1) = \deg \Psi(\mu_2(H_1))$$

of part (c). It follows from the definition (72) that the graph $\Psi(\mu_2(H))$ can be visualized as a graph obtained from $\Psi(H)$ by subdivision of edges of $\Psi(H)$ into

paths in accordance with formula (72) and subsequent “mergers” of edges that have labels in $G_2 \cup \dots \cup G_m$. In particular, for every vertex $v \in V\Psi(H)$ with $\deg v > 2$, there will be a unique vertex $u = u(v) \in V_S\Psi(\mu_2(H))$ of degree $\deg u = \deg v$ and this map $v \mapsto u(v)$ is bijective on the sets of all vertices of $\Psi(H)$, $\Psi(\mu_2(H))$ of degree > 2 . Hence, the maximal degree of vertices of $\Psi(\mu_2(H))$ is equal to that of $\Psi(H)$, as claimed.

(d) By part (c), the subgroups $\mu_2(K_1), \mu_2(K_2)$ of $\mathcal{F}_2(1)$ are factor-free and the subgroups $\mu(K_1), \mu(K_2)$ of \mathcal{F} are also factor-free. Let

$$T(\mu_2(K_1), \mu_2(K_2))$$

be a set of representatives of those double cosets $\mu_2(K_1)U\mu_2(K_2)$ of $\mathcal{F}_2(1)$, where $U \in \mathcal{F}_2(1)$, that have the property

$$\mu_2(K_1) \cap U\mu_2(K_2)U^{-1} \neq \{1\}.$$

If $T \in T(\mu_2(K_1), \mu_2(K_2))$, then it follows from the definition of the set $T(\mu_2(K_1), \mu_2(K_2))$ that there are nontrivial $V_i \in K_i, i = 1, 2$, such that

$$T\mu_2(V_2)T^{-1} = \mu_2(V_1) \neq 1$$

in $\mathcal{F}_2(1)$. By part (b), such an equality implies $T \in \mu_2(\mathcal{F})$ (note μ_2 could be replaced with μ). Now we can see that there is a set $S(K_1, K_2) \subseteq \mathcal{F}$ such that

$$\mu_2(S(K_1, K_2)) = T(\mu_2(K_1), \mu_2(K_2))$$

and $S(K_1, K_2)$ is a set of representatives of those double cosets K_1SK_2 of \mathcal{F} , $S \in \mathcal{F}$, that have the property $K_1 \cap SK_2S^{-1} \neq \{1\}$. Therefore,

$$\bar{r}(K_1, K_2) := \sum_{S \in S(K_1, K_2)} \bar{r}(K_1 \cap SK_2S^{-1}) = \bar{r}(\mu_2(K_1), \mu_2(K_2)),$$

as desired.

(e) This follows from Lemma 5.2, parts (c)–(d) and definitions. □

6. Proofs of theorems

For the reader’s convenience, we restate Theorems 1.1–1.3 before proving them.

Theorem 1.1. *Suppose that $\mathcal{F} = G_1 * G_2$ is the free product of two nontrivial groups G_1, G_2 and H_1 is a finitely generated factor-free noncyclic subgroup of \mathcal{F} . Then the following are true.*

- (a) For every integer $d \geq 3$, there exists a linear programming problem (LP-problem)

$$\mathcal{P}(H_1, d) = \max\{c(d)x(d) \mid A(d)x(d) \leq b(d)\} \tag{8}$$

with integer coefficients whose solution is equal to $-\sigma_d(H_1)\bar{r}(H_1)$.

- (b) There is a finitely generated factor-free subgroup H_2^* of \mathcal{F} , $H_2^* = H_2^*(H_1)$, such that H_2^* corresponds to a vertex solution of the dual problem

$$\mathcal{P}^*(H_1, d) = \min\{b(d)^\top y(d) \mid A(d)^\top y(d) = c(d)^\top, y(d) \geq 0\}$$

of the primal LP-problem (8) of part (a) and

$$\bar{r}(H_1, H_2^*) = \sigma_d(H_1)\bar{r}(H_1)\bar{r}(H_2^*).$$

In particular, the WN_d -coefficient $\sigma_d(H_1)$ of H_1 is rational.

Furthermore, if $\Psi(H_1)$ and $\Psi(H_2^*)$ denote irreducible core graphs representing subgroups H_1 and H_2^* , resp., and $|E\Psi|$ is the number of oriented edges in the graph Ψ , then

$$|E\Psi(H_2^*)| < 2^{2^{|E\Psi(H_1)|/4 + \log_2 \log_2(4d)}}.$$

- (c) There exists a linear semi-infinite programming problem (LSIP-problem) $\mathcal{P}(H_1) = \sup\{cx \mid Ax \leq b\}$ with finitely many variables in x and with countably many constraints in the system $Ax \leq b$ whose dual problem

$$\mathcal{P}^*(H_1) = \inf\{b^\top y \mid A^\top y = c^\top, y \geq 0\}$$

has a solution equal to $-\sigma(H_1)\bar{r}(H_1)$.

- (d) Let the word problem for both groups G_1, G_2 be solvable and let an irreducible core graph $\Psi(H_1)$ of H_1 be given. Then the LP-problem (8) of part (a) can be algorithmically written down and the WN_d -coefficient $\sigma_d(H_1)$ for H_1 can be computed. In addition, an irreducible core graph $\Psi(H_2^*)$ of the subgroup H_2^* of part (b) can be algorithmically constructed.
- (e) Let both groups G_1 and G_2 be finite, let $d_m := \max(|G_1|, |G_2|) \geq 3$, and let an irreducible core graph $\Psi(H_1)$ of H_1 be given. Then the LP-problem (8) of part (a) for $d = d_m$ coincides with the LSIP-problem $\mathcal{P}(H_1)$ of part (c) and the WN -coefficient $\sigma(H_1)$ for H_1 is rational and computable.

Proof of Theorem 1.1. We start with part (a). Assume that

$$I = \{1, 2\}, \quad \mathcal{F} = G_1 * G_2$$

and H_1 is a finitely generated factor-free noncyclic subgroup of \mathcal{F} . As in Section 2, let $\Psi_o(H_1)$ denote a finite irreducible \mathcal{A} -graph of H_1 and let $\Psi(H_1)$ denote the core of $\Psi_o(H_1)$. Conjugating H_1 if necessary, we may assume that $\Psi_o(H_1) = \Psi(H_1)$.

Denote $Y_1 := \Psi(H_1)$ and pick an integer $d \geq 3$. As in Sections 3 and 4, consider the system of linear inequalities $\text{SLI}_d[Y_1]$, see (16), and the LP-problem

$$\max\{-x_s \mid \text{SLI}_d[Y_1]\}. \tag{74}$$

According to Lemma 4.2, the maximum of the LP-problem (74) is equal to

$$-\sigma_d(Y_1)\bar{r}(Y_1),$$

where

$$\sigma_d(Y_1)\bar{r}(Y_1) = \sup_{Y_2} \left\{ \frac{\bar{r}(\text{core}(Y_1 \times Y_2))}{\bar{r}(Y_2)} \right\}$$

over all finite irreducible \mathcal{A} -graphs Y_2 with property (Bd) relative to Y_1 . By Lemma 5.2, we have

$$\sigma_d(Y_1)\bar{r}(Y_1) = \sigma_d(H_1)\bar{r}(H_1),$$

as desired in part (a). Part (a) is proven.

We will continue to use below the notation introduced in the proof of part (a).

Part (b) follows from Lemmas 4.2, 4.3 and their proofs in which the construction of the graph Y_{2, \mathcal{Q}_V} is based on a vertex solution y_V to the dual LP-problem (30). To define the desired subgroup H_2^* of \mathcal{F} for H_1 , we can use the graph Y_{2, \mathcal{Q}_V} of Lemma 4.3 as an irreducible \mathcal{A} -graph $\Psi_{o^*}(H_2^*)$. By Lemmas 4.2, 4.3, and 5.2, the subgroup H_2^* has all of the desired properties. Part (b) is proven.

To prove part (c), we note that it follows from Lemmas 4.2 and 5.2 that the dual problem (29) of the LSIP-problem $\sup\{-x_s \mid \text{SLI}[Y_1]\}$, where $Y_1 = \Psi(H_1)$ as above, has the infimum equal to $-\sigma(Y_1)\bar{r}(Y_1) = -\sigma(H_1)\bar{r}(H_1)$. This proves part (c).

Now we turn to parts (d)–(e) of Theorem 1.1. First we discuss how to algorithmically write down inequalities of the system $\text{SLI}_d[Y_1]$, where $d \geq 3$ is a fixed integer. Recall that every inequality of $\text{SLI}[Y_1]$ is written in the form (13)–(14) and there are finitely many subsets $A \subseteq S_1(V_P Y_1)$ that are indices of k variables $\pm x_A$ in the left hand sides of inequalities (13) and (14), where $2 \leq k = |T| \leq d$.

The coefficient of x_s is the integer $-(k-2)$ and the right hand side of (13) and (14) is an integer $-N(\Omega_T)_\alpha$, where

$$0 \leq N(\Omega_T)_\alpha \leq (d-2)|V_P Y_1|,$$

see (12). This information is sufficient to conclude that the set of inequalities in the system $SLI_d[Y_1]$ is finite. However, this information is not sufficient to algorithmically write down inequalities of $SLI_d[Y_1]$ because the set of available sets T is infinite whenever the union $G_1 \cup G_2$ is infinite.

To algorithmically write down the system $SLI_d[Y_1]$, we assume that the word problem for both groups G_1, G_2 is solvable and we will look more closely into the definition of inequalities (13) and (14).

Recall that inequalities (13) and (14) are defined in Section 3 by using an α -admissible function $\Omega_T: T \rightarrow S_1(V_P Y_1)$, where $T \in S_2(G_\alpha)$ and $\alpha \in I = \{1, 2\}$.

We also recall that \sim_{Ω_T} denotes an equivalence relation on the set of all pairs (a, u) , where $a \in T$ and $u \in \Omega_T(a)$, see Section 3. Making use of the equivalence relation \sim_{Ω_T} , we define a relation \approx on the set T so that $a \approx b$ if and only if there are $u \in \Omega_T(a)$ and $v \in \Omega_T(b)$ such that

$$(a, u) \sim_{\Omega_T} (b, v).$$

Note that this relation \approx is reflexive and symmetric. The transitive closure of the relation \approx is an equivalence relation on T which we denote by \approx_{Ω_T} . The equivalence class of $a \in T$ is denoted $[a]_{\approx_{\Omega_T}}$. It follows from the definition of $[a]_{\approx_{\Omega_T}}$ and from the property of being α -admissible for Ω_T that, for every $b_1 \in [a]_{\approx_{\Omega_T}}$, there is an element $b_2 \in [a]_{\approx_{\Omega_T}}$ such that $b_2 \neq b_1$ and there are edges $e_1, e_2 \in EY_1$ such that $(e_1)_+ = (e_2)_+ \in V_S Y_1$, the vertex $(e_1)_+$ has type α , $(e_i)_- \in \Omega_T(b_i), i = 1, 2$, and

$$b_1 b_2^{-1} = \varphi(e_1)\varphi(e_2)^{-1} \tag{75}$$

in G_α . Note that if we connect every two such elements $b_1, b_2 \in [a]_{\approx_{\Omega_T}}$ by an edge, then the graph $\Gamma(\Omega_T)$, whose vertex set is T , will have connected components whose vertex sets are equivalence classes $[a]_{\approx_{\Omega_T}}$ of T . This connectedness of subgraphs of $\Gamma(\Omega_T)$ on vertex sets $[a]_{\approx_{\Omega_T}}$ obviously implies the following.

Lemma 6.1. *Equations (75) can be used to determine all elements of the equivalence class $[a]_{\approx_{\Omega_T}}$ for given $a \in G_\alpha$.*

Proof. This easily follows from the definitions. Recall that the word problem is solvable in G_α . △

Let $C(\alpha, d)$ be a subset of G_α of cardinality

$$|C(\alpha, d)| = d^2 + d,$$

where $\alpha = 1, 2$ and $d \geq 3$ is a fixed integer. In the arguments below, this set $C(\alpha, d)$ will be held fixed. Note that if $|G_\alpha| < d^2 + d$, so it is not possible to choose $d^2 + d$ distinct elements in G_α , then all inequalities (13) if $\alpha = 1$ or (14) if $\alpha = 2$ for $k \leq d$, where as before $k = |T|$, can be written down effectively for the following reasons. The sets

$$S_2(G_\alpha) \quad \text{and} \quad \{\Omega_T \mid \Omega_T: T \longrightarrow S_1(V_P Y_1), T \in S_2(G_\alpha)\}$$

are finite, they can be written down explicitly, and it is possible to verify whether given function

$$\Omega_T: T \longrightarrow S_1(V_P Y_1)$$

is α -admissible.

Clearly, the same conclusion as above holds if both G_1, G_2 are finite but in the arguments below we will only need the equality $|C(\alpha, d)| = d^2 + d$, hence we can just assume that $|G_\alpha| \geq d^2 + d$.

Consider a subset $C \subset C(\alpha, d)$, where $1 \leq |C| \leq k \leq d$, and let $Z = \{z_1, \dots, z_{k-|C|}\}$ be a set of indeterminates. Note that $|C \cup Z| = k$. Consider a function

$$\Omega_{C \cup Z}: C \cup Z \longrightarrow S_1(V_P Y_1). \quad (76)$$

Similarly to the relation \sim_{Ω_T} defined in Section 3, we introduce a relation $\sim_{\Omega_{C \cup Z}}$ on the set of all pairs (a, u) , where $a \in C \cup Z$ and $u \in \Omega_{C \cup Z}(a)$, defined as follows. Two pairs (a, u) and (b, v) are related by $\sim_{\Omega_{C \cup Z}}$ if and only if either $(a, u) = (b, v)$ or, otherwise, there exist edges $e, f \in EY_1$ such that $e_- = u$, $f_- = v$ and the secondary vertex $e_+ = f_+$ has type α .

We also consider an analogue \approx_Z of the relation \approx defined above so that $a \approx_Z b$, where $a, b \in C \cup Z$, if and only if there are

$$u \in \Omega_{C \cup Z}(a), \quad v \in \Omega_{C \cup Z}(b)$$

such that $(a, u) \sim_{\Omega_{C \cup Z}} (b, v)$. As before, the relation \approx_Z is reflexive and symmetric. By taking the transitive closure of the relation \approx_Z we obtain an equivalence relation on the set $C \cup Z$ which is denoted by $\approx_{\Omega_{C \cup Z}}$.

We will say that a function $\Omega_{C \cup Z}$, as in (76), is *unacceptable* if there is an equivalence class $[(a, u)]_{\sim_{\Omega_{C \cup Z}}}$ of $\sim_{\Omega_{C \cup Z}}$ with a single element in it or there is an equivalence class $[a]_{\approx_{\Omega_{C \cup Z}}}$ of the relation $\approx_{\Omega_{C \cup Z}}$ that contains no elements of C . Note that, when given a function $\Omega_{C \cup Z}$ as in (76), we can algorithmically check whether or not $\Omega_{C \cup Z}$ is unacceptable.

If now the function $\Omega_{C \cup Z}$ is not found to be unacceptable, then we attempt to construct a function

$$\zeta: Z \longrightarrow G_\alpha$$

by using the following algorithm.

First, we set $\zeta_0(c) := c$ if $c \in C$ and let

$$C_0 := C, \quad Z_0 := \emptyset.$$

Consider the set of all triples (a, u, ℓ) , where $a \in C \cup Z$, $u \in \Omega_{C \cup Z}(a)$, $1 \leq \ell \leq d + 1$, and do the following. By induction on $i \geq 0$, assume that the sets

$$C_i \subseteq G_\alpha, \quad Z_i \subseteq Z$$

are constructed and a bijective function

$$\zeta_i: C_0 \cup Z_i \longrightarrow C_i$$

is defined so that the restriction of ζ_i on C_0 is ζ_0 . For every unordered pair $\{(a, u, \ell), (b, v, \ell)\}$ of distinct triples with a fixed ℓ (first we use $\ell = 1$, then $\ell = 2$ and so on up to $\ell = d + 1$), we check whether there are edges $e, f \in EY_1$ such that

$$e_- = u, \quad f_- = v, \quad e_+ = f_+,$$

and $e_+ = f_+ \in V_S Y_1$ has type α . If there are no such edges, then we pass on to the next pair $\{(a, u, \ell), (b, v, \ell)\}$. If there are such edges e, f , then we consider three Cases 1–3 below, perform the described actions and pass on to the next pair. We remark that these actions can be algorithmically implemented as follows from the solvability of the word problem for groups G_1, G_2 and the availability of the graph $Y_1 = \Psi(H_1)$.

Case 1. If both $a, b \in C \cup Z_i$, then we check whether the equality

$$\zeta_i(a)\zeta_i(b)^{-1} = \varphi(e)\varphi(f)^{-1}$$

holds in G_α . If this equality is false, then we conclude that the function $\Omega_{C \cup Z}$ is unacceptable and stop. Otherwise, we set

$$Z_{i+1} := Z_i, \quad C_{i+1} := C_i, \quad \zeta_{i+1} := \zeta_i.$$

Case 2. Suppose that exactly one of a, b is in $C \cup Z_i$, say $b \in C \cup Z_i$. Then it is clear that $a \in Z \setminus Z_i$ and we can uniquely determine an element $\xi(a)$ by solving the equation $\xi(a)\zeta_i(b)^{-1} = \varphi(e)\varphi(f)^{-1}$. If $\xi(a) \in C_i$, then we conclude that the function $\Omega_{C \cup Z}$ is unacceptable and stop. Otherwise, we set

$$Z_{i+1} := Z_i \cup \{a\}, \quad C_{i+1} := C_i \cup \{\xi(a)\}$$

and define a function ζ_{i+1} on the set $C \cup Z_{i+1}$ so that $\zeta_{i+1}(a) := \xi(a)$ and the restriction of ζ_{i+1} on $C \cup Z_i$ is ζ_i .

Case 3. If both $a, b \notin C \cup Z_i$, then we set

$$Z_{i+1} := Z_i, \quad C_{i+1} := C_i, \quad \zeta_{i+1} := \zeta_i.$$

Cases 1–3 are complete.

Since every equivalence class $[a]_{\approx_{\Omega_{C \cup Z}}}$ contains an element of C , it follows from the definitions that while this algorithm runs over all pairs for a fixed $\ell' = 1, \dots, d$, one of the following three Cases (C1)–(C3) will occur.

(C1) For some i , $|Z_{i+1}| = |Z_i| + 1$.

(C2) The set $\Omega_{C \cup Z}$ is found to be unacceptable.

(C3) For the index i , corresponding to the last pair $\{(a, u, \ell'), (b, v, \ell')\}$ for parameter ℓ equal to ℓ' , one has $Z_i = Z$.

Since $|Z| \leq d - 1$, we can see that it is not possible for Case (C1) to occur for all $\ell' = 1, \dots, d$. Hence, running this algorithm consecutively for $\ell' = 1, \dots, d$, results either in conclusion that the function $\Omega_{C \cup Z}$ is unacceptable or in construction of a bijective function

$$\zeta = \zeta_i: C \cup Z \longrightarrow C_i \subseteq G_\alpha,$$

where $Z_i = Z$, in which case we say that the function $\Omega_{C \cup Z}$ is *acceptable*. Furthermore, setting

$$T := \zeta(C \cup Z) \quad \text{and} \quad \Omega_T(\zeta(a)) := \Omega_{C \cup Z}(a)$$

for every $a \in C \cup Z$, we obtain an α -admissible function Ω_T on the set T , $T \subseteq G_\alpha$.

Observe that the set of all such functions

$$\Omega_{C \cup Z}: C \cup Z \longrightarrow S_1(V_P Y_1),$$

where $C \subseteq C(\alpha, d)$ and $Z = \{z_1, \dots, z_{k-|C|}\}$, see (76), is finite (recall the set $C(\alpha, d)$ is fixed) and that all such functions can be written down explicitly. Moreover, using the foregoing algorithm, we can verify whether a function $\Omega_{C \cup Z}$ is acceptable and, when doing so, construct a unique function

$$\zeta: C \cup Z \longrightarrow S_1(V_P Y_1),$$

where $T := \zeta(C \cup Z)$, so that $\Omega_T(\zeta(a)) := \Omega_{C \cup Z}(a)$ for every $a \in C \cup Z$ and $\zeta(c) = c$ if $c \in C$. Therefore, in order to establish that inequalities (13)–(14) can be algorithmically written down, it remains to prove the following.

Lemma 6.2. *For every α -admissible function*

$$\Omega_{T'}: T' \longrightarrow S_1(V_P Y_1),$$

where $T' \subset G_\alpha$ and $2 \leq |T'| = k \leq d$, there exists an acceptable function

$$\Omega_{C \cup Z}: C \cup Z \longrightarrow S_1(V_P Y_1),$$

where $C \subset C(\alpha, d)$ and $Z = \{z_1, \dots, z_{k-|C|}\}$, with the following property.

Let $T := \zeta(C \cup Z)$ and let

$$\Omega_T: T \longrightarrow S_1(V_P Y_1)$$

be the α -admissible function, defined by $\Omega_T(\zeta(a)) := \Omega_{C \cup Z}(a)$ for every $a \in C \cup Z$ and $\zeta(c) = c$ for $c \in C$. Then the two inequalities (13), that correspond to $\Omega_{T'}$ and to Ω_T if $\alpha = 1$, or the two inequalities (14), that correspond to $\Omega_{T'}$ and to Ω_T if $\alpha = 2$, are identical.

To prove Lemma 6.2, we first establish an auxiliary lemma.

Lemma 6.3. *Suppose*

$$\Omega_{T'}: T' \longrightarrow S_1(V_P Y_1)$$

is an α -admissible function, where $2 \leq |T'| \leq d$, and $T' = E_1 \cup \dots \cup E_r$ is a partition of T' into equivalence classes $[a]_{\approx_{\Omega_{T'}}$ of the equivalence relation $\approx_{\Omega_{T'}}$. Then there are elements $h_1, \dots, h_r \in G_\alpha$ such that the set

$$T := E_1 h_1 \cup \dots \cup E_r h_r$$

has the cardinality $|T| = |T'|$ and every set $E_i h_i$, $i = 1, \dots, r$, contains an element from the set $C(\alpha, d)$.

Proof. By induction on i , where $1 \leq i \leq r$, we will prove the existence of elements $h_1, \dots, h_i \in G_\alpha$ with the property that the set $E_1 h_1 \cup \dots \cup E_i h_i$ has the cardinality

$$\sum_{j=1}^i |E_j h_j|$$

and every set $E_j h_j$, $j = 1, \dots, i$, contains an element from $C(\alpha, d)$.

If $i = 1$, then we set $h_1 := b^{-1}c$, where $b \in E_1$ and $c \in C(\alpha, d)$.

Making the induction hypothesis, assume that there are elements $h_1, \dots, h_i \in G_\alpha$ with the desired properties.

To make the induction step from i to $i + 1$, denote

$$C_i(\alpha, d) := C(\alpha, d) \cap (E_1 h_1 \cup \cdots \cup E_i h_i)$$

and let $b \in E_{i+1}$. For an element $c \in C(\alpha, d) \setminus C_i(\alpha, d)$, we consider the set

$$R_c := E_{i+1} b^{-1} c.$$

Clearly, R_c contains an element from $C(\alpha, d)$ and if R_c is disjoint from the set $E_1 h_1 \cup \cdots \cup E_i h_i$, then we can set

$$h_{i+1} := b^{-1} c.$$

Therefore, we may assume that R_c contains an element from $E_1 h_1 \cup \cdots \cup E_i h_i$ for every $c \in C(\alpha, d) \setminus C_i(\alpha, d)$.

Suppose that elements in $E_1 h_1 \cup \cdots \cup E_i h_i$ are indexed by integers from 1 to $|E_1 h_1 \cup \cdots \cup E_i h_i|$ and elements in $R_c = E_{i+1} b^{-1} c$, where b and c are chosen as above, are indexed by integers from 1 to $|E_{i+1}|$ so that, for every $e \in E_{i+1}$, the index of $eb^{-1}c \in R_c$ is equal to that of $e \in E_{i+1}$. In other words, we wish to keep indices stable when multiplying E_{i+1} by $b^{-1}c$.

Making use of these indices, we fix an element $b \in E_{i+1}$ and, for every

$$c \in C(\alpha, d) \setminus C_i(\alpha, d),$$

we consider the pair $(j_R(c), j_E(c))$ of indices $j_R(c), j_E(c)$ in $R_c = E_{i+1} b^{-1} c$ and in $E_1 h_1 \cup \cdots \cup E_i h_i$, resp., of an element of the intersection

$$R_c \cap (E_1 h_1 \cup \cdots \cup E_i h_i)$$

which is not empty as was assumed above.

Suppose that $(j_R(c_1), j_E(c_1)) = (j_R(c_2), j_E(c_2))$. Then it follows from the definitions that if $e_1, e_2 \in E_{i+1}$ are such that

$$e_1 b^{-1} c_1 \in R_{c_1} \cap (E_1 h_1 \cup \cdots \cup E_i h_i),$$

$$e_2 b^{-1} c_2 \in R_{c_2} \cap (E_1 h_1 \cup \cdots \cup E_i h_i),$$

then $e_1 = e_2$ and $e_1 b^{-1} c_1 = e_2 b^{-1} c_2$ in G_α . These equalities imply that $c_1 = c_2$. Therefore, for distinct elements $c_1, c_2 \in C(\alpha, d) \setminus C_i(\alpha, d)$, the pairs

$$(j_R(c_1), j_E(c_1)), \quad (j_R(c_2), j_E(c_2))$$

are also distinct. However, the number of elements c in $C(\alpha, d) \setminus C_i(\alpha, d)$ is

$$|C(\alpha, d)| - |C_i(\alpha, d)| \geq (d^2 + d) - d = d^2$$

and the number of all such pairs $(j_R(c), j_E(c))$ is less than d^2 . This contradiction completes the induction step and Lemma 6.3 is proved. \triangle

Proof of Lemma 6.2. Utilizing the notation of Lemma 6.3, we let

$$T' = E_1 \cup \dots \cup E_r$$

and let $h_1, \dots, h_r \in G_\alpha$ be elements such that the set

$$T := E_1 h_1 \cup \dots \cup E_r h_r$$

has cardinality $|T| = |T'| = k$ and every set $E_i h_i, i = 1, \dots, r$, contains an element from $C(\alpha, d)$.

Define a function

$$\widehat{\Omega}: T \longrightarrow S_1(V_P Y_1)$$

so that if $a \in E_i, i = 1, \dots, r$, then $\widehat{\Omega}(ah_i) := \Omega_{T'}(a)$.

Define $C := C(\alpha, d) \cap T$ and let $C = \{c_1, \dots, c_{|C|}\}$. Introducing more notation, denote

$$T = \{c_1, \dots, c_{|C|}, b_1, \dots, b_{k-|C|}\}$$

and $Z = \{z_1, \dots, z_{k-|C|}\}$.

We also define a function

$$\Omega_{C \cup Z}: C \cup Z \longrightarrow S_1(V_P Y_1)$$

by setting $\Omega_{C \cup Z}(c_i) := \widehat{\Omega}(c_i)$ and $\Omega_{C \cup Z}(z_j) := \widehat{\Omega}(b_j)$ for all i, j . In view of Lemma 6.1, it is not difficult to see that the function $\Omega_{C \cup Z}$ is acceptable, $\zeta(C \cup Z) = T$, and if

$$\Omega_T: T \longrightarrow S_1(V_P Y_1)$$

is the function defined by $\Omega_T(\zeta(a)) := \Omega_{C \cup Z}(a)$ for every $a \in C \cup Z$, where $\zeta(c) = c$ for $c \in C$, then the following hold true. The function Ω_T is α -admissible, $\Omega_T = \widehat{\Omega}$, and the two inequalities (13) if $\alpha = 1$ or the two inequalities (14) if $\alpha = 2$, corresponding to $\Omega_{T'}$ and to Ω_T , are identical. Lemma 6.2 is proved. \triangle

To finish the proof of part (d) of Theorem 1.1, we remark that, by Lemma 6.2, the LP-problem $\max\{-x_s \mid \text{SLI}_d[Y_1]\}$ can be algorithmically written down. Solving this LP-problem we obtain, by Lemma 4.2, the number $-\sigma_d(Y_1)\bar{r}(Y_1)$ which is equal to $-\sigma_d(H_1)\bar{r}(H_1)$ by Lemma 5.2. Since the number $\bar{r}(Y_1) = \bar{r}(H_1)$ is readily computable off the graph Y_1 (recall $\bar{r}(Y_1) = |EY_1|/2 - |VY_1|$), it follows that the coefficient $\sigma_d(Y_1)$ is also computable.

Since the LP-problem $\max\{-x_s \mid \text{SLI}_d[Y_1]\}$ can be effectively written down, its dual problem (30) can also be effectively constructed. Using the notation of the foregoing proof of parts (a)–(c), we observe that a vertex solution $y_V = y_V(d)$ to (30) can be computed, see [26]. Hence, a combination with repetitions Q_V , such that $\text{sol}_d(Q_V) = y_V$ and all entries in $\eta(Q_V)$ are coprime, is also computable, see Lemma 4.1.

Now, as in the proof of Lemma 3.2, we can construct a graph $Y_{2, Q_V} = \Psi(H_2^*)$ from Q_V and observe that this construction can be done algorithmically. The proof of part (d) is complete.

To show part (e), we note that if both groups G_1, G_2 are finite then any irreducible finite \mathcal{A} -graph Ψ has the property that $\deg u \leq \max\{|G_1|, |G_2|\}$ for every secondary vertex $u \in V_S \Psi$. Hence, setting

$$d_m := \max\{|G_1|, |G_2|\},$$

we obtain that $\text{SLI}[Y_1] = \text{SLI}_{d_m}[Y_1]$ and so, by Lemma 4.2, $\sigma(Y_1) = \sigma_{d_m}(Y_1)$. Since the coefficient $\sigma_{d_m}(Y_1) = \sigma_{d_m}(H_1)$ is rational and computable by part (d), the number $\sigma(H_1) = \sigma(Y_1)$ is also rational and computable. Theorem 1.1 is proved. \square

Theorem 1.2. *Suppose that $\mathcal{F} = G_1 * G_2$ is the free product of two nontrivial finite groups G_1, G_2 and H_1 is a subgroup of \mathcal{F} given by a finite generating set \mathcal{S} of words over the alphabet $G_1 \cup G_2$. Then the following are true.*

- (a) *In deterministic polynomial time in the size of \mathcal{S} , one can detect whether H_1 is factor-free and noncyclic and, if so, one can construct an irreducible graph $\Psi_o(H_1)$ of H_1 .*
- (b) *If H_1 is factor-free and noncyclic, then, in deterministic exponential time in the size of \mathcal{S} , one can write down and solve an LP-problem $\mathcal{P} = \max\{cx \mid Ax \leq b\}$ whose solution is equal to $-\sigma(H_1)\bar{r}(H_1)$. In particular, the WN-coefficient $\sigma(H_1)$ of H_1 is computable in exponential time in the size of \mathcal{S} .*
- (c) *If H_1 is factor-free and noncyclic, then there exists a finitely generated factor-free subgroup $H_2^* = H_2^*(H_1)$ of \mathcal{F} such that*

$$\bar{r}(H_1, H_2^*) = \sigma(H_1)\bar{r}(H_1)\bar{r}(H_2^*)$$

and the size of an irreducible core graph $\Psi(H_2^)$ of H_2^* is at most doubly exponential in the size of $\Psi(H_1)$. Specifically,*

$$|E\Psi(H_2^*)| < 2^{2^{E\Psi(H_1)/4 + \log_2 \log_2(4d_m)}},$$

where $\Psi(H_1)$ is an irreducible core graph of H_1 , $|E\Psi|$ denotes the number of oriented edges of the graph Ψ , and $d_m := \max\{|G_1|, |G_2|\}$.

In addition, an irreducible core graph $\Psi(H_2^)$ of H_2^* can be constructed in deterministic exponential time in the size of \mathcal{S} or $\Psi(H_1)$.*

Proof of Theorem 1.2. Part (a) follows from Lemma 2.1.

To show part (b), we first observe that, in the case when G_1 and G_2 are finite, we can effectively write down the system $\text{SLI}_{d_m}[Y_1]$, where $d_m = \max\{|G_1|, |G_2|\}$, and this can be done in exponential time in the size of $Y_1 := \Psi(H_1)$. Indeed, the number of all functions

$$\Omega_T: T \longrightarrow S_1(V_P Y_1),$$

where $T \in S_2(G_\alpha)$ and $|T| \leq d_m$, is bounded above by $2^{d_m} 2^{|V_P Y_1| d_m} = 2^{d_m} 2^{(|E Y_1|/4)^{d_m}}$. Hence, we can construct all such functions in exponential time. We can also check whether every such function is α -admissible in polynomial time in the size of Y_1 . Note that the input is the generating set \mathcal{S} while the orders of finite groups G_α , $\alpha = 1, 2$, and the parameter d_m are regarded as constants. Hence, all inequalities of the system $\text{SLI}_{d_m}[Y_1]$ that are defined by means of α -admissible functions Ω as above, see definitions (13)–(14), can be computed in exponential time in the size of Y_1 .

Furthermore, by Lemma 2.1, the size of the graph Y_1 is polynomial in the size of the generating set \mathcal{S} . By Theorem 1.1(e), $\text{SLI}[Y_1] = \text{SLI}_{d_m}[Y_1]$. Hence, the size of the system $\text{SLI}[Y_1] = \text{SLI}_{d_m}[Y_1]$ is exponential in the size of \mathcal{S} . It is clear that the size of the primal LP-problem $\max\{-x_s \mid \text{SLI}_{d_m}[Y_1]\}$ as well as the size of the dual problem (30) are also exponential in the size of Y_1 or in the size of \mathcal{S} . By Theorem 1.1 and Lemma 5.2, an optimal solution to the dual problem (30) is equal to

$$-\sigma_{d_m}(Y_1)\bar{r}(Y_1) = -\sigma_{d_m}(H_1)\bar{r}(H_1) = -\sigma(Y_1)\bar{r}(Y_1) = -\sigma(H_1)\bar{r}(H_1).$$

It remains to mention that an LP-problem $\max\{cx \mid Ax \leq b\}$ can be solved in polynomial time in the size of the problem, see [26], and that the reduced rank $\bar{r}(Y_1) = \bar{r}(H_1)$ can be computed in polynomial time in the size of Y_1 .

To prove part (c), we recall that the size of the dual LP-problem (30), similarly to the size of the primal LP-problem $\max\{-x_s \mid \text{SLI}_{d_m}[Y_1]\}$, is exponential (in the size of Y_1 or \mathcal{S}) that a vertex solution $y_V = y_V(d_m)$ to (30) can be computed in polynomial time in the size of the dual LP-problem (30), see [26]. Note that here and below we use the notation of the proofs of proofs of Lemmas 3.2 and 4.3. Hence, a vertex solution y_V to (30) can be computed in exponential time (in the size of Y_1 or \mathcal{S}). Using the function sol_{d_m} , we can compute a combination with repetitions Q_V , such that $\text{sol}_{d_m}(Q_V) = y_V$ and entries of Q_V are coprime, in polynomial time in the size of y_V . The size of the vertex y_V , as was established in the proof of Lemma 4.3, see (51)–(53), (56), is exponential. Hence, the combination Q_V can also be computed in exponential time.

The inequality

$$|E\Psi(H_2^*)| < 2^{2^{|E\Psi(H_1)|/4 + \log_2 \log_2(4d_m)}},$$

where, as above, $d_m = \max\{|G_1|, |G_2|\}$, follows from part (d) of Theorem 1.1.

In view of inequalities (54) and (57), we obtain that

$$|Q_V| < r(2d_m)^{r-1} < 2^{2^{|EY_1|/4 + \log_2 \log_2(4d_m)}}. \tag{77}$$

This bound, in particular, means that every inequality $q \in \text{SLI}_{d_m}(Y_1)$ occurs in Q_V less than

$$2^{2^{|EY_1|/4 + \log_2 \log_2(4d_m)}}$$

times, hence, the number $n_{Q_V}(q)$ of occurrences of q in Q_V can be written by using at most $2^{|EY_1|/4 + \log_2 \log_2(4d_m)}$ bits.

As in the proofs of Lemmas 3.2 and 4.3, we construct a graph Y_{2,Q_V} whose secondary vertices are in bijective correspondence with inequalities of Q_V and whose primary vertices are defined by means of an involution ι_V on the set of terms $\pm x_D$ of the left hand sides q^L of the inequalities $q \in Q_V$.

Lemma 6.4. *The graph Y_{2,Q_V} can be constructed in deterministic exponential time in the size of Y_1 .*

Proof. We need to explain how to compute the involution ι_V as above in exponential time (in the size of Y_1). To do this, for each variable x_D of the system $\text{SLI}_{d_m}(Y_1)$, see (15), we consider a graph Λ_D whose set of vertices is the subset

$$R_V := \{q \mid q \in Q_V\}$$

of $\text{SLI}_{d_m}(Y_1)$ formed with the inequalities of Q_V . If $q_1, q_2 \in R_V$ are distinct, q_1^L contains the term x_D and q_2^L contains the term $-x_D$, then we draw an edge in Λ_D that connects q_1 and q_2 . In other words, if there is a potential cancellation between terms $\pm x_D$ in the sum $q_1^L + q_2^L$ then Λ_D contains an edge that connects q_1 and q_2 .

It is clear that Λ_D is a bipartite graph so that every edge connects a vertex of type (13) and a vertex of type (14).

Consider a weight function

$$\omega_D: E\Lambda_D \longrightarrow \mathbb{Z}, \tag{78}$$

where \mathbb{Z} is the set of integers, such that $\omega_D(e^{-1}) = \omega_D(e) \geq 0$ and

$$\sum_{e=q} \omega_D(e) = n_q(x_D)n_{Q_V}(q),$$

where $n_q(x_D)$ is the number of times the term x_D or $-x_D$ occurs in q^L and $n_{Q_V}(q)$ is the number of occurrences of q in Q_V . Clearly, $n_q(x_D)n_{Q_V}(q)$ is the total number of occurrences of terms $\pm x_D$ in the subsum

$$\underbrace{q^L + \dots + q^L}_{n_{Q_V}(q) \text{ times}}$$

of the sum $\sum_{q' \in Q_V} (q')^L$. Note that $n_{Q_V}(q) = \eta_j(Q_V)$ if $q = q_j$ in the notation of (35).

Our nearest goal is to show that such a weight function ω_D can be computed in exponential time for every index D .

Let the edge set

$$E\Lambda_D = \{e_1, e_1^{-1}, e_2, e_2^{-1}, \dots, e_{|E\Lambda_D|/2}, e_{|E\Lambda_D|/2}^{-1}\}$$

of the graph Λ_D be indexed as indicated and let $(e_i)_-$ be a vertex of type (13) for every i .

We will define the numbers $\omega_D(e_i)$ by induction for $i = 1, 2, \dots, |E\Lambda_D|/2$ by the following procedure which also assigns intermediate weights $\omega_D(q)$ to vertices $q \in R_V$ of Λ_D .

Originally, we set

$$\omega_D(q) := n_q(x_D)n_{Q_V}(q)$$

for every $q \in R_V$. For $i \geq 1$, if the edge e_i connects q_1 and q_2 then we set

$$\omega_D(e_i) := \min(\omega_D(q_1), \omega_D(q_2))$$

and redefine the weights of q_1 and q_2 by setting

$$\omega'_D(q_1) := \omega_D(q_1) - \min(\omega_D(q_1), \omega_D(q_2)),$$

$$\omega'_D(q_2) := \omega_D(q_2) - \min(\omega_D(q_1), \omega_D(q_2)),$$

where $\omega'_D(q_1)$ denotes the new weight.

Note that the assignment of a nonnegative weight $\omega_D(e_i)$ to the edge e_i , connecting q_1 and q_2 , can be interpreted as making $\omega_D(e_i)$ cancellations between terms $\pm x_D$ of the subsums

$$\underbrace{q_1^L + \dots + q_1^L}_{n_{Q_V}(q_1) \text{ times}} \quad \text{and} \quad \underbrace{q_2^L + \dots + q_2^L}_{n_{Q_V}(q_2) \text{ times}}$$

of the sum in the left hand side of the equality

$$\sum_{q \in Q_V} q^L = -2\bar{\Gamma}(Y_1)x_s. \tag{79}$$

Analogously, the intermediate weight $\omega_D(q_1)$ of a vertex $q_1 \in V\Lambda_D$ can be interpreted as the number of terms $\pm x_D$ of the subsum

$$\underbrace{q_1^L + \dots + q_1^L}_{n_{Q_V}(q_1) \text{ times}}$$

which are still uncanceled in the left hand side of (79).

Therefore, in view of the equality (79), in the end of this process, we will obtain that the weights $\omega_D(q)$ of all vertices $q \in R_V$ are zeros, i.e., cancellations of the terms $\pm x_D$ are complete, and the weights $\omega_D(e_i)$ of all edges e_i have desired properties.

Clearly, the foregoing inductive procedure makes it possible to compute such a weight function ω_D in polynomial time in the size of the graph Λ_D and in the size of numbers $n_{Q_V}(q)$, $q \in R_V$, written in binary. Hence, we can compute weight functions ω_D for all D in exponential time.

Now we will define the involution ι_V based on the weight functions ω_D .

Let elements of the set $R_V = \{q_1, \dots, q_{|R_V|}\}$ be indexed as indicated and let elements of the combination

$$\begin{aligned}
 Q_V = & \llbracket [q_{1,1}, q_{1,2}, \dots, q_{1,n_{Q_V}(q_1)}, \\
 & \dots, \\
 & q_{i,1}, q_{i,2}, \dots, q_{i,n_{Q_V}(q_i)}, \\
 & \dots, \\
 & q_{|R_V|,1}, q_{|R_V|,2}, \dots, q_{|R_V|,n_{Q_V}(q_{|R_V|})}] \rrbracket,
 \end{aligned} \tag{80}$$

where $q_{i,j} = q_i \in R_V$ for all possible i, j , be double indexed as indicated according to the indices introduced on elements of R_V .

Since the secondary vertices of the graph Y_{2,Q_V} are in bijective correspondence with elements of Q_V , see the proof of Lemma 3.2, we can also write

$$V_S Y_{2,Q_V} = \{u_{i,j} \mid 1 \leq i \leq |R_V|, 1 \leq j \leq n_{Q_V}(q_i)\},$$

where

$$u_{i,j} \longmapsto q_{i,j} \tag{81}$$

under this correspondence.

Let $q_i \in R_V$ be fixed and let

$$q_{m_1(i)}, \dots, q_{m_{t_i}(i)}$$

be all vertices of Λ_D , where $m_1(i) < \dots < m_{t_i}(i)$, that are connected to q_i by edges f_1, \dots, f_{t_i} , resp., in Λ_D with positive weights $\omega_D(f_1), \dots, \omega_D(f_{t_i})$, resp. We assume that q_i is the terminal vertex of the edges f_1, \dots, f_{t_i} .

Recall that q_i^L contains $n_{q_i}(x_D) \geq 1$ terms $\pm x_D$, here the sign is a minus if q_i has type (13) and the sign is a plus if q_i has type (14).

According to the weights $\omega_D(f_1), \dots, \omega_D(f_{t_i})$, we will define (D, i, t) -blocks of consecutive terms $\pm x_D$ in the sum

$$q_{i,1}^L + q_{i,2}^L + \dots + q_{i,n_{Q_V}(q_i)}^L, \tag{82}$$

see (80), in the following manner. (Here and below we disregard all terms $\pm x_B$, where $B \neq D$, in (82) when we talk about consecutive terms $\pm x_D$ in (82).)

The $(D, i, 1)$ -block consists of the first $\omega_D(f_1)$ consecutive terms $\pm x_D$ in the sum (82). The $(D, i, 2)$ -block consists of the next $\omega_D(f_2)$ consecutive terms $\pm x_D$ in the sum (82) and so on. Note that the first term $\pm x_D$ of the $(D, i, 2)$ -block is $(\omega_D(t_1) + 1)$ st term $\pm x_D$ in the sum (82) and the last term $\pm x_D$ of the $(D, i, 2)$ -block is the $(\omega_D(t_1) + \omega_D(t_2))$ th term $\pm x_D$ in the sum (82).

The (D, i, t_i) -block consists of the last $\omega_D(f_{t_i})$ consecutive terms $\pm x_D$ in the sum (82). Since

$$\sum_{t=1}^{t_i} \omega_D(f_t) = n_{q_i}(x_D)n_{Q_V}(q_i)$$

and $\omega_D(f_t) > 0$ for every t , it follows that these (D, i, t) -blocks, where $t = 1, \dots, t_i$ and D, i are fixed, will form a partition of the sequence of terms $\pm x_D$ of the sum (82) into t_i subsequences. Note that the terms $\pm x_D$ of the same summand $q_{i,j}^L$ of (82) could be in different blocks when $n_{q_i}(x_D) > 1$.

We emphasize that every (D, i, t) -block is associated with a vertex $q_i \in V\Lambda_D = R_V$ and with an edge f_t of Λ_D so that f_t ends in q_i and $\omega_D(f_t) > 0$. In particular, for every (D, i, t) -block, associated with a vertex $q_i \in R_V$ and with an edge f_t of Λ_D , we have another (D, i', t') -block, associated with a vertex $q_{i'} \in R_V$ and with an edge $f'_{t'}$ of Λ_D , so that $q_{i'} \neq q_i$ and $f'_{t'} = f_t^{-1}$. Here $f'_1, \dots, f'_{t'_i}$ are the edges of Λ_D defined for $q_{i'}$ in the same fashion as the edges f_1, \dots, f_{t_i} of Λ_D were defined for q_i . Note that $i'' = i$ and $f''_{t''} = f_t$ in this notation.

We define the involution ι_V so that all the terms $\pm x_D$ of the (D, i, t) -block are mapped by ι_V to the terms $\mp x_D$ of the (D, i', t') -block in the natural increasing order of elements in the block.

In other words, this definition of the involution ι_V means that the primary vertices of the graph Y_{2,Q_V} , for details see the proof of Lemma 3.2, that are connected by edges to the secondary vertices

$$u_{i,1}, u_{i,2}, \dots, u_{i,n_{Q_V}(q_i)} \tag{83}$$

of Y_{2,Q_V} , see (81), and that correspond to the terms $\pm x_D$ of the (D, i, t) -block, will be identified, in the increasing order, with the primary vertices that are connected by edges to the secondary vertices

$$u_{i',1}, u_{i',2}, \dots, u_{i',n_{Q_V}(q_{i'})} \tag{84}$$

of Y_{2,Q_V} and that correspond to the terms $\mp x_D$ of the (D, i', t') -block.

The labels to the edges of the graph Y_{2,Q_V} are assigned as described in the proof of Lemma 3.2. Specifically, let $e_{1,j}, \dots, e_{k_i,j}$ be all the edges of Y_{2,Q_V} that end in a secondary vertex $u_{i,j}$, i.e.,

$$(e_{1,j})_+ = \dots = (e_{k_i,j})_+ = u_{i,j},$$

where $k_i = k(q_i)$. Furthermore, let $\{b_1, \dots, b_{k_i}\}$ denote the domain of an α_i -admissible function

$$\Omega_{T_i}: \{b_1, \dots, b_{k_i}\} \longrightarrow S_1(V_P Y_1)$$

that defines the inequality q_i . Then we set

$$\varphi(e_{1,j}) := b_1, \dots, \varphi(e_{k_i,j}) := b_{k_i}.$$

Note that the primary vertices that are discussed above and that are connected by edges to vertices (83) will be precisely those $e_{\ell,j}$, among $(e_{1,j})_-, \dots, (e_{k_i,j})_-$ over all $j = 1, \dots, n_{Q_V}(q_i)$, for which

$$\Omega_{T_i}(\varphi(e_{\ell,j})) = \Omega_{T_i}(b_\ell) = D.$$

Similar remark can be made about the primary vertices that are discussed above and that are connected by edges to vertices (84).

It is clear that the foregoing construction of the involution ι_V can be done in polynomial time in the total size of graphs Λ_D , weights $\omega_D(e)$, $e \in E\Lambda_D$, and numbers $n_{Q_V}(q)$, $q \in R_V$, written in binary. Therefore, we can compute ι_V in exponential time in the size of Y_1 (or \mathcal{S}). Thus the graph Y_{2,Q_V} can also be constructed in exponential time, as required. The proof of Lemma 6.4 is complete. △

Since the graph Y_{2,Q_V} can be constructed in exponential time in the size of the generating set \mathcal{S} , it follows from Lemma 4.3 that we can use Y_{2,Q_V} as an irreducible \mathcal{A} -graph $\Psi(H_2^*)$ of the subgroup H_2^* . Theorem 1.2 is proved. □

It is worthwhile to mention that our construction of the graph Y_{2,Q_V} is somewhat succinct (cf. the definition of succinct representations of graphs in [25]) in the sense that, despite the fact that the size of Y_{2,Q_V} could be doubly exponential, we are able to give a description of Y_{2,Q_V} in exponential time (in the size of Y_1). In particular, vertices of Y_{2,Q_V} are represented by exponentially long bit strings and edges of Y_{2,Q_V} are drawn in blocks. As a result, we can find out in exponential time whether two given vertices of Y_{2,Q_V} are connected by an edge labelled by given letter $g \in G_\alpha$.

Theorem 1.3. *Suppose that $\mathcal{F} = \prod_{\alpha \in I}^* G_\alpha$ is the free product of nontrivial groups G_α , $\alpha \in I$, and H_1 is a finitely generated factor-free noncyclic subgroup of \mathcal{F} . Then there are two disjoint finite subsets I_1, I_2 of the index set I such that if $\hat{G}_1 := \prod_{\alpha \in I_1}^* G_\alpha$, $\hat{G}_2 := \prod_{\alpha \in I_2}^* G_\alpha$, and $\hat{\mathcal{F}} := \hat{G}_1 * \hat{G}_2$, then there exists a finitely generated factor-free subgroup \hat{H}_1 of $\hat{\mathcal{F}}$ with the following properties.*

- (a) $\bar{r}(\hat{H}_1) = \bar{r}(H_1)$, $\sigma_d(\hat{H}_1) \geq \sigma_d(H_1)$ for every $d \geq 3$, and $\sigma(\hat{H}_1) \geq \sigma(H_1)$. In particular, if the conjecture (5) fails for H_1 then the conjecture (5) also fails for \hat{H}_1 .
- (b) If the word problem for every group G_α , where $\alpha \in I_1 \cup I_2$, is solvable and a finite irreducible graph of H_1 is given, then the LP-problem $\mathcal{P}(\hat{H}_1, d)$ for \hat{H}_1 of part (a) of Theorem 1.1 can be algorithmically written down and the WN_d -coefficient $\sigma_d(\hat{H}_1)$ for \hat{H}_1 can be computed.
- (c) Let every group G_α , where $\alpha \in I_1 \cup I_2$, be finite, let H_1 be given either by a finite irreducible graph or by a finite generating set, and let

$$d_M := \max\{|I_1 \cup I_2|, \max\{|G_\alpha| \mid \alpha \in I_1 \cup I_2\}\}.$$

Then $\sigma_{d_M}(\hat{H}_1) \geq \sigma(H_1)$ and there is an algorithm that decides whether the conjecture (5) holds for H_1 .

Proof of Theorem 1.3. (a) As in the proof of Theorem 1.1, we assume that the subgroup H_1 is given by an irreducible \mathcal{A} -graph $\Psi(H_1)$ with $\text{core}(\Psi(H_1)) = \Psi(H_1)$, now the alphabet is $\mathcal{A} = \bigcup_{\alpha \in I} G_\alpha$. Note that it is also possible to assume that H_1 is defined by a finite generating set \mathcal{S} whose elements are words over the alphabet \mathcal{A} . In the latter case, we could apply Lemma 2.1 which, when given a finite generating set of a subgroup H of \mathcal{F} , verifies that H is a factor-free subgroup of \mathcal{F} and, if so, constructs an irreducible \mathcal{A} -graph of H .

Making use of the graph $\Psi(H_1)$ of H_1 , we switch from the original index set I to its finite subset $I(H_1)$ and rename it by $\{1, \dots, m\}$. Here and below we use the notation introduced in Section 5. Without loss of generality, we may assume that $m \geq 3$, otherwise, we set $\hat{H}_1 := H_1$.

Consider the embedding

$$\mu_2: \mathcal{F} \longrightarrow \mathcal{F}_2(1)$$

defined by means of the map (72), where

$$\mathcal{F}_2(1) = G_1 * G(2, m) \quad \text{and} \quad G(2, m) = G_2 * \dots * G_m.$$

Denote $\hat{H}_1 := \mu_2(H_1)$. By Lemma 5.3, μ_2 is a monomorphism, hence, $\bar{r}(\hat{H}_1) = \bar{r}(H_1)$ and, by Lemma 5.3(e),

$$\sigma_d(H_1) \leq \sigma_d(\hat{H}_1)$$

for every $d \geq 3$. Consequently, $\sigma(H_1) \leq \sigma(\hat{H}_1)$ as well. This proves part (a).

(b) Assume that the word problem is solvable in groups G_α , $\alpha \in I(H_1)$. Then the word problem is also solvable in factors $G_1, G(2, m)$ of the free product $\mathcal{F}_2(1) = G_1 * G(2, m)$. Furthermore, using the graph $\Psi(H_1)$ of H_1 and the map (72), we can algorithmically construct a finite irreducible graph $\Psi(\hat{H}_1)$ with $\text{core}(\Psi(\hat{H}_1)) = \Psi(\hat{H}_1)$. By Theorem 1.1(d), the LP-problem $\mathcal{P}(\hat{H}_1, d) = \mathcal{P}(\Psi(\hat{H}_1), d)$, associated with \hat{H}_1 , can be effectively constructed and the coefficient $\sigma_d(\hat{H}_1)$ can be computed, as claimed in part (b).

(c) We will continue to use the notation introduced above. Suppose that all factors G_α , where $\alpha \in I(H_1) = \{1, \dots, m\}$, are finite. We also assume that H_1 is given by an irreducible graph $\Psi(H_1)$ with $\text{core}(\Psi(H_1)) = \Psi(H_1)$ or H_1 is given by a finite generating set. Note that Lemma 2.1 reduces the latter case to the former one. By Lemma 5.2, when computing the number

$$\sigma(H_1) = \sup_{H_2} \left\{ \frac{\bar{r}(H_1, H_2)}{\bar{r}(H_1)\bar{r}(H_2)} \right\}$$

over all finitely generated factor-free subgroups H_2 with $\bar{r}(H_2) > 0$, we may assume that the subgroup H_2 has property (B) and satisfies the condition $I(H_2) \subseteq I(H_1)$. The condition $I(H_2) \subseteq I(H_1)$ implies that the degree of every primary vertex of $\Psi(H_2)$ does not exceed $|I(H_1)|$. On the other hand, the degree of every secondary vertex of $\Psi(H_2)$ does not exceed

$$\max\{|G_\alpha| \mid \alpha \in I(H_1)\}.$$

Hence, the degree $\deg v$ of every vertex v of $\Psi(H_2)$ satisfies

$$\deg v \leq d_M := \max\{|I(H_1)|, \max\{|G_\alpha| \mid \alpha \in I(H_1)\}\}. \tag{85}$$

Thus, by Lemma 5.2, we may conclude that

$$\sigma(H_1) = \sigma_{d_M}(H_1) = \sup_{H_2} \left\{ \frac{\bar{r}(H_1, H_2)}{\bar{r}(H_1)\bar{r}(H_2)} \right\},$$

where the supremum is taken over all subgroups H_2 with property (Bd) in which $d = d_M$. Applying Lemma 5.3(e) to H_1 , we obtain

$$\sigma(H_1) = \sigma_{d_M}(H_1) \leq \sigma_{d_M}(\hat{H}_1).$$

Recall that an irreducible graph $\Psi(\hat{H}_1)$ of $\hat{H}_1 = \mu_2(H_1)$ can be algorithmically constructed from $\Psi(H_1)$ (for details see the proof of Lemma 5.3(c)) and that the word problem is solvable for factors of the free product

$$\mathcal{F}_2(1) = G_1 * G(2, m).$$

Invoking Theorem 1.1(d), we see that the LP-problem

$$\mathcal{P}(\hat{H}_1, d_M) = \mathcal{P}(\Psi(\hat{H}_1), d_M)$$

can be algorithmically written down and hence the coefficient $\sigma_{d_M}(\hat{H}_1)$ can be computed. The proof of Theorem 1.3 is complete. □

In conclusion, we mention that it is not clear whether there is a duality gap between the LSIP-problem $\sup\{-x_s \mid \text{SLI}[Y_1]\}$, introduced in Section 4, and its dual problem (29) and it would be of interest to find this out. Another natural problem is to find an algorithm that solves the dual problem (29) of the LSIP-problem $\sup\{-x_s \mid \text{SLI}[Y_1]\}$ and thereby effectively computes the WN-coefficient $\sigma(\Psi(H_1)) = \sigma(H_1)$ for a finitely generated factor-free subgroup H_1 of the free product of two groups (and, perhaps, more than two groups) which are not necessarily finite. It would also be interesting to find an algorithm that computes the Hanna Neumann coefficient $\bar{\sigma}(H_1)$ for a finitely generated factor-free noncyclic subgroup H_1 of the free product \mathcal{F} of two finite groups which is defined as

$$\bar{\sigma}(H_1) := \sup_{H_2} \left\{ \frac{\bar{r}(H_1 \cap H_2)}{\bar{r}(H_1)\bar{r}(H_2)} \right\}$$

over all finitely generated factor-free noncyclic subgroups H_2 of \mathcal{F} .

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