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Abstract operator-valued Fourier transforms over homogeneous spaces of compact groups

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Abstract. This paper presents a systematic theoretical study for the abstract notion of operator-valued Fourier transforms over homogeneous spaces of compact groups. Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure over the left coset space G/H associated to the Weil's formula. We introduce the generalized notions of abstract dual homogeneous space $\widehat{G/H}$ for the compact homogeneous space G/H and also the operator-valued Fourier transform over the Banach function space $L^1(G/H,\mu)$. We prove that the abstract Fourier transform over G/H satisfies the Plancherel formula and the Poisson summation formula.

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Contents

1. Introduction

The abstract aspect of harmonic analysis over homogeneous spaces of compact non-Abelian groups, more precisely, left (resp. right) coset spaces of non-normal subgroups of compact non-Abelian groups is placed as building blocks for coherent states analysis, constructive approximation, and particle physics, see [4, 11, 15, 16, 20, 18, 19, 17] and references therein. Over the last decades, abstract and computational aspects of Fourier transforms and Plancherel formulas over symmetric spaces have achieved significant popularity in geometric analysis, mathematical physics, and scientific computing, see [6, 12, 13, 14, 10, 25, 24, 23, 22, 21] and references therein.

Let G be a compact group and H be a closed subgroup of G. Then the left coset space G/H is a compact homogeneous space, where G acts on it via the left action. Let μ be the normalized G-invariant measure on the compact homogeneous space G/H associated to the Weil's formula with respect to the probability measures of G and H. This article consists of theoretical aspects of a unified approach to the nature of abstract operator-valued Fourier transform over the Banach function space $L^1(G/H,\mu)$. We aim to further develop the abstract notions of dual homogeneous spaces and also Fourier transforms over homogeneous spaces of compact groups, which has not been studied when compared to the Fourier analysis over (reductive) symmetric spaces [29].

This paper is organized as follows. Section 2 is devoted to fixing notations and preliminaries including a summary on non-Abelian Fourier analysis over compact groups and classical results on abstract harmonic analysis over compact homogeneous spaces. In Section 3, we present some abstract aspects of harmonic analysis on the Banach functions spaces over the compact homogeneous space G/H. We then introduce the abstract notion of dual homogeneous space $\widehat{G/H}$ for the compact homogeneous space G/H. Next we present the theoretical definition of the abstract operator-valued Fourier transform over the compact homogeneous space G/H, and we study analytic-algebraic properties of this transform. The paper is concluded by presenting an abstract generalized version of the Poisson summation formula.

2. Preliminaries and notations

2.1. Operator theory. Let $\mathcal{H}$ be a separable Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is called a Hilbert-Schmidt operator if for one, hence for any orthonormal basis $\{e_k\}$ of $\mathcal{H}$ we have $\sum_k \|Te_k\|^2 < \infty$. The set of all Hilbert-Schmidt operators on $\mathcal{H}$ is denoted by $\mathrm{HS}(\mathcal{H})$, and for $T \in \mathrm{HS}(\mathcal{H})$ the Hilbert-Schmidt norm of T is $\|T\|_{\mathrm{HS}}^2 = \sum_k \|Te_k\|^2$. The set $\mathrm{HS}(\mathcal{H})$ is a self adjoint two sided ideal in $\mathcal{B}(\mathcal{H})$, and if $\mathcal{H}$ is finite-dimensional, we have $\mathrm{HS}(\mathcal{H}) = \mathcal{B}(\mathcal{H})$. An operator $T \in \mathcal{B}(\mathcal{H})$ is trace-class, if $\|T\|_{\mathrm{tr}} = \mathrm{tr}[|T|] < \infty$, where $\mathrm{tr}[T] = \sum_k \langle Te_k, e_k\rangle$ and $|T| = (TT^*)^{1/2}$ [16, 26].

Let $\mathcal{H}$ be a finite dimensional Hilbert space of dimension d and $1 \leq p \leq \infty$. For a (bounded) linear operator $T \in \mathcal{B}(\mathcal{H})$ the Schatten operator p-norm of T, denoted by $\|T\|_p$, is defined as [16]

$$\|T\|_p := \Big(\sum_{j=1}^{d} |s_j(T)|^p\Big)^{1/p},$$

where $\{s_j(T) : 1 \leq j \leq d\}$ is the sequence of singular values of T, that is the sequence of eigenvalues of the positive-definite operator $|T|$. It is a well-known result that $\|T\|_p^p = \mathrm{tr}[|T|^p]$. Thus, $\|T\|_1 = \|T\|_{\mathrm{tr}}$ and $\|T\|_2 = \|T\|_{\mathrm{HS}}$.

2.2. Abstract Fourier analysis over compact groups. Let G be a compact group with the probability Haar measure dx. Then each irreducible representation of G is finite-dimensional, and every unitary representation of G is a direct sum of irreducible representations, see [4, 15, 16]. The set of all unitary equivalence classes of irreducible unitary representations of G is denoted by $\widehat{G}$. This definition of $\widehat{G}$ is in essential agreement with the classical definition when G is Abelian, see [4, 15]. For a subset $\Omega \subseteq \widehat{G}$, the $*$-algebra $\prod_{\pi\in\Omega} \mathcal{B}(\mathcal{H}_\pi)$ is denoted by $\mathfrak{C}(\Omega)$, where scalar multiplication, addition multiplication, and the adjoint of an element are defined coordinatewise. For $1 \leq p \leq \infty$, $\mathfrak{C}^p(\Omega)$ is defined by

$$\mathfrak{C}^p(\Omega) = \Big\{\mathbf{T} = (T_\pi)_{\pi\in\Omega} \in \mathfrak{C}(\Omega) \;\Big|\; \|\mathbf{T}\|_{\mathfrak{C}^p(\Omega)}^p := \sum_{[\pi]\in\Omega} d_\pi \|T_\pi\|_p^p < \infty\Big\},$$

and

$$\mathfrak{C}^\infty(\Omega) = \{\mathbf{T} = (T_\pi)_{\pi\in\Omega} \in \mathfrak{C}(\Omega) \mid \|\mathbf{T}\|_{\mathfrak{C}^\infty(\Omega)} := \sup_{[\pi]\in\Omega} \|T_\pi\|_\infty < \infty\},$$

where $\|T_\pi\|_p$ with $\pi \in \widehat{\Omega}$ are the Schatten operator p-norms. The set of all $\mathbf{T} \in \mathfrak{C}(\Omega)$ such that $\{\pi \in \Omega : T_\pi \neq 0\}$ is finite, is denoted by $\mathfrak{C}_{00}(\Omega)$, and $\mathfrak{C}_0(\Omega)$ is defined as the set of all $\mathbf{T} \in \mathfrak{C}(\Omega)$ such that $\mathbf{T}_\epsilon := \{\pi \in \Omega : \|T_\pi\|_\infty \geq \epsilon\}$ is finite for all $\epsilon > 0$, see [16, §28.24 and §28.34] and [26].

If π is any unitary representation of G, for $\zeta, \xi \in \mathcal{H}_\pi$, the functions $\pi_{\zeta,\xi}(x) = \langle \pi(x)\zeta, \xi\rangle$ are called matrix elements of π. If $\{e_j\}$ is an orthonormal basis for $\mathcal{H}_\pi$, then π_{ij} means π_{e_i,e_j}. The Fourier transform of $f \in L^1(G)$ at $\pi \in \widehat{G}$ is defined in the weak sense as an operator in $\mathcal{B}(\mathcal{H}_\pi)$ by

$$\hat{f}(\pi) = \int_G f(x)\pi(x)^* dx \tag{2.1}$$

If $\pi(x)$ is represented by the matrix $(\pi_{ij}(x)) \in \mathbb{C}^{d_\pi \times d_\pi}$, then $\hat{f}(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ is the matrix with entries given by $\hat{f}(\pi)_{ij} = d_\pi^{-1} c_{ji}^\pi(f)$ satisfying

$$\sum_{i,j=1}^{d_\pi} c_{ij}^\pi(f)\pi_{ij}(x) = d_\pi \sum_{i,j=1}^{d_\pi} \hat{f}(\pi)_{ji}\pi_{ij}(x) = d_\pi \operatorname{tr}[\hat{f}(\pi)\pi(x)],$$

where $c_{i,j}^\pi(f) = d_\pi \langle f, \pi_{ij}\rangle_{L^2(G)}$. Then the Peter–Weyl Theorem implies

$$\|f\|_{L^2(G)}^2 = \sum_{[\pi]\in\widehat{G}} d_\pi \|\hat{f}(\pi)\|_2^2 = \sum_{[\pi]\in\widehat{G}} \sum_{j,k=1}^{d_\pi} |\langle f, d_\pi^{1/2}\pi_{jk}\rangle_{L^2(G)}|^2, \tag{2.2}$$

and

$$f(x) = \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}[\hat{f}(\pi)\pi(x)] \quad \text{for a.e. } x \in G. \tag{2.3}$$

Thus, the abstract Fourier transform

$$\mathcal{F}_G : L^2(G) \longrightarrow \mathfrak{C}^2(\widehat{G}),$$

given by

$$f \mapsto \mathcal{F}_G(f) = (\hat{f}(\pi))_{\pi\in\widehat{G}},$$

is a unitary transform, see [16, §27.40 and §28.43] and [4, 21].

2.3. Classical harmonic analysis over homogeneous spaces of compact groups. Let H be a closed subgroup of G with the probability Haar measure dh. The left coset space G/H is considered as a locally compact homogeneous space that G acts on it from the left, and $q : G \to G/H$ given by $x \mapsto q(x) := xH$ is the surjective canonical map. The classical aspects of abstract harmonic analysis on locally compact homogeneous spaces are quite well studied by several authors, see [4, 5, 15, 16, 27] and references therein. The function space $\mathcal{C}(G/H)$ consists of all functions $T_H(f)$, where $f \in \mathcal{C}(G)$ and

$$T_H(f)(xH) = \int_H f(xh)dh. \tag{2.4}$$

Let μ be a Radon measure on G/H and $x \in G$. The translation μ_x of μ is defined by $\mu_x(E) = \mu(xE)$, for all Borel subsets E of G/H. The measure μ is called G-invariant if $\mu_x = \mu$, for all $x \in G$. The compact homogeneous space G/H has a normalized G-invariant measure μ associated to the following Weil's formula

$$\int_{G/H} T_H(f)(xH)d\mu(xH) = \int_G f(x)dx, \tag{2.5}$$

and hence the linear map T_H is norm-decreasing, that is

$$\|T_H(f)\|_{L^1(G/H,\mu)} \leq \|f\|_{L^1(G)},$$

for all $f \in L^1(G)$, see [4, 15, 27].

3. Abstract harmonic analysis over homogeneous spaces of compact groups

This section is devoted to present a classical study for abstract harmonic analysis of Banach function spaces over homogeneous spaces of compact groups [8, 7]. Throughout this paper we assume that G is a compact group with the probability Haar measure dx, H is a closed subgroup of G with the probability Haar measure dh, and μ is the normalized G-invariant measure on the compact homogeneous space G/H associated to the Weil's formula (2.5) with respect to the probability measures of G and H. From now on, we may say μ is the normalized G-invariant measure over the compact homogeneous space G/H, at times.

The following proposition shows that the linear map $T_H\colon \mathcal{C}(G) \to \mathcal{C}(G/H)$ is uniformly continuous.

Proposition 3.1. *Let H be a closed subgroup of a compact group G. The linear map $T_H\colon \mathcal{C}(G) \to \mathcal{C}(G/H)$ is uniformly continuous.*

Proof. Let $f \in \mathcal{C}(G)$ and $x \in G$. Then we have

$$\begin{aligned} |T_H(f)(xH)| &= \left|\int_H f(xh)dh\right| \\ &\leq \int_H |f(xh)|dh \\ &\leq \|f\|_{\sup}\left(\int_H dh\right) \\ &= \|f\|_{\sup}, \end{aligned}$$

which implies $\|T_H(f)\|_{\sup} \leq \|f\|_{\sup}$. □

Next theorem proves that the linear map T_H is norm-decreasing in other L^p-spaces, when $p > 1$.

Theorem 3.2. *Let H be a closed subgroup of a compact group G, μ be the normalized G-invariant measure on G/H, and $p \geq 1$. The linear map*

$$T_H\colon \mathcal{C}(G) \longrightarrow \mathcal{C}(G/H)$$

satisfies

$$\|T_H(f)\|_{L^p(G/H,\mu)} \leq \|f\|_{L^p(G)} \quad \text{for } f \in \mathcal{C}(G), \tag{3.1}$$

and hence it has a unique extension to a norm-decreasing linear map from $L^p(G)$ onto $L^p(G/H, \mu)$.

Proof. Let $f \in \mathcal{C}(G)$. Using compactness of H, and the Weil's formula, we can write

$$\begin{aligned}
\|T_H(f)\|^p_{L^p(G/H,\mu)} &= \int_{G/H} |T_H(f)(xH)|^p d\mu(xH) \\
&= \int_{G/H} \left|\int_H f(xh)dh\right|^p d\mu(xH) \\
&\leq \int_{G/H} \left(\int_H |f(xh)|dh\right)^p d\mu(xH) \\
&\leq \int_{G/H} \int_H |f(xh)|^p dh d\mu(xH) \\
&= \int_{G/H} \int_H |f|^p(xh) dh d\mu(xH) \\
&= \int_{G/H} T_H(|f|^p)(xH) d\mu(xH) \\
&= \int_G |f(x)|^p dx \\
&= \|f\|^p_{L^p(G)},
\end{aligned}$$

which implies (3.1). Thus, we can extend T_H uniquely to a norm-decreasing linear operator from $L^p(G)$ onto $L^p(G/H,\mu)$, which still will be denoted by T_H. □

As an immediate consequence of Theorem 3.2 we deduce the following corollary.

Corollary 3.3. *Let H be a closed subgroup of a compact group G, μ be the normalized G-invariant measure on G/H, and $p \geq 1$. Let $\varphi \in L^p(G/H,\mu)$ and $\varphi_q := \varphi \circ q$. Then $\varphi_q \in L^p(G)$ with*

$$\|\varphi_q\|_{L^p(G)} = \|\varphi\|_{L^p(G/H,\mu)}. \tag{3.2}$$

Proof. Using the Weil's formula, compactness of H, and since dh is a probability measure, we get

$$\begin{aligned}
\|\varphi_q\|^p_{L^p(G)} &= \int_G |\varphi_q(x)|^p dx \\
&= \int_{G/H} T_H(|\varphi_q|^p)(xH) d\mu(xH) \\
&= \int_{G/H} \left(\int_H |\varphi_q(xh)|^p dh\right) d\mu(xH)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{G/H} \left(\int_H |\varphi(xhH)|^p dh \right) d\mu(xH) \\
&= \int_{G/H} \left(\int_H |\varphi(xH)|^p dh \right) d\mu(xH) \\
&= \int_{G/H} |\varphi(xH)|^p \left(\int_H dh \right) d\mu(xH) \\
&= \int_{G/H} |\varphi(xH)|^p d\mu(xH) \\
&= \|\varphi\|^p_{L^p(G/H,\mu)},
\end{aligned}
$$

which implies (3.2). □

For $\varphi, \psi \in L^1(G/H, \mu)$, one can define $\varphi * \psi \in L^1(G/H, \mu)$ by

$$
\varphi *_{G/H} \psi(xH) = T_H(\varphi_q *_G \psi_q)(xH) = \int_H \varphi_q *_G \psi_q(xh)dh, \tag{3.3}
$$

where $\varphi_q := \varphi \circ q$, $\psi_q := \psi \circ q$, and $\varphi_q *_G \psi_q \in L^1(G)$ is the convolution of $L^1(G)$.

Then we have

$$
(\varphi *_{G/H} \psi)_q = \varphi_q *_G \psi_q, \quad \text{for } \varphi, \psi \in L^1(G/H, \mu). \tag{3.4}
$$

The following theorem [9] states basic property of the convolution defined by (3.3).

Theorem 3.4. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. The Banach function space $L^1(G/H, \mu)$ equipped with the convolution given in (3.3) is a Banach algebra.*

Proof. It is obvious that $(\varphi, \psi) \mapsto \varphi *_{G/H} \psi$ is a well-defined bilinear map on $L^1(G/H, \mu)$. Then using (3.2), and (3.4), we get

$$
\begin{aligned}
\|\varphi *_{G/H} \psi\|_{L^1(G/H,\mu)} &= \|(\varphi *_{G/H} \psi)_q\|_{L^1(G)} \\
&= \|(\varphi_q *_G \psi_q\|_{L^1(G)} \\
&\leq \|\varphi_q\|_{L^1(G)} \|\psi_q\|_{L^1(G)} \\
&= \|\varphi\|_{L^1(G/H,\mu)} \|\psi\|_{L^1(G/H,\mu)},
\end{aligned}
$$

which completes the proof. □

Remark 3.5. It is straightforward to check that, if H is a closed normal subgroup of G then the convolution defined in (3.3) coincides with the standard convolution of the function algebra of the compact quotient group G/H, see [1, 2, 3, 9, 8, 27].

Let $\mathcal{J}^2(G,H) := \{f \in L^2(G) \colon T_H(f) = 0\}$ and $\mathcal{J}^2(G,H)^\perp$ be the orthogonal complement of the closed subspace $\mathcal{J}^2(G,H)$ in $L^2(G)$.

Next result shows that the linear operator $T_H \colon L^2(G) \to L^2(G/H,\mu)$ is a partial isometric.

Proposition 3.6. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Then $T_H \colon L^2(G) \to L^2(G/H,\mu)$ is a partial isometric linear map.*

Proof. Let $\varphi \in L^2(G/H,\mu)$. Invoking Corollary 3.3 we claim that $T_H^*(\varphi) = \varphi_q$, and hence $T_H T_H^*(\varphi) = \varphi$. Indeed, using the Weil's formula, we can write

$$\begin{aligned}
\langle T_H^*(\varphi), f\rangle_{L^2(G)} &= \langle \varphi, T_H(f)\rangle_{L^2(G/H,\mu)} \\
&= \int_{G/H} \varphi(xH)\overline{T_H(f)(xH)}d\mu(xH) \\
&= \int_{G/H} \varphi(xH)T_H(\overline{f})(xH)d\mu(xH) \\
&= \int_{G/H} T_H(\varphi_q.\overline{f})(xH)d\mu(xH) \\
&= \int_G \varphi_q(x)\overline{f(x)}dx \\
&= \langle \varphi_q, f\rangle_{L^2(G)},
\end{aligned}$$

for all $f \in L^2(G)$, which implies that $T_H^*(\varphi) = \varphi_q$. Now a straightforward calculation shows that $T_H = T_H T_H^* T_H$. Then, by Theorem 2.3.3 of [26], T_H is a partial isometric operator. □

The following corollaries are straightforward consequences of Proposition 3.6.

Corollary 3.7. *Let H be a closed subgroup of a compact group G. Let $P_{\mathcal{J}^2(G,H)}$ and $P_{\mathcal{J}^2(G,H)^\perp}$ be the orthogonal projections onto the closed subspaces $\mathcal{J}^2(G,H)$ and $\mathcal{J}^2(G,H)^\perp$ respectively. Then, for $f \in L^2(G)$ and a.e. $x \in G$, we have*

$$P_{\mathcal{J}^2(G,H)^\perp}(f)(x) = T_H(f)(xH), \quad P_{\mathcal{J}^2(G,H)}(f)(x) = f(x) - T_H(f)(xH). \tag{3.5}$$

Corollary 3.8. *Let H be a compact subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Then*

(1) $\mathcal{J}^2(G,H)^\perp = \{\psi_q = \psi \circ q \colon \psi \in L^2(G/H,\mu)\}$;
(2) *for $f \in \mathcal{J}^2(G,H)^\perp$ and $h \in H$, we have $R_h f = f$;*
(3) *for $\psi \in L^2(G/H,\mu)$, we have $\|\psi_q\|_{L^2(G)} = \|\psi\|_{L^2(G/H,\mu)}$;*
(4) *for $f, g \in \mathcal{J}^2(G,H)^\perp$, we have $\langle T_H(f), T_H(g)\rangle_{L^2(G/H,\mu)} = \langle f, g\rangle_{L^2(G)}$.*

Remark 3.9. Invoking Corollary 3.8, one can regard the Hilbert function space $L^2(G/H, \mu)$ as a closed subspace of the Hilbert function space $L^2(G)$, that is the closed subspace consists of all $f \in L^2(G)$ which satisfies $R_h f = f$, for all $h \in H$. Then Theorem 3.2 and Proposition 3.6 guarantee that the linear map

$$T_H \colon L^2(G) \longrightarrow L^2(G/H, \mu) \subset L^2(G)$$

is an orthogonal projection onto $L^2(G/H, \mu)$.

4. Abstract duality over homogeneous spaces of compact groups

In this section, we present the abstract notion of dual homogeneous spaces associated to homogeneous spaces of compact groups.

For a closed subgroup H of G, define

$$H^{\perp} := \{[\pi] \in \widehat{G} \colon \pi(h) = I \text{ for all } h \in H\}, \tag{4.1}$$

If G is Abelian, each closed subgroup H of G is normal and the locally compact group G/H is Abelian, then the character group $\widehat{G/H}$ is the set of all characters (one dimensional irreducible representations) of G which are constant on H, that is precisely $H^{\perp}$. If G is a non-Abelian group and H is a closed normal subgroup of G, then the dual space $\widehat{G/H}$ which is the set of all unitary equivalence classes of unitary representations of G/H, has meaning and it is well-defined. Indeed, G/H is a non-Abelian group. In this case, the map $\Phi \colon \widehat{G/H} \to H^{\perp}$ defined by $\sigma \mapsto \Phi(\sigma) := \sigma \circ q$ is a Borel isomorphism and $\widehat{G/H} = H^{\perp}$, see [4, 15, 21, 28]. Thus, if H is normal, $H^{\perp}$ coincides with the classic definitions of the dual space either when G is Abelian or non-Abelian.

Definition 4.1. Let H be a closed subgroup of a compact group G and $\Omega \subseteq \widehat{G}$. The homogeneous space G/H satisfies the Ω-separation property, if $\pi(x) = I_{\mathcal{H}_\pi}$ for all $[\pi] \in \Omega$ imply $x \in H$. In this case, we say that Ω separates points of G/H, at times. It is evident to check that, Ω separates points of G/H if and only if for any $xH \neq yH$ there exists $[\pi] \in \Omega$ such that $\pi(x) \neq \pi(y)$.

The following theorem shows that the homogeneous space G/H does not satisfy the $H^{\perp}$-separation property, if H is not normal in G.

Theorem 4.2. *Let H be a closed subgroup of a compact group G. Then $H^{\perp}$ separates points of G/H if and only if H is normal.*

Proof. Let H be a normal subgroup of G. Then G/H is a compact group. Thus, Gelfand-Raikove Theorem guarantees that the dual space of G/H, which is precisely $H^\perp$, separates points of G/H. Conversely, assume that $H^\perp$ separates points of G/H. Let $x \in G$ and $h \in H$. Then we can write

$$\pi(x^{-1}hx) = \pi(x^{-1})\pi(h)\pi(x) = \pi(x)^*\pi(h)\pi(x) = \pi(x)^*\pi(x) = I_{\mathcal{H}_\pi},$$

for all $[\pi] \in H^\perp$. Hence, the $H^\perp$-separation property of G/H implies $x^{-1}hx \in H$. Since $x \in G$ and $h \in H$ are arbitrary, we obtain that H is normal in G. □

Remark 4.3. Theorem 4.2 guarantees that $H^\perp$ is not the appropriate candidate to be considered as the dual space of the homogeneous space G/H, when H is not a normal subgroup of G. In fact, $H^\perp$ is not large enough with respect to the left coset space G/H if H is a given closed subgroup of G, unless H be a normal subgroup of G.

For a closed subgroup H of G and a continuous unitary representation $(\pi, \mathcal{H}_\pi)$ of G, define

$$T_H^\pi := \int_H \pi(h)dh, \tag{4.2}$$

where the operator valued integral (4.2) is considered in the weak sense.

In other words,

$$\langle T_H^\pi \zeta, \xi\rangle = \int_H \langle \pi(h)\zeta, \xi\rangle dh, \quad \text{for } \zeta, \xi \in \mathcal{H}_\pi. \tag{4.3}$$

The function $h \mapsto \langle \pi(h)\zeta, \xi\rangle$ is bounded and continuous on H. Since H is compact, the right integral is the ordinary integral of a function in $L^1(H)$. Therefore, T_H^π is a bounded linear operator on $\mathcal{H}_\pi$ with $\|T_H^\pi\| \le 1$.

The following proposition presents basic properties of the linear operator T_H^π.

Proposition 4.4. *Let H be a closed subgroup of a compact group G and $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G with $T_H^\pi \neq 0$. Then the linear operator T_H^π is an orthogonal projection.*

Proof. Using compactness of H, we have

$$(T_H^\pi)^* = \left(\int_H \pi(h)dh\right)^* = \int_H \pi(h)^* dh = \int_H \pi(h^{-1})dh = T_H^\pi.$$

As well as, we can write

$$\begin{aligned} T_H^\pi T_H^\pi &= \left(\int_H \pi(h)dh\right)\left(\int_H \pi(t)dt\right) \\ &= \int_H \pi(h)\left(\int_H \pi(t)dt\right)dh \\ &= \int_H \left(\int_H \pi(h)\pi(t)dt\right)dh \\ &= \int_H \left(\int_H \pi(ht)dt\right)dh \\ &= \int_H T_H^\pi dt \\ &= T_H^\pi. \end{aligned}$$

□

Remark 4.5. Let H be a closed subgroup of a compact group G. Let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G on the Hilbert space $\mathcal{H}_\pi$. A vector $\zeta \in \mathcal{H}_\pi$ is called H-constant if $\pi(h)\zeta = \zeta$ for all $h \in H$. Then T_H^π is the orthogonal projection onto the closed subspace $\mathcal{K}_\pi^H$ consists of all H-constant vectors of $\mathcal{H}_\pi$, that is

$$\mathcal{K}_\pi^H := \{\zeta \in \mathcal{H}_\pi : \pi(h)\zeta = \zeta \text{ for all } h \in H\}. \tag{4.4}$$

Invoking Remark 4.3, we can suggest the following dual space for G/H.

Definition 4.6. Let H be a closed subgroup of a compact group G. Then we define the dual space of G/H, as the subset of $\widehat{G}$ which is given by

$$\widehat{G/H} := \{[\pi] \in \widehat{G} : T_H^\pi \neq 0\} = \left\{[\pi] \in \widehat{G} : \int_H \pi(h)dh \neq 0\right\}. \tag{4.5}$$

Let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary irreducible representation of a compact group G on the Hilbert space $\mathcal{H}_\pi$. Invoking Remark 4.5, we deduce that $[\pi] \in \widehat{G/H}$ if and only if there exists a non-zero vector $\zeta \in \mathcal{H}_\pi$ such that $\pi(h)\zeta = \zeta$ for all $h \in H$, or equivalently $\mathcal{K}_\pi^H$ be a non-zero subspace of $\mathcal{H}_\pi$.

Evidently, any closed subgroup H of G satisfies

$$H^\perp \subseteq \widehat{G/H}. \tag{4.6}$$

Let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G such that $T_H^\pi \neq 0$. Then the functions $\pi_{\zeta,\xi}^H : G/H \to \mathbb{C}$ defined by

$$\pi_{\zeta,\xi}^H(xH) := \langle \pi(x)T_H^\pi \zeta, \xi\rangle \quad \text{for } xH \in G/H, \tag{4.7}$$

for $\xi, \zeta \in \mathcal{H}_\pi$, are called H-matrix elements of $(\pi, \mathcal{H}_\pi)$.

For $xH \in G/H$ and $\zeta, \xi \in \mathcal{H}_\pi$, we have

$$\begin{aligned} |\pi^H_{\zeta,\xi}(xH)| &= |\langle \pi(x)T^\pi_H\zeta, \xi\rangle| \\ &\leq \|\pi(x)T^\pi_H\zeta\|\|\xi\| \\ &\leq \|T^\pi_H\zeta\|\|\xi\| \\ &\leq \|\zeta\|\|\xi\|. \end{aligned}$$

Also, we can write

$$\pi^H_{\zeta,\xi}(xH) = \langle \pi(x)T^\pi_H\zeta, \xi\rangle = \pi_{T^\pi_H\zeta,\xi}(x). \tag{4.8}$$

Invoking definition of the linear map T_H and T^π_H, we have

$$\begin{aligned} T_H(\pi_{\zeta,\xi})(xH) &= \int_H \pi_{\zeta,\xi}(xh)dh \\ &= \int_H \langle \pi(xh)\zeta, \xi\rangle dh \\ &= \int_H \langle \pi(x)\pi(h)\zeta, \xi\rangle dh \\ &= \left\langle \pi(x)\left(\int_H \pi(h)dh\right)\zeta, \xi\right\rangle \\ &= \langle \pi(x)T^\pi_H\zeta, \xi\rangle, \end{aligned}$$

which implies

$$T_H(\pi_{\zeta,\xi}) = \pi^H_{\zeta,\xi}. \tag{4.9}$$

The linear span of the H-matrix elements of a continuous unitary representation $(\pi, \mathcal{H}_\pi)$ satisfying $T^\pi_H \neq 0$, is denoted by $\mathrm{Trig}_\pi(G/H)$ which is a subspace of $\mathcal{C}(G/H)$.

Then we define

$$\mathrm{Trig}(G/H) := \text{the linear span of} \bigcup_{[\pi]\in\widehat{G/H}} \mathrm{Trig}_\pi(G/H). \tag{4.10}$$

Functions in $\mathrm{Trig}(G/H)$ are called trigonometric polynomials over G/H.

In the followings we study abstract harmonic analysis aspects of trigonometric polynomials.

Proposition 4.7. *The linear operator T_H maps* $\mathrm{Trig}(G)$ *onto* $\mathrm{Trig}(G/H)$.

Next theorem presents some analytic aspects of trigonometric polynomials over G/H as a function space.

Theorem 4.8. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Then*

(1) $\mathrm{Trig}(G/H)$ *is* $\|.\|_{L^p(G/H,\mu)}$*-dense in* $L^p(G/H,\mu)$;

(2) $\mathrm{Trig}(G/H)$ *is* $\|.\|_{\sup}$*-dense in* $\mathcal{C}(G/H)$.

Proof. (1) Let $p \geq 1$ and $\phi \in L^p(G/H,\mu)$. Let $f \in L^p(G)$ with $T_H(f) = \phi$. By $\|.\|_{L^p(G)}$-density of $\mathrm{Trig}(G)$ in $L^p(G)$ we can pick a sequence $\{f_n\}$ in $\mathrm{Trig}(G)$ such that $f = \|.\|_{L^p(G)} - \lim_n f_n$. Then continuity of the linear map $T_H\colon L^p(G) \to L^p(G/H,\mu)$, implies

$$\phi = T_H(f) = \|.\|_{L^p(G/H,\mu)} - \lim_n T_H(f_n),$$

which completes the proof.

(2) Invoking uniformly boundedness of T_H, uniformly density of $\mathrm{Trig}(G)$ in $\mathcal{C}(G)$, and the same argument as used in (1), we get $\|.\|_{\sup}$-density of $\mathrm{Trig}(G/H)$ in $\mathcal{C}(G/H)$. □

The following theorem guarantees the $\widehat{G/H}$-separation property of the homogeneous space G/H.

Theorem 4.9. *Let H be a closed subgroup of a compact group G. Then $\widehat{G/H}$ separates points of G/H.*

Proof. Let $x, y \in G$ with $xH \neq yH$. Let $\psi \in \mathcal{C}(G/H)$ such that $\psi(xH) \neq \psi(yH)$. Since $\mathrm{Trig}(G/H)$ is $\|.\|_{\sup}$-dense in $\mathcal{C}(G/H)$, we can uniformly approximate ψ with elements of $T_H(\mathrm{Trig}(G)) = \mathrm{Trig}(G/H)$. Thus, there exists an irreducible representation $(\pi, \mathcal{H}_\pi)$ of G and non-zero vectors $\zeta, \xi \in \mathcal{H}_\pi$ such that

$$T_H(\pi_{\zeta,\xi})(xH) \neq T_H(\pi_{\zeta,\xi})(yH).$$

This automatically guarantees that $T_H^\pi \neq 0$, and $\pi(x) \neq \pi(y)$ as well. □

Next we show that the reverse inclusion of (4.6) holds, if and only if H is a normal subgroup of G.

Theorem 4.10. *Let H be a closed subgroup of a compact group G. Then H is normal in G if and only if $\widehat{G/H} = H^{\perp}$.*

Proof. Let $\widehat{G/H} = H^{\perp}$. Then Theorem 4.9 guarantees that G/H satisfies the $H^{\perp}$-separation property. Thus, Theorem 4.2 implies that H is normal in G. Conversely, let H be a closed normal subgroup of a compact group G. It is sufficient to show that $\widehat{G/H} \subseteq H^{\perp}$. Let $[\pi] \in \widehat{G/H}$ be given. Due to the normality of H in G, for all $x \in G$ the map $\tau_x\colon H \to H$ given by $h \mapsto \tau_x(h) := x^{-1}hx$

belongs to $Aut(H)$ and $x^{-1}Hx = H$. Invoking compactness of G we have $d(\tau_x(h)) = dh$, for $x \in G$. Now, for $x \in G$ we get

$$\begin{aligned}\int_H \pi(h)dh &= \int_{xHx^{-1}} \pi(\tau_x(h))d(\tau_x(h))\\ &= \int_H \pi(\tau_x(h))dh\\ &= \int_H \pi(x)^*\pi(h)\pi(x)dh\\ &= \pi(x)^*\left(\int_H \pi(h)dh\right)\pi(x)\\ &= \pi(x)^* T_H^\pi \pi(x).\end{aligned}$$

Therefore, $\pi(x)T_H^\pi = T_H^\pi\pi(x)$ for $x \in G$, which implies $T_H^\pi \in \mathcal{C}(\pi)$. Irreducibility of π guarantees that $T_H^\pi = \alpha I$ for some non-zero $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$. Thus, for $t \in H$, we can write

$$\begin{aligned}\pi(t) &= \alpha^{-1}\pi(t)\alpha I\\ &= \alpha^{-1}\pi(t)T_H^\pi\\ &= \alpha^{-1}\int_H \pi(th)dh\\ &= \alpha^{-1}\int_H \pi(h)dh\\ &= \alpha^{-1}T_H^\pi = I,\end{aligned}$$

which implies $[\pi] \in H^\perp$. □

For $\mathbf{T} = (T_\pi)_{[\pi]\in\widehat{G}} \in \mathfrak{C}(\widehat{G})$, let

$$\mathbf{T}_H := (T_H^\pi T_\pi)_{[\pi]\in\widehat{G/H}} \in \mathfrak{C}(\widehat{G/H}).$$

Then

$$\widehat{T_H}\colon \mathfrak{C}(\widehat{G}) \longrightarrow \mathfrak{C}(\widehat{G/H})$$

given by $\mathbf{T} \mapsto \mathbf{T}_H$ is a well-defined linear operator.

Theorem 4.11. *Let H be a closed subgroup of a compact group G and $p \geq 1$. Then*

$$\widehat{T_H}\colon \mathfrak{C}^p(\widehat{G}) \longrightarrow \mathfrak{C}^p(\widehat{G/H}),$$

is a norm-decreasing linear operator.

Proof. Let $\mathbf{T} = (T_\pi)_{[\pi]\in\widehat{G}} \in \mathfrak{C}(\widehat{G})$. Then we can write

$$\begin{aligned}\|\widehat{T_H}(\mathbf{T})\|^p_{\mathfrak{C}^p(\widehat{G/H})} &= \sum_{[\pi]\in\widehat{G/H}} d_\pi \|T_H^\pi T_\pi\|_p^p \\ &\leq \sum_{[\pi]\in\widehat{G/H}} d_\pi \|T_\pi\|_p^p \\ &\leq \sum_{[\pi]\in\widehat{G}} d_\pi \|T_\pi\|_p^p \\ &= \|\mathbf{T}\|^p_{\mathfrak{C}^p(\widehat{G})}.\end{aligned}$$

□

5. Abstract operator-valued Fourier transforms over homogeneous spaces of compact groups

Throughout this section, we present the abstract notion of operator-valued Fourier transforms over homogeneous spaces of compact groups. It is still assumed that H is a closed subgroup of a compact group G and μ is the normalized G-invariant measure on the compact homogeneous space G/H.

Let $\varphi \in L^1(G/H,\mu)$ and $[\pi] \in \widehat{G/H}$. The Fourier transform of φ at $[\pi]$ is defined as the operator

$$\mathcal{F}_{G/H}(\varphi)(\pi) = \hat{\varphi}(\pi) := \int_{G/H} \varphi(xH)\Gamma_\pi(xH)^* d\mu(xH), \tag{5.1}$$

on the Hilbert space $\mathcal{H}_\pi$, where for $xH \in G/H$ the notation $\Gamma_\pi(xH)$ stands for the bounded linear operator on $\mathcal{H}_\pi$ satisfying

$$\langle \zeta, \Gamma_\pi(xH)\xi\rangle = \langle \zeta, \pi(x)T_H^\pi \xi\rangle, \tag{5.2}$$

for all $\zeta, \xi \in \mathcal{H}_\pi$.

Remark 5.1. Let H be a closed normal subgroup of a compact group G and μ be the normalized G-invariant measure over the left coset space G/H associated to the Weil's formula. Then it is easy to check that μ is a Haar measure of the compact quotient group G/H and by Theorem 4.10 we have $\widehat{G/H} = H^\perp$. Also, for each $\varphi \in L^1(G/H,\mu)$ and $[\pi] \in H^\perp$, we have

$$\hat{\varphi}(\pi) = \int_{G/H} \varphi(xH)\pi(x)^* d\mu(xH).$$

Thus, we deduce that the abstract Fourier transform defined by (5.1) coincides with the classical Fourier transform over the compact quotient group G/H if H is normal in G.

The operator-valued integral (5.1) is also considered in the weak sense. That is

$$\langle\hat{\varphi}(\pi)\zeta,\xi\rangle = \int_{G/H}\varphi(xH)\langle\Gamma_\pi(xH)^*\zeta,\xi\rangle d\mu(xH), \tag{5.3}$$

for all $\zeta,\xi \in \mathcal{H}_\pi$.

In other words, for $[\pi] \in \widehat{G/H}$ and $\zeta,\xi \in \mathcal{H}_\pi$, we have

$$\langle\hat{\varphi}(\pi)\zeta,\xi\rangle = \int_{G/H}\varphi(xH)\langle\zeta,\pi(x)T_H^\pi\xi\rangle d\mu(xH). \tag{5.4}$$

Indeed, we can write

$$\begin{aligned}\langle\hat{\varphi}(\pi)\zeta,\xi\rangle &= \int_{G/H}\varphi(xH)\langle\Gamma_\pi(xH)^*\zeta,\xi\rangle d\mu(xH)\\ &= \int_{G/H}\varphi(xH)\langle\zeta,\Gamma_\pi(xH)\xi\rangle d\mu(xH)\\ &= \int_{G/H}\varphi(xH)\langle\zeta,\pi(x)T_H^\pi\xi\rangle d\mu(xH).\end{aligned}$$

Thus, for $\zeta,\xi \in \mathcal{H}_\pi$, we get

$$\begin{aligned}|\langle\hat{\varphi}(\pi)\zeta,\xi\rangle| &= \left|\int_{G/H}\varphi(xH)\langle\zeta,\pi(x)T_H^\pi\xi\rangle d\mu(xH)\right|\\ &\leq \int_{G/H}|\varphi(xH)||\langle\zeta,\pi(x)T_H^\pi\xi\rangle|d\mu(xH)\\ &\leq \int_{G/H}|\varphi(xH)|\|\zeta\|\|\pi(x)T_H^\pi\xi\|d\mu(xH)\\ &= \int_{G/H}|\varphi(xH)|\|\zeta\|\|T_H^\pi\xi\|d\mu(xH)\\ &\leq \int_{G/H}|\varphi(xH)|\|\zeta\|\|\xi\|d\mu(xH)\\ &= \|\zeta\|\|\xi\|\|\varphi\|_{L^1(G/H,\mu)}.\end{aligned}$$

Therefore, $\hat{\varphi}(\pi)$ is a bounded linear operator on $\mathcal{H}_\pi$ satisfying

$$\|\hat{\varphi}(\pi)\| \leq \|\varphi\|_{L^1(G/H,\mu)}. \tag{5.5}$$

From now on, we may use $\hat{\varphi}(\pi)$ or $\mathcal{F}_{G/H}(\varphi)(\pi)$ at times.

The following proposition gives us the connection of the Fourier transform over the homogeneous space G/H with the Fourier transform on the group G.

Proposition 5.2. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Let $\varphi \in L^1(G/H,\mu)$ and $[\pi] \in \widehat{G/H}$. Then*

$$\mathcal{F}_{G/H}(\varphi)(\pi) = \mathcal{F}_G(\varphi_q)(\pi). \tag{5.6}$$

$$\mathcal{F}_{G/H}(L_g\varphi)(\pi) = \mathcal{F}_{G/H}(\varphi)(\pi)\pi(g)^*, \quad \textit{for } g \in G. \tag{5.7}$$

$$\mathcal{F}_{G/H}(\varphi)(\pi)^* = \int_{G/H} \overline{\varphi(xH)}\Gamma_\pi(xH)d\mu(xH). \tag{5.8}$$

Proof. Using (5.4) and also the Weil's formula, for $\zeta, \xi \in \mathcal{H}_\pi$, we can write

$$\begin{aligned}
\langle\hat{\varphi}(\pi)\zeta,\xi\rangle &= \int_{G/H} \varphi(xH)\langle\zeta,\pi(x)T_H^\pi\xi\rangle d\mu(xH) \\
&= \int_{G/H} \varphi(xH)\overline{T_H(\pi_{\xi,\zeta})(xH)}d\mu(xH) \\
&= \int_{G/H} \varphi(xH)T_H(\overline{\pi_{\xi,\zeta}})(xH)d\mu(xH) \\
&= \int_{G/H} T_H(\varphi_q.\overline{\pi_{\xi,\zeta}})(xH)d\mu(xH) \\
&= \int_G \varphi_q(x)\overline{\pi_{\xi,\zeta}(x)}dx \\
&= \int_G \varphi_q(x)\langle\zeta,\pi(x)\xi\rangle dx \\
&= \int_G \varphi_q(x)\langle\pi(x)^*\zeta,\xi\rangle dx \\
&= \langle\widehat{\varphi_q}(\pi)\zeta,\xi\rangle,
\end{aligned}$$

which implies (5.6).

Then using (5.6), we get

$$\begin{aligned}
\mathcal{F}_{G/H}(L_g\varphi)(\pi) &= \mathcal{F}_G(L_g\varphi_q)(\pi) \\
&= \mathcal{F}_G(\varphi_q)(\pi)\pi(g)^* \\
&= \mathcal{F}_{G/H}(\varphi)(\pi)\pi(g)^*.
\end{aligned}$$

Let $\zeta,\xi\in\mathcal{H}_\pi$. Then, using (5.4) and the fact that μ is a positive measure, we can write

$$\begin{aligned}\langle\hat{\varphi}(\pi)^*\zeta,\xi\rangle &= \langle\zeta,\hat{\varphi}(\pi)\xi\rangle\\ &= \overline{\langle\hat{\varphi}(\pi)\xi,\zeta\rangle}\\ &= \left(\int_{G/H}\varphi(xH)\langle\xi,\pi(x)T_H^\pi\zeta\rangle d\mu(xH)\right)^-\\ &= \int_{G/H}\overline{\varphi(xH)}\langle\pi(x)T_H^\pi\zeta,\xi\rangle d\mu(xH)\\ &= \int_{G/H}\overline{\varphi(xH)}\langle\Gamma_\pi(xH)\zeta,\xi\rangle d\mu(xH),\end{aligned}$$

implying that (5.8), holds. □

The following proposition states some generic properties of the Fourier transform (5.1).

Proposition 5.3. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Then*

(1) *for $f\in L^1(G)$ and $[\pi]\in\widehat{G/H}$ we have $\widehat{T_H(f)}(\pi)=T_H^\pi\hat{f}(\pi)$;*
(2) *for $\varphi\in L^1(G/H,\mu)$ and $[\pi]\in\widehat{G/H}$ we have $T_H^\pi\hat{\varphi}(\pi)=\hat{\varphi}(\pi)$;*
(3) *for $\varphi\in L^1(G/H,\mu)$ and $[\pi]\in\widehat{G/H}$ we have $\mathrm{tr}\left[\hat{\varphi}(\pi)T_H^\pi\right]=\mathrm{tr}\left[\hat{\varphi}(\pi)\right]$.*

Proof. (1) Let $[\pi]\in\widehat{G/H}$ and $\zeta,\xi\in\mathcal{H}_\pi$. Invoking definition of the Fourier transform (5.4), and using the Weil's formula in the weak sense, we can write

$$\begin{aligned}\langle\widehat{T_H(f)}(\pi)\zeta,\xi\rangle &= \int_{G/H}T_H(f)(xH)\langle\zeta,\pi(x)T_H^\pi\xi\rangle d\mu(xH)\\ &= \int_{G/H}T_H(f.g_{\zeta,\xi})(xH)d\mu(xH)\\ &= \int_G f(x)\langle\zeta,\pi(x)T_H^\pi\xi\rangle dx\\ &= \int_G f(x)\langle\pi(x)^*\zeta,T_H^\pi\xi\rangle dx\\ &= \int_G f(x)\langle T_H^\pi\pi(x)^*\zeta,\xi\rangle dx\\ &= \langle T_H^\pi\hat{f}(\pi)\zeta,\xi\rangle,\end{aligned}$$

where $g_{\zeta,\xi}\colon G\to\mathbb{C}$ is given by $g_{\zeta,\xi}(x):=\langle\zeta,\pi(x)T_H^\pi\xi\rangle$ for $x\in G$.

(2) Let $\varphi \in L^1(G/H,\mu)$ and $[\pi] \in \widehat{G/H}$. Then, $\varphi_q \in L^1(G)$ and $T_H(\varphi_q) = \varphi$. Using (1) and since T_H^π is an orthogonal projection, we get

$$T_H^\pi \hat{\varphi}(\pi) = T_H^\pi \widehat{T_H(\varphi_q)}(\pi) = T_H^\pi T_H^\pi \widehat{\varphi_q}(\pi) = T_H^\pi \widehat{\varphi_q}(\pi) = \widehat{T_H(\varphi_q)}(\pi) = \hat{\varphi}(\pi).$$

(3) Let $\varphi \in L^1(G/H,\mu)$ and $[\pi] \in \widehat{G/H}$. Then, using (2), we have

$$\mathrm{tr}\left[\hat{\varphi}(\pi)T_H^\pi\right] = \mathrm{tr}\left[T_H^\pi\hat{\varphi}(\pi)\right] = \mathrm{tr}[\hat{\varphi}(\pi)]. \square$$

Definition 5.4. Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. The linear map

$$\mathcal{F}_{G/H}: L^1(G/H,\mu) \longrightarrow \mathfrak{C}(\widehat{G/H}), \tag{5.9}$$

is called as *abstract operator-valued Fourier transform* over the compact homogeneous space G/H.

The abstract operator-valued Fourier transform over G/H satisfies

$$\mathcal{F}_{G/H} \circ T_H = \widehat{T_H} \circ \mathcal{F}_G. \tag{5.10}$$

We here study some analytic properties of the Fourier transform $\mathcal{F}_{G/H}$.

Theorem 5.5. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. The Fourier transform*

$$\mathcal{F}_{G/H}: L^1(G/H,\mu) \longrightarrow \mathfrak{C}_0(\widehat{G/H})$$

is a norm-decreasing homomorphism onto a subalgebra of $\mathfrak{C}_0(\widehat{G/H})$.

Proof. Let $\varphi \in L^1(G/H,\mu)$. Using (5.5), we get

$$\|\hat{\varphi}\|_\infty = \sup_{[\pi]\in\widehat{G/H}} \|\hat{\varphi}(\pi)\|_\infty = \sup_{[\pi]\in\widehat{G/H}} \|\hat{\varphi}(\pi)\| \le \|\varphi\|_{L^1(G/H,\mu)},$$

which means that $\mathcal{F}_{G/H}$ is a norm-decreasing linear operator. Invoking (3.3) and (5.6), we obtain

$$\begin{aligned}\mathcal{F}_{G/H}(\varphi *_{G/H} \psi)(\pi) &= \mathcal{F}_G((\varphi *_{G/H} \psi)_q)(\pi)\\ &= \mathcal{F}_G(\varphi_q *_G \psi_q)(\pi)\\ &= \mathcal{F}_G(\varphi_q)(\pi).\mathcal{F}_G(\psi_q)(\pi)\\ &= \mathcal{F}_{G/H}(\varphi)(\pi).\mathcal{F}_{G/H}(\psi)(\pi),\end{aligned}$$

which guarantees that $\mathcal{F}_{G/H}$ is a homomorphism as well. $\square$

Since $L^2(G/H,\mu) \subset L^1(G/H,\mu)$, the Fourier transform defined in (5.1) is well-defined for L^2-functions over the homogeneous space G/H.

In the next result, we show that the Fourier transform defined in (5.1) satisfies a generalized version of the Plancherel formula in L^2-sense.

Theorem 5.6. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Each $\varphi \in L^2(G/H,\mu)$ satisfies the Plancherel formula*

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi \|\hat{\varphi}(\pi)\|_2^2 = \|\varphi\|^2_{L^2(G/H,\mu)}. \tag{5.11}$$

Proof. Let $\varphi \in L^2(G/H,\mu)$ be given. Then we have $\varphi_q \in L^2(G)$. If $[\pi] \in \widehat{G}$ with $[\pi] \notin \widehat{G/H}$, then we have $T_H^\pi = 0$. Hence, for $\zeta,\xi \in \mathcal{H}_\pi$, we have $T_H(\pi_{\xi,\zeta}) = 0$. Therefore, we get

$$\widehat{\varphi_q}(\pi) = 0. \tag{5.12}$$

Indeed, using the Weil's formula, for $\zeta,\xi \in \mathcal{H}_\pi$, we can write

$$\begin{aligned}
\langle \widehat{\varphi_q}(\pi)\zeta,\xi\rangle &= \int_G \varphi_q(x)\langle \pi(x)^*\zeta,\xi\rangle dx \\
&= \int_G \varphi_q(x)\langle \zeta,\pi(x)\xi\rangle dx \\
&= \int_G \varphi_q(x)\overline{\langle \pi(x)\xi,\zeta\rangle} dx \\
&= \int_G \varphi_q(x)\overline{\pi_{\xi,\zeta}(x)} dx \\
&= \int_{G/H} T_H(\varphi_q.\overline{\pi_{\xi,\zeta}})(xH)d\mu(xH) \\
&= \int_{G/H} \varphi(xH)\overline{T_H(\pi_{\xi,\zeta})(xH)}d\mu(xH) \\
&= 0.
\end{aligned}$$

Using (5.6), (5.12), invoking the Plancherel formula (2.2), and (3.2) we obtain

$$\begin{aligned}
\sum_{[\pi]\in\widehat{G/H}} d_\pi \|\hat{\varphi}(\pi)\|_2^2 &= \sum_{[\pi]\in\widehat{G/H}} d_\pi \|\widehat{\varphi_q}(\pi)\|_2^2 \\
&= \sum_{[\pi]\in\widehat{G/H}} d_\pi \|\widehat{\varphi_q}(\pi)\|_2^2 + \sum_{\{[\pi]\in\widehat{G}:[\pi]\notin\widehat{G/H}\}} d_\pi \|\widehat{\varphi_q}(\pi)\|_2^2
\end{aligned}$$

$$= \sum_{[\pi]\in\widehat{G}} d_\pi \|\widehat{\varphi_q}(\pi)\|_2^2$$

$$= \|\varphi_q\|^2_{L^2(G)}$$

$$= \|\varphi\|^2_{L^2(G/H,\mu)},$$

which implies (5.11). □

The following corollary is a consequence of Theorem 5.6.

Corollary 5.7. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Then*

(1) *For $\varphi, \psi \in L^2(G/H,\mu)$, we have*

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\hat{\varphi}(\pi)\hat{\psi}(\pi)^*] = \langle\varphi,\psi\rangle_{L^2(G/H,\mu)}. \tag{5.13}$$

(2) *For $f, g \in \mathcal{J}^2(G,H)^\perp$, we have*

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\hat{f}(\pi)\hat{g}(\pi)^*] = \langle f, g\rangle_{L^2(G)}. \tag{5.14}$$

Proof. (1) Using the polarization identity and Theorem 5.6, we get (5.13).

(2) Applying the Weil's formula, (5.13), and Corollary 3.8, we get (5.14). □

Remark 5.8. Let H be a closed normal subgroup of a compact group G and μ be the normalized G-invariant measure over the left coset space G/H associated to the Weil's formula. Then Theorem 4.10 implies that $\widehat{G/H} = H^\perp$ and hence the Plancherel (trace) formula (5.11) reads as follows:

$$\sum_{[\pi]\in H^\perp} d_\pi \|\hat{\varphi}(\pi)\|_2^2 = \|\varphi\|^2_{L^2(G/H,\mu)},$$

for all $\varphi \in L^2(G/H,\mu)$, where

$$\hat{\varphi}(\pi) = \int_{G/H} \varphi(xH)\pi(x)^* d\mu(xH),$$

for all $[\pi] \in H^\perp$, see Remark 5.1.

As a consequence of Theorem 5.6, we can also prove the following results.

Theorem 5.9. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. The range of the Fourier transform $\mathfrak{F}_{G/H}: L^2(G/H,\mu) \to \mathfrak{C}^2(\widehat{G/H})$ is the closed subspace*

$$\mathfrak{R}^2(G/H) := \{\mathbf{T} = (T_\pi)_{\pi\in\widehat{G/H}} \in \mathfrak{C}^2(\widehat{G/H}) \mid T_H^\pi T_\pi = T_\pi \text{ for all } \pi \in \widehat{G/H}\}.$$

Proof. Using Proposition 5.2 and Theorem 5.6, we can deduce that the subspace $\{\mathfrak{F}_{G/H}(\varphi): \varphi \in L^2(G/H,\mu)\}$ is contained in $\mathfrak{R}^2(G/H)$. Let $\mathbf{T} = (T_\pi)_{\pi\in\widehat{G/H}} \in \mathfrak{R}^2(G/H)$ be given. Let $\widetilde{\mathbf{T}} = (\widetilde{T}_\pi)_{\pi\in\widehat{G}} \in \mathfrak{C}^2(\widehat{G})$ such that $\widetilde{T}_\pi = 0$ for $[\pi] \in \widehat{G}$ with $[\pi] \notin \widehat{G/H}$ and $\widetilde{T}_\pi = T_\pi$ for all $[\pi] \in \widehat{G/H}$. Since the Fourier transform $\mathfrak{F}_G: L^2(G) \to \mathfrak{C}^2(\widehat{G})$ is unitary, there exists a unique $f \in L^2(G)$ such that $\mathfrak{F}_G(f) = \widetilde{\mathbf{T}}$. Let $\varphi := T_H(f) \in L^2(G/H,\mu)$ and $[\pi] \in \widehat{G/H}$. Then using Proposition 5.2, we can write

$$\mathfrak{F}_{G/H}(\varphi)(\pi) = \mathfrak{F}_{G/H}(T_H(f))(\pi) = T_H^\pi \mathfrak{F}_G(f)(\pi) = T_H^\pi T_\pi = T_\pi,$$

which implies that $\mathfrak{F}_{G/H}(\varphi) = \mathbf{T}$. □

We then can also present the following consequences.

Corollary 5.10. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. The abstract Fourier transform $\mathfrak{F}_{G/H}: L^2(G/H,\mu) \to \mathfrak{R}^2(G/H)$ is a unitary isomorphism of Hilbert spaces.*

Corollary 5.11. *Let H be a closed subgroup of a compact group G. Then $\widehat{T_H}$ maps $\mathfrak{C}^2(\widehat{G})$ onto $\mathfrak{R}^2(G/H)$.*

We then present the abstract definition of the inverse Fourier transform over homogeneous spaces of compact groups.

Definition 5.12. The inverse Fourier transform $\mathbf{T} \to \check{\mathbf{T}}$ from $\mathfrak{R}^2(G/H)$ onto $L^2(G/H,\mu)$, is defined as the inverse of the Fourier transform $\varphi \to \hat{\varphi}$, which maps $L^2(G/H,\mu)$ isometrically onto $\mathfrak{R}^2(G/H)$.

Plainly $\mathbf{T} \to \check{\mathbf{T}}$ is a unitary isomorphism of Hilbert spaces.

The following theorem shows that the abstract Fourier transform

$$\mathfrak{F}_{G/H}: L^2(G/H,\mu) \longrightarrow \mathfrak{C}^2(\widehat{G/H})$$

is surjective, only when H is normal in G.

Theorem 5.13. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. The abstract Fourier transform $\mathcal{F}_{G/H}$ given by (5.9) maps $L^2(G/H,\mu)$ onto $\mathfrak{C}^2(\widehat{G/H})$ if and only if H is normal in G.*

Proof. Let H be a closed normal subgroup of G. Then G/H is a compact group, μ is automatically the normalized Haar measure of G/H, and $\widehat{G/H} = H^\perp$ as well. Then by Theorem 28.43 of [16], the abstract Fourier transform

$$\mathcal{F}_{G/H}\colon L^2(G/H) \longrightarrow \mathfrak{C}^2(H^\perp)$$

is a unitary isomorphism of Hilbert spaces. Conversely, assume that $\mathfrak{R}^2(G/H) = \mathfrak{C}^2(\widehat{G/H})$. Then we get $T_H^\pi = I$, for all $[\pi] \in \widehat{G/H}$. Let $\varphi \in \mathcal{C}(G/H)$ be arbitrary and $h \in H$. Using Proposition 5.2 for $[\pi] \in \widehat{G/H}$, we can write

$$\mathcal{F}_{G/H}(L_h\varphi)(\pi) = \mathcal{F}_{G/H}(\varphi)(\pi)\pi(h)^* = \mathcal{F}_{G/H}(\varphi)(\pi).$$

Thus, we get $\mathcal{F}_{G/H}(L_h\varphi) = \mathcal{F}_{G/H}(\varphi)$. Since the abstract Fourier transform

$$\mathcal{F}_{G/H}\colon L^2(G/H,\mu) \longrightarrow \mathfrak{C}^2(\widehat{G/H})$$

is injective by Theorem 5.6, we get $L_h\varphi = \varphi$ in the L^2-sense. Then continuity of φ implies that $L_h\varphi(xH) = \varphi(xH)$, for all $x \in G$ and $h \in H$. Since φ is arbitrary, we get $hxH = xH$, for all $x \in G$ and $h \in H$, which means that H is normal in G. □

Remark 5.14. Theorem 5.13 can be considered as an abstract characterization of the algebraic structure of the homogeneous space G/H via analytic and topological aspects.

The following theorem is the Hausdorff–Young inequality for the abstract operator-valued Fourier transform over the compact homogeneous spaces.

Theorem 5.15. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure over G/H. Let $1 < p < 2$ and p' be the Hölder conjugate of p. Then the Fourier transform $\mathcal{F}_{G/H}\colon L^p(G/H,\mu) \to \mathfrak{C}^{p'}(\widehat{G/H})$ is a norm-decreasing linear operator.*

Proof. Let $\varphi \in L^p(G/H,\mu)$. Since $1 < p$ and G/H is compact we have $L^p(G/H,\mu) \subseteq L^1(G/H,\mu)$. Thus, $\hat{\varphi}(\pi)$ is already defined and hence it is a bounded linear operator on $\mathcal{H}_\pi$ for all $[\pi] \in \widehat{G/H}$. Then we claim that

$$\|\hat{\varphi}\|_{\mathfrak{C}^{p'}(\widehat{G/H})} \leq \|\varphi\|_{L^p(G/H,\mu)}. \tag{5.15}$$

To this end, using (2.2), (5.12), and the Hausdorff–Young inequality over the compact group G (Theorem 31.22 of [16]), we can write

$$
\begin{aligned}
\|\hat{\varphi}\|_{\mathfrak{C}^{p'}(\widehat{G/H})}^{p'} &= \sum_{[\pi]\in\widehat{G/H}} d_\pi \|\hat{\varphi}(\pi)\|_{p'}^{p'} \\
&= \sum_{[\pi]\in\widehat{G/H}} d_\pi \|\widehat{\varphi_q}(\pi)\|_{p'}^{p'} \\
&= \sum_{[\pi]\in\widehat{G}} d_\pi \|\widehat{\varphi_q}(\pi)\|_{p'}^{p'} \\
&= \|\widehat{\varphi_q}\|_{\mathfrak{C}^{p'}(\widehat{G})}^{p'} \\
&\leq \|\varphi_q\|_{L^p(G)}^{p'} \\
&= \|\varphi\|_{L^p(G/H,\mu)}^{p'},
\end{aligned}
$$

which implies (5.15). □

For $1 < p < 2$, one can define $\mathfrak{R}^p(G/H)$ as the subset of $\mathfrak{C}^p(\widehat{G/H})$ given by

$$
\mathfrak{R}^p(G/H) := \{\mathbf{T} = (T_\pi)_{\pi\in\widehat{G/H}} \in \mathfrak{C}^p(\widehat{G/H}) \mid T_H^\pi T_\pi = T_\pi \text{ for all } \pi \in \widehat{G/H}\}. \tag{5.16}
$$

Then the inclusion $\mathfrak{R}^p(G/H) \subset \mathfrak{R}^2(G/H)$ holds, and hence the inverse Fourier transform $\mathbf{T} \mapsto \check{\mathbf{T}}$ is already defined on $\mathfrak{R}^p(G/H)$. Then it obviously maps $\mathfrak{R}^p(G/H)$ into $L^2(G/H,\mu)$.

Theorem 5.16. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure over G/H. Let $1 < p < 2$ and p' be the Hölder conjugate of p. Then the inverse Fourier transform is a norm-decreasing linear operator from $\mathfrak{R}^p(G/H)$ into $L^{p'}(G/H,\mu)$.*

Proof. Let $\mathbf{T} \in \mathfrak{R}^p(G/H)$. Let $\widetilde{\mathbf{T}} = (\widetilde{T}_\pi)_{\pi\in\widehat{G}} \in \mathfrak{C}^p(\widehat{G})$ such that $\widetilde{T}_\pi = 0$ for $[\pi] \in \widehat{G}$ with $[\pi] \notin \widehat{G/H}$ and $\widetilde{T}_\pi = T_\pi$ for all $[\pi] \in \widehat{G/H}$. Then we get $\|\mathbf{T}\|_{\mathfrak{C}^p(\widehat{G/H})} = \|\widetilde{\mathbf{T}}\|_{\mathfrak{C}^p(\widehat{G})}$. By Theorem 31.24 of [16], we have $\check{\widetilde{\mathbf{T}}} \in L^{p'}(G)$ and $\|\check{\widetilde{\mathbf{T}}}\|_{L^{p'}(G)} \leq \|\widetilde{\mathbf{T}}\|_{\mathfrak{C}^p(\widehat{G})}$. Then using Theorem 4.11 and (5.10), we can write

$$
\|\check{\mathbf{T}}\|_{L^{p'}(G/H,\mu)} \leq \|\check{\widetilde{\mathbf{T}}}\|_{L^{p'}(G)} \leq \|\widetilde{\mathbf{T}}\|_{\mathfrak{C}^p(\widehat{G})} = \|\mathbf{T}\|_{\mathfrak{C}^p(\widehat{G/H})}.
$$
□

Next theorem presents an explicit L^2-inversion formula for the Fourier transform given in (5.1).

Theorem 5.17. *Let H be a closed subgroup of a compact group G, μ be the normalized G-invariant measure on G/H, and $\varphi \in L^2(G/H,\mu)$. Then the function $\breve{\varphi}\colon G/H \to \mathbb{C}$ defined by*

$$xH \mapsto \breve{\varphi}(xH) := \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{\varphi}(\pi)\pi(x)], \tag{5.17}$$

belongs to $L^2(G/H,\mu)$, and we have

$$\varphi(xH) = \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{\varphi}(\pi)\pi(x)] \quad \text{for } \mu-a.e.\ xH \in G/H. \tag{5.18}$$

Proof. Let $\varphi \in L^2(G/H)$ and $x \in G$. Then, $\varphi_q \in L^2(G)$ and hence the series

$$\sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}[\widehat{\varphi_q}(\pi)\pi(x)],$$

converges and also $y \mapsto \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}[\widehat{\varphi_q}(\pi)\pi(y)]$ defines a function in $L^2(G)$. Thus, we deduce that the series $\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{\varphi}(\pi)\pi(x)]$, is converges, because

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{\varphi}(\pi)\pi(x)] = \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{\varphi_q}(\pi)\pi(x)] = \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}[\widehat{\varphi_q}(\pi)\pi(x)].$$

Now let $h \in H$. Using Proposition 5.3, we can write

$$\begin{aligned}
\operatorname{tr}[\widehat{\varphi}(\pi)\pi(xh)] &= \operatorname{tr}[T_H^\pi\widehat{\varphi}(\pi)\pi(xh)] \\
&= \operatorname{tr}[\widehat{\varphi}(\pi)\pi(x)\pi(h)T_H^\pi] \\
&= \operatorname{tr}[\widehat{\varphi}(\pi)\pi(x)T_H^\pi] \\
&= \operatorname{tr}[T_H^\pi\widehat{\varphi}(\pi)\pi(x)] \\
&= \operatorname{tr}[\widehat{\varphi}(\pi)\pi(x)],
\end{aligned}$$

which guarantees that

$$yH \longmapsto \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{\varphi}(\pi)\pi(y)]$$

gives a well-defined function on the left coset space G/H. Using (2.3), for $\varphi_q \in L^2(G)$, we get

$$\varphi_q(x) = \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}[\widehat{\varphi_q}(\pi)\pi(x)], \quad \text{for a.e. } x \in G. \tag{5.19}$$

Then applying the linear map T_H to (5.19), implies (5.18). Indeed, for $x \in G$, we can write

$$\begin{aligned}
\varphi(xH) &= T_H(\varphi_q)(xH) \\
&= \int_H \Big(\sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}[\widehat{\varphi_q}(\pi)\pi(xh)] \Big) dh \\
&= \sum_{[\pi]\in\widehat{G}} d_\pi \left(\int_H \operatorname{tr}[\widehat{\varphi_q}(\pi)\pi(xh)] dh \right) \\
&= \sum_{[\pi]\in\widehat{G}} d_\pi \left(\int_H \operatorname{tr}[\widehat{\varphi_q}(\pi)\pi(x)\pi(h)] dh \right) \\
&= \sum_{[\pi]\in\widehat{G}} d_\pi \left(\int_H \operatorname{tr}[\pi(h)\widehat{\varphi_q}(\pi)\pi(x)] dh \right) \\
&= \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}[T_H^\pi \widehat{\varphi_q}(\pi)\pi(x)] \\
&= \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[T_H^\pi \widehat{\varphi_q}(\pi)\pi(x)] \\
&= \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[T_H^\pi \hat{\varphi}(\pi)\pi(x)] \\
&= \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\hat{\varphi}(\pi)\pi(x)].
\end{aligned}$$

□

Remark 5.18. Let H be a closed normal subgroup of a compact group G and μ be the normalized G-invariant measure over the left coset space G/H associated to the Weil's formula. Then Theorem 4.10 implies that $\widehat{G/H} = H^\perp$ and hence the L^2-inversion formula (5.18) reads as follows:

$$\varphi(xH) = \sum_{[\pi]\in H^\perp} d_\pi \operatorname{tr}[\hat{\varphi}(\pi)\pi(x)]$$

for all $\varphi \in L^2(G/H, \mu)$ and $\mu -$ a.e. $xH \in G/H$, where

$$\hat{\varphi}(\pi) = \int_{G/H} \varphi(xH)\pi(x)^* d\mu(xH),$$

for all $[\pi] \in H^\perp$.

Using the Weil's formula and Theorem 5.17, we conclude the following corollary.

Corollary 5.19. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Then*

(1) *For $\varphi \in L^2(G/H,\mu)$, we have*

$$\int_G \varphi(xH)dx = \int_{G/H}\Big(\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\hat{\varphi}(\pi)\pi(x)]\Big)d\mu(xH). \tag{5.20}$$

(2) *For $f \in \mathfrak{J}^2(G,H)^\perp$, we have*

$$\int_G f(x)dx = \int_{G/H}\Big(\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\hat{f}(\pi)\pi(x)]\Big)d\mu(xH). \tag{5.21}$$

Proof. (1) Let $\varphi \in L^2(G/H,\mu)$. Then we have $\varphi_q \in L^2(G)$ with $T_H(\varphi_q) = \varphi$. Using the Weil's formula and (5.18), we have

$$\begin{aligned}\int_G \varphi(xH)dx &= \int_G \varphi_q(x)dx\\ &= \int_{G/H} T_H(\varphi_q)(xH)d\mu(xH)\\ &= \int_{G/H}\varphi(xH)d\mu(xH)\\ &= \int_{G/H}\Big(\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\hat{\varphi}(\pi)\pi(x)]\Big)d\mu(xH),\end{aligned}$$

which proves (5.20).

(2) It is straightforward by (1). □

Remark 5.20. Let H be a closed normal subgroup of a compact group G and μ be the normalized G-invariant measure over the left coset space G/H associated to the Weil's formula. Then Theorem 4.10 implies that $\widehat{G/H} = H^\perp$ and hence the formulas (5.20) and (5.21) read as follows:

$$\int_G \varphi(xH)dx = \int_{G/H}\Big(\sum_{[\pi]\in H^\perp} d_\pi \operatorname{tr}[\hat{\varphi}(\pi)\pi(x)]\Big)d\mu(xH),$$

and

$$\int_G f(x)dx = \int_{G/H}\Big(\sum_{[\pi]\in H^\perp} d_\pi \operatorname{tr}[\hat{f}(\pi)\pi(x)]\Big)d\mu(xH),$$

for all $\varphi \in L^2(G/H,\mu)$ and $f \in \mathcal{J}^2(G,H)^\perp$, where

$$\hat{\varphi}(\pi) = \int_{G/H} \varphi(xH)\pi(x)^* d\mu(xH),$$

and

$$\hat{f}(\pi) = \int_G f(x)\pi(x)^* dx,$$

for all $[\pi] \in H^\perp$.

The following result can be considered as an abstract generalization of the Poisson summation formula for homogeneous spaces of compact groups.

Theorem 5.21. *Let H be a closed subgroup of a compact group G. Each $f \in \mathcal{C}(G)$ satisfies the following Poisson summation type formula*

$$\int_H f(xh)dh = \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\hat{f}(\pi)\pi(x)T_H^\pi], \quad \textit{for } x \in G. \tag{5.22}$$

Proof. Let μ be the normalized G-invariant measure on G/H and $f \in \mathcal{C}(G)$. Then by (5.18), we have

$$T_H(f)(xH) = \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{T_H(f)}(\pi)\pi(x)] \quad \text{for a.e. } xH \in G/H. \tag{5.23}$$

Let $x \in G$. Using Proposition 5.3 and (5.23), we get

$$\begin{aligned}\int_H f(xh)dh &= T_H(f)(xH)\\ &= \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{T_H(f)}(\pi)\pi(x)]\\ &= \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[T_H^\pi \hat{f}(\pi)\pi(x)]\\ &= \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\hat{f}(\pi)\pi(x)T_H^\pi],\end{aligned}$$

which proves (5.22). □

Remark 5.22. Let H be a closed normal subgroup of a compact group G and μ be the normalized G-invariant measure over the left coset space G/H associated to the Weil's formula. Then Theorem 4.10 implies that $\widehat{G/H} = H^\perp$ and hence the Poisson summation formula (5.22) reads as follows:

$$\int_H f(xh)dh = \sum_{[\pi]\in H^\perp} d_\pi \operatorname{tr}[\hat{f}(\pi)\pi(x)],$$

for all $f \in \mathcal{C}(G)$ and $x \in G$, where

$$\hat{f}(\pi) = \int_G f(x)\pi(x)^* dx,$$

for all $[\pi] \in H^{\perp}$.

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