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Fibered commensurability and arithmeticity of random mapping tori

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Abstract. We consider a random walk on the mapping class group of a surface of finite type. We assume that the random walk is determined by a probability measure whose support is finite and generates a non-elementary subgroup H. We further assume that H is not consisting only of lifts with respect to any one covering. Then we prove that the probability that such a random walk gives a non-minimal mapping class in its fibered commensurability class decays exponentially. As an application of the minimality, we prove that for the case where a surface has at least one puncture, the probability that a random walk gives mapping classes with arithmetic mapping tori decays exponentially. We also prove that a random walk gives rise to asymmetric mapping tori with exponentially high probability for closed case.

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1. Introduction

Let S be an orientable surface of finite type (g, n), where g is the genus and n is the number of punctures. We consider a random walk on the mapping class group $G := \mathrm{Mod}(S)$ which is determined by a probability measure on G whose support generates a non-elementary subgroup. It has been shown that such a random walk gives rise to pseudo-Anosov elements with asymptotic probability one [18, 21, 22, 30]. Let μ be a probability measure on G. A subset $A \subset G$ is said to be exponentially small (with respect to μ) if the probability that the random walk determined by μ visits A decays exponentially with the number of steps. A subset is called exponentially large (with respect to μ) if its complement is exponentially small. The work of Maher [22] can be stated as "the set of pseudo-Anosov elements is exponentially large." In this paper, we consider fibered commensurability, a notion introduced by Calegari, Sun, and Wang [7], of random mapping classes. Roughly, a mapping class ϕ is said to cover another mapping class φ if ϕ is a power of some lift of φ with respect to some finite covering of underlying surfaces. The commensurability with respect to this covering relation is called fibered commensurability. Each commensurability class enjoys an order by the covering relation. It has been shown [7, 24] that for pseudo-Anosov case, each commensurability class contains a unique minimal (orbifold) element (see Theorem 2.3). Our aim is to prove that the set of minimal elements is exponentially large with respect to any measure which satisfies a suitable condition (Condition 1.2). As an application of the minimality, we also show a result on arithmeticity of random mapping tori. By using random walks on G, we may generate randomly 3-manifolds by taking mapping tori. The work of Thurston [31] together with [22] shows that the set of mapping classes with hyperbolic mapping tori is exponentially large. A cusped hyperbolic 3-manifold is called arithmetic if it is commensurable to a Bianchi orbifold (see §5.1). Several distinguished hyperbolic 3-manifolds, for example the complement of the figure eight knot or the Whitehead link, are known to be arithmetic. However, a "generic" hyperbolic 3-manifold is believed to be non-arithmetic. The minimality of random mapping classes together with the work by Bowditch, Maclachlan, and Reid [5] enables us to prove that the set of mapping classes with arithmetic mapping tori is exponentially small if S has at least one puncture. We also prove that the set of mapping classes with asymmetric mapping tori is exponentially large for closed case.

The paper is organized as follows. In §2, we prepare several definitions and facts about random walks on groups and mapping class groups. Note that to prove that a given mapping class ϕ is minimal, it suffices to show that ϕ is primitive and not symmetric. In §3 we prove the primitivity of random mapping classes.

Theorem 1.1. *Let μ be a probability measure on G whose support is finite and generates a non-elementary subgroup. Then the set of primitive elements in G is exponentially large with respect to μ.*

Next, we prove that random mapping classes are not symmetric in §4. We call a mapping class *symmetric* if it is a lift with respect to some finite covering $\pi\colon S \to S'$. We need further assumption for the measure μ to avoid the case that there is some finite covering $\pi\colon S \to S'$ such that every element in the support of μ is a lift of a mapping class on S'. Let $\mathcal{PMF}(S)$ denote the set of projective measured foliations on S, where in the case of orbifolds, we consider the one for the surface we get by puncturing the orbifold points. Each covering $\pi\colon S \to S'$ determines a map $\Pi\colon \mathcal{PMF}(S') \to \mathcal{PMF}(S)$ so that $a \in \Pi(\mathcal{PMF}(S'))$ if and only if $\pi(a) \in \mathcal{PMF}(S')$. Let gr (resp. $\mathrm{sgr}(\mu)$) denote the group (resp. semigroup) generated by the support of μ. The condition for the measure μ which we need is the following.

Condition 1.2. The support is finite, and generates a non-elementary subgroup of G. Moreover, for any (possibly orbifold) covering $\pi\colon S \to S'$, $\mathrm{sgr}(\mu)$ contains a pseudo-Anosov element whose fixed points set is disjoint from $\Pi(\mathcal{PMF}(S'))$.

Note that since $\Pi(\mathcal{PMF}(S'))$ is closed (see e.g. [15, §2.2]) and the set of all stable and unstable foliations of all pseudo-Anosov elements are dense [20, Lemma 3.4], Condition 1.2 is satisfied if μ has finite support which generates G. In §4, we prove:

Theorem 1.3. *Let μ be a probability measure on G which satisfies Condition 1.2. Then the set of symmetric elements is exponentially small with respect to μ.*

Putting Theorems 1.1 and 1.3 together, we have:

Theorem 1.4. *Let μ be a probability measure on G which satisfies Condition 1.2. Then the set of minimal elements in their fibered commensurability class is exponentially large with respect to μ.*

Finally in §5.1, we prove the following theorem.

Theorem 1.5. *Suppose that S has at least one puncture. Let μ be a probability measure on G which satisfies Condition 1.2. Then the set of mapping classes with arithmetic mapping tori is exponentially small with respect to μ.*

In §5.2, it is proved that closed random mapping tori are asymmetric.

2. Preliminary

In this section, we summarize several definitions and facts that we use throughout the paper. Interested readers may refer to several papers regarding to random walks on the mapping class groups (for example [16, 21]) in which there are detailed expositions of basic theory of both random walks and mapping class groups.

2.1. Random walks on groups. We recall the definitions and terminologies of random walks. See [33] for more details about random walks on groups. Let G be a countable group. A (possibly infinite) matrix $\mathbb{P} = (p_{g,h})_{g,h\in G}$ is called *stochastic* if every element is non-negative and

$$\sum_{h\in G} p_{g,h} = 1$$

for all $g \in G$. For a given probability measure μ on G, by putting $p_{g,h} = \mu(g^{-1}h)$, we have a stochastic matrix $\mathbb{P}_\mu = (p_{g,h})_{g,h\in G}$. Let P_n denote the probability measure on $(G^n, 2^{G^n})$ defined by

$$P_n(A) = \sum_{(g_1,\dots,g_n)\in A} p_{\mathrm{id},g_1} p_{g_1,g_2} \cdots p_{g_{n-1},g_n} \quad \text{for } A \in 2^{G^n}.$$

Note that by definition, we have $P_{n+1}(A \times G) = P_n(A)$ for any $A \in 2^{G^n}$. Let $\mathcal{B}(G^{\mathbb{N}})$ denote the σ-algebra generated by cylinder sets, where a cylinder set is a subset defined as

$$\{\omega = (\omega_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}} \mid (\omega_1, \ldots, \omega_n) \in A\}$$

for some $A \subset G^n$. Then by the Kolmogorov extension theorem, there exists a unique measure P on $(G^{\mathbb{N}}, \mathcal{B}(G^{\mathbb{N}}))$ which satisfies

$$P(A \times G^{\mathbb{N}}) = P_n(A) \quad \text{for all } n \in \mathbb{N} \text{ and } A \in 2^{G^n}.$$

For $\omega = (\omega_n) \in G^{\mathbb{N}}$, we define G-valued random variables X_n on $(G^{\mathbb{N}}, \mathcal{B}(G^{\mathbb{N}}))$ by $X_n(\omega) = \omega_n$. Thus we have a stochastic process $\{X_n\}_{n\in\mathbb{N}}$ which is a Markov chain with the transition matrix $\mathbb{P}_\mu$. We call this Markov chain $\{X_n\}_{n\in\mathbb{N}}$ the *random walk* determined by μ.

Let us fix a probability measure μ and the random walk determined by μ. Each element $(\omega_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ is called a *sample path*. Let $A \subset G$. By abbreviation of notations, we write $\mathbb{P}(\omega_n \in A)$ to mean $P(G^{n-1} \times A \times G^{\mathbb{N}})$. A subset $A \subset G$ is called *exponentially small* (with respect to μ) if there exist $c < 1, K > 0$ which depend only on μ and A such that $\mathbb{P}(\omega_n \in A) < Kc^n$. A subset is called *exponentially large* (with respect to μ) if its complement is exponentially small. Let Q be a property for elements in G. We say that the random walks determined by μ has property Q *with exponentially high probability* if $S_Q := \{g \in G \mid g \text{ is } Q\}$ is exponentially large. It can be readily seen that if $A, B \subset G$ are both exponentially small (resp. large), then so is $A \cup B$ (resp. $A \cap B$).

2.2. Mapping class groups and curve graphs. For more details about topics in this subsection, one may refer to the books [3, 9]. Let $S := S_{g,n}$ be an orientable surface of finite type (g, n) where g is the genus and n is the number of punctures. In this paper, we always suppose $3g-3+n > 0$ unless otherwise stated. The *mapping class group* $\mathrm{Mod}(S)$ is the group of isotopy classes of orientation preserving automorphisms on S. A mapping class is called *pseudo-Anosov* if it is aperiodic and has no fixed 1-dimensional submanifold of S. Thurston [32] showed that each pseudo-Anosov mapping class has exactly two fixed points $\mathcal{F}_s, \mathcal{F}_u$ in the space $\mathcal{PMF}(S)$ of projective measured foliations. A subgroup of $\mathrm{Mod}(S)$ is called *non-elementary* if it contains two pseudo-Anosov mapping classes with disjoint fixed points in $\mathcal{PMF}(S)$.

The *curve graph* $\mathcal{C}(S)$ of S is a graph whose vertices consist of isotopy classes of simple closed curves, and two vertices are connected by an edge if the corresponding curves can be disjointly represented on S. By giving length 1 to every edge, the curve graph enjoys a metric $d_{\mathcal{C}(S)}(\cdot, \cdot)$. If S is an annulus, then vertices of $\mathcal{C}(S)$ are essential arcs, considered up to isotopy relative to their boundary. Edges are placed between vertices with representatives having disjoint interiors.

Let (X, d_X) be a metric space. For a fixed point $p \in X$, the *Gromov product* $(x \cdot x')_p$ of two points $x, x' \in X$ is defined by

$$(x \cdot x')_p = \frac{1}{2}(d_X(x, p) + d_X(x', p) - d_X(x, x')).$$

Then for $r > 0$, a *shadow* $S_p(x, r) \subset X$ is defined by

$$S_p(x, r) := \{y \in X \mid (x \cdot y)_p \geq r\}.$$

Note that this definition of shadow is not standard in the literature (see e.g. [23] for the standard one). We use this definition for the convenience of the proof (c.f. [22]). If we have another metric space (Y, d_Y), a map $f : X \to Y$ is said to be *Q-quasi-isometric* if for any $x, x' \in X$,

$$d_X(x, x')/Q - Q \leq d_Y(f(x), f(x')) \leq Q d_X(x, x') + Q.$$

Such f is called *Q-quasi-isometry* if it further satisfies that for any $y \in Y$, there exists $x \in X$ such that $d_Y(y, f(x)) < Q$. Two metric spaces are said to be *quasi-isometric* if there is a Q-quasi-isometry between the two. Suppose further that X is a geodesic space. Then X is called *δ-hyperbolic* if every geodesic triangle is *δ-thin*; one side of a geodesic triangle is contained in the δ-neighborhood of the other two sides. X is called *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$. Two geodesics in X are said to be *asymptotic* if they are finite Hausdorff distance apart. We define *the Gromov boundary* as the set of asymptotic classes of geodesics. Hyperbolicity is invariant under quasi-isometries, and a quasi-isometry induces a homeomorphism of the Gromov boundaries. For two points x, x' in a geodesic space X, we denote by $[x, x']$ a geodesic connecting x and x'. Note that there can be many such geodesics, and $[x, x']$ is an arbitrarily chosen one. We suppose that if $a, b \in [x, x']$, then $[a, b] \subset [x, x']$.

Remark 2.1. It is well known that if X is δ-hyperbolic, the Gromov product $(x, x')_p$ is equal to the distance from p to $[x, x']$ up to additive constant K which depends only on δ (c.f. Lemma 4.13). By this fact, a shadow $S_p(x, r)$ for $x \in X$ and $r > 0$ can be (coarse equivalently) regarded as the set of $x' \in X$ such that every geodesic connecting p and x' passes through a point in the $(d_X(x, p) - r + C)$-neighborhood of x for some C depending only on δ.

In [25], Masur-Minsky proved that the curve graph $\mathcal{C}(S)$ is hyperbolic. The mapping class group $G := \mathrm{Mod}(S)$ acts isometrically on $\mathcal{C}(S)$. Using this action, by fixing a base point $p \in \mathcal{C}(S)$, G admits a δ-hyperbolic (improper) metric which we denote again by $d_{\mathcal{C}(S)}$;

$$d_{\mathcal{C}(S)}(g, h) = d_{\mathcal{C}(S)}(gp, hp).$$

2.3. Commensurability of mapping classes. In [7], Calegari, Sun, and Wang defined commensurability of mapping classes on possibly distinct surfaces as follows.

Definition 2.2 ([7]). Let S_1 and S_2 be orientable surfaces of finite type. A mapping class $\phi_1 \in \mathrm{Mod}(S_1)$ *covers* $\phi_2 \in \mathrm{Mod}(S_2)$ if there exists a finite covering $\pi\colon S_1 \to S_2$ and $k \in \mathbb{Z} \setminus \{0\}$ such that a lift φ of ϕ_2 with respect to π satisfies $\varphi^k = \phi_1$. Two mapping classes are said to be *commensurable* if there exists a mapping class that covers both.

Since this gives commensurability of the monodromies of fibers on orientable surface bundles over the circle, this notion is also called *fibered commensurability*. Commensurability gives rise to an equivalence relation by taking transitive closure. We consider conjugacy classes in order to have each commensurability class enjoy an order by covering relation (see [7] for a detail). We call a mapping class *minimal* if it is a minimal element with respect to the order in its commensurability class. By extending our category to the orbifolds and orbifold automorphisms, for the cases where mapping classes are pseudo-Anosov, we have the following uniqueness of minimal element.

Theorem 2.3 ([7, 24]). *If $\phi \in \mathrm{Mod}(S)$ is pseudo-Anosov, then the commensurability class of ϕ contains a unique minimal* (*orbifold*) *element.*

Note that a mapping class ϕ is minimal if it is *primitive* (i.e. if $\varphi^k = \phi$, then $k = 1$ and $\phi = \varphi$, or $k = -1$ and $\phi = \varphi^{-1}$) and it is not a lift of any orbifold automorphism.

3. Random mapping classes are primitive

Throughout this section, let us fix an orientable surface S of finite type and denote by G the mapping class group $\mathrm{Mod}(S)$. To prove the primitivity, we consider the action of G on the curve graph $\mathcal{C}(S)$. We shall fix a base point $p \in \mathcal{C}(S)$. For $g \in G$, the translate $gp \in \mathcal{C}(S)$ is also denoted by g by abuse of notation. We abbreviate the distance on $\mathcal{C}(S)$ to $d_{\mathcal{C}}(\cdot,\cdot)$. In this section, unless otherwise stated, we consider the random walk determined by a probability measure μ on G with finite support which generates a non-elementary subgroup.

3.1. Random mapping classes do not (anti-)align. We first recall the work of Calegari and Maher [6].

Definition 3.1. Let $p_0, \ldots, p_n$ be points in $\mathcal{C}(S)$ and $\gamma = [p_0, p_n]$. A point $y \in \gamma$ is *D-proximal* (with respect to $p_0, \ldots, p_n$) if $d_{\mathcal{C}}(y, p_i) < D$ for some $0 \le i \le n$. Let γ_D denote the subset of D-proximal points of γ.

Let $\omega = (\omega_n)$ be a sample path in $G^{\mathbb{N}}$, then for large enough n, Calegari-Maher proved that most part of $[\omega_0, \omega_n]$ should be D-proximal with exponentially high probability.

Lemma 3.2 ([6, Lemma 5.14]). *There are constants $C_1, K > 0$ and $c < 1$ so that for any $\epsilon > 0$, there is a further constant D depending on C_1 and ϵ with the following property. Let $\gamma := [\omega_0, \omega_n]$ and γ_D denote the set of D-proximal points on γ with respect to $\omega_0, \dots \omega_n \in \mathcal{C}(S)$. Then*

$$\mathbb{P}((\text{length}(\gamma) \geq C_1 n) \wedge (\text{length}(\gamma_D)/\text{length}(\gamma) \geq 1 - \epsilon)) \geq 1 - Kc^n.$$

Lemma 3.2 shows that coarsely, any random walk fellow travels with a geodesic connecting the endpoints with exponentially high probability.

We also recall the work of Maher which shows that each shadow is exponentially small.

Lemma 3.3 ([22]). *There are constants $K > 0$ and $c < 1$ such that for any $q \in \mathcal{C}(S)$ and any r,*

$$\mathbb{P}(\omega_n \in S_1(q, r)) < Kc^r.$$

Throughout in this section, we suppose that a path in $\mathcal{C}(S)$ is a continuous map $[0, 1] \to \mathcal{C}(S)$. Hence for a given path γ, $\gamma(0)$ denotes the initial point and $\gamma(1)$ denotes the terminal point. Two paths γ_1 and γ_2 are said to be *D-aligned* (resp. *D-anti-aligned*) if there exists $h \in G$ such that $d_{\mathcal{C}}(h\gamma_1(0), \gamma_2(0)) < D$ and $d_{\mathcal{C}}(h\gamma_1(1), \gamma_2(1)) < D$ (resp. $d_{\mathcal{C}}(h\gamma_1(1), \gamma_2(0)) < D$ and $d_{\mathcal{C}}(h\gamma_1(0), \gamma_2(1)) < D$). Lemma 3.4 below looks quite similar to [6, Lemma 5.26] showing the probability that a random walk has two anti-aligned subpaths decays polynomially. Lemma 3.4 shows the probability that a random walk has aligned subpaths decays exponentially. The order of the decay is exponential since we consider the case that a random walk has aligned subpaths of length of linear order (see property (1) of Lemma 3.4) while in [6], the order was of logarithm. Although one can prove Lemma 3.4 by almost the same argument as in [6], we include a proof for completeness. Recall that by the work of Bowditch [4], the action of G on $\mathcal{C}(S)$ is *acylindrical*; for any $C_1 > 0$, there are constants C_2, C_3 such that for $a, b \in \mathcal{C}(S)$ with $d_{\mathcal{C}}(a, b) \geq C_2$, there are at most C_3 elements $h \in G$ with $d_{\mathcal{C}}(a, ha) \leq C_1$ and $d_{\mathcal{C}}(b, hb) \leq C_1$.

Lemma 3.4 (c.f. [6, Lemma 5.26]). *Fix $D, M > 0$. Then there is a constant $c_1 < 1$, $K > 0$ such that the following holds. Consider the collection of indices $a < a' < b < c < c' < d$ for which there are geodesics $\alpha := [\omega_a, \omega_b]$ and $\beta := [\omega_c, \omega_d]$ with the following properties*:

(1) length$(\alpha) \geq Mn$ *and similarly for* β;

(2) *there is* $t \in [0.1, 0.2]$ *so that* $d_{\mathcal{C}}(\omega_{a'}, \alpha(t)) \leq D$ *and* $d_{\mathcal{C}}(\omega_{c'}, \beta(t)) \leq D$;

(3) *there is some* $h \in G$ *so that* $d_{\mathcal{C}}(h\alpha(0), \beta(0)) \leq D$, *and* $d_{\mathcal{C}}(h\alpha(1), \beta(1)) \leq D$.

The probability that this collection of indices is non-empty is at most Kc_1^n.

Proof. We first fix $a < a' < b < c < c'$. To satisfy conditions (2) and (3), we need to have $h \in G$ such that $d_{\mathcal{C}}(h\omega_a, \omega_c) \le C_1$ and $d_{\mathcal{C}}(h\omega_{a'}, \omega_{c'}) \le C_1$ for some constant C_1 depending only on D and the hyperbolicity constant δ. Hence, the acylindricity of the action of G on $\mathcal{C}(S)$ implies that if $\alpha = [\omega_a, \omega_b]$ is long enough, there is a set $A \subset \mathcal{C}(S)$ of at most C_3 points so that ω_d should be in D neighborhood of some point $x \in A$ where C_3 depends only on D and δ. By Remark 2.1, it follows that $\omega_d \in S_{\omega_{c'}}(x, d_{\mathcal{C}}(\omega_{c'}, x) - C)$ for some C depending only on δ. Then by Lemma 3.3, the probability that a random walk from $\omega_{c'}$ is in $S_{\omega_{c'}}(x, d_{\mathcal{C}}(\omega_{c'}, x) - C))$ decays exponentially since $d_{\mathcal{C}}(\omega_{c'}, x)$ is at least $8Mn/10$ by the conditions (1) and (2). Since the number of elements of A is universally bounded, the probability that $a < a' < b < c < c'$ will be followed by some d which satisfies (1)-(3) is less than $K'c_2^n$ for some $K' > 0$ and $c_2 < 1$ which depend only on D, δ and M but not on n and $a < a' < b < c < c'$. The number of all possible choices of $a < a' < b < c < c'$ is of order n^5. We may find some $K > 0$ and $c_1 < 1$ such that $n^5 K' c_2^n < K c_1^n$. Thus we complete the proof. □

Remark 3.5. As shown in [6], almost the same argument shows anti-aligned version of Lemma 3.4. Namely, we may replace the conditions (2) and (3) of Lemma 3.4 with

(2)′ there is $t \in [0.1, 0.2]$ so that $d_{\mathcal{C}}(\omega_{a'}, \alpha(1-t)) \le D$ and $d_{\mathcal{C}}(\omega_{c'}, \beta(t)) \le D$,

(3)′ there is some $h \in G$ so that $d_{\mathcal{C}}(h\alpha(0), \beta(1)) \le D$, and $d_{\mathcal{C}}(h\alpha(1), \beta(0)) \le D$,

to have the probability that we have indices satisfying (1), (2)′, and (3)′ decays exponentially.

3.2. Proof of Theorem 1.1.

For $g \in G$, let $\tau(g)$ denote the *translation length*

$$\tau(g) := \lim_{n\to\infty} \frac{d_{\mathcal{C}}(g^n(p), p)}{n}$$

of g on the curve graph $\mathcal{C}(S)$. Maher and Tiozzo proved that the translation length grows linearly [23].

Lemma 3.6 ([23]). *There exist $L > 0$, $K > 0$ and $c < 1$ which only depend on S and μ such that*

$$\mathbb{P}(\tau(\omega_n) < Ln) < Kc^n.$$

We first prepare an elementary observation for an action of a group on a δ-hyperbolic space.

Proposition 3.7 (c.f. [21, Lemma 3.3]). *Let H be a group acting isometrically on a δ-hyperbolic space (Y, d_Y) with a base point x. Fix $h \in H$. Suppose that h has a geodesic axis α, i.e. a geodesic satisfying $h^n(\alpha) \subset \mathcal{N}_{2\delta}(\alpha)$ for all $n \in \mathbb{Z}$ where $\mathcal{N}_{2\delta}(\alpha)$ denotes the 2δ neighborhood of α. Let q be a nearest point projection of x to α. If $d_Y(q, hq) > 28\delta$, the following holds. There exist $D_1, D_2 \geq 0$ which depend only on δ such that the geodesic $\gamma = [x, hx]$ can be decomposed into three subsegments $\gamma = \gamma_1\gamma_2\gamma_3$ so that*

- *the distance $d_Y(\gamma_1(1), q) \leq D_1$ and $d_Y(\gamma_3(0), hq) \leq D_1$, and*
- *$\gamma_2 \subset \mathcal{N}_{D_2}(\alpha)$ and* $\text{length}(\gamma_2) \geq d_Y(q, hq) - 28\delta$.

Proof. Any side of a geodesic quadrilateral in a δ-hyperbolic space is in the 2δ neighborhood of the other three sides. We consider a geodesic quadrilateral whose vertices are x, q, hq, hx. Since q, hq are nearest point projections, if a point $s \in [q, hq]$ is at least 4δ apart from q and hq, then $d_Y(s, \gamma) \leq 2\delta$. This is because if $d_Y(s, \gamma) > 2\delta$, then there must be $s' \in [x, q] \cup [hq, hx]$ such that $d_Y(s, s') \leq 2\delta$, which contradicts the fact that q and hq are nearest point projections to α. Let q_1 (resp. q_2) denote the point on $[q, hq]$ that is exactly 4δ apart from q (resp. hq). Let x_1' (resp. x_2') be a nearest point projection to γ of the point q_1 (resp. q_2). Then $d_Y(x_i', q) \leq 6\delta$ for $i = 1, 2$. By δ-hyperbolicity, if a point $a \in [x_1', x_2']$ is at least 4δ away from both x_1' and x_2', then $d_Y(a, [q, hq]) \leq 2\delta$. Let x_1 (resp. x_2) denote the point on $[x_1', x_2']$ exactly 4δ away from x_1' (resp. x_2'). Put $\gamma_1 := [x, x_1]$, $\gamma_2 := [x_1, x_2]$ and $\gamma_3 := [x_2, hx]$. Note that $d_Y(x_i, q) \leq 10\delta$ for $i = 1, 2$, so we put $D_1 := 10\delta$. By δ-hyperbolicity, except for the 3δ neighborhood of hq, points on $[q, hq]$ is in the δ neighborhood of α. Hence by putting $D_2 := 3\delta$, we have $\gamma_2 \subset \mathcal{N}_{D_2}(\alpha)$. Let q_1', q_2' be nearest point projections of x_1, x_2 to $[q, hq]$ respectively. Then $d_Y(q, q_1') \leq d_Y(q, q_1) + d_Y(q_1, x_1') + d_Y(x_1', x_1) + 2\delta \leq 12\delta$. By symmetry we have $d_Y(q_2', hq) \leq 12\delta$. By triangle inequality, we have $\text{length}(\gamma_2) \geq d_Y(q_1', q_2') - d_Y(x_1, q_1') - d_Y(x_2, q_2') \geq d_Y(q, hq) - 28\delta$. Thus we have a required decomposition. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose $\omega_n = \phi^k$ for some $\phi \in G$ and $k > 1$. Let η be a geodesic axis of ϕ, and $\gamma = [\omega_0, \omega_n]$. By Lemma 3.6, γ_2 of the decomposition of $\gamma = \gamma_1\gamma_2\gamma_3$ from Proposition 3.7 has length at least Ln for some $L > 0$ with exponentially high probability. Let $L' := \text{length}(\gamma_2)$. Then by applying Lemma 3.2 for small enough ϵ, say $1/100$, we may find $D' > 0$ such that $\text{length}(\gamma_{D'})/\text{length}(\gamma) \geq 1 - \epsilon$ with exponentially high probability. Then we can find a D'-proximal point $q_a \in \gamma_2$ such that $d_{\mathcal{C}}(q_a, \gamma_2(0)) \leq L'\epsilon$. Let a denote the index that $d_{\mathcal{C}}(\omega_a, q_a) \leq D'$. Similarly we can find a point $q_b \in \gamma_2$ such that

- $\frac{Ln}{4} \leq d_{\mathcal{C}}(q_a, q_b) \leq \frac{Ln}{4} + L'\epsilon$,
- q_b is D'-proximal so that $d_{\mathcal{C}}(\omega_b, q_b) \leq D'$ for $a < b$.

We consider translating $[q_a, q_b] \subset \gamma_2$ by $\varphi := \phi^{\lfloor k/2 \rfloor}$ where $\lfloor k/2 \rfloor$ is the largest integer among all integers smaller than $k/2$. Note that

$$\frac{\tau(\omega_n)}{3} \leq \tau(\varphi) \leq \frac{\tau(\omega_n)}{2}.$$

By perturbing at most $L'\epsilon$ if necessary, we may assume that both $\varphi(\omega_a)$ and $\varphi(\omega_b)$ are within at most $2D_2 + 2\delta$ distance from D'-proximal points $q_c, q_d \in \gamma_2$ respectively. The constant D_2 is from Proposition 3.7. Hence there exist indices c, d with $a < b < c < d$ such that $d_{\mathcal{C}}(\omega_i, q_i) \leq D' + 2D_2 + 2\delta$ for $i \in \{a, b, c, d\}$. Let $\alpha := [\omega_a, \omega_b]$ and $\beta := [\omega_c, \omega_d]$. By δ-hyperbolicity, we can decompose $\alpha = \alpha_1\alpha_2\alpha_3$ so that length(α_1), length$(\alpha_3) < D' + 2D_2 + 4\delta$ and $\alpha_2 \subset \mathcal{N}_{2\delta}(\gamma)$. Hence if n is large enough, then for some $t \in [0.1, 0.2]$ we can find a D'-proximal point $q_{a'} \in \gamma_2$ with $d_{\mathcal{C}}(q_{a'}, \alpha(t)) \leq 2\delta$. Similarly, we can also find a D'-proximal point $q_{c'}$ such that $d_{\mathcal{C}}(q_{c'}, \beta(t)) \leq 2\delta$. Thus we have indices a' and c' such that $d_{\mathcal{C}}(\omega_{a'}, \alpha(t)) \leq D' + 2\delta$ and $d_{\mathcal{C}}(\omega_{c'}, \beta(t)) < D' + 2\delta$. Thus if ω_n is not primitive we can find indices satisfying conditions (1)-(3) of Lemma 3.4 for $M = L/4$ and $D = D' + 2D_2 + 2\delta$. Therefore the probability that ω_n is not primitive decays exponentially. $\square$

4. Random mapping classes are not symmetric

The goal in this section is to prove Theorem 1.3. We fix a (possibly orbifold) finite covering $\pi\colon S \to S'$. A simple closed curve $a \in \mathcal{C}(S)$ is called *symmetric* if $\pi(a)$ is also a simple closed curve on S' (see §4.2 for more detail). The first step is to show the exponential decay of the shadow of the set of symmetric curves in $\mathcal{C}(S)$. To show the exponential decay, we prepare two lemmas (Lemma 4.2 and Lemma 4.5) in §4.1 and §4.2. Then §4.3 will be devoted to the proof of the exponential decay. Finally we prove Theorem 1.3 in §4.4.

4.1. Set of symmetric projective measured foliations has μ-stationary measure zero. Let μ be a probability measure on the mapping class group G of surface S of finite type. In this section, we suppose that μ satisfies Condition 1.2. A measure ν on $\mathcal{PMF}(S)$ is called μ-stationary if

$$\nu(X) = \sum_{g \in G} \mu(g)\nu(g^{-1}X)$$

for any measurable subset $X \subset \mathcal{PMF}(S)$. We first recall the work of Kaimanovich-Masur. Recall that a projective measured foliation is said to be *uniquely ergodic* if its supporting foliation admits only one transverse measure up to scale. We denote by $\mathcal{UE}(S) \subset \mathcal{PMF}(S)$ the space of uniquely ergodic foliations with unique projective measures.

Theorem 4.1 ([16, Theorem 2.2.4(1)]). *There exists a unique μ-stationary probability measure ν on $\mathcal{PMF}(S)$. The measure ν is non-atomic and concentrated on the set of uniquely ergodic foliations $\mathcal{UE}(S)$.*

Similarly as simple closed curves, a projective measured foliation λ is said to be *symmetric* if $\pi(\lambda)$ is also a projective measured foliation on S'. In this subsection, we will measure by ν the set of symmetric projective measured foliations.

We now recall the Teichmüller space of S. The Teichmüller space $\mathcal{T}(S)$ is the space of conformal structures on S. In this paper we consider the Teichmüller metric on $\mathcal{T}(S)$;

$$d_{\mathcal{T}}(\sigma_1, \sigma_2) = \frac{1}{2} \log \inf_h K(h), \quad \sigma_1, \sigma_2 \in \mathcal{T}(S),$$

where the infimum is taken over all quasi-conformal maps $h\colon \sigma_1 \to \sigma_2$ homotopic to the identity, and $K(h)$ is the maximal dilatation of h. Thurston (c.f. [9]) showed that $\mathcal{PMF}(S)$ compactifies $\mathcal{T}(S)$ so that the action of $G := \mathrm{Mod}(S)$ extends continuously. This compactification is called the *Thurston compactification*. Let $\overline{\mathcal{T}}(S) := \mathcal{T}(S) \cup \mathcal{PMF}(S)$.

Note that our covering $\pi\colon S \to S'$ may be an orbifold covering. If S' is an orbifold, $\mathcal{PMF}(S')$ and $\mathcal{T}(S')$ are defined to be the ones on the surface that we get by puncturing the orbifold points of S'. The covering π determines $\Pi\colon \overline{\mathcal{T}}(S') \to \overline{\mathcal{T}}(S)$ so that $X \in \Pi(\mathcal{T}(S'))$ if $\pi(X) \in \mathcal{T}(S)$, and $\lambda \in \Pi(\mathcal{PMF}(S'))$ if $\pi(\lambda) \in \mathcal{PMF}(S')$. As pointed out in [27, Section 7], Π is an isometric embedding of $\mathcal{T}(S')$. We may also extend the μ-stationary measure ν in Theorem 4.1 to $\overline{\mathcal{T}}(S)$ by $\nu(A) = \nu(A \cap \mathcal{PMF}(S))$ for each subset $A \subset \overline{\mathcal{T}}(S)$. Let $E_{\mathcal{T}} := \Pi(\overline{\mathcal{T}}(S'))$. Our goal in this subsection is the following lemma.

Lemma 4.2. *Let μ be a probability measure on G which satisfies Condition 1.2, and ν the μ-stationary measure on $\overline{\mathcal{T}}(S)$ from Theorem 4.1. Then for any finite covering $\pi\colon S \to S'$, we have for all $g \in G$,*

$$\nu(gE_{\mathcal{T}}) = 0.$$

Recall that $\mathcal{PMF}(S)$ is homeomorphic to the sphere $\mathbb{S}^{6g-7+2n}$. Although the image $\Pi(\mathcal{PMF}(S'))$ is a sphere of lower dimension, Lemma 4.2 is non-trivial. This is because the μ-stationary measure ν is singular to the standard Lebesgue measure on the sphere by the work of Gadre [10].

First, we give a sufficient condition for a subset of $\mathcal{PMF}(S)$ to have ν measure zero.

Proposition 4.3 (c.f. [16, Lemma 2.2]). *Let A be a measurable subset of $\mathcal{PMF}(S)$. Suppose there exist infinitely many disjoint translations of A by elements in* $\mathrm{gr}(\mu)$. *Suppose further that*

$(*)$ $\nu(g_1 A \cap g_2 A) = 0$ *or* $\nu(g_1 A) = \nu(g_2 A)$ *for all* $g_1, g_2 \in G$.

Then $\nu(A) = 0$.

Proof. By $(*)$, we see that there is some $h \in G$ such that $A' := hA$ satisfies $\nu(A') \geq \nu(gA)$ for all $g \in G$. Then since ν is μ-stationary, we have

$$\nu(A') = \sum_{g \in G} \mu(g)\nu(g^{-1}A') \leq \sum_{g \in G} \mu(g)\nu(A') = \nu(A').$$

Thus we see that $\nu(g^{-1}A') = \nu(A')$ for every g in the support of μ. By discussing the n-convolution μ^n of μ, we see that $\nu(g^{-1}A') = \nu(A')$ for every $g \in \mathrm{sgr}(\mu)$. Since we have infinitely many disjoint translates of A' by elements of $\mathrm{gr}(\mu)$, we see that we also have infinitely many disjoint translates by elements of $\mathrm{sgr}(\mu)^{-1}$. Hence we have $\nu(A') = \nu(A) = 0$. □

To prove Lemma 4.2, we recall Teichmüller geodesics on the Teichmüller space, see for example [9, 11, 16] for more details. Recall that S is a surface of finite type (g, n). Teichmüller showed that for any given point $\sigma \in \mathcal{T}(S)$, a holomorphic quadratic differential q determines a geodesic $\Gamma(q)$ with respect to Teichmüller metric. It is also proved that given two points $\sigma_1, \sigma_2 \in \mathcal{T}(S)$, there exists a unique Teichmüller geodesic $\Gamma(\sigma_1, \sigma_2)$ that connects the two.

For $\sigma \in \mathcal{T}(S)$, let $\mathrm{QD}(\sigma)$ denote the Banach space of holomorphic quadratic differentials on σ with $\| \varphi \| = \int_\sigma |\varphi|$. Each $\varphi \in \mathrm{QD}(\sigma)$ determines two measured foliations, called the horizontal foliation and the vertical foliation. By Riemann-Roch theorem, $\mathrm{QD}(\sigma)$ has complex dimension $3g - 3 + n$. Let $\mathcal{Q}_0 \subset \mathrm{QD}(\sigma)$ denote the unit sphere. This $\mathcal{Q}_0$ compactifies $T(\sigma)$ which is called the *Teichmüller compactification*.

By the work of Hubbard-Masur (compact) and Gardiner (finite type), we see:

Lemma 4.4 ([14], [11, Chapter 11]). *For any $\sigma \in \mathcal{T}(S)$ and $F \in \mathcal{PMF}(S)$, there is a unique $\varphi \in \mathrm{QD}(\sigma)$ whose horizontal foliation is F up to scale.*

Proof of Lemma 4.2. The proof goes by induction. Let $E'_{\mathcal{T}} := gE_{\mathcal{T}}$ and d the complex dimension of $\mathrm{QD}(\sigma')$ for any $\sigma' \in \mathcal{T}(S')$. We consider intersection $E' := g_1E'_{\mathcal{T}} \cap g_2E'_{\mathcal{T}} \cap \cdots \cap g_nE'_{\mathcal{T}}$. We first define $d(E') \in \mathbb{N}$. If $E' \cap \mathcal{PMF}(S)$ contains at most one uniquely ergodic foliation, then we define $d(E') = 0$. In this case we also have $\nu(E') = 0$ since ν is non-atomic. If $E' \cap \mathcal{PMF}(S)$ contains at least two uniquely ergodic foliations $\mathcal{E}_1, \mathcal{E}_2$, then there is a unique Teichmüller geodesic γ connecting $\mathcal{E}_1$ and $\mathcal{E}_2$ by [12]. Since covering maps induce isometric embeddings of Teichmüller spaces [27, Section 7], any point of γ is in E'. In particular $E' \cap \mathcal{T}(S)$ is non-empty. For any $\sigma \in E' \cap \mathcal{T}(S)$, each $g_iE'_{\mathcal{T}}$ determines a subspace of $S_i(\sigma) \subset \mathrm{QD}(\sigma)$ which consists of the lifts of holomorphic quadratic differentials with respect to the covering $\pi \circ g_i^{-1}$. Let $S(\sigma) := \cap_{i=1,\dots,n} S_i(\sigma)$ and $d(\sigma) := \dim S(\sigma)$. Since $d(\sigma) \in \mathbb{N}$, there exists $\sigma' \in E' \cap \mathcal{T}(S)$ such that $d(\sigma') \geq d(\sigma)$ for any $\sigma \in E' \cap \mathcal{T}(S)$. We define $d(E') := d(\sigma')$. Then we explain how the induction works by using a style of inductive algorithm, see Algorithm 1 which is named MνIT. By MνIT(E, d), we have $\nu(E) = 0$. Note that although the depth of Algorithm 1 is finite, the width is infinite. □

Algorithm 1 MνIT(Measure by ν the Intersection of Translates)

Input: $(E' := g_1 E'_{\mathcal{T}} \cap g_2 E'_{\mathcal{T}} \cap \cdots \cap g_n E'_{\mathcal{T}}, d(E'))$.
Ensure: $\nu(E') = 0$.
  if $d(E') = 0$ **then**
    By the definition of $d(E')$ and Lemma 4.4, we have $\nu(E') = 0$.
  end if
  for $h_1, h_2 \in G$ **do**
    Let $E'_1 := h_1 E'$ and $E'_2 := h_2 E'$. Note that since each $g \in G$ induces a vector isomorphism between $\mathrm{QD}(\sigma)$ and $\mathrm{QD}(g\sigma)$, we have $d(E'_1) = d(E'_2) = d(E')$.
    if $d(E'_1 \cap E'_2) = d(E')$ **then**
      We see that $\nu(E'_1) = \nu(E'_2)$ by Lemma 4.4.
    else
      In this case we have $d(E') > d(E'_1 \cap E'_2)$. Then we apply M$\nu$IT$(E'_1 \cap E'_2, d(E'_1 \cap E'_2))$, which proves $\nu(E'_1 \cap E'_2) = 0$.
    end if
  end for

  We have seen that the condition $(*)$ of Proposition 4.3 is satisfied. By Condition 1.2 and the north-south dynamics of pseudo-Anosov maps (see [32]), we see that there are infinitely many disjoint translates of E'. Thus by Proposition 4.3, we have $\nu(E') = 0$.

4.2. Upper bound for the number of parallel translates of the set of symmetric curves. Recall that we have fixed a (possibly orbifold) covering $\pi: S \to S'$. If S' is an orbifold, we define $\mathcal{C}(S')$ as the curve graph of the surface that we get by puncturing every orbifold point of S'. We define one to finite relation $\Pi_{\mathcal{C}}: \mathcal{C}(S') \to \mathcal{C}(S)$ as follows. A curve $b \in \mathcal{C}(S)$ is in $\Pi_{\mathcal{C}}(a)$ for some $a \in \mathcal{C}(S')$ if $\pi(a) = b$ as isotopy classes of simple closed curves. In [27], Rafi-Schleimer showed that $\Pi_{\mathcal{C}}$ is quasi-isometric (Theorem 4.9). Hence the map $\Pi_{\mathcal{C}}$ extends continuously to the Gromov boundary $\partial\mathcal{C}(S')$. Let E denote $\Pi_{\mathcal{C}}(\mathcal{C}(S') \cup \partial\mathcal{C}(S'))$. We call elements in E symmetric. We consider translates gE's of E by $g \in G$. Our aim in this subsection is to prove the following lemma.

Lemma 4.5. *For any $D_0 > 0$, there exist $D_1, D_2 > 0$ which depend only on S and D_0 such that for any $a, b \in \mathcal{C}(S)$ with $d_{\mathcal{C}}(a,b) > D_1$, the number of elements in*

$$\mathcal{P}(a,b,D_0) := \{gE \mid d_{\mathcal{C}}(a,gE) < D_0 \text{ and } d_{\mathcal{C}}(b,gE) < D_0\}$$

is bounded from above by D_2. Here we count the number of images i.e. if $g_1 E = g_2 E$ as subsets, we just count one time.

For a proof, we need the notion of subsurface projection. A subsurface $Y \subset S$ is called *essential* if each component of ∂Y is an essential simple closed curve. Unless otherwise stated, we always assume that subsurfaces are essential. Given a subsurface $Y \subset S$ which is not an annulus nor three holed sphere, we define subsurface projection $\pi_Y : \mathcal{C}(S) \to \mathcal{C}(Y)$ as follows: given a curve $a \in \mathcal{C}(S)$ on S, arrange a so that it has minimal intersection with Y. Take a component a' of $Y \cap a$ and consider a small neighborhood N of $a' \cup \partial Y$. Then $\pi_Y(a)$ is defined to be a component of N which is in $\mathcal{C}(Y)$. If a does not intersect with Y, then we define $\pi_Y(a) = \emptyset$. If Y is an annulus, we need special care, however we do not need the detail for the proof, so we omit the definition. See for example [26, 29] for the detail. If Y is a three holed sphere, subsurface projection is not defined. We call a subsurface $Y \subset S$ *symmetric* if it is a component of $p^{-1}(Y')$ for some $Y' \subset S'$. Given two curves $a, b \in \mathcal{C}(S)$, we let $d_Y(a, b) := \mathrm{diam}(\pi_Y(a), \pi_Y(b))$. If Y is an annulus with core curve α, we often use d_α to denote d_Y. Rafi-Schleimer showed the following lemma.

Lemma 4.6 ([27, Lemma 7.2]). *There exists T_1 which depends only on S and the degree of $\pi : S \to S'$ such that for any subsurface $Y \subset S$ and $a, b \in E$, if $d_Y(a, b) \geq T_1$ then Y is symmetric.*

We recall the work of Masur and Minsky [26].

Theorem 4.7 ([26, Theorem 3.1], bounded geodesic image). *There exists a constant $M_1 > 0$ which depends only on S with the following property. Let $Y \subset S$ be a proper subsurface Y which is not a three holed sphere. Let γ be a geodesic in $\mathcal{C}(S)$ with $\pi_Y(v) \neq \emptyset$ for all vertex v on γ. Then*

$$\mathrm{diam}_Y(\gamma) \leq M_1,$$

where $\mathrm{diam}_Y(\gamma)$ *is the diameter of $\pi_Y(\gamma)$ in $\mathcal{C}(Y)$.*

Combining Lemma 4.6 and Theorem 4.7, we have the following.

Lemma 4.8. *Given $g \in G$, fix $a, b \in gE$ and $D > 0$. Suppose there exists a subsurface $Y \subset S$ such that $d_Y(a, b) \geq T + 2M_1$ for $T \geq T_1$, and $d_{\mathcal{C}}(\{a, b\}, \partial Y) \geq D + 2$. Then if there are $c, d \in hE$ for some $h \in G$ such that $d_{\mathcal{C}}(c, a) \leq D$ and $d_{\mathcal{C}}(d, b) \leq D$ then we have $d_Y(c, d) \geq T$. In particular Y is symmetric for both πg^{-1} and πh^{-1}.*

Proof. Since we assume that ∂Y is far from a, b, c, d, we see that every vertex on geodesics $[a, c]$ and $[b, d]$ intersects Y non-trivially. Hence by Theorem 4.7, we see that $d_Y(c, d) \geq d_Y(a, b) - d_Y(c, a) - d_Y(b, d) \geq T_1$. The last assertion follows from Lemma 4.6. □

We recall the following work of Rafi and Schleimer [27] and Rafi [29].

Theorem 4.9 ([27]). *The covering relation $\Pi\colon \mathcal{C}(S') \to \mathcal{C}(S)$ is a Q-quasi-isometric embedding. The constant Q depends only on S and the degree of $\pi\colon S \to S'$.*

To state the work of Rafi [29], we need the notion of *shortest markings*. Given a point $\sigma \in \mathcal{T}(S)$, a shortest marking of σ is the set of curves chosen as follows. Consider σ as a hyperbolic structure of S. First we greedily choose shortest curves; let α_1 be a shortest curve, then we choose α_2 as a shortest curve on $S \setminus \alpha_1$. We proceed until $\alpha_i' s$ give a pants decomposition of S. Then we choose β_i as a shortest curve among curves intersecting only α_i. We denote a shortest marking of σ by $\mu(\sigma)$. In the statement of Theorem 4.10, the function $[x]_k$ is equal to zero when $x < k$ and is equal to x when $x \geq k$. We also modify the log in the statement so that $\log x = 0$ for $x \in [0, 1]$.

Theorem 4.10 ([29]). *There exists $k' > 0$ such that for $k > k'$, for any $\sigma_1, \sigma_2 \in \mathcal{T}(S)$, the following holds. Let $A := d_{\mathcal{T}}(\sigma_1, \sigma_2)$, and*

$$B := \sum_Y [d_Y(\mu(\sigma_1), \mu(\sigma_2))]_k + \sum_\alpha \log[d_\alpha(\mu(\sigma_1), \mu(\sigma_2))]_k.$$

Where in the first sum, Y is taken over all subsurfaces of S which are not three holed spheres nor annuli, and in the second sum α is taken over all essential simple closed curves. Then there exist constants $C, c > 0$ which depend only on S and k such that

$$\frac{1}{C}A - c \leq B \leq CA + c.$$

We prepare one more lemma in order to prove Lemma 4.5.

Lemma 4.11. *There exists $K > 0$ which depends only on S such that for any $q \in \mathrm{QD}(S)$, $\sharp\{gE_{\mathcal{T}} \mid \Gamma(q) \subset gE_{\mathcal{T}}\} < K$.*

Proof. By taking conjugation if necessary, we may suppose $E_{\mathcal{T}} \in \{gE_{\mathcal{T}} \mid \Gamma(q) \subset gE_{\mathcal{T}}\}$. Recall that by integrating the square root, each non-zero element $q \in \mathrm{QD}(S)$ determines a singular Euclidean structure with horizontal and vertical foliation. Let $\mathrm{Sing}(q)$ denote the set of singular points of the singular Euclidean structure. This $\mathrm{Sing}(q)$ is finite. Pick any $s \in \mathrm{Sing}(q)$, then we define $\Sigma_1(s) := \pi^{-1}\pi(s)$ where $\pi\colon S \to S'$ is the finite covering we fixed above. Inductively define $\Sigma_{i+1}(s) := g \circ \pi^{-1}(\pi \circ g^{-1}(\Sigma_i(s)))$. Since $\Sigma_i(s) \subset \Sigma_{i+1}(s) \subset \mathrm{Sing}(q)$, we eventually have $\Sigma_i(s) = \Sigma_{i+1}(s)(=: \Sigma(s))$ for large enough i. Next, we pick any $x \in S \setminus \mathrm{Sing}(q)$. There is a point $s' \in \mathrm{Sing}(q)$ such that we can connect x and s' by a single Euclidean geodesic γ. The geodesic γ has well defined angle

$\theta_\gamma \mod \pi$. Let $l_q(\gamma)$ denote the Euclidean length of γ. Since there are only finitely many points from $\Sigma(s')$ with angle θ_γ and Euclidean distance $l_q(\gamma)$, we get $\Sigma(x) \subset S \setminus \mathrm{Sing}(q)$ in the same way as above. Thus we get an equivalence relation $x \sim y \iff y \in \Sigma(x)$ on S. Since this relation is defined by composing local homeomorphisms g and π, the quotient map $\pi' \colon S \to S/\!\sim$ is a covering. By construction, π' factors through $\pi \colon S \to S'$ and we have two coverings $p, p_g \colon S' \to S/\!\sim$ such that $p \circ \pi = p_g \circ \pi \circ g$ as covering maps. Furthermore, since for each $x \in S/\sim$, we may find a small open neighborhood U_x such that on all component of $(\pi')^{-1}(U_x)$, we can identify the quadratic differentials via π and πg^{-1}, we have a quadratic differential q' with $(\pi')^{-1}(q') = q$. Let us suppose that for $p, p_g \colon S' \to S/\sim$, we have $p_*(\pi_1(S')) = (p_g)_*(\pi_1(S'))$. Then there exists a homeomorphism $f_g \colon S' \to S'$ such that $p_g = p \circ f_g$. We further suppose that $(f_g \circ \pi)_*(\pi_1(S)) = \pi_*\pi_1(S)$ in $\pi_1(S')$. Then we can lift f_g to $\tilde{f_g} \colon S \to S$. By construction, $(\tilde{f_g})^{-1} g$ preserves q. Suppose there is $hE_{\mathcal{T}} \neq gE_{\mathcal{T}} \in \{gE_{\mathcal{T}} \mid \Gamma(q) \subset gE_{\mathcal{T}}\}$ such that $hE_{\mathcal{T}}$ determines the same equivalence relation $\sim$ and similarly as g, we have a map $\tilde{f_h}$ which is a lift of a homeomorphism $f_h \colon S' \to S'$ and $(\tilde{f_h})^{-1} h$ preserves q. If $(\tilde{f_g})^{-1} g = (\tilde{f_h})^{-1} h$, we have $h^{-1} g = (\tilde{f_h})^{-1} \tilde{f_g}$ and hence $hE_{\mathcal{T}} = gE_{\mathcal{T}}$. Hence $(\tilde{f_g})^{-1} g \neq (\tilde{f_h})^{-1} h$. The number of mapping classes that preserve q is universally bounded. Furthermore the number of possibility of $S/\sim$ is universally bounded and for each case the number of coverings $S' \to S/\!\sim$ is also universally bounded in terms of S. Since the number of subgroups of a fixed degree in $\pi_1(S')$ is also bounded, there is an upper bound which depends only on S for the number of $\{gE_{\mathcal{T}} \mid \Gamma(q) \subset gE_{\mathcal{T}}\}$. The number of possible coverings from S is finite and thus we complete the proof. $\square$

We are now ready to prove Lemma 4.5.

Proof of Lemma 4.5. Throughout the proof, we call a constant *universal* if it only depends on S and the degree of $\pi \colon S \to S'$. For a given $s \in \mathcal{C}(S)$, let $\sigma(s, gE)$ denote a point $\mathcal{T}(S)$ so that a shortest marking $\mu(s, gE) := \mu(\sigma(s, gE))$ contains a closest point projection of s to gE. Note that there may be several closest projections. We choose one of them, and fix it. By considering the conjugacy of the covering, we may suppose $E \in \mathcal{P}(a, b, D_0)$.

We proceed by induction. First note that the statement of Lemma 4.5 is true for annuli. Then we may suppose Lemma 4.5 holds for any subsurface of S which is not a three holed sphere. Let D_3' be the constant so that Lemma 4.5 holds for $D_0 := M_1$ and $D_1 := D_3'$ for any subsurface of S. Then let $D_3 := \max\{D_3', T_1\}$.

We now consider two cases. The first case is where we can find four subsurfaces $Y_i (i = 1, 2, 3, 4)$ such that

(1) $d_{\mathcal{C}}(a, \partial Y_i) > 2D_0 + 2, d_{\mathcal{C}}(b, \partial Y_i) > 2D_0 + 2$,

(2) $d_{\mathcal{C}}(\partial Y_i, \partial Y_j) > 3$,

(3) $d_{Y_i}(\mu(a, E), \mu(b, E)) \geq D_3 + 6M_1$.

Note that by the third condition and Theorem 4.7, any geodesic in $\mathcal{C}(S)$ connecting $\mu(a, E)$ and $\mu(b, E)$ passes close to ∂Y_i's, and we also suppose ∂Y_i's appear in the order of the index. By Lemma 4.8 we see that for any $gE \in \mathcal{P}(a, b, D_0)$, we have $d_{Y_i}(\mu(a, gE), \mu(b, gE)) \geq D_3 + 4M_1$ and hence ∂Y_i are all contained in every $gE \in \mathcal{P}(a, b, D_0)$. Then again by Theorem 4.7, we may choose components y_1 and y_4 of ∂Y_1 and ∂Y_4 respectively so that $d_{Y_2}(y_1, y_4) \geq D_3$. Hence for any $gE \in \mathcal{P}(a, b, D_0)$, we have $d_{Y_2}(\mu(y_1, gE), \mu(y_4, gE)) \geq D_3$. By induction, for $gE \in \mathcal{P}(a, b, D_0)$, the restriction $\pi \circ g^{-1}|Y_2$ of the covering is one of the universally bounded number of coverings from Y_2. Moreover, once we fix the topology of a subsurface Y', there are only finitely many possible embedding $Y' \hookrightarrow S'$ up to the action of mapping classes on S' that can be lifted to S via $\pi: S \to S'$. Let $\mathcal{P}(a, b, D_0, Y_2)$ denote the subset of $\mathcal{P}(a, b, D_0)$ whose elements correspond g's that satisfy the following; $\pi|Y_2$ and $\pi \circ g^{-1}|Y_2$ are the same as coverings from Y_2, and the images of Y_2 in S' by the coverings are related without permuting boundary components by a mapping class ϕ_g which can be lifted to S. We see that $\mathcal{P}(a, b, D_0)$ can be decomposed into universally bounded number of subsets of type $\mathcal{P}(a, b, D_0, Y_2)$. To find a universal bound for the cardinality of $\mathcal{P}(a, b, D_0)$, we only need to find a universal bound for the cardinality of each subset in the decomposition. Hence, it suffices to find a universal bound for the number of elements in $\mathcal{P}(a, b, D_0, Y_2)$.

Let $gE \in \mathcal{P}(a, b, D_0, Y_2)$. Note that if $\widetilde{\varphi}: S \to S$ is a lift with respect to π, then $\widetilde{\varphi}E = E$. Hence by precomposing suitable lift $\widetilde{\phi_g}$ of ϕ_g to g, which we again denote by g by abuse of notation, we may suppose that $\pi(g^{-1}(Y_2)) = \pi(Y_2)$ and $\pi(y_j) \cap \pi(Y_2) = \pi g^{-1}(y_j) \cap \pi(Y_2)$ for both $j = 1, 4$. Let y_2' be a component of $\partial\pi(Y_2)$. Then by the above observation, we have for any component z_2 of $\pi^{-1}(y_2')$,

$$\frac{i(y_2', \pi(y_1))}{i(y_2', \pi(y_4))} = \frac{i(z_2, \pi^{-1}\pi(y_1))}{i(z_2, \pi^{-1}\pi(y_4))} = \frac{i(g(z_2), \pi^{-1}\pi(y_1))}{i(g(z_2), \pi^{-1}\pi(y_4))}. \tag{1}$$

Note that since $\pi^{-1}(\pi(y_1))$ and $\pi^{-1}(\pi(y_4))$ fill the surface S, it determines a quadratic differential q with horizontal foliation $\pi^{-1}(\pi(y_1))$ and vertical foliation $\pi^{-1}(\pi(y_4))$. Given a Teichmüller geodesic $\Gamma: \mathbb{R} \to \mathcal{T}(S)$, a simple closed curve is said to be *balanced* at time t on Γ if the intersection number with horizontal foliation and vertical foliation of quadratic differential determined by Γ and t coincide. By (1), on the Teichmüller geodesic $\Gamma(q)$ determined by q, all components of $\pi^{-1}(y_2') \cup g\pi^{-1}(y_2')$ are balanced at the same time. By Theorem [28, Theorem 6.1] applied to $\Gamma(q)$, we see that $\pi^{-1}(y_2') \cup g\pi^{-1}(y_2')$ is a disjoint union of simple closed curves. Since the number of disjoint simple closed curves is universally bounded from above, we may further decompose $\mathcal{P}(a, b, D_0, Y_2)$ into universally bounded number of subsets so that if g_1E and g_2E are in the same subset, we have $g_1\pi^{-1}(y_2') = g_2\pi^{-1}(y_2')$. We argue similarly for Y_3 and hence all we need is to find a universal bound for the number of $\{g_iE\}_{i \in I}$ with

- $g_i\pi^{-1}(y'_2) = g_j\pi^{-1}(y'_2)$, and
- $g_i\pi^{-1}(y'_3) = g_j\pi^{-1}(y'_3)$ for some component y'_3 of $\partial\pi(Y_3)$

for any $i, j \in I$. In this case, the Teichmüller geodesic determined by the quadratic differential with horizontal foliation $g_i\pi^{-1}(y'_2)$ and vertical foliation $g_i\pi^{-1}(y'_3)$ is contained in $g_i E_{\mathcal{T}}$ for all $i \in I$. Then by Lemma 4.11, we have a desired bound. Thus we are done for the first case.

We now consider the case where we can not find subsurfaces satisfying the conditions of the first case. The shadow of a Teichmüller geodesic Γ is the set of all curves which are the shortest at some point on Γ. In [25], it is proved that the shadow of any Teichmüller geodesic is a (unparametrized) quasi-geodesic. Hence there exists a universal constant D_4 such that for $hE \in \mathcal{P}(a, b, D_0)$, the shadow of the Teichmüller geodesic $\Gamma(\sigma(a, hE), \sigma(b, hE))$ is contained in D_4-neighborhood of the shadow of $\Gamma(\sigma(a, E), \sigma(b, E))$. Then by assuming D_1 in the statement large enough, we may suppose that there are points σ_1, σ_2 on the Teichmüller geodesic $\Gamma(\sigma(a, E), \sigma(b, E))$ such that

- $d_{\mathcal{C}}(\mu(\sigma_1), \mu(\sigma_2)) > D_4 + 2$, and
- $d_Y(\mu(\sigma_1), \mu(\sigma_2)) < D_3 + 6M_1$ for all proper subsurface $Y \subset S$,
- $d_Y(\mu(\mu(\sigma_1), hE), \mu(\mu(\sigma_2), hE)) < D_3 + 6M_1$ for all proper subsurface $Y \subset S$.

Let $D_5 := D_3 + 6M_1$ and $\sigma'_1 := \sigma(\mu(\sigma_1), hE)$. These condition together with Theorem 4.7 imply that $d_Y(\mu(\sigma_1), \mu(\sigma'_1), hE)) < 2D_5 + M_1$ for any proper subsurface $Y \subset S$. Hence by Theorem 4.10, we see that there exists a universal constant D_6 and ϵ such that $d_{\mathcal{T}}(\sigma_1, \sigma'_1) < D_6$ and the Teichmüller geodesic connecting σ_1 and σ'_1 is contained in the ϵ-thick part of $\mathcal{T}(S)$.

Recall that the subgroup of $\mathrm{Mod}(S')$ that can be lifted via $\pi\colon S \to S'$ is of finite index, and the thick part of the moduli space of S' is compact. Hence we can find $h' \in G$ such that $h'\sigma_1 \in hE_{\mathcal{T}}$ and we can connect σ'_1 and $h'\sigma_1$ by a Teichmüller geodesic which is contained in the ϵ-thick part and of length universally bounded from above. Thus we have a universal constant D_7 such that σ_1 and $h'\sigma_1$ can be connected by a path in the $d\epsilon$-thick part of length less than D_7. Again, by the compactness of the thick part of the moduli space of S, the number of such mapping classes are universally bounded. Hence we see that if $hE \in \mathcal{P}(a, b, D_0)$, hE has to pass through one of the universally bounded number of points. By assuming D_1 large enough we may apply the same argument to different subarc of $\Gamma(\sigma(a, E), \sigma(b, E))$, and hence we see that there are two disjoint finite subsets of $\mathcal{T}(S)$ so that if $hE \in \mathcal{P}(a, b, D_0)$, hE has to pass through both subsets. Since two points in Teichmüller space determines a quadratic differential, again by Lemma 4.11, we have a desired bound. This completes the proof of Lemma 4.5. □

4.3. Exponential decay for the shadow of E. By the work of Klarreich [17] (see also Hamenstädt [13]), the Gromov boundary $\partial\mathcal{C}(S)$ of $\mathcal{C}(S)$ is identified with the space $\mathcal{F}_{\min}(S)$ of minimal foliations. There is a natural measure forgetting map from $\mathcal{UE}(S)$ to $\mathcal{F}_{\min}(S)$. Hence we may consider the push forward of ν to $\mathcal{F}_{\min}(S)$, which we again write as ν by abuse of notation. This ν extends to $\overline{\mathcal{C}}(S) := \mathcal{C}(S) \cup \partial\mathcal{C}(S)$ by $\nu(A) = \nu(A \cap \partial\mathcal{C}(S))$ for $A \subset \overline{\mathcal{C}}(S)$.

For a subset $A \subset \overline{\mathcal{C}}(S)$, we define the shadow $S_p(A,r)$ for $r > 0$ and $p \in \overline{\mathcal{C}}(S)$ by

$$S_p(A,r) := \bigcup_{a\in A} S_p(a,r).$$

We first prove the following lemma, which is a key step for showing Theorem 1.3.

Lemma 4.12 (c.f. [22, Lemma 2.10]). *Let μ be a probability measure on G which satisfies Condition 1.2, and ν the μ-stationary measure on $\overline{\mathcal{T}}(S)$ from Theorem 4.1. Then there are constants $K > 0$ and $c < 1$, such that for any $r > 0$ and $g \in G$,*

$$\nu(S_1(gE,r)) < c^r, \mathbb{P}(\omega_n \in (S_1(gE,r))) < Kc^r,$$

and the constants K and c depend on μ and $\pi\colon S \to S'$ but not on r, g and n.

We prove Lemma 4.12 by borrowing several arguments from the proof of [22, Lemma 2.10]. In [22], Maher uses several lemmas from [6], which are applications of Lemma 4.13 below. Instead of using those lemmas, we only use Lemma 4.13 since the proof of each lemma in [6] that we need is short and elementary.

Lemma 4.13 (see for example [3, Proposition 6.7]). *Let (X,d_X) be a δ-hyperbolic space. Then there is a constant K_1 which depends only on δ with the following property. For any four points $x_1,x_2,x_3,x_4 \in X$, there is an embedded tree T connecting the four point such that*

$$d_T(x_i,x_j) \le d_X(x,y) + K_1, \tag{2a}$$

$$(x_i\cdot x_j)_{x_k} - 2K_1 \le (x_i\cdot x_j)^T_{x_k} \le (x_i\cdot x_j)_{x_k} + K_1, \tag{2b}$$

for $1 \le i,j \le 4$. Where d_T denotes the distance in T, and for $a,b,c \in T$, $(a,b)^T_c$ denotes the Gromov product with respect to d_T.

Note that the only combinatorial type of the tree up to reindexing is as depicted in Figure 1.

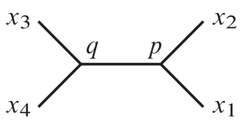

Figure 1. Approximate tree.

We will use the following lemma in [22].

Lemma 4.14 ([22, Proposition 2.12]). *For any $\epsilon > 0$, there is a constant $K_2(\epsilon)$ which depends on ϵ and μ, such that if $r \geq K_2(\epsilon)$, then $\nu(S_1(x, r)) < \epsilon$.*

For the proof of Lemma 4.12, we also prepare the following lemma.

Lemma 4.15. *Let μ be a probability measure on G which satisfies Condition 1.2, and ν the μ-stationary measure on $\overline{\mathcal{T}}(S)$ from Theorem 4.1. Then there exist $\epsilon > 0$ and a constant K_3 which depend only on μ and $\pi\colon S \to S'$, such that if $r \geq K_3$, then $\nu(S_1(gE, r)) < 1 - \epsilon$ for any $g \in G$.*

Proof. First we prove that for a given $g \in G$ and $\epsilon > 0$, there is some number $r_{g,\epsilon}$ such that $\nu(S_1(gE, r_{g,\epsilon})) < \epsilon$. The argument below is almost the same as [22, Proposition 2.12]. Suppose contrary that there exists a sequence $\{r_i\}_{i \in I}$ with $r_i \to \infty$ and $\nu(S_1(gE, r_i)) \geq \epsilon$. Then since $S_1(gE, r_i) \subset S_1(gE, r_j)$ if $j > i$, we have $U := \cap_{i \in I} S_1(gE, r_i) \geq \epsilon$. But if $\lambda \in U$, there is a sequence $\{x_i\}$ of points in gE such that $(x_i \cdot \lambda)_1 \to \infty$. Since gE is a quasi-isometrically embedded image of $\mathcal{C}(S')$, we have $\lambda \in gE$. Hence $U \subset gE$. However, by Condition 1.2, we may apply Lemma 4.2 to conclude $\nu(gE) = 0$ and hence $\nu(U) = 0$. Thus we have a contradiction.

Then recall that $S_1(a, r)$ can also be regarded as the set of points with geodesics connecting 1 and those points pass through certain neighborhood of a. This, together with the hyperbolicity of $\mathcal{C}(S)$, and the fact gE is a quasi-isometrically embedded image of $\mathcal{C}(S')$, implies for sufficiently large r, K, if $S_1(gE, r)$ intersects both $S_1(a_1, K)$ and $S_1(a_2, K)$ with a_1 and a_2 sufficiently far apart, then $gE \in \mathcal{P}(a_1, a_2, D_0)$ for some constant D_0 depending only on S but not on $d_{\mathcal{C}}(a_1, a_2)$. Let us take a_1 and a_2 so that $d_{\mathcal{C}}(a_1, a_2)$ is sufficiently large so that we can apply Lemma 4.5. Furthermore, we may choose a_1, a_2 from the semigroup generated by the support of μ. Then by [23, Proposition 5.4], we have $\nu(S_1(a_i, K)) > 0$ for $i = 1, 2$ after taking K large enough if necessary. Let $\epsilon := \min(\nu(S_1(a_1, K)), \nu(S_1(a_2, K)))$. Then suppose $S_1(gE, r)$ is disjoint from $S_1(a_i, K)$ for one of $i = 1, 2$, then $\nu(S_1(gE, r)) < 1 - \epsilon$. On the other hand, by Lemma 4.5, there are only bounded number of many $g_j E$ such that $S_1(g_j E, r)$ intersects $S_1(a_i, K)$ for both $i = 1, 2$. For each j there exists r_j such that $\nu(S_1(g_j E, r_j)) < 1 - \epsilon$. Then for $K_3 := \max(r_j, r)$, we have $\nu(S_1(gE, r)) < 1 - \epsilon$ for any $r > K_3$ and $g \in G$. □

We recall the following lemma which is a version of the lemma due to Maher. For later convenience, we slightly modify the constants in (4) and (5), but the proof goes exactly the same way as [22].

Lemma 4.16 ([22, Lemma 2.11]). *Let μ be a probability distribution of finite support of diameter D. Let $X_0 \supset X_1 \supset X_2 \supset \dots$ be a sequence of nested closed subsets of $\overline{\mathcal{C}}(S)$ with the following properties*:

$$1 \notin X_0, \tag{1}$$

$$(\mathcal{C}(S) \setminus X_n) \cap X_{n+1} = \emptyset, \tag{2}$$

$$d_{\mathcal{C}}(\mathcal{C}(S) \setminus X_n, X_{n+1}) \geq D. \tag{3}$$

Furthermore, suppose there is a constant $0 < \epsilon < 1$ such that, for any $x \in X_n \setminus X_{n+1}$ which is the translate of the base point p by $x \in G$,

$$\nu_x(X_{n+2}) \leq 1 - \epsilon, \tag{4}$$

$$\nu_x(\mathcal{C}(S) \setminus X_{n-1}) \leq \epsilon/2, \tag{5}$$

where $\nu_x(A) := \nu(x^{-1}A)$ for any $A \subset \overline{\mathcal{C}}(S)$. Then there are constants $c < 1$ and K, which depend only on ϵ and μ, such that $\nu(X_n) < c^n$ and $\mathbb{P}(\omega_i \in X_n) < Kc^n$ for all $i \in \mathbb{N}$.

Then, to prove Lemma 4.12, it suffices to prove

Lemma 4.17. *There exists L which depends on μ, δ with the following property. The sets $X_n := S_1(gE, L(n+1))$ for all $n \in \mathbb{N}$ form a sequence of nested sets which satisfies* (1)–(5) *in Lemma* 4.16.

Proof. The proof goes in a similar way to [22, Lemma 2.13]. Let D be the diameter of μ. We use the constants $K_1, \dots, K_3$ from Lemma 4.13-4.15. Let $L := 4K_1 + \max\{D, K_2(\epsilon/2), K_3, 2\delta\}$.

(1) The Gromov product $(1 \cdot a)_1 = 0$ for all $a \in \overline{\mathcal{C}}(S)$. For all $y \in X_0$, there is $e_y \in gE$ such that $(e_y \cdot y)_1 \geq L > 0$, hence $1 \notin X_0$.

(2) If $y_i \to y \in \partial G$, then by the property of the Gromov product (see for example [2, III.H 3.17(5)]), $\liminf (x \cdot y_i)_1 \geq (x \cdot y)_1 - 2\delta$. This implies if $y \in X_{n+1}$, then for any sequence $y_i \to y$, all but finitely many y_i's are in $X_n = S_1(gE, L(n+1))$ since $L > 2\delta$. Thus we have $X_{n+1} \cap (\mathcal{C}(S) \setminus X_n) = \emptyset$.

(3) Let $a \in X_{n+1}$, then there exists $e_a \in gE$ such that $a \in S_1(e_a, L(n+2))$. Let $b \in \mathcal{C}(S) \setminus X_n$, then for all $e \in gE$, we have $b \notin S_1(e, L(n+1))$. In particular $b \notin S_1(e_a, L(n+1))$. Then we consider a tree T_1 from Lemma 4.13 that connects $\{1, b, a, e_a\}$. Since $(a \cdot e_a)_1 \geq L(n+2)$ and $(b \cdot e_a)_1 < L(n+1)$, by (2b), the only possible combinatorial type of T_1 is the one we get by

substituting $(x_1, x_2, x_3, x_4) = (1, b, a, e_a)$ in Figure 1. Then we see that

$$\begin{aligned} d_{\mathcal{C}}(a,b) &\geq d_{T_1}(a,b) - K_1 \\ &\geq d_{T_1}(p,q) - K_1 \\ &\geq (a \cdot e_a)_1 - (b \cdot e_a)_1 - 4K_1 \\ &\geq L - 4K_1, \end{aligned}$$

where p, q are the trivalent vertices as depicted in Figure 1. Thus by the definition of L, we have $d_{\mathcal{C}}(a,b) \geq D$.

(4) Let $x \in X_n \setminus X_{n+1}$ and $y \in X_{n+2}$. Then there exists $e_y \in gE$ such that $(e_y \cdot y)_1 \geq L(n+3)$ and $(x \cdot e_y) < L(n+2)$. Then, similarly as (3), by Lemma 4.13, we see that there is a tree T_2 with $(x_1, x_2, x_3, x_4) = (1, x, y, e_y)$ in Figure 1. Then we have

$$\begin{aligned} (e_y \cdot y)_x &\geq (e_y \cdot y)_x^{T_2} - K_1 \\ &\geq d_{T_2}(p,q) - K_1 \\ &\geq (e_y \cdot y)_1 - (e_y \cdot x)_1 - 4K_1 \\ &\geq L - 4K_1. \end{aligned}$$

Hence $S_x(gE, L - 4K_1) \supset X_{n+2}$. This implies that

$$\nu_x(X_{n+2}) \leq \nu_x(S_x(gE, L - 4K_1)) = \nu(S_1(x^{-1}gE, L - 4K_1)).$$

Then by Lemma 4.15, we have $\nu_x(X_{n+2}) \leq \nu(S_1(x^{-1}gE, L - 4K_1)) < 1 - \epsilon$ since $L - 4K_1 \geq K_3$.

(5) Since $x \in X_n \setminus X_{n+1}$, there is $e \in gE$ such that $(x \cdot e)_1 \geq L(n+1)$. Let $y \notin X_{n-1}$, which implies $(y \cdot e)_1 < Ln$. Similarly as (3) and (4), we have a tree T_3 for $(x_1, x_2, x_3, x_4) = (1, y, x, e)$ in Figure 1. Then we have

$$(1 \cdot y)_x \geq (1 \cdot y)_x^{T_3} - K_1 \geq d_{T_3}(p,q) - 4K_1 \geq L - 4K_1.$$

Thus, we see $y \in S_x(1, L - 4K_1)$. Hence we have

$$\overline{\mathcal{C}}(S) \setminus X_{n-1} \subset S_x(1, L - 4K_1).$$

Since we have chosen $L \geq 4K_1 + K_2(\epsilon/2)$, we see that by Lemma 4.14

$$\nu_x(\overline{\mathcal{C}}(S) \setminus X_{n-1}) \leq \nu_x(S_x(1, L - 4K_1)) = \nu(S_1(x^{-1}, L - 4K_1)) < \epsilon/2. \qquad \square$$

Proof of Lemma 4.12. By Lemma 3.6, we may suppose for some $L' > 0$, $\tau(\omega_n) \geq L'n$ with exponentially high probability. This implies that if $\omega_n \in gE$, then ω_n must be in $X_{\lfloor L'/L \rfloor n - 1}$, where $X_i := S_1(gE, L(i+1))$ as in Lemma 4.17. Therefore by Lemma 4.17, we have

$$\mathbb{P}(\omega_n \in gE) \leq \mathbb{P}(\omega_n \in X_{\lfloor L'/L \rfloor n - 1}) + \mathbb{P}(d_{\mathcal{C}}(\omega_n, \omega_0) < L'n) \leq Kc^n$$

for some $K > 0$ and $c < 1$. $\square$

4.4. Proof of Theorem 1.3. We are now ready to prove Theorem 1.3. The proof goes similarly as the proof of Theorem 1.1. First we prepare an alternative of Lemma 3.4.

Lemma 4.18. *Fix $D_0, M > 0$. Then there is a constant $D_1 > 0$, $c_1 < 1$, $K > 0$ such that the following holds. Consider the collection of indices $a < b < c$ with the following properties*:

(1) $d_{\mathcal{C}}(\omega_a, \omega_b) \geq D_1$,

(2) $d_{\mathcal{C}}(\omega_b, \omega_c) \geq Mn$, *and*

(3) *there exists a covering $\pi: S \to S'$ such that $d_{\mathcal{C}}(\omega_i, \Pi(\mathcal{C}(S'))) \leq D_0$ for all $i \in \{a, b, c\}$.*

Then the probability that this collection of indices is non-empty is at most Kc_1^n.

Proof. The number of possible types of orbifolds which may be covered by S is finite. Furthermore, for each such an orbifold, there are only finitely many possible covering maps up to conjugacy. This is because the number of subgroups of bounded index in a finitely generated group is finite. Hence it suffices to fix a covering $\pi: S \to S'$ and consider only its conjugates. Let $E := \Pi(\mathcal{C}(S'))$ and D be a constant that Lemma 4.5 works for $\mathcal{P}(x, y, D)$ of any covering from S.

Suppose we have indices a, b which satisfy condition (1). Then by Lemma 4.5, the cardinality of $\mathcal{P}(\omega_a, \omega_b, D)$ is universally bounded. Hence to have a index c which satisfies condition (2) and (3), the random walk that starts from ω_b must get into $S_{\omega_b}(gE, Mn)$ for some $gE \in \mathcal{P}(\omega_a, \omega_b, D)$. Since number of elements in $\mathcal{P}(\omega_a, \omega_b, D)$ is universally bounded, and the number of possible choices of indices a, b is of order n^2, by Lemma 4.12 we complete the proof. □

Proof of Theorem 1.3. Similarly to the proof of Theorem 1.1, we consider the action of G on $\mathcal{C}(S)$. For the readability of the proof we will not explicitly write the constants. One can compute constants in a similar way to the proof of Theorem 1.1.

We consider $\omega = (\omega_n) \in G^{\mathbb{N}}$. We may suppose ω_n is pseudo-Anosov. Suppose ω_n is symmetric. Since the stable and unstable measured foliations of ω_n are in some $gE \cap \partial\mathcal{C}(S)$, and gE is quasi-convex, any geodesic axis of ω_n fellow travels with gE. By Lemma 3.2 and Proposition 3.7, we see that we can find some indices that satisfies the conditions of Lemma 4.18 for suitable constants. Hence ω_n is not symmetric with exponentially high probability. □

5. Applications

5.1. Cusped random mapping tori are non-arithmetic. First, we recall the definition of non-compact arithmetic 3-manifolds, see [19] for more details and properties of arithmetic 3-manifolds. Let d be a positive square-free integer and

$\mathcal{O}_d$ denote the ring of integers of $\mathbb{Q}(\sqrt{-d})$. A *Bianchi group* is a subgroup of $\mathrm{PSL}(2,\mathbb{C})$ which is of the form $\mathrm{PSL}(2,\mathcal{O}_d)$. One can show that every Bianchi group is a lattice. The quotient $\mathbb{H}^3/\mathrm{PSL}(2,\mathcal{O}_d)$ is called a *Bianchi orbifold*, where $\mathbb{H}^3$ is the hyperbolic 3-space. A non-compact hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ of finite volume is *arithmetic* if a conjugate of Γ in $\mathrm{PSL}(2,\mathbb{C})$ is commensurable to some Bianchi group $\mathrm{PSL}(2,\mathcal{O}_d)$. Recall that two subgroups of $\mathrm{PSL}(2,\mathbb{C})$ are said to be commensurable if their intersection is a finite index subgroup in both. Let S be an orientable surface of finite type with at least one puncture. For $\phi \in \mathrm{Mod}(S)$, the mapping torus $M(S,\phi)$ is defined by

$$M(S,\phi) = S \times [0,1]/(x,1) \sim (\phi(x),0).$$

Two mapping tori $M(S,\phi_1)$ and $M(S,\phi_2)$ are said to be *cyclic commensurable* if there exists $k_1, k_2 \in \mathbb{Z} \setminus \{0\}$ such that $M(S,\phi_1^{k_1}) = M(S,\phi_2^{k_2})$. Bowditch-Maclachlan-Reid proved the following theorem.

Theorem 5.1 ([5, Theorem 4.2]). *Let S be an orientable surface of finite type with at least one puncture. There are at most finitely many cyclic commensurability classes of arithmetic mapping tori with fiber S.*

Now we are in a position to prove Theorem 1.5.

Proof of Theorem 1.5. Note that if two mapping classes give rise to cyclic commensurable mapping tori, then they are fibered commensurable. By Theorem 1.4, it suffices to discuss minimal mapping classes in their commensurability classes. The uniqueness of the minimal element (Theorem 2.3) implies that two minimal mapping classes give rise to cyclic commensurable mapping tori if and only if they are conjugate. Hence there are at most finitely many conjugacy classes of minimal elements that give arithmetic mapping tori by Theorem 5.1. Hence there is an upper bound of the translation length for minimal mapping classes to have arithmetic mapping tori. Then Lemma 3.6 applies to complete the proof. □

Remark 5.2. For S closed, one can prove similar statement as Theorem 5.1 with upper bound for the degree of the invariant trace fields, see [5, Corollary 4.4]. For S closed, we do not know if the set of a random mapping classes with arithmetic mapping tori is exponentially small or not.

5.2. Closed random mapping tori are asymmetric. It is well known that the isometry group of any closed hyperbolic 3-manifold is finite. A closed hyperbolic 3-manifold is called *asymmetric* if the isometry group is trivial. As a corollary of Theorem 1.4 and the work of Bachman and Schleimer [1], we have the following.

Theorem 5.3. *Let μ be a probability measure on G which satisfies Condition 1.2. Then the set of mapping classes with asymmetric mapping tori is exponentially large with respect to μ.*

Proof. By Lemma 3.6, the translation length $\tau(\omega_n)$ grows linearly with n with exponentially high probability. By the work of Bachman and Schleimer [1, Theorem 3.1], we see that if the translation distance of $\tau(\phi)$ of ϕ is greater than $-\chi(S)$ then any isometry of $M(S,\phi)$ must map each fiber to a fiber. Theorem 1.4 implies that the probability that $M(S,\omega_n)$ has an isometry h which maps each fiber to a fiber and the quotient by $\langle h\rangle$ is a 2-orbifold bundle over the circle decays exponentially. The only case remained is when $M(S,\omega_n)$ admits an isometry of type $(x,t)\mapsto(\beta x, 1-t)$ for some involution $\beta\colon S\to S$, in other words when $M(S,\omega_n)$ admits a quotient which is a 2-orbifold bundle over the 1-orbifold $S^1/\mathbb{Z}_2$. Note that we may suppose ω_n is pseudo-Anosov. In this case we have $\beta\omega_n\beta=\omega_n^{-1}$ and especially β permutes the elements in $\mathrm{Fix}(\omega_n)\subset\mathcal{PMF}(S)$. Hence around geodesic axes of ω_n in the curve complex $\mathcal{C}(S)$, β coarsely acts as a reflection. Then by taking conjugate by ω_n^k for some $k\in\mathbb{Z}$ if necessary, we may suppose coarse fixed points of β are on the γ_2 of the decomposition of $[\omega_0,\omega_n]$ from Proposition 3.7. Then with Remark 3.5, we have desired conclusion by a similar argument to the proof of Theorem 1.1. □

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References

[1] D. Bachman and S. Schleimer, Surface bundles versus Heegaard splittings. *Comm. Anal. Geom.* **13** (2005), no. 5, 903–928. Zbl 1138.57026 MR 2216145

[2] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature.* Grundlehren der Mathematischen Wissenschaften, 319. Springer Verlag, Berlin, 1999. Zbl 0988.53001 MR 1744486

[3] B. Bowditch, *A course on geometric group theory.* MSJ Memoirs, 16. Mathematical Society of Japan, Tokyo, 2006. Zbl 1103.20037 MR 2243589

[4] B. Bowditch, Tight geodesics in the curve complex. *Invent. Math.* **171** (2008), no. 2, 281–300. Zbl 1185.57011 MR 2367021

[5] B. Bowditch, C. Maclachlan, and A. Reid, Arithmetic hyperbolic surface bundles. *Math. Ann.* **302** (1995), no. 1, 31–60. Zbl 0830.57008 MR 1329446

[6] D. Calegari and J. Maher, Statistics and compression of scl. *Ergodic Theory Dynam. Systems* **35** (2015), no. 1, 64–110. Zbl 1351.37214 MR 3294292

[7] D. Calegari, H. Sun, and S. Wang, On fibered commensurability. *Pacific J. Math.* **250** (2011), no. 2, 287–317. Zbl 1236.57022 MR 2794601

[8] A. Fathi, F. Laudenbach, V. Poénaru, et al., *Travaux de Thurston sur les surfaces.* Astérisque 66-67, Société Mathématique de France, Paris, 1979. Zbl 0406.00016 MR 0568308

[9] B. Farb and D. Margalit, *A primer on mapping class groups.* Princeton Mathematical Series, 49. Princeton University Press, Princeton, N.J., 2012. Zbl 1245.57002 MR 2850125

[10] V. Gadre, Harmonic measures for distributions with finite support on the mapping class group are singular. *Duke Math. J.* **163** (2014), no. 2, 309–368. Zbl 1285.30025 MR 3161316

[11] F. P. Gardiner, *Teichmüller theory and quadratic differentials.* Pure and Applied Mathematics (New York). John Wiley & Sons, New York, 1987. Zbl 0629.30002 MR 0903027

[12] F. P. Gardiner and H. Masur, Extremal length geometry of Teichmüller space. *Complex Variables Theory Appl.* **16** (1991), no. 2-3, 209–237. Zbl 0702.32019 MR 1099913

[13] U. Hamenstädt, Train tracks and the Gromov boundary of the complex of curves. In Y. N. Minsky, M. Sakuma and C. Series (eds.), *Spaces of Kleinian groups.* (Cambridge, 2003) London Mathematical Society Lecture Note Series, 329. Cambridge University Press, Cambridge, 2006, 187–207. Zbl 1117.30036 MR 2258749

[14] J. Hubbard and H. Masur, Quadratic differentials and foliations. *Acta Math.* **142** (1979), no. 3-4, 221–274. Zbl 0415.30038 MR 0523212

[15] R. Kent and C. Leininger, Subgroups of mapping class groups from the geometrical viewpoint. In D. Canary, J. Gilman, J. Heinonen, and H. Masur (eds.), *In the tradition of Ahlfors–Bers.* IV. Contemporary Mathematics, 432. American Mathematical Society, Providence, R.I., 2007, 119–141. Zbl 1140.30017 MR 2342811

[16] V. Kaimanovich and H. Masur, The Poisson boundary of the mapping class group. *Invent. Math.* **125** (1996), no. 2, 221–264. Zbl 0864.57014 MR 1395719

[17] E. Klarreich, The boundary at infinity of the curve complex and the relative Teichmüller space. Preprint 1999. http://www.math.unicaen.fr/~levitt/klarreich.pdf

[18] E. Kowalski, *The large sieve and its applications.* Arithmetic geometry, random walks and discrete groups. Cambridge Tracts in Mathematics, 175. Cambridge University Press, Cambridge, 2008. Zbl 1177.11080 MR 2426239

[19] C. Maclachlan and A. Reid. *The arithmetic of hyperbolic* 3*-manifolds.* Graduate Texts in Mathematics, 219. Springer Verlag, New York, 2003. Zbl 1025.57001 MR 1937957

[20] J. Maher, Random Heegaard splittings. *J. Topol.* **3** (2010), no. 4, 997–1025. Zbl 1207.37027 MR 2746344

[21] J. Maher, Random walks on the mapping class group. *Duke Math. J.* **156** (2011), no. 3, 429–468. Zbl 1213.37072 MR 2772067

[22] J. Maher, Exponential decay in the mapping class group. *J. Lond. Math. Soc.* (2) **86** (2012), no. 2, 366–386. A correction for the proof of Lemma 2.11 can be found in Maher's webpage: http://www.math.csi.cuny.edu/maher/research/index.html Zbl 1350.37010 MR 2980916

[23] J. Maher and G. Tiozzo, Random walks on weakly hyperbolic groups. To appear in *J. Reine Angew. Math.* Preprint 2014. arXiv:1410.4173 [math.GT]

[24] H. Masai, On commensurability of fibrations on a hyperbolic 3-manifold. *Pacific J. Math.* **266** (2013), no. 2, 313–327. Zbl 1311.57023 MR 3130626

[25] H. Masur and Y. Minsky, Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.* **138** (1999), no. 1, 103–149. Zbl 0941.32012 MR 1714338

[26] H. Masur and Y. Minsky, Geometry of the complex of curves. II. Hierarchical structure. *Geom. Funct. Anal.* **10** (2000), no. 4, 902–974. Zbl 0972.32011 MR 1791145

[27] K. Rafi and S. Schleimer, Covers and the curve complex. *Geom. Topol.* **13** (2009), no. 4, 2141–2162. Zbl 1166.57013 MR 2507116

[28] K. Rafi, A characterization of short curves of a Teichmüller geodesic. *Geom. Topol.* **9** (2005), 179–202. Zbl 1082.30037 MR 2115672

[29] K. Rafi, A combinatorial model for the Teichmüller metric. *Geom. Funct. Anal.* **17** (2007), no. 3, 936–959. Zbl 1129.30031 MR 2346280

[30] I. Rivin, Walks on groups, counting reducible matrices, polynomials, and surface and free group automorphisms. *Duke Math. J.* **142** (2008), no. 2, 353–379. Zbl 1207.20068 MR 2401624

[31] W. P. Thurston, Hyperbolic Structures on 3-manifolds. II: Surface groups and 3-manifolds which fiber over the circle. Preprint 1998. arXiv:math/9801045 [math.GT]

[32] W. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc.* (*N.S.*) **19** (1988), no. 2, 417–431. Zbl 0674.57008 MR 0956596

[33] W. Woess, *Random walks on infinite graphs and groups.* Cambridge Tracts in Mathematics, 138. Cambridge University Press, Cambridge, 2000. Zbl 951.60002 MR 1743100

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