Groups Geom. Dyn. 12 (2018), 1–64
DOI 10.4171/GGD/437

Groups, Geometry, and Dynamics

Dynamics of Teichmüller modular groups and topology of moduli spaces of Riemann surfaces of infinite type

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Abstract. We investigate the dynamics of the Teichmüller modular group on the Teichmüller space of a Riemann surface of infinite topological type. Since the modular group does not necessarily act discontinuously, the quotient space cannot inherit a rich geometric structure from the Teichmüller space. However, we introduce the set of points where the action of the Teichmüller modular group is stable, and we prove that this region of stability is generic in the Teichmüller space. By taking the quotient and completion with respect to the Teichmüller distance, we obtain a geometric object that we regard as an appropriate moduli space of the quasiconformally equivalent complex structures admitted on a topologically infinite Riemann surface.

Mathematics Subject Classification (2010). Primary: 30F60; Secondary: 37F30, 32G15.

Keywords. Teichmüller modular group, moduli space, Riemann surface of infinite type, quasiconformal deformation, hyperbolic geometry, length spectrum, limit set, region of discontinuity.

Contents

1. Introduction

The moduli space of an analytically finite Riemann surface (i.e., a compact Riemann surface from which at most a finite number of points are removed) is a complex analytic space whose singular points are normal. It has long been studied in various fields of mathematics. However, once we extend our interest to analytically infinite surfaces, we recognize that their moduli spaces in the quasiconformal category no longer have such a researchable structure. In fact, they have rarely appeared in the literature. This is in contrast to the situation of a Teichmüller space, which is the universal covering of a moduli space. The Teichmüller space $T(R)$ can also be defined for an analytically infinite Riemann surface R even though it has an infinite-dimensional complex structure. From the viewpoint of complex analysis, the complex structure of $T(R)$ is considered via an infinite-dimensional Banach space of Beltrami differentials on R, and we are able to develop their theories in common for both finite and infinite Riemann surfaces. The moduli space $M(R)$ is the quotient space of the Teichmüller space $T(R)$ by the Teichmüller modular group $\mathrm{Mod}(R)$, which is the covering transformation group for the projection $T(R) \to M(R)$ and is induced by the action of the quasiconformal mapping class group $\mathrm{MCG}(R)$. When introducing the moduli space for an analytically infinite Riemann surface, a problem arises because $\mathrm{Mod}(R)$ does not necessarily act discontinuously, but it acts discontinuously on the Teichmüller space of an analytically finite Riemann surface.

In the first part of this paper, we investigate the action of modular groups on infinite-dimensional Teichmüller spaces for analytically infinite Riemann surfaces. We generalize this analysis to a purely topological consideration of the dynamics of isometries acting on a complete metric space. In this general situation, the comparison between countability and uncountability serves as a fundamental basis for our arguments. This appears practically as the Baire category theorem and formulates our fundamental principles (Theorems 4.4 and 4.5). When we apply these facts to the Teichmüller modular group, the countable compactness of a Riemann surface represents the countable side, whereas the cardinality of the mapping class group represents the uncountable side. In general, we first show some consequences deduced from this topological structure of Riemann surfaces. Then, we claim more specific results based on the hyperbolic geometric structure on Riemann surfaces. For instance, if we impose boundedness on the hyperbolic geometry of R, which is roughly a condition that the injectivity radii are uniformly bounded from below and above, the analysis of the dynamics of $\mathrm{Mod}(R)$ becomes simplified. In particular, we will see that the discontinuity of the action of $\mathrm{Mod}(R)$ is the same as its stability explained below (Theorem 5.3).

As in the case of Kleinian groups, we consider the set $\Omega(\Gamma)$ of points in $T(R)$, where a subgroup Γ of $\mathrm{Mod}(R)$ acts discontinuously, and we call it the region of discontinuity. Its complement is defined to be the limit set $\Lambda(\Gamma)$. The action of Γ is desirable on $\Omega(\Gamma)$ in the sense that the quotient space $\Omega(\Gamma)/\Gamma$ inherits the geometric structure from $T(R)$. On the other hand, to investigate the complicated action on the complement, we first classify the limit points and examine their distribution. In contrast to the case of Kleinian groups, the set $\Lambda_\infty(\Gamma)$ of stabilized limit points, which are fixed by infinitely many elements of Γ, is nowhere dense in the limit set $\Lambda(\Gamma)$ unless $\Lambda(\Gamma)$ coincides with a certain exceptional set (this result itself is proved later on in Theorem 12.1). Such limit points are due to infinite groups of conformal automorphisms of some Riemann surfaces quasiconformally equivalent to R. Around these limit points, unusual phenomena occur, such as the existence of non-closed orbits. This makes the analysis of the dynamics difficult and the quotient space exotic. Accordingly, the topological moduli space $M(R)$ is not a T_1-space in many cases (Corollary 6.5). In addition, we consider the problem of determining whether an isolated limit point exists or not. If it exists, we see that its isotropy subgroup is a very special group in a group theoretical sense, which is related to the Burnside problem (Theorem 7.1). We conjecture that an isolated limit point exists, but we only present evidence for it. By excluding such exceptional limit points, we can conclude that, as in the case of Kleinian groups, the accumulation points of orbits $\Lambda_0(\Gamma)$ (called generic limit points) are dense in the limit set (Theorem 8.3).

It would be preferable if we could always make use of the region of discontinuity $\Omega = \Omega(\mathrm{Mod}(R))$ for providing a geometric structure with the topological moduli space $M(R)$. However, Ω may be empty. Instead, we introduce another criterion of manageable action, i.e., stability. We say that a subgroup Γ of $\mathrm{Mod}(R)$ acts at $p \in T(R)$ stably if the orbit $\Gamma(p)$ is closed and the isotropy subgroup $\mathrm{Stab}_\Gamma(p)$ is finite. Under this condition, the quotient space has separability at this point. The set $\Phi(\Gamma)$ of points where Γ acts stably is called the region of stability. Although stability is a weaker condition than discontinuity, we can prove that $\Phi = \Phi(\mathrm{Mod}(R))$ is open (Theorem 5.2), dense, and connected in $T(R)$ for every Riemann surface R. This genericity of Φ in $T(R)$ ensures that the metric completion of the quotient space $M_\Phi(R) = \Phi/\mathrm{Mod}(R)$ captures all points in the moduli space. These properties (except openness) are demonstrated in the second part of this paper (Corollaries 10.2 and 11.2). The main tool for their proofs is the length spectrum $\mathrm{LS}(p)$ at $p \in T(R)$, which is the closure of the set of lengths of all simple closed geodesics on the hyperbolic Riemann surface corresponding to $p \in T(R)$. The essential spectrum $\mathrm{LS}_{\mathrm{ess}}(p)$ is a subset of $\mathrm{LS}(p)$ consisting of all accumulation points of the spectra. We see that, if there is a discrete point spectrum in $\mathrm{LS}(p) - \mathrm{LS}_{\mathrm{ess}}(p)$, then $\mathrm{Mod}(R)$ acts at p stably (Theorem 9.2). Moreover, we consider the variation of $\mathrm{LS}_{\mathrm{ess}}(p)$ under a quasiconformal deformation and prove that it is invariant under any quasiconformal homeomorphism having a compact support of the deformation (Theorem 9.5). By using such a deformation, we produce a discrete point spectrum to claim the stability.

The closure equivalence relation is stronger than the orbit equivalence relation, and two points p and q in $T(R)$ are related by the closure equivalence if they are both contained in the closure of the same orbit under $\mathrm{Mod}(R)$. The quotient space

$T(R)//\operatorname{Mod}(R)$ by the closure equivalence is called the geometric moduli space, and it is denoted by $M_*(R)$. The quotient distance is induced on $M_*(R)$ from the Teichmüller distance of $T(R)$. Since the closure equivalence and the orbit equivalence are the same on the region of stability Φ, we see that the moduli space of the stable points $M_\Phi(R)$ is isometrically embedded in $M_*(R)$. From the above-mentioned properties of Φ, we prove that $M_*(R)$ coincides with the metric completion of $M_\Phi(R)$ (Theorem 14.3). We can regard this space as an appropriate moduli space for a topologically infinite Riemann surface R (in other words, R is of infinite topological type, which means that the fundamental group $\pi_1(R)$ is infinitely generated). In fact, it is possible to introduce a certain structure of a complex analytic space to $M_*(R)$ by a general theory, and when we assume that R satisfies the bounded geometry condition, $M_*(R)$ is the completion of the complex Banach orbifold $\Omega/\operatorname{Mod}(R)$. Moreover, by construction, $M_*(R)$ is a type of universal space for the geometric invariants of the moduli. On the other hand, $M_*(R)$ is so large that it does not satisfy the second countability axiom. Actually, we prove that the topological moduli space $M(R)$ does not have a countable dense subset in it (Theorem 13.1). Although the Teichmüller space $T(R)$ is non-separable in this sense, it is not straightforward to show this property for $M(R)$.

The next two sections are devoted to introducing preliminaries for theories of Teichmüller spaces and hyperbolic geometry on Riemann surfaces. A conformal automorphism group G of a Riemann surface R defines an embedding of the Teichmüller space of the orbifold R/G into the Teichmüller space of R, and we will see that the embedded space $T(R/G)$ is a proper subset of $T(R)$ at many places in our arguments. Lemma 2.1 will serve as a basic fact for this claim. By a quasiconformal deformation of R, the geodesic length of each simple closed curve c changes. We have to estimate this variation frequently in this paper, especially in the case where the support of the quasiconformal deformation is far from c. Theorem 3.3 serves as a powerful tool for this purpose.

This research has been developed over many years, and preprint versions have been extended and revised several times. The current revision remains largely unchanged since 2010. A primary announcement of this research appeared in [21]. A survey partially based on the results of this paper was presented in [25].

2. Teichmüller spaces and modular groups

Throughout this paper, we assume that a Riemann surface R is hyperbolic, i.e., it is represented by a quotient space $\mathbb{H}/H$ of the hyperbolic plane $\mathbb{H}$ by a torsion-free Fuchsian group H. Moreover, we are mainly interested in the case where $H \cong \pi_1(R)$ is infinitely generated, i.e., R is topologically infinite (R is of infinite topological type).

The Teichmüller space $T(R)$ of R is the set of all equivalence classes of quasiconformal homeomorphisms f of R onto other Riemann surfaces. Two quasiconformal homeomorphisms f_1 and f_2 are defined to be equivalent if there is a conformal homeomorphism $g\colon f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ g \circ f_1$ is homotopic to the identity on R. Here, the homotopy is considered to be relative to the boundary at infinity $\partial_\infty R = (\partial_\infty \mathbb{H} - \Lambda(H))/H$ of $R = \mathbb{H}/H$ when the limit set $\Lambda(H)$ of the Fuchsian group H is a proper subset of the circle at infinity $\partial_\infty \mathbb{H}$ of the hyperbolic plane. Earle and McMullen [7] proved that the existence of the homotopy is equivalent to the existence of an isotopy to the identity on R relative to $\partial_\infty R$ through uniformly quasiconformal automorphisms. The equivalence class of f is called the *Teichmüller class* and denoted by $[f]$. We often represent the Riemann surface $f(R)$ as R_p for $p = [f] \in T(R)$. In this case, a certain quasiconformal homeomorphism f in the Teichmüller class p is assigned implicitly or the argument depends only on p.

The Teichmüller space $T(R)$ has a complex Banach manifold structure, which is shown below. Moreover, it has a metric structure such that the distance between $p_1 = [f_1]$ and $p_2 = [f_2]$ in $T(R)$ is defined by $d_T(p_1, p_2) = \log K(f)$, where f is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$ relative to the boundary at infinity. This is called the *Teichmüller distance*. By virtue of the compactness property of quasiconformal maps, the Teichmüller distance d_T is complete on $T(R)$. This coincides with the Kobayashi distance on $T(R)$ with respect to the complex Banach manifold structure. For further details on Teichmüller spaces, readers may refer to monographs by Gardiner and Lakic [16] and Lehto [18].

The *quasiconformal mapping class group* $\mathrm{MCG}(R)$ is a group of all homotopy classes $[g]$ of quasiconformal automorphisms g of R, where the homotopy is again relative to the boundary at infinity $\partial_\infty R$ if it is not empty. Each element $[g]$ is called a mapping class and acts on $T(R)$ from the left such that $[g]_*\colon [f] \mapsto [f \circ g^{-1}]$. It is evident from the definition that $\mathrm{MCG}(R)$ acts on $T(R)$ isometrically with respect to the Teichmüller distance. It also acts biholomorphically on $T(R)$. Let $\iota\colon \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$ be the homomorphism defined by $[g] \mapsto \gamma = [g]_*$, where $\mathrm{Aut}(T(R))$ denotes the group of all isometric and biholomorphic automorphisms of $T(R)$. The image $\mathrm{Im}\,\iota \subset \mathrm{Aut}(T(R))$ is called the *Teichmüller modular group* and denoted by $\mathrm{Mod}(R)$. Except for a few low-dimensional cases, ι is injective. In particular, if R is topologically infinite, then ι is always injective. This was first proved by Earle, Gardiner, and Lakic [6], and another proof was given by Epstein [8]. Moreover, ι is surjective except for the case of $\dim T(R) = 1$, which was finally proved by Markovic [19] after a series of pioneering studies. Hence, when there is no confusion, we identify $\mathrm{MCG}(R)$ with $\mathrm{Mod}(R) = \mathrm{Aut}(T(R))$ if R is topologically infinite.

The group of all conformal automorphisms of R is denoted by $\mathrm{Conf}(R)$. Since each element of $\mathrm{Conf}(R)$ determines a mapping class of R and each mapping class contains at most one conformal automorphism, we can identify $\mathrm{Conf}(R)$ with a subgroup of $\mathrm{MCG}(R)$. In general, for each $p = [f] \in T(R)$, the group of all conformal automorphisms of R_p is denoted by $\mathrm{Conf}(R_p)$ and the mapping classes in $f^{-1}\,\mathrm{Conf}(R_p) f$ determine a subgroup $\mathrm{MCG}_p(R)$ of $\mathrm{MCG}(R)$. If $[g]$ belongs to $\mathrm{MCG}_p(R)$ for some $p \in T(R)$, we say that $[g]$ is a *conformal mapping class*. Note that the group $\mathrm{MCG}_p(R)$ itself is determined by the Teichmüller class p, but the correspondence between the elements in $\mathrm{Conf}(R_p)$ and $\mathrm{MCG}_p(R)$ depends on the homotopy class of f; only the conjugacy class is well defined by the Teichmüller class p. We denote the isomorphism defined by the inverse of this correspondence by

$$e_f \colon \mathrm{MCG}_p(R) \longrightarrow \mathrm{Conf}(R_p).$$

Furthermore, under the identification $\iota \colon \mathrm{MCG}(R) \to \mathrm{Mod}(R)$, the subgroup $\mathrm{MCG}_p(R)$ is identified with the isotropy (stabilizer) subgroup $\mathrm{Stab}(p)$ of $\mathrm{Mod}(R)$ for $p \in T(R)$. We remark that, since the action of $\mathrm{Mod}(R)$ on $T(R)$ is not necessarily transitive, isotropy subgroups are not conjugate to each other in general.

Teichmüller spaces can be realized in certain Banach spaces by the Bers embedding. For an arbitrary hyperbolic Riemann surface R, take a torsion-free Fuchsian group H acting on the upper half-plane model $\mathbb{U}$ of the hyperbolic plane such that $\mathbb{U}/H = R$. For an element $p = [f]$ of the Teichmüller space $T(R)$, lift the quasiconformal homeomorphism f to $\mathbb{U}$ such that it extends to a quasiconformal automorphism F of the Riemann sphere $\widehat{\mathbb{C}}$ mapping the lower half-plane $\mathbb{L}$ conformally. Then, the Schwarzian derivative $\varphi(z) = \mathcal{S}_F(z)$ of the restriction of F to $\mathbb{L}$ is a holomorphic function satisfying the automorphic condition

$$(h_*\varphi)(z) := \varphi(h^{-1}(z))(h^{-1})'(z)^2 = \varphi(z)$$

for every $h \in H$ and the norm condition

$$\|\varphi\|_B := \sup \rho^{-2}(z)|\varphi(z)| \le \frac{3}{2}$$

for the hyperbolic metric $\rho(z)|dz|$ on $\mathbb{L}$. Let $B(H)$ be the Banach space of all holomorphic functions φ on $\mathbb{L}$ satisfying the automorphic condition for H and $\|\varphi\|_B < \infty$. Then, the correspondence $\beta \colon T(R) \to B(H)$ by $[f] \mapsto \mathcal{S}_F$ gives a homeomorphism of $T(R)$ onto a bounded contractible domain in $B(H)$ containing the origin, which is called the *Bers embedding*.

The Banach space $B(H)$ is a subspace of the Banach space $B(1)$ of all holomorphic functions φ on L with $\|\varphi\|_B < \infty$. For a conformal automorphism of $R = \mathbb{U}/H$, its lift to $\mathbb{U}$ is the restriction of a Möbius transformation g of $\widehat{\mathbb{C}}$, which is also regarded as a conformal automorphism of $\mathbb{L}$. Then, g belongs to the normalizer $N(H)$ of H in $\mathrm{Conf}(\mathbb{L})$, and vice versa. Thus, we identify the group $\mathrm{Conf}(R)$ of all conformal automorphisms of R with the quotient group $N(H)/H$. It is known that $N(H)$ is discrete if H is non-elementary. Consequently, we see that $\mathrm{Conf}(R)$ is also discrete. For every $g \in N(H)$, the linear isometry $g_*\colon B(1) \to B(1)$ keeps the subspace $B(H)$ invariant. For any subgroup G of $\mathrm{Conf}(R)$, there is a subgroup $\widehat{H}$ of $N(H)$ containing H such that $\widehat{H}/H$ is isomorphic to G. Then, $B(\widehat{H})$ coincides with a subspace consisting of all elements in $B(H)$ that are fixed by g_* for any lift $g \in N(H)$ of each conformal automorphism of R.

The Teichmüller space $T = T(\mathbb{U})$ is called the *universal Teichmüller space*. It is known (see [18]) that, for any Riemann surface $R = \mathbb{U}/H$, the Bers embedding $\beta(T(R)) \subset B(H)$ coincides with $\beta(T) \cap B(H)$, where $\beta(T) \subset B(1)$ is the Bers embedding of the universal Teichmüller space T. The covering relation $R_2 \to R_1$ of Riemann surfaces gives the inclusion relation $H_1 \supset H_2$ of their Fuchsian groups. Hence, $B(H_1) \subset B(H_2)$ induces the inclusion relation $T(R_1) \subset T(R_2)$ of the Teichmüller spaces via the Bers embedding. Moreover, for a subgroup G of $\mathrm{Conf}(R)$, we can consider the Teichmüller space $T(R/G)$ of the orbifold R/G. The image of $T(R/G)$ under the Bers embedding $\beta\colon T(R) \to B(H)$ coincides with $\beta(T) \cap B(\widehat{H})$, where $\widehat{H}$ is the intermediate subgroup between H and $N(H)$ satisfying $\widehat{H}/H \cong G$. If we identify $G \subset \mathrm{Conf}(R)$ with a subgroup of $\mathrm{MCG}(R)$ and define a subgroup $\Gamma = \iota(G)$ of $\mathrm{Mod}(R)$, then the subspace $T(R/G)$ coincides with a locus $\mathrm{Fix}(\Gamma)$ of $T(R)$ consisting of the points fixed by all $\gamma \in \Gamma$.

With regard to the properness of the inclusion relation stated above, we can show the following lemma as a consequence of Theorem 1 in [22].

Lemma 2.1. *Let G_1 and G_2 be subgroups of* $\mathrm{Conf}(R)$ *for a Riemann surface R. If $G_1 \supsetneqq G_2$ and if the orbifold R/G_2 is of non-exceptional type, then $T(R/G_1) \subsetneqq T(R/G_2)$. In particular, if* $\dim T(R/G_2) \geq 4$ *or if the index $[G_1 : G_2]$ is sufficiently large, then $T(R/G_1) \subsetneqq T(R/G_2)$ is satisfied.*

Proof. We can choose Fuchsian groups H_1 and H_2 such that $\mathbb{U}/H_1 = R/G_1$, $\mathbb{U}/H_2 = R/G_2$ with $H_1 \supsetneqq H_2$ and $[H_1 : H_2] = [G_1 : G_2]$. If R/G_2 is of non-exceptional type, then H_2 and hence H_1 are non-exceptional. By Theorem 1 in [22], we have $B(H_1) \subsetneqq B(H_2)$; thus, $T(\mathbb{U}/H_1) \subsetneqq T(\mathbb{U}/H_2)$ follows. This proves the first statement. Exceptional Fuchsian groups are listed (Proposition 1 in [22]); in particular, we see that the orbifold $R/G_2 = \mathbb{U}/H_2$ is of exceptional type only when $0 \leq \dim T(R/G_2) \leq 3$. In this case, e.g., if $[G_1 : G_2] > 84$, then $T(R/G_1) \subsetneqq T(R/G_2)$ can be verified directly. □

3. Geometry of hyperbolic surfaces

The hyperbolic geometrical aspects of Riemann surfaces reflect certain properties of Teichmüller spaces and their modular groups. In this section, we prepare several assertions concerning the geometry of topologically infinite Riemann surfaces, which are utilized later. Let d_h denote the hyperbolic distance on the hyperbolic plane $\mathbb{H}$ as well as on a hyperbolic Riemann surface $R = \mathbb{H}/H$.

Let c be a free homotopy class of non-trivial, non-cuspidal, simple closed curves on R, and let be $\mathcal{S}(R)$ the family of all such free homotopy classes. We always ignore the orientation of c and identify c and c^{-1}. In each class c in $\mathcal{S}(R)$, there is a unique geodesic representative, which we denote by the same letter c. Let $\ell(c)$ be the geodesic length of the free homotopy class c on R. By fixing an arbitrary $c \in \mathcal{S}(R)$, we have a function $\ell_p(c) := \ell(f(c))$ on the Teichmüller space $T(R)$, where $p = [f] \in T(R)$ is the Teichmüller class of a quasiconformal homeomorphism f and $f(c)$ is the corresponding free homotopy class on $R_p = f(R)$. This is called the length function. We remark that, even though the free homotopy class $f(c)$ is determined by the homotopy class of f, its geodesic length is well defined by the Teichmüller class p.

By taking the union over all $c \in \mathcal{S}(R)$, we have a family of the lengths of all simple closed geodesics on R_p (counting multiplicity). Moreover, we define the closure of the set of their logarithmic lengths as

$$\mathrm{LS}(p) = \mathrm{Cl}\,\{\log \ell_p(c) \mid c \in \mathcal{S}(R)\} \subset \mathbb{R},$$

and we call it the *length spectrum* for $p \in T(R)$. Actually, $\mathrm{LS}(p)$ is determined by the underlying complex structure of p. If R is topologically finite (i.e., $\pi_1(R)$ is finitely generated), then the lengths of all simple closed geodesics are known to be discrete; hence, so is the length spectrum $\mathrm{LS}(p)$. In fact, the lengths of all closed geodesics that are not necessarily simple are also discrete (see Buser [5]).

The length spectrum defines a new distance on $T(R)$. For $p, q \in T(R)$, set

$$d_{\mathrm{LS}}(p,q) := \sup\{\,|\log \ell_p(c) - \log \ell_q(c)| \mid c \in \mathcal{S}(R)\},$$

which is called the length spectrum distance. It is known that $d_{\mathrm{LS}}(p,q)$ satisfies the axiom of distance. Since $\mathrm{LS}(p)$ is determined by the underlying complex structure, the distance d_{LS} is invariant under the action of $\mathrm{Mod}(R)$. The following formula attributed to Sorvali [30] and Wolpert [32] gives the inequality $d_{\mathrm{LS}}(p,q) \le d_T(p,q)$ between the Teichmüller distance and the length spectrum distance.

Proposition 3.1. *Let $f: R \to R'$ be a K-quasiconformal homeomorphism for $K \ge 1$. Then, for every simple closed geodesic c, the geodesic lengths satisfy*

$$\frac{1}{K}\ell(c) \le \ell(f(c)) \le K\ell(c).$$

It follows that

$$e^{-d_T(p,q)}\ell_p(c) \le \ell_q(c) \le e^{d_T(p,q)}\ell_p(c)$$

for each $c \in \mathcal{S}(R)$ and for any p and q in $T(R)$.

On the other hand, Basmajian [2] provided the following estimate for the distance between two simple closed geodesics.

Proposition 3.2. *Let $f: R \to R'$ be a K-quasiconformal homeomorphism for $K \geq 1$. Then, for any simple closed geodesics c and c', the distance between the corresponding simple closed geodesics $f(c)$ and $f(c')$ satisfies*

$$\frac{1}{K} d_h(c, c') - b \leq d_h(f(c), f(c')) \leq K d_h(c, c') + b,$$

where $b \geq 0$ is a constant depending only on K continuously such that $b \to 0$ as $K \to 1$ monotonously.

For a simple closed geodesic c on R, a subdomain $\{a \in R \mid d_h(a, c) < \omega\}$ is called a *collar* of c with width $\omega > 0$ if it becomes an annular neighborhood of c. The collar lemma asserts that a collar always exists for the width

$$\omega = \operatorname{arcsinh} \frac{1}{\sinh(\ell(c)/2)},$$

which we call the *canonical collar*, denoted by $A^*(c)$. Actually, the collar lemma further claims that, if we take a family of mutually disjoint simple closed geodesics, then their canonical collars are mutually disjoint (see [5]).

We may assume that the simple closed geodesic c corresponds to a hyperbolic element $h(z) = kz$ ($k > 1$) acting on the upper half-plane model $\mathbb{U}$ of the hyperbolic plane. Let $A(c) = \mathbb{U}/\langle h \rangle$ be the annular cover of $R = \mathbb{U}/H$, where H is the Fuchsian group containing h. Let (l, θ) be the (logarithmic) polar coordinate of $\mathbb{U}$; the canonical coordinate (x, y) for $y > 0$ and the polar coordinate (l, θ) for $0 < \theta < \pi$ are transformed by $x + iy = \exp(l + i\theta)$. Since the polar coordinate is conformal to the canonical coordinate, we have a conformal coordinate (l, θ) on A, where l is taken modulo $\ell(c) = \log k$. Let

$$A^{\psi}(c) = \left\{(l, \theta) \in A(c) \;\middle|\; \frac{\pi}{2} - \frac{\psi}{2} < \theta < \frac{\pi}{2} + \frac{\psi}{2}\right\}$$

be a subdomain of $A(c)$ with a positive angle ψ ($\leq \pi$). The collar lemma asserts that $A^{\psi}(c)$ is conformally embedded in R by the covering projection $A(c) \to R$ for any angle $\psi \leq 2\arctan(\sinh \omega)$, where $\sinh \omega = 1/\sinh(\ell(c)/2)$. Hence, every collar in the canonical collar $A^*(c) \subset R$ can be identified with $A^{\psi}(c) \subset A(c)$ for an angle ψ.

For an annular domain A, the conformal modulus $m(A)$ is defined to be $\log r$ if A is conformally mapped onto $\{z \in \mathbb{C} \mid 1 < |z| < r\}$. Then, the *conformal modulus* $m(A^{\psi}(c))$ of $A^{\psi}(c)$ is $2\pi\psi/\ell(c)$. In particular, $m(A(c)) = 2\pi^2/\ell(c)$ and

$$m(A^*(c)) = \frac{4\pi}{\ell(c)} \arctan \frac{1}{\sinh(\ell(c)/2)}.$$

Let $f: R \to R'$ be a K-quasiconformal homeomorphism. Since f lifts to a K-quasiconformal homeomorphism $\tilde{f}: A(c) \to A(f(c))$ between the annular covers of R and R' and since any K-quasiconformal homeomorphism between annuli changes their moduli at most by a factor of K, we have

$$K^{-1}m(A(c)) \le m(A(f(c))) \le Km(A(c)).$$

This is equivalent to the estimate given in Proposition 3.1.

Let $\mathcal{F}$ be the family of all closed rectifiable curves in an annular domain A that separate the two boundary components of A. Then, the *extremal length* for $\mathcal{F}$ is defined by

$$L(\mathcal{F}) = \sup_{\rho} \frac{(\inf_{\beta\in\mathcal{F}} \int_{\beta} \rho(z)|dz|)^2}{\iint_A \rho(z)^2 dxdy},$$

where the supremum is taken over all measurable conformal metrics $\rho(z)|dz|$ on A. When $\rho(z)|dz|$ attains the supremum, it is called an *extremal metric*. It is known that the extremal length $L(\mathcal{F})$ for the curve family $\mathcal{F}$ in $A(c)$ is directly proportional to the geodesic length $\ell(c)$ and hence inversely proportional to the conformal modulus $m(A(c))$.

We refine Proposition 3.1 as follows. The argument originally presented in this paper has been further developed in [11], [24], and [14].

Theorem 3.3. *Let $\tilde{f}$ be a lift of a K-quasiconformal homeomorphism f of R to its annular cover $A(c)$ with respect to $c \in \mathcal{S}(R)$. If $\tilde{f}$ is conformal on $A^{\psi}(c)$, then*

$$\frac{\pi}{K(\pi-\psi)+\psi}\ell(c) \le \ell(f(c)) \le \frac{\pi}{K^{-1}(\pi-\psi)+\psi}\ell(c)$$

is satisfied.

Proof. For the first inequality in the statement, since the geodesic length $\ell(c)$ is proportional to the extremal length $L(\mathcal{F})$ for the family $\mathcal{F}$ of all closed rectifiable curves in $A(c)$ that separate the two boundary components of $A(c)$, we consider $L(\mathcal{F})$ instead. The extremal metric $\rho_0(z)|dz|$ on $A(c)$ for this extremal length is the Euclidean metric with respect to the polar coordinate (l, θ).

Set a conformal metric $\rho'(\zeta)|d\zeta|$ on $A(f(c))$ with respect to the canonical coordinate $\zeta = \xi + i\eta$ on $\mathbb{U}$ by

$$\rho'(\zeta) := \frac{\rho_0(\tilde{f}^{-1}(\zeta))}{|\partial\tilde{f}(\tilde{f}^{-1}(\zeta))| - |\bar{\partial}\tilde{f}(\tilde{f}^{-1}(\zeta))|}.$$

Since $|d\zeta| \ge (|\partial\tilde{f}(z)| - |\bar{\partial}\tilde{f}(z)|)|dz|$, we have

$$\int_{\tilde{f}(\beta)} \rho'(\zeta)|d\zeta| \ge \int_{\beta} \rho_0(z)|dz|$$

for an arbitrary curve $\beta \in \mathcal{F}$. On the other hand,

$$\begin{aligned}\iint_{A(f(c))} \rho'(\zeta)^2 d\xi d\eta &= \iint_{A(c)} \rho_0(z)^2 \frac{|\partial \tilde{f}(z)| + |\bar{\partial} \tilde{f}(z)|}{|\partial \tilde{f}(z)| - |\bar{\partial} \tilde{f}(z)|} dxdy \\ &\leq \iint_{A^{\psi}(c)} \rho_0(z)^2 dxdy + K \iint_{A(c)-A^{\psi}(c)} \rho_0(z)^2 dxdy \\ &= \Big(\frac{\psi}{\pi} + K\frac{\pi - \psi}{\pi}\Big) \iint_{A(c)} \rho_0(z)^2 dxdy.\end{aligned}$$

Thus, we have

$$L(f(\mathcal{F})) \geq \frac{\pi}{K(\pi - \psi) + \psi} L(\mathcal{F}),$$

which yields the first inequality.

For the second inequality in the statement, we consider the modulus $m(A(c))$ instead, as it is inversely proportional to the geodesic length $\ell(c)$. By well-known inequalities on modulus (see Vasil′ev [31]), we have

$$m(\tilde{f}(A^{\psi}(c))) = m(A^{\psi}(c)) = \frac{\psi}{\pi} m(A(c));$$

$$m(\tilde{f}(A(c) - A^{\psi}(c))) \geq K^{-1} m(A(c) - A^{\psi}(c)) = \frac{K^{-1}(\pi - \psi)}{\pi} m(A(c));$$

$$m(A(f(c))) \geq m(\tilde{f}(A^{\psi}(c))) + m(\tilde{f}(A(c) - A^{\psi}(c))).$$

Then,

$$m(A(f(c))) \geq \frac{K^{-1}(\pi - \psi) + \psi}{\pi} m(A(c)),$$

which yields the second inequality. □

Corollary 3.4. *Let $\omega = d_h(c, E)$ be the hyperbolic distance between a simple closed geodesic c and a compact subset E in R. If f is a K-quasiconformal homeomorphism of R that is conformal on $R - E$, then*

$$\frac{1}{\alpha} \ell(c) \leq \ell(f(c)) \leq \alpha' \ell(c)$$

is satisfied for constants

$$\alpha = K + (1 - K)\frac{2}{\pi} \arctan(\sinh \omega) \geq 1,$$

$$\alpha' = \Big[\frac{1}{K} + \Big(1 - \frac{1}{K}\Big)\frac{2}{\pi} \arctan(\sinh \omega)\Big]^{-1} \geq 1$$

depending only on K and ω. These constants tend to 1 as $\omega \to \infty$.

Proof. If the distance between c and E is ω, then no lift of E intersects the annulus $A^{\psi}(c)$ in the annular cover $A(c)$ of R, where $\psi = 2\arctan(\sinh\omega)$. Then, Theorem 3.3 yields the assertion. □

Grafting by an amount $\phi > 0$ (or ϕ-grafting in brief) with respect to $c \in \mathcal{S}(R)$ is a procedure for inserting an annulus after cutting a hyperbolic surface R along a simple closed geodesic c. Here, the inserted annulus occupies the portion $\mathbb{R}/\ell(c)\times(-\phi/2, \phi/2)$ in the original polar coordinate (l, θ) on the canonical collar $A^*(c)$. The resulting Riemann surface is denoted by $R(c, \phi)$ and the extended collar in $R(c, \phi)$ is defined by

$$A^*(c, \phi) := \mathbb{R}/\ell(c) \times \Big(-\frac{\psi+\phi}{2}, \frac{\psi+\phi}{2}\Big), \quad \psi = 2\arctan\frac{1}{\sinh(\ell(c)/2)}.$$

A canonical quasiconformal homeomorphism $\chi_{c,\phi}\colon R \to R(c, \phi)$ for this grafting, which itself is called a grafting, is defined by linearly stretching $A^*(c)$ to $A^*(c, \phi)$ along the direction of θ and by leaving $R - A^*(c)$ identical. The maximal dilatation $K(\chi_{c,\phi})$ of $\chi_{c,\phi}$ is $(\psi + \phi)/\psi$.

We estimate the maximal dilatation $K(f)$ of an extremal quasiconformal homeomorphism $f\colon R \to R(c, \phi)$ homotopic to $\chi_{c,\phi}$. A lower estimate is given by an upper estimate of the geodesic length $\ell(f(c))$. This has been proved by McMullen [26] as follows.

Lemma 3.5. *Let $\chi_{c,\phi}\colon R \to R(c, \phi)$ be the ϕ-grafting with respect to $c \in \mathcal{S}(R)$. Then, the geodesic length $\ell_p(c)$ for $p = [\chi_{c,\phi}] \in T(R)$ satisfies*

$$\ell_p(c) \le \frac{\pi}{\pi+\phi}\ell(c).$$

Hence, for an extremal quasiconformal homeomorphism $f\colon R \to R(c, \phi)$ homotopic to $\chi_{c,\phi}$, the maximal dilatation $K(f)$ satisfies

$$\frac{\pi+\phi}{\pi} \le K(f) \le \frac{\psi+\phi}{\psi},$$

where $\psi = 2\arctan\{1/\sinh(\ell(c)/2)\}$.

Proof. Consider the annular cover $A(c)$ of R with respect to c. If we graft $A(c)$ along c by an amount $\phi > 0$, we have a new annulus $A(c, \phi)$. Its conformal modulus is $m(A(c, \phi)) = 2\pi(\pi + \phi)/\ell(c)$. Hence, the geodesic length of the core curve c in the hyperbolic annulus $A(c, \phi)$ is equal to

$$\frac{2\pi^2}{m(A(c,\phi))} = \frac{\pi}{\pi+\phi}\ell(c).$$

By considering projective universal covers of $R(c, \phi)$ and $A(c, \phi)$ such that the former contain the latter, we see from the monotonicity of the hyperbolic metric that the geodesic length $\ell_p(c)$ of c in $R(c,\phi)$ is not greater than that of c in $A(c,\phi)$ (see [26]). Hence, we have the first statement. Then, by Proposition 3.1, we have

$$K(f) \geq \frac{\ell(c)}{\ell_p(c)} \geq \frac{\pi+\phi}{\pi},$$

which is the lower estimate in the second statement. The upper estimate obviously follows from $K(f) \leq K(\chi_{c,\phi})$. □

Remark. In Lemma 3.5, we consider the grafting with respect to a single simple closed geodesic. However, if a quasiconformal homeomorphism f is obtained by multiple graftings with respect to mutually disjoint, and possibly infinitely many, simple closed geodesics $\{c_i\}$ by amounts $\{\phi_i\}$, then we have the same length inequality as that in Lemma 3.5 for each i by the same proof.

Next, we consider moderate assumptions concerning the geometry on hyperbolic Riemann surfaces, which make the analysis of Teichmüller modular groups easier. Typical conditions of this type are as follows.

Definition. We say that a hyperbolic Riemann surface R satisfies *the lower boundedness condition* if the injectivity radius at every point of R is uniformly bounded away from 0 except in horocyclic cusp neighborhoods of area 1. We say that R satisfies *the upper boundedness condition* if there exists a connected subsurface R^* of R such that the injectivity radius at every point of R^* is uniformly bounded from above and the inclusion $R^* \to R$ induces a surjection $\pi_1(R^*) \to \pi_1(R)$. We say that R satisfies *the bound geometry condition* if both the lower and the upper boundedness conditions are satisfied and if the boundary at infinity $\partial_\infty R$ is empty.

These conditions are quasiconformally invariant; hence, we may regard them as conditions for the Teichmüller space $T(R)$. For example, a non-universal normal cover of an analytically finite hyperbolic Riemann surface satisfies the bound geometry condition.

The virtue of assuming the bounded geometry condition lies in the next theorem, which was proved by Fujikawa, Shiga, and Taniguchi [15] and [10]. For any $c \in \mathcal{S}(R)$, we define a subgroup of $\mathrm{MCG}(R)$ consisting of all mapping classes that preserve c:

$$\mathrm{MCG}_c(R) = \{[g] \in \mathrm{MCG}(R) \mid g(c) \sim c\},$$

where $\sim$ denotes the free homotopy equivalence. The corresponding subgroup $\iota(\mathrm{MCG}_c(R))$ of $\mathrm{Mod}(R)$ is denoted by $\mathrm{Mod}_c(R)$.

Theorem 3.6. *Assume that a Riemann surface R satisfies the bounded geometry condition. Then, no sequence of distinct elements $\gamma_n \in \mathrm{Mod}_c(R)$ for $c \in \mathcal{S}(R)$ satisfies $\gamma_n(p) \to p$ as $n \to \infty$ for some $p \in T(R)$.*

Thus, under the bounded geometry condition, $\mathrm{Mod}_c(R)$ acts on $T(R)$ discontinuously. A precise definition for this property will be given in the next section.

4. Isometries on complete metric spaces

Let $X = (X, d)$ be a complete metric space with a distance d in general, and let $\mathrm{Isom}(X)$ be the group of all isometric automorphisms of X. For a subgroup $\Gamma \subset \mathrm{Isom}(X)$, the orbit of $x \in X$ under Γ is denoted by $\Gamma(x)$ and the isotropy (stabilizer) subgroup of $x \in X$ in Γ is denoted by $\mathrm{Stab}_\Gamma(x)$. For an element $\gamma \in \mathrm{Isom}(X)$, the set of all fixed points of γ is denoted by $\mathrm{Fix}(\gamma)$.

For a subgroup $\Gamma \subset \mathrm{Isom}(X)$ and for a point $x \in X$, a point $y \in X$ is a *limit point* of x for Γ if there exists a sequence $\{\gamma_n\}$ of distinct elements of Γ such that $\gamma_n(x)$ converges to y as $n \to \infty$. The set of all limit points of x for Γ is denoted by $\Lambda(\Gamma, x)$, and the *limit set* for Γ is defined by $\Lambda(\Gamma) = \bigcup_{x \in X} \Lambda(\Gamma, x)$. From this definition, it is clear that if Γ' is of finite index in Γ, then $\Lambda(\Gamma') = \Lambda(\Gamma)$. It is said that $x \in X$ is a *recurrent point* for Γ if $x \in \Lambda(\Gamma, x)$, and the set of all recurrent points for Γ is denoted by $\mathrm{Rec}(\Gamma)$. It is evident that $\mathrm{Rec}(\Gamma) \subset \Lambda(\Gamma)$, and these sets are Γ-invariant.

The following fact appeared in Fujikawa [9] and [13].

Proposition 4.1. *For a subgroup* $\Gamma \subset \mathrm{Isom}(X)$*, the limit set* $\Lambda(\Gamma)$ *coincides with* $\mathrm{Rec}(\Gamma)$ *and it is a closed set. Moreover,* $x \in X$ *is a limit point of* Γ *if and only if either the orbit* $\Gamma(x)$ *is not discrete or the isotropy subgroup* $\mathrm{Stab}_\Gamma(x)$ *consists of infinitely many elements.*

A limit point $x \in \Lambda(\Gamma)$ is called a *generic limit point* if $\Gamma(x)$ is not a discrete set and a *stabilized limit point* if $\mathrm{Stab}_\Gamma(x)$ is infinite. The set of all generic limit points is denoted by $\Lambda_0(\Gamma)$ and the set of all stabilized limit points is denoted by $\Lambda_\infty(\Gamma)$. By Proposition 4.1, we see that $\Lambda(\Gamma) = \Lambda_0(\Gamma) \cup \Lambda_\infty(\Gamma)$; however, the intersection $\Lambda_0(\Gamma) \cap \Lambda_\infty(\Gamma)$ can be non-empty. Furthermore, $\Lambda_\infty(\Gamma)$ is divided into two disjoint subsets $\Lambda^1_\infty(\Gamma)$ and $\Lambda^2_\infty(\Gamma)$ as in [9]. A limit point $x \in \Lambda_\infty(\Gamma)$ belongs to $\Lambda^1_\infty(\Gamma)$ if there is an element of infinite order in $\mathrm{Stab}_\Gamma(x)$; otherwise, it belongs to $\Lambda^2_\infty(\Gamma)$. In other words, $\Lambda^1_\infty(\Gamma) = \bigcup \mathrm{Fix}(\gamma)$, where the union is taken over all elements $\gamma \in \Gamma$ of infinite order.

Here, we introduce discontinuity and a weaker property defined as stability for the action of Γ.

Definition. Let Γ be a subgroup of $\mathrm{Isom}(X)$. We say that Γ acts at $x \in X$

(a) *discontinuously* if $\Gamma(x)$ is discrete and $\mathrm{Stab}_\Gamma(x)$ is finite;

(b) *weakly discontinuously* if $\Gamma(x)$ is discrete;

(c) *stably* if $\Gamma(x)$ is closed and $\mathrm{Stab}_\Gamma(x)$ is finite;

(d) *weakly stably* if $\Gamma(x)$ is closed.

If Γ acts at every point x in X (weakly) discontinuously or (weakly) stably, then we say that Γ acts on X (weakly) discontinuously or stably. The set of points $x \in X$ where Γ acts discontinuously is denoted by $\Omega(\Gamma)$ and called the *region of discontinuity* for Γ. The set of points $x \in X$ where Γ acts stably is denoted by $\Phi(\Gamma)$ and called the *region of stability* for Γ.

The inclusion relation $\Omega(\Gamma) \subset \Phi(\Gamma)$ is immediately known from the corresponding definitions. Furthermore, it is clear that, if $\Gamma_1 \subset \Gamma_2$, then $\Omega(\Gamma_1) \supset \Omega(\Gamma_2)$. However, for the region of stability, $\Gamma_1 \subset \Gamma_2$ does not necessarily imply that $\Phi(\Gamma_1) \supset \Phi(\Gamma_2)$. A counter-example will be given in the next section.

The discontinuity of the action is usually defined in another way (as condition (2) below, which is equivalent to proper discontinuity if X is locally compact); however, as stated by the following proposition, these definitions are all equivalent (see [9]).

Proposition 4.2. *For a subgroup* $\Gamma \subset \mathrm{Isom}(X)$ *and a point* $x \in X$*, the following conditions are equivalent*:

(1) Γ *acts at* x *discontinuously*;

(2) *there exists an open ball* U *centered at* x *such that the number of elements* $\gamma \in \Gamma$ *satisfying* $\gamma(U) \cap U \neq \emptyset$ *is finite*;

(3) x *is not a limit point of* Γ.

Hence, the region of discontinuity $\Omega(\Gamma)$ *is the complement of the limit set* $\Lambda(\Gamma)$*, which is an open subset of* X.

Similar statements hold for weak discontinuity.

Proposition 4.3. *For a subgroup* $\Gamma \subset \mathrm{Isom}(X)$ *and a point* $x \in X$*, the following conditions are equivalent*:

(1) Γ *acts at* x *weakly discontinuously*;

(2) *there exists an open ball* U *centered at* x *such that* $\gamma(U) = U$ *for every* $\gamma \in \mathrm{Stab}_\Gamma(x)$ *and* $\gamma(U) \cap U = \emptyset$ *for every* $\gamma \in \Gamma - \mathrm{Stab}_\Gamma(x)$;

(3) x *is not a generic limit point of* Γ.

Discontinuity and stability have the obvious inclusion relation mentioned above. The following theorem states that the converse inclusion holds under a certain countability assumption. This fact is based on the Baire category theorem and uncountability of perfect closed sets.

Theorem 4.4. *Assume that* $\Gamma \subset \mathrm{Isom}(X)$ *contains a subgroup* Γ_0 *of countable index, i.e., the cardinality of the cosets* Γ/Γ_0 *is countable, such that* Γ_0 *acts at* $x \in X$ *weakly discontinuously. If* Γ *acts at* x (*weakly*) *stably, then* Γ *acts at* x (*weakly*) *discontinuously. In particular, this claim is always satisfied if* Γ *itself is countable.*

Proof. We consider the coset decomposition of Γ by Γ_0:

$$\Gamma = \gamma_1\Gamma_0 \sqcup \gamma_2\Gamma_0 \sqcup \gamma_3\Gamma_0 \sqcup \cdots .$$

Then, $\Gamma(x) = \bigcup_{i=1}^{\infty} \gamma_i\Gamma_0(x)$, where each $\gamma_i\Gamma_0(x)$ is discrete and especially closed because Γ_0 acts at x weakly discontinuously. Since $\Gamma(x)$ is closed by assumption, we can regard $\Gamma(x)$ as a complete metric space by the restriction of the distance d on X. By the Baire category theorem, there exists an integer $i \in \mathbb{N}$, say $i = 1$, such that $\gamma_1\Gamma_0(x)$ has an interior point y in $\Gamma(x)$. Since $\gamma_1\Gamma_0(x)$ is discrete, y is an isolated point of $\Gamma(x)$. By the group invariance, this implies that $\Gamma(x)$ is discrete. □

While the region of discontinuity $\Omega(\Gamma)$ is always an open set, the region of stability $\Phi(\Gamma)$ becomes an open set under a certain condition upon Γ. This is also based on the Baire category theorem.

Theorem 4.5. *If* $\Gamma \subset \mathrm{Isom}(X)$ *contains a subgroup* Γ_0 *of countable index such that* Γ_0 *acts on* X *stably, then the region of stability* $\Phi(\Gamma)$ *is open. In particular, this claim is always satisfied if* Γ *itself is countable.*

Proof. Take a point $x \in \Phi(\Gamma)$ and consider the isotropy subgroup $H_x = \mathrm{Stab}_\Gamma(x)$, which is a finite group. We consider the two-sided coset decomposition of Γ by Γ_0 and H_x:

$$\Gamma = \Gamma_0\gamma_1 H_x \sqcup \Gamma_0\gamma_2 H_x \sqcup \Gamma_0\gamma_3 H_x \sqcup \cdots .$$

Since Γ_0 is of countable index in Γ, we see that the cardinality of the cosets $\Gamma_0\backslash\Gamma/H_x$ is also countable.

According to this coset decomposition, the closed orbit $\Gamma(x)$ is decomposed into the disjoint union

$$\Gamma(x) = \Gamma_0\gamma_1(x) \sqcup \Gamma_0\gamma_2(x) \sqcup \Gamma_0\gamma_3(x) \sqcup \cdots .$$

Here, each $\Gamma_0\gamma_i(x)$ is closed because Γ_0 acts on X stably. Then, by the Baire category theorem, at least one orbit, say $\Gamma_0\gamma_1(x)$, has an interior point with respect to the relative topology on $\Gamma(x)$. This means that there exists a neighborhood $U \subset X$ of $\gamma_1'(x)$ for some $\gamma_1' \in \Gamma_0\gamma_1$ satisfying $U \cap \Gamma(x) = U \cap \Gamma_0\gamma_1(x)$.

Since the action is isometric, we can choose a neighborhood V of x and a smaller neighborhood $U' \subset U$ of $\gamma_1'(x)$ such that $\gamma_1'(y) \in U'$ and $U' \cap \Gamma(y) = U' \cap \Gamma_0\gamma_1 H_x(y)$ for every $y \in V$. Here, $\Gamma_0\gamma_1 H_x(y)$ is closed because it is the finite union of the closed sets $\Gamma_0\gamma_1\gamma(y)$ taken over $\gamma \in H_x$. In other words, the orbit $\Gamma(y)$ is closed if it is restricted to U'. However, by the group invariance, this implies that the entire orbit $\Gamma(y)$ is closed itself.

Moreover, we see that $H_y = \mathrm{Stab}_\Gamma(y)$ is finite. Indeed, every element $\gamma' \in H_y$ satisfies $\gamma_1'\gamma'(y) \in U'$; hence, γ' is in $(\gamma_1')^{-1}\Gamma_0\gamma_1 H_x$. In particular, $\gamma_1'(y)$ is in the orbit $\Gamma_0\gamma_1\gamma(y)$ for some $\gamma \in H_x$. The stability of Γ_0 then implies that there are only finitely many choices for $\gamma_0 \in \Gamma_0$ to satisfy this relation. Hence, the number of elements γ' that belong to $(\gamma_1')^{-1}\Gamma_0\gamma_1 H_x$ is finite, which means that H_y is finite.

We have seen that, for every $y \in V$, $\Gamma(y)$ is closed and H_y is finite. This implies that $y \in \Phi(\Gamma)$; hence, $\Phi(\Gamma)$ is open. □

Next, we consider certain quotient spaces of the complete metric space (X, d) by the isometric group action. For an arbitrary subgroup Γ of $\mathrm{Isom}(X)$, we define two points x and y in X to be equivalent, which is denoted by $x \sim y$, if there exists a sequence of elements γ_n of Γ not necessarily distinct such that $\gamma_n(x)$ converges to y. In particular, all points in the same orbit of Γ are mutually equivalent. It is easy to check that this satisfies the axiom of equivalence relation, which is called the *closure equivalence*. An equivalence class coincides with the closure of the orbit $\overline{\Gamma(x)}$ of some point $x \in X$. This means that $\overline{\Gamma(x_1)} \cap \overline{\Gamma(x_2)} \neq \emptyset$ is equivalent to $\overline{\Gamma(x_1)} = \overline{\Gamma(x_2)}$ as well as $x_1 \sim x_2$.

The closure equivalence is stronger than the ordinary orbit equivalence under the group action of Γ. The ordinary quotient space by Γ is denoted by X/Γ and the quotient space by the closure equivalence is denoted by $X//\Gamma$. The projections are denoted by $\pi_1 \colon X \to X/\Gamma$ and $\pi_2 \colon X \to X//\Gamma$, respectively. Then, the projection $\bar{\pi} \colon X/\Gamma \to X//\Gamma$ is well defined by $\pi_2 \circ (\pi_1)^{-1}$. The inverse image $\bar{\pi}^{-1}(s)$ for $s \in X//\Gamma$ coincides with the closure $\overline{\{\sigma\}} \subset X/\Gamma$ for any point $\sigma \in \bar{\pi}^{-1}(s)$. Clearly, $\overline{\{\sigma\}} = \{\sigma\}$ if and only if the corresponding orbit $\Gamma(x)$ is closed for any $x \in \pi_1^{-1}(\sigma)$.

The distance d induces pseudo-distances d_1 on X/Γ and d_2 on $X//\Gamma$ as

$$d_1(\pi_1(x), \pi_1(y)) := \inf\{d(x', y') \mid x' \in \Gamma(x),\ y' \in \Gamma(y)\};$$

$$d_2(\pi_2(x), \pi_2(y)) := \inf\{d(x', y') \mid x' \in \overline{\Gamma(x)},\ y' \in \overline{\Gamma(y)}\}.$$

Here, d_2 always becomes a distance by virtue of the manner of defining the closure equivalence. Hence, $(X//\Gamma, d_2)$ is a complete metric space.

A theorem on general topology implies the following.

Proposition 4.6. *For a subgroup* $\Gamma \subset \mathrm{Isom}(X)$ *and a point* $x \in X$, *the following conditions are equivalent*:

(a) Γ *acts at* x *weakly stably*;

(b) *there exists no point* $\pi_1(y)$ *different from* $\pi_1(x)$ *such that*

$$d_1(\pi_1(x), \pi_1(y)) = 0;$$

(c) *for every point* $\pi_1(y)$ *different from* $\pi_1(x)$, *there exists a neighborhood of* $\pi_1(y)$ *that separates* $\pi_1(x)$, *or equivalently, the one-point set* $\{\pi_1(x)\}$ *is closed in* X/Γ.

Corollary 4.7. *For a subgroup $\Gamma \subset \mathrm{Isom}(X)$ and the quotient space X/Γ, the following conditions are equivalent*:

(1) *Γ acts on X weakly stably*;

(2) *the pseudo-distance d_1 on X/Γ is a distance*;

(3) *X/Γ satisfies the first separation (T_1) axiom, or equivalently, every point constitutes a closed set in X/Γ.*

In these cases, the closure equivalence is the same as the orbit equivalence; *hence, $\bar{\pi}\colon X/\Gamma \to X//\Gamma$ is a homeomorphism.*

5. Dynamics of Teichmüller modular groups and moduli spaces

For an analytically finite Riemann surface R, the Teichmüller modular group $\mathrm{Mod}(R)$ acts on $T(R)$ discontinuously. Although $\mathrm{Mod}(R)$ has fixed points on $T(R)$, each orbit is discrete and each isotropy subgroup is finite. Hence, an orbifold structure on the moduli space $M(R)$ is induced from $T(R)$ as the quotient space by $\mathrm{Mod}(R)$. However, these are not always satisfied for analytically infinite Riemann surfaces.

We introduce the concepts (limit set and so on) defined in the previous sections for the Teichmüller space $X = T(R)$ with the Teichmüller distance $d = d_T$ and the Teichmüller modular group $\mathrm{Mod}(R) \subset \mathrm{Isom}(X)$. Then, the results presented in the previous section are all applicable to this case. Moreover, the following property of $\mathrm{Mod}(R)$ enables us to draw more interesting conclusions from Theorems 4.4 and 4.5.

Theorem 5.1. *The subgroup* $\mathrm{Mod}_c(R)$ *for each $c \in \mathcal{S}(R)$ is of countable index in* $\mathrm{Mod}(R)$. *Moreover,* $\mathrm{Mod}_c(R)$ *acts stably on $T(R)$.*

Proof. Number all free homotopy classes of $\mathcal{S}(R)$ by $\{c_i\}_{i=1}^{\infty}$. For each i, consider a subset

$$\{[g] \in \mathrm{MCG}(R) \mid g(c) \sim c_i\} = [g_i] \cdot \mathrm{MCG}_c(R),$$

where $[g_i]$ is any element of $\mathrm{MCG}(R)$ satisfying $g_i(c) \sim c_i$. Since $\mathrm{MCG}(R)$ is the disjoint union of all these subsets taken over i, we get the coset decomposition of $\mathrm{MCG}(R)$ by $\mathrm{MCG}_c(R)$, whose cardinality is countable. Hence, $\mathrm{Mod}_c(R)$ is also of countable index in $\mathrm{Mod}(R)$.

For $p = [f] \in T(R)$, consider the orbit $\Gamma(p)$ for $\Gamma = \mathrm{Mod}_c(R)$. Suppose that a sequence $p_n = \gamma_n(p)$ for $\gamma_n = [g_n]_* \in \Gamma$ converges to a point $q = [f_\infty] \in T(R)$. Then, we may choose f, f_∞, and g_n in each Teichmüller and mapping class such that the maximal dilatation $K(h_n)$ of $h_n := f \circ g_n^{-1} \circ f_\infty^{-1}$ converges to 1. On the other hand, every g_n preserves the free homotopy class c. Hence, a subsequence of h_n converges locally uniformly to a quasiconformal homeomorphism

$h: f_\infty(R) \to f(R)$ such that $h \circ f_\infty(c) \sim f(c)$ and $K(h) = 1$, i.e., h is conformal. Consider a quasiconformal automorphism $g = f_\infty^{-1} \circ h^{-1} \circ f$ of R, which preserves c, and set $\gamma := [g]_* \in \Gamma$. Then, $f \circ g^{-1} = h \circ f_\infty$; thus, $\gamma(p) = [f \circ g^{-1}] = [f_\infty] = q$. This proves that the orbit $\Gamma(p)$ is closed.

Next, we consider the case where all p_n and q coincide with p in the above proof, i.e., we assume that all h_n are conformal automorphisms of R_p. They have a convergent subsequence, as we have seen above. On the other hand, $\mathrm{Conf}(R_p)$ is discrete (we have seen this claim before by the Fuchsian model, but another explanation for it is to use the fact that a conformal automorphism fixing the homotopy class of a pair of pants is the identity). This means that $\mathrm{Stab}_\Gamma(p)$ consists only of finitely many elements. Therefore, $\Gamma = \mathrm{Mod}_c(R)$ acts stably on $T(R)$. □

Remark. We call a subgroup $G \subset \mathrm{MCG}(R)$ and its representation $\Gamma = \iota(G) \subset \mathrm{Mod}(R)$ *stationary* if there exists a compact subsurface V with boundary in R such that every representative g of every mapping class $[g] \in G$ satisfies $g(V) \cap V \neq \emptyset$. The subgroup $\mathrm{Mod}_c(R)$ in Theorem 5.1 is stationary. In general, for an arbitrary stationary subgroup Γ, there exists a minimal stationary subgroup $\overline{\Gamma} \subset \mathrm{Mod}(R)$ that contains Γ and acts on $T(R)$ stably, which can be defined as the *closure* of Γ in $\mathrm{Mod}(R)$. A proof of this fact can be given similarly as in the arguments above (see also Corollary 2.24 in [25]).

By virtue of the existence of the subgroup $\mathrm{Mod}_c(R)$, Theorem 4.5 becomes the following assertion in our case.

Theorem 5.2. *The region of stability $\Phi(\Gamma)$ for $\Gamma = \mathrm{Mod}(R)$ is an open subset of $T(R)$.*

Proof. By Theorem 5.1, $\Gamma = \mathrm{Mod}(R)$ has the subgroup $\mathrm{Mod}_c(R)$ of countable index, which acts stably. Then, by Theorem 4.5, $\Phi(\Gamma)$ is open. □

If R satisfies the bounded geometry condition, then Theorem 3.6 states that $\mathrm{Mod}_c(R)$ acts on $T(R)$ discontinuously. Hence, Theorem 4.4 yields the following.

Theorem 5.3. *Assume that R satisfies the bounded geometry condition or a subgroup Γ of $\mathrm{Mod}(R)$ is countable. If Γ acts at $p \in T(R)$ (weakly) stably, then Γ acts at p (weakly) discontinuously. In other words, the stability for Γ is equivalent to the discontinuity.*

Proof. The intersection of Γ with the subgroup $\mathrm{Mod}_c(R)$ is of countable index in Γ by Theorem 5.1, and it acts on $T(R)$ discontinuously by Theorem 3.6. Hence, the stability and the discontinuity are equivalent by Theorem 4.4. □

Corollary 5.4. *If R satisfies the bounded geometry condition, then $\Phi(\Gamma) = \Omega(\Gamma)$ for every subgroup Γ of $\mathrm{Mod}(R)$.*

Note that one cannot remove the assumptions on R and Γ in Theorem 5.3. In other words, there is an example of an uncountable subgroup $\Gamma \subset \mathrm{Mod}(R)$ for some R without the bounded geometry condition that acts on $T(R)$ stably but not discontinuously.

Example. Assume that R has a sequence of mutually disjoint, simple closed geodesics $\{c_i\}_{i=1}^{\infty}$ whose geodesic lengths $\ell(c_i)$ tend to 0. Let G be a stationary subgroup of $\mathrm{MCG}(R)$ consisting of all mapping classes represented by the composition of simultaneous Dehn twists along possibly infinitely many curves in $\{c_i\}$. Set the subgroup $\iota(G)$ of $\mathrm{Mod}(R)$ by Γ. Then, the orbit $\Gamma(p)$ for every $p \in T(R)$ is closed but not discrete. Since $\mathrm{Stab}_{\Gamma}(p) = \{\mathrm{id}\}$, this group Γ acts on $T(R)$ stably but not discontinuously.

There exists a subgroup $\Gamma' \subset \Gamma$ that does not act stably on $T(R)$. Indeed, let G' be a countable subgroup of G that is generated by all Dehn twists along each c_i and set $\Gamma' = \iota(G')$. Then, Γ' does not act discontinuously either; hence, it does not act stably by Theorem 5.3. Here, Γ is actually the closure $\overline{\Gamma'}$ of Γ' in the sense of the definition in the remark above. For these groups, we see that $\Phi(\Gamma) \not\subset \Phi(\Gamma')$; thus, this is an example where the inclusion of the regions of stability does not conversely follow the inclusion of the subgroups.

The bounded geometry condition is satisfied for any non-universal normal cover R of an analytically finite Riemann surface (see [10]). In this case, $\mathrm{Mod}(R)$ acts weakly discontinuously at the origin $o = [\mathrm{id}]$ of $T(R)$.

Lemma 5.5. *Let Γ be a subgroup of* $\mathrm{Mod}(R)$. *Assume that the isotropy subgroup* $\mathrm{Stab}_{\Gamma}(o)$ *at the origin $o \in T(R)$ is identified with $G_0 \subset \mathrm{Conf}(R)$ and the orbifold R/G_0 is analytically finite. Then, Γ acts at o weakly discontinuously.*

Proof. Suppose that Γ does not act at o weakly discontinuously. Then, there is a sequence of elements $\gamma_n = [g_n]_* \in \Gamma$ such that $p_n = \gamma_n(o) \neq o$ converges to o as $n \to \infty$. For a simple closed geodesic $c \in \mathcal{S}(R)$, Proposition 3.1 implies that $\ell_{p_n}(c) = \ell(g_n^{-1}(c))$ converges to $\ell(c)$. Since the lengths of all closed geodesics not necessarily simple on R/G_0 are still discrete, we see that $\ell(g_n^{-1}(c)) = \ell(c)$ for all sufficiently large n. Then, there are a finite number of simple closed geodesics $\{c_i\}_{i=1}^{k} \subset \mathcal{S}(R)$ with $\ell(c_i) = \ell(c)$ such that, for each sufficiently large n, there are an element $h_n \in G_0$ and an integer $i(n)$ with $1 \leq i(n) \leq k$ satisfying $h_n \circ g_n^{-1}(c) = c_{i(n)}$. By passing to a subsequence, we may assume that $h_n \circ g_n^{-1}(c) = h_1 \circ g_1^{-1}(c)$ for all n. Then, $[h_n \circ g_n^{-1}] \in [h_1 \circ g_1^{-1}] \cdot \mathrm{MCG}_c(R)$. Since R satisfies the bounded geometry condition, $\mathrm{Mod}_c(R)$ acts discontinuously on $T(R)$ by Theorem 3.6. However, from $p_n = [g_n]_*(o) \to o$, it follows that $[g_n \circ h_n^{-1}]_*(o) \to o$ as $n \to \infty$. This contradicts the discontinuity of $\mathrm{Mod}_c(R)$. □

In the remainder of this section, we consider moduli spaces associated with a Riemann surface R. Regardless of how far the action of $\mathrm{Mod}(R)$ is from discontinuity, the moduli space $M(R)$ is a topological space by the quotient topology induced by the projection

$$\pi_1 = \pi_M : T(R) \longrightarrow M(R) = T(R)/\,\mathrm{Mod}(R).$$

We call $M(R)$ the *topological* moduli space. Moreover, a pseudo-distance $d_1 = d_M$ on $M(R)$ is induced from the Teichmüller distance $d = d_T$ on $T(R)$.

We define two open subregions in $M(R)$: $M_\Omega(R) = \Omega(\Gamma)/\Gamma$ and $M_\Phi(R) = \Phi(\Gamma)/\Gamma$ for $\Gamma = \mathrm{Mod}(R)$. The region $M_\Omega(R)$ inherits the geometric structure from $\Omega(\Gamma) \subset T(R)$. In particular, $M_\Omega(R)$ is a complex Banach orbifold. The moduli space of the stable points $M_\Phi(R)$ is the maximal open subset of $M(R)$ where the restriction of the pseudo-distance d_M becomes a distance.

The *geometric* moduli space $M_*(R)$ is a complete metric space, which is the quotient by the closure equivalence with the projection

$$\pi_2 = \pi_{M_*} : T(R) \longrightarrow M_*(R) = T(R)/\!/\,\mathrm{Mod}(R).$$

The distance $d_2 = d_{M_*}$ is induced from $d = d_T$. Let $\bar{\pi}: M(R) \to M_*(R)$ be the canonical projection. We will consider the projection $\bar{\pi}$ in further detail later on and see that the metric completion of $M_\Phi(R)$ is isometric to $M_*(R)$.

By Corollary 4.7, we have the following theorem. Note that a sufficient condition for $\mathrm{Mod}(R)$ to not act on $T(R)$ weakly stably will be given in Section 6.

Theorem 5.6. *For the Teichmüller modular group* $\mathrm{Mod}(R)$ *acting on* $T(R)$ *and for the moduli spaces* $M(R)$ *and* $M_*(R)$*, the following conditions are equivalent*:

(a) $\mathrm{Mod}(R)$ *acts on* $T(R)$ *weakly stably*;

(b) *the pseudo-distance* d_M *on* $M(R)$ *is a distance*;

(c) $M(R)$ *satisfies the first separation* (T_1) *axiom, or equivalently, every point constitutes a closed set in* $M(R)$;

(d) *the projection* $\bar{\pi}: M(R) \to M_*(R)$ *is an isometric homeomorphism.*

We can also consider quotient spaces defined by certain proper subgroups Γ of $\mathrm{Mod}(R)$. The following space has been defined in [14] for the investigation of the asymptotic Teichmüller space, which is a deformation space of the complex structures outside any compact subsurfaces in R.

Example. For a topologically infinite Riemann surface R, let $\mathrm{MCG}_\infty(R)$ be the subgroup of $\mathrm{MCG}(R)$ consisting of all mapping classes $[g]$ such that a representative g is the identity outside some topologically finite subsurface with boundary in R. This is called the *stable mapping class group*, which is countable and normal in $\mathrm{MCG}(R)$. The corresponding subgroup in $\mathrm{Mod}(R)$ is denoted by $\mathrm{Mod}_\infty(R)$. If we assume that R satisfies the bounded geometry condition, then $\mathrm{Mod}_\infty(R)$ acts on $T(R)$ discontinuously and freely. Then, the quotient space $T^\infty(R) = T(R)/\mathrm{Mod}_\infty(R)$ is defined to be the *enlarged moduli space*. The quotient group $\mathrm{Mod}^\infty(R) = \mathrm{Mod}(R)/\mathrm{Mod}_\infty(R)$ is isomorphic to the asymptotic Teichmüller modular group.

For an arbitrary $c \in \mathcal{S}(R)$, Theorem 5.1 states that $\mathrm{Mod}_c(R)$ is a subgroup of countable index in $\mathrm{Mod}(R)$ and it acts stably on $T(R)$. Moreover, if R satisfies the bounded geometry condition, then it acts discontinuously on $T(R)$ by Theorem 5.3. We consider the quotient space $T^c(R) = T(R)/\mathrm{Mod}_c(R)$, which we call the *relative Teichmüller space* with respect to c. This is a complete metric space with the quotient distance $\hat{d}$. The relative Teichmüller space $T^c(R)$ divides the action of $\mathrm{Mod}(R)$ on $T(R)$ into the stable part $\mathrm{Mod}_c(R)$ and the countable part $\mathrm{Mod}(R)/\mathrm{Mod}_c(R)$. In Section 13, we will investigate $T^c(R)$ in order to see the non-separability of the topological moduli space $M(R)$.

6. Elliptic subgroups

We say that a modular transformation in $\mathrm{Mod}(R)$ is *elliptic* if it has a fixed point p on the Teichmüller space $T(R)$. A mapping class corresponding to an elliptic element is realized as a conformal automorphism of the Riemann surface corresponding to $p \in T(R)$. We call this a conformal mapping class at p. Therefore, the following sentences have the same meaning for $p = [f] \in T(R)$ and $\gamma = [g]_* \in \mathrm{Mod}(R)$: the Teichmüller class p belongs to $\mathrm{Fix}(\gamma)$; the modular transformation γ belongs to $\mathrm{Stab}(p)$; the mapping class $[g]$ belongs to $\mathrm{MCG}_p(R) \cong \mathrm{Conf}(R_p)$. When R is topologically finite, every elliptic element of $\mathrm{Mod}(R)$ is of finite order because every conformal automorphism of R is of finite order. However, when R is topologically infinite, an elliptic element of $\mathrm{Mod}(R)$ can be of infinite order.

Remark. For an analytically finite Riemann surface R, there are two types of classification of the elements in $\mathrm{MCG}(R)$ and $\mathrm{Mod}(R)$ related to each other: one is topological classification of the mapping classes according to Thurston and the other is analytical classification of the modular transformations according to Bers [3]. We adopt the definition of ellipticity from the latter. For analytically infinite Riemann surfaces, we have attempted to classify the modular transformations in [23].

We say that a subgroup $\Gamma \subset \mathrm{Mod}(R)$ is *elliptic* if it has a common fixed point on $T(R)$. Let $\mathrm{Fix}(\Gamma)$ denote the set of all common fixed points of Γ in $T(R)$. As before, the following notations are equivalent for $p = [f]$ and $\Gamma = \iota(G)$: $p \in \mathrm{Fix}(\Gamma)$; $\Gamma \subset \mathrm{Stab}(p)$; $G \subset \mathrm{MCG}_p(R)$. Note that every elliptic subgroup is countable because so is every conformal automorphism group of a Riemann surface. Assume that the origin $o \in T(R)$ belongs to the fixed point locus $\mathrm{Fix}(\Gamma)$. Then, $\mathrm{Fix}(\Gamma)$ coincides with the Teichmüller space $T(R/G)$ embedded in $T(R)$, which has been explained in Section 2. In general, if $p \in \mathrm{Fix}(\Gamma)$, then $\mathrm{Fix}(\Gamma)$ is identified with $T(R_p/G_p)$ for $G_p = e_f(G)$.

In the analytically finite case, the solution of the Nielsen realization problem given by Kerckhoff [17] asserts that $\Gamma \subset \mathrm{Mod}(R)$ is elliptic if and only if Γ is a finite group. We generalize this fact to the analytically infinite case. Here, we do not have to restrict ourselves to finite groups in this case. Note that, if Γ has a fixed point on $T(R)$, then the orbit of Γ is clearly bounded since Γ acts isometrically.

Theorem 6.1. *A subgroup Γ of $\mathrm{Mod}(R)$ is elliptic if and only if the orbit $\Gamma(p)$ is bounded for any $p \in T(R)$.*

Let $\mathbb{D} \to R$ be the universal cover of a Riemann surface R and let H be the corresponding Fuchsian group acting on the unit disk model $\mathbb{D}$ of the hyperbolic plane. Let G be a subgroup of $\mathrm{MCG}(R)$ and assume that the orbit $\Gamma(p)$ for $\Gamma = \iota(G)$ is bounded for any $p \in T(R)$. We lift a quasiconformal automorphism g of R representing each $[g] \in G$ to $\mathbb{D}$ as a quasiconformal automorphism. We take all such lifts for all $[g] \in G$ and extend them to quasisymmetric automorphisms of the boundary $\partial\mathbb{D}$. Thus, we have a group $\tilde{H}$ of quasisymmetric automorphisms that contain the Fuchsian group H as a normal subgroup such that $\tilde{H}/H$ is isomorphic to G. Since the orbit $\Gamma(p)$ is bounded, we see that there exists a uniform bound for the quasisymmetric constants of all elements of $\tilde{H}$, i.e., $\tilde{H}$ is a *uniformly quasisymmetric* group. Then, Theorem 6.1 is a consequence of the following theorem proved by Markovic [20].

Theorem 6.2. *For a uniformly quasisymmetric group $\tilde{H}$ acting on the unit circle $\partial\mathbb{D}$, there exists a quasisymmetric automorphism f of $\partial\mathbb{D}$ such that $f\tilde{H}f^{-1}$ is the restriction of a Fuchsian group to $\partial\mathbb{D}$.*

Next, we consider the discreteness of the orbit for an elliptic subgroup of $\mathrm{Mod}(R)$. Note that, by Theorem 5.3, the discreteness is equivalent to the closedness of the orbit for an elliptic subgroup since it is countable. For an elliptic cyclic group of infinite order, we have the following, which has appeared in [23].

Proposition 6.3. *Let $\gamma \in \mathrm{Mod}(R)$ be an elliptic transformation of infinite order. Then, the cyclic group $\langle\gamma\rangle$ does not act weakly discontinuously on $T(R)$. In fact, in every neighborhood U of a fixed point $p \in \mathrm{Fix}(\gamma)$, there exists $q \neq p$ such that the orbit of q under $\langle\gamma\rangle$ is not a discrete set.*

This is easily seen from the following more general assertion if we observe that an infinite cyclic group $\langle\gamma\rangle$ contains an infinite descending sequence $\langle\gamma\rangle \supsetneqq \langle\gamma^2\rangle \supsetneqq \langle\gamma^4\rangle \supsetneqq \cdots$.

Theorem 6.4. *Let Γ_0 be a subgroup of the isotropy subgroup* $\mathrm{Stab}(p) \subset \mathrm{Mod}(R)$ *at $p \in T(R)$. If there exists an infinite descending sequence*

$$\Gamma_0 \supsetneqq \Gamma_1 \supsetneqq \Gamma_2 \supsetneqq \cdots \supsetneqq \Gamma_n \supsetneqq \cdots$$

of subgroups of Γ_0, then, in every neighborhood U of p, there exists $q \neq p$ such that $\Gamma_0(q)$ is not a discrete set, i.e., $q \in \Lambda_0(\Gamma_0)$.

Proof. We may assume that p is the origin of $T(R)$. Let $G_0 \subset \mathrm{Conf}(R)$ be the group of conformal automorphisms of R identified with Γ_0. We also define G_n to be the corresponding subgroup to Γ_n for each $n \geq 1$. By Lemma 2.1, if two subgroups Γ_1 and Γ_2 satisfy the inclusion relation $\Gamma_1 \supsetneqq \Gamma_2$ and if the index of Γ_2 in Γ_1 is sufficiently large, then the fixed point loci $\mathrm{Fix}(\Gamma_1) = T(R/G_1)$ and $\mathrm{Fix}(\Gamma_2) = T(R/G_2)$ in $T(R)$ satisfy the inclusion relation $\mathrm{Fix}(\Gamma_1) \subsetneqq \mathrm{Fix}(\Gamma_2)$. Hence, by choosing a subsequence if necessary, we have an infinite ascending sequence

$$\mathrm{Fix}(\Gamma_0) \subsetneqq \mathrm{Fix}(\Gamma_1) \subsetneqq \mathrm{Fix}(\Gamma_2) \subsetneqq \cdots \subsetneqq \mathrm{Fix}(\Gamma_n) \subsetneqq \cdots$$

of the fixed point loci. By considering the Bers embedding β, we may regard each of these fixed point loci $\mathrm{Fix}(\Gamma_n) = T(R/G_n)$ as the intersection of the open subset $\beta(T)$ with the closed subspace $B(H_n)$ in the Banach space $B(1)$, where H_n is the Fuchsian group such that $\mathbb{U}/H_n = R/G_n$.

Take the union $F = \bigcup_{n=1}^{\infty} \mathrm{Fix}(\Gamma_n)$ of all these sets and consider its closure $\overline{F}$ in $T(R)$. Then, $\overline{F} - F$ is not an empty set (in fact, this is a dense subset). Indeed, if it is empty, then the complete metric space $\overline{F}$ is composed of the countable union of the closed subsets $\mathrm{Fix}(\Gamma_n)$. By the Baire category theorem, at least one of them has a non-empty interior; however, this is impossible for the infinite ascending sequence of linear subspaces of a Banach space restricted to some open subset in it. Hence, we have a point $q \in \overline{F} - F$.

Take a point $q_n \in \mathrm{Fix}(\Gamma_n)$ for each $n \geq 1$ such that the Teichmüller distances $d_T(q, q_n)$ converge to 0 as $n \to \infty$. Then, since every element γ_n of Γ_n fixes q_n, we have

$$d_T(q, \gamma_n(q)) \leq d_T(q, q_n) + d_T(q_n, \gamma_n(q_n)) + d_T(\gamma_n(q_n), \gamma_n(q)) = 2d_T(q, q_n).$$

Here, $\gamma_n(q)$ is distinct from q because q does not belong to $\mathrm{Fix}(\Gamma_n)$. Hence, $\gamma_n(q) \neq q$ converges to q, which means that the orbit $\Gamma_0(q)$ is not a discrete set. □

By applying Theorem 5.3 to the above theorem, we see that the topological moduli space $M(R) = T(R)/\mathrm{Mod}(R)$ for a certain topologically infinite Riemann surface R is not a T_1-space.

Corollary 6.5. *We assume that R satisfies the bounded geometry condition and* $\mathrm{Mod}(R)$ *contains an elliptic element of infinite order. Then, $M(R)$ does not satisfy the first separation (T_1) axiom. In particular, for an infinite cyclic cover R of an analytically finite Riemann surface, $M(R)$ is not a T_1-space.*

Proof. Since $\mathrm{Mod}(R)$ contains an elliptic element of infinite order, it does not act weakly discontinuously by Proposition 6.3. Since R satisfies the bounded geometry condition, this implies that $\mathrm{Mod}(R)$ does not act weakly stably by Theorem 5.3. Then, Theorem 5.6 asserts that $M(R)$ does not satisfy the first separation axiom. □

7. Isolated limit points and Tarski monsters

We investigate the dynamics of Teichmüller modular groups by attempting to find an isolated point of the limit set. This problem itself does not affect the succeeding arguments; however, it opens up an interesting group theoretical problem. First, we give the necessary conditions for a limit point to be isolated in the limit set.

Theorem 7.1. *Assume that $p \in T(R)$ is an isolated point of the limit set $\Lambda(\Gamma)$ for a subgroup $\Gamma \subset \mathrm{Mod}(R)$. Then, the isotropy subgroup $\mathrm{Stab}_\Gamma(p)$ satisfies the following conditions*:

(1) *the common fixed point of each infinite subgroup in* $\mathrm{Stab}_\Gamma(p)$ *is only* p;

(2) $\mathrm{Stab}_\Gamma(p)$ *is a finitely generated infinite group but does not contain an element of infinite order*;

(3) *every subgroup of* $\mathrm{Stab}_\Gamma(p)$ *is either of finite order or of finite index.*

Proof. Without loss of generality, we may assume that p is the origin of $T(R)$. Let Γ_0 be an infinite subgroup of $\mathrm{Stab}_\Gamma(p)$ and G_0 be the corresponding subgroup of $\mathrm{Conf}(R)$. Then, the fixed point locus $\mathrm{Fix}(\Gamma_0)$ coincides with the Teichmüller space $T(R/G_0)$ embedded in $T(R)$. Clearly, $p \in \mathrm{Fix}(\Gamma_0)$ and $\mathrm{Fix}(\Gamma_0) \subset \Lambda(\Gamma)$. Since p is isolated in $\Lambda(\Gamma)$, we see that $\mathrm{Fix}(\Gamma_0) = \{p\}$, which gives condition (1).

Suppose that p is a generic limit point, i.e., $p \in \Lambda_0(\Gamma)$. Then, there exists a sequence $\{\gamma_n\}$ in Γ such that $p_n = \gamma_n(p) \neq p$ converge to p as $n \to \infty$. However, since $p_n \in \Lambda(\Gamma)$, this violates the assumption that p is isolated in $\Lambda(\Gamma)$. Hence, p must belong to $\Lambda_\infty(\Gamma)$, i.e., $\mathrm{Stab}_\Gamma(p)$ is an infinite group. Assume that $\mathrm{Stab}_\Gamma(p)$ contains an element γ of infinite order. Then, by condition (1), $\mathrm{Fix}(\Gamma_1) = \{p\}$ for the infinite cyclic group $\Gamma_1 = \langle\gamma\rangle \subset \mathrm{Stab}_\Gamma(p)$ and $\mathrm{Fix}(\Gamma_k) = \{p\}$ for its proper subgroup $\Gamma_k = \langle\gamma^k\rangle$ with $k \geq 2$. However, this contradicts Lemma 2.1 for a sufficiently large k. Thus, we see that $\mathrm{Stab}_\Gamma(p)$ has no element of infinite order.

Moreover, assume that $\mathrm{Stab}_\Gamma(p)$ is infinitely generated. Since an infinitely generated group always contains an infinitely generated proper subgroup, we have an infinitely generated proper subgroup Γ_1' of $\mathrm{Stab}_\Gamma(p)$. Then, by applying the above fact again to this Γ_1', we have an infinitely generated proper subgroup Γ_2' of Γ_1'. By repeating this process several times, we can find an infinite subgroup Γ_k' of $\mathrm{Stab}_\Gamma(p)$ with a sufficiently large index. Since $\mathrm{Fix}(\mathrm{Stab}_\Gamma(p)) = \{p\}$ and

$\mathrm{Fix}(\Gamma'_k) = \{p\}$ by condition (1), this also contradicts Lemma 2.1. Hence, we see that $\mathrm{Stab}_\Gamma(p)$ is finitely generated. Thus, we obtain condition (2).

Assume that $\mathrm{Stab}_\Gamma(p)$ contains an infinite subgroup Γ_0 of infinite index. Then, $\mathrm{Fix}(\mathrm{Stab}_\Gamma(p)) = \mathrm{Fix}(\Gamma_0) = \{p\}$ by condition (1) as before. However, since the index of Γ_0 in $\mathrm{Stab}_\Gamma(p)$ is infinite, this again contradicts Lemma 2.1 and hence yields condition (3). □

In particular, condition (2) of this theorem implies the following.

Corollary 7.2. *If $p \in \Lambda(\Gamma)$ is an isolated limit point for $\Gamma \subset \mathrm{Mod}(R)$, then $p \in \Lambda^2_\infty(\Gamma)$.*

We cannot determine whether an isolated limit point exists or not. In this section, we will see that an abstract group satisfying conditions (2) and (3) in Theorem 7.1 actually exists and can be realized as a group of conformal automorphisms of a certain Riemann surface. The corresponding isotropy subgroup also satisfies condition (1). Then, we examine the dynamics of the Teichmüller modular group of this Riemann surface to seek an isolated limit point.

A finitely generated group G is called a *periodic group* if the order of each element of G is finite and a *bounded periodic group* if the order is uniformly bounded. For integers $m \geq 2$ and $n \geq 2$, let F_m be a free group of rank m and let $F_m^{(n)}$ be the characteristic subgroup of F_m generated by all the elements of the form f^n for $f \in F_m$. Then, the quotient group $B(m,n) = F_m/F_m^{(n)}$ is an m-generator group, all of whose elements are the identity by n-times composition. This is called a Burnside group or a *free periodic group*. It is easy to see that, for every bounded periodic group G, there exists a free periodic group $B(m,n)$ for some positive integers m and n such that G is the image of a homomorphism of $B(m,n)$. For $m = 2$, it is known that $B(2,2)$, $B(2,3)$, $B(2,4)$, and $B(2,6)$ are finite groups. On the other hand, Novikov and Adjan [27] proved the following.

Theorem 7.3. *For all sufficiently large odd integers $n \in \mathbb{N}$, the free periodic group $B(2,n)$ is infinite.*

A problem for seeking a stronger example of the finiteness aspect in an infinite group is whether there is an infinite group G, all of whose proper subgroups are finite. For this problem, the strongest example was obtained so that every proper subgroup is a cyclic group of prime order n. This was constructed as the quotient of $B(m,n)$ by giving certain extra relations (see Adjan and Lysionok [1] and Ol′shanskii [28]). Such a group is sometimes called a *Tarski monster*.

Theorem 7.4. *For all sufficiently large primes $n \in \mathbb{N}$, there exists a 2-generator Tarski monster of exponent n.*

A free periodic group and its quotient $\widehat{B}(m,n) = \langle x_1, \ldots, x_m \mid r_1, r_2, \ldots \rangle$, such as a Tarski monster, can be realized as a group of conformal automorphisms of some Riemann surface. Indeed, since the fundamental group of an $(m+1)$-times punctured sphere is isomorphic to the free group F_m, a covering Riemann surface R corresponding to the normal closure N of the relators $r_1, r_2, \ldots$ has the covering transformation group $\widehat{B}(m,n) = F_m/N$. This means that a subgroup of $\mathrm{Conf}(R)$ is isomorphic to $\widehat{B}(m,n)$. Therefore, we consider the following Riemann surface as a potential candidate for proving the existence of an isolated limit point of $\mathrm{Mod}(R)$.

Proposition 7.5. *Let R be a Riemann surface that covers the three-times punctured sphere with the covering transformation group $G_0 \subset \mathrm{Conf}(R)$ isomorphic to a bounded periodic group $\widehat{B}(2,n)$, all of whose proper subgroups are finite. Then, the isotropy subgroup $\mathrm{Stab}(o)$ of $\mathrm{Mod}(R)$ at the origin $o \in T(R)$ satisfies the three conditions stated in Theorem 7.1.*

Proof. Let Γ_0 be the subgroup of $\mathrm{Stab}(o)$ corresponding to $G_0 \subset \mathrm{Conf}(R)$. From the condition that R/G_0 is the three-times punctured sphere whose Teichmüller space is trivial, we see that

$$\mathrm{Fix}(\mathrm{Stab}(o)) = \mathrm{Fix}(\Gamma_0) = \{o\}$$

and $G_0 \cong \Gamma_0$ is of finite index in $\mathrm{Conf}(R) \cong \mathrm{Stab}(o)$. Since G_0 has no infinite proper subgroup, every infinite subgroup of $\mathrm{Stab}(o)$ contains Γ_0; hence, condition (1) in Theorem 7.1 is satisfied. Since G_0 satisfies algebraic conditions (2) and (3), so does $\mathrm{Stab}(o)$. □

We expect that, in the circumstances of Proposition 7.5 with an additional assumption that the bounded periodic group $\widehat{B}(2,n)$ is a Tarski monster given by Theorem 7.4, the origin $o \in T(R)$ should be an isolated limit point. In the next lemma, we show that this statement is true under a certain extra hypothesis.

Lemma 7.6. *Let R be a Riemann surface that covers the three-times punctured sphere with the covering transformation group $G_0 \subset \mathrm{Conf}(R)$ isomorphic to a 2-generator Tarski monster of prime exponent n. Let Γ_0 be the subgroup of $\mathrm{Stab}(o)$ corresponding to G_0. Then, the origin $o \in T(R)$ is an isolated limit point of $\mathrm{Mod}(R)$ if the union $\bigcup_{\gamma \in \Gamma_0 - \{\mathrm{id}\}} \mathrm{Fix}(\gamma)$ of the sets of all fixed points of the non-trivial elements of Γ_0 is closed in $T(R)$.*

Proof. By Proposition 4.3 and Lemma 5.5, we see that there exists a neighborhood U of the origin o that is precisely invariant under $\mathrm{Stab}(o)$; in other words, $\gamma(U) = U$ for every $\gamma \in \mathrm{Stab}(o)$ and $\gamma(U) \cap U = \emptyset$ for every $\gamma \in \mathrm{Mod}(R) - \mathrm{Stab}(o)$. Hence, we have only to prove that o is an isolated limit point of $\mathrm{Stab}(o)$. Furthermore, since $\Lambda(\mathrm{Stab}(o)) = \Lambda(\Gamma_0)$ for the subgroup

$\Gamma_0 \subset \mathrm{Stab}(o)$ of finite index, it suffices to prove the same statement for Γ_0. Assume that there exist a point $p \in U - \{o\}$ and a sequence $\{\gamma_k\}_{k=1}^{\infty}$ of Γ_0 such that $\gamma_k(p)$ converges to p as $k \to \infty$. We will show that p is a fixed point of some non-trivial element of Γ_0 that is accumulated by fixed points of other elements of Γ_0. Note that, since the exponent n is prime, every non-trivial element of G_0 has order n.

By the Bers embedding $\beta\colon T(R) \to B(H)$, where we represent $R = \mathbb{U}/H$ by a Fuchsian group H, the Teichmüller space $T(R)$ is regarded as a bounded domain of the Banach space $B(H)$ and $G_0 \subset \mathrm{Conf}(R)$ acts on $B(H)$ as a group of linear isometries. Let $g_k \in G_0$ be the element corresponding to $\gamma_k \in \Gamma_0$ for each $k \in \mathbb{N}$. Then, for $\varphi = \beta(p) \in B(H) - \{0\}$, we have a sequence $\{(g_k)_*(\varphi)\}_{k=1}^{\infty}$ in $B(H)$ that converges to φ.

For each k, we take the average of the orbit $\{\varphi, (g_k)_*(\varphi), \dots, (g_k)_*^{n-1}(\varphi)\}$ under the cyclic group $\langle g_k \rangle$ of order n, i.e., $\psi_k = \frac{1}{n}\sum_{j=0}^{n-1}(g_k)_*^j(\varphi)$. This satisfies $(g_k)_*(\psi_k) = \psi_k$, i.e., ψ_k is a fixed point of $(g_k)_*$. Moreover, we see that ψ_k converges to φ as $k \to \infty$. Indeed, the difference between ψ_k and φ is estimated by

$$\|\psi_k - \varphi\|_B \le \frac{1}{n}\sum_{j=0}^{n-1}\|(g_k)_*^j(\varphi) - \varphi\|_B \le \frac{\sum_{j=0}^{n-1} j}{n}\|(g_k)_*(\varphi) - \varphi\|_B.$$

In particular, this shows that $\psi_k \in \beta(T(R))$, i.e., ψ_k represents a point of $T(R)$, for any sufficiently large k because $\beta(T(R))$ is an open subset of $B(H)$. From the assumption that the set of all fixed points for $\Gamma_0 - \{\mathrm{id}\}$ is closed, we see that $\varphi = \beta(p)$ is a fixed point of some non-trivial element $g_0 \in G_0$.

For any non-trivial elements g and g' of G_0, we have $\mathrm{Fix}(g_*) \cap \mathrm{Fix}(g'_*) = \{o\}$ if $\langle g \rangle \neq \langle g' \rangle$. This is because $\langle g, g' \rangle = G_0$ by the property of Tarski monsters and because the origin o is the only common fixed point for G_0. For $r = \|\varphi\|_B > 0$, let S_r be a sphere of radius r in $B(H)$ centered at the origin. Set $I(g) = \mathrm{Fix}(g_*) \cap S_r$ for a non-trivial $g \in G_0$, which is a closed subset of S_r. Then, $I(g)$ and $I(g')$ are disjoint if and only if $\langle g \rangle \neq \langle g' \rangle$.

From the fact proved above, the set $I(g_0)$ containing φ is accumulated by other $I(g_k)$. By the group invariance, the situation is the same for every $I(g)$. Hence, for the same reason as the fact that a perfect closed set in a complete metric space is uncountable, the cardinality of $\{I(g)\}$ taken over all non-trivial cyclic subgroups $\langle g \rangle \subset G_0$ is uncountable. However, this is impossible, as G_0 is countable. □

We will comment about the extra assumption on the closedness of the fixed point set in Lemma 7.6 later on at the end of Section 12. Then, we will wait for further arguments to complete the proof of the existence of an isolated limit point.

8. Exceptional limit points and density of generic limit points

We wish to claim that the set $\Lambda_0(\Gamma)$ of all generic limit points for a subgroup $\Gamma \subset \mathrm{Mod}(R)$ is dense in $\Lambda(\Gamma)$. However, for instance, since an isolated limit point is not in the closure of $\Lambda_0(\Gamma)$, we have to make a certain modification to justify this density problem.

We have seen in Theorem 7.1 that, if $p \in \Lambda(\Gamma)$ is an isolated limit point of $\Gamma \subset \mathrm{Mod}(R)$, then $\mathrm{Stab}_\Gamma(p)$ must satisfy certain algebraic conditions. By focusing on the occupation of limit points satisfying these algebraic conditions, we present the following concept for limit points.

Definition. A limit point $p \in \Lambda(\Gamma)$ for $\Gamma \subset \mathrm{Mod}(R)$ is defined to be *exceptional* if $p \notin \Lambda_0(\Gamma)$ and if there exists a neighborhood U of p in $T(R)$ such that $U \cap \Lambda(\Gamma) \subset \Lambda^2_\infty(\Gamma)$. The set of all exceptional limit points is called the exceptional limit set and denoted by $E(\Gamma)$.

By this definition and Corollary 7.2, it is clear that

$$\{\text{isolated limit points}\} \subset E(\Gamma) \subset \Lambda^2_\infty(\Gamma).$$

However, thus far, we are unaware of the existence of exceptional limit points, not to mention isolated limit points.

First, we give a condition for a limit point to be exceptional in Lemma 8.2 below. The following lemma is crucial for that argument.

Lemma 8.1. *For a countable subgroup* Γ *of* $\mathrm{Mod}(R)$*, if* $\Lambda(\Gamma) = \Lambda_\infty(\Gamma)$*, then they coincide with* $\Lambda^2_\infty(\Gamma)$*. More generally, for an open subset* U *in* $T(R)$*, if* $U \cap \Lambda(\Gamma) = U \cap \Lambda_\infty(\Gamma)$*, then they coincide with* $U \cap \Lambda^2_\infty(\Gamma)$*.*

Proof. We number the elements of infinite order of Γ by $\{\gamma_i\}_{i \in \mathbb{N}}$ and the elements of finite order of $\Gamma - \{\mathrm{id}\}$ by $\{e_j\}_{j \in \mathbb{N}}$. Set $X_i = \mathrm{Fix}(\gamma_i)$ for each $i \in \mathbb{N}$, which is a closed subset of $T(R)$. Consider the union

$$\bigcup_{i=1}^{\infty} X_i = \Lambda^1_\infty(\Gamma) = \Lambda_\infty(\Gamma) - \Lambda^2_\infty(\Gamma).$$

To prove that this is an empty set, we assume that $\bigcup_{i=1}^{\infty} X_i \neq \emptyset$ and draw a contradiction.

Set $Y_j = \mathrm{Fix}(e_j) \cap \overline{\Lambda^1_\infty(\Gamma)}$ for each $j \in \mathbb{N}$, which is also closed. Since $\overline{\Lambda^1_\infty(\Gamma)} \subset \Lambda(\Gamma) = \Lambda_\infty(\Gamma)$ by assumption, we have

$$\overline{\Lambda^1_\infty(\Gamma)} = \bigcup_{i=1}^{\infty} X_i \cup \bigcup_{j=1}^{\infty} Y_j.$$

We regard $\overline{\Lambda^1_\infty(\Gamma)}$ as a complete metric space by the restriction of the Teichmüller distance. Then, by the Baire category theorem, there exists at least one X_i or Y_j that contains an interior point p. This means that there exists an open neighborhood V of p in $T(R)$ such that $\overline{\Lambda^1_\infty(\Gamma)} \cap V$ is contained in X_i or Y_j.

First, assume that $X_i = \mathrm{Fix}(\gamma_i)$ contains an interior point p for some i. Consider $\mathrm{Fix}(\gamma_i^k)$ for a sufficiently large integer k, i.e., $X_{i'}$ for some different integer $i' \neq i$. Since $X_i \subsetneqq X_{i'}$ by Lemma 2.1, any neighborhood V of p contains a point in $\overline{\Lambda^1_\infty(\Gamma)} - X_i$, which is a contradiction. Next, assume that $Y_j = \mathrm{Fix}(e_j) \cap \overline{\Lambda^1_\infty(\Gamma)}$ contains an interior point p for some j. Then, there exists an open subset V of $T(R)$ such that $p \in \overline{\Lambda^1_\infty(\Gamma)} \cap V \subset \mathrm{Fix}(e_j)$. We choose some $X_i = \mathrm{Fix}(\gamma_i) \subset \Lambda^1_\infty(\Gamma)$ that intersects V. In this situation, the cyclic group $\langle \gamma_i \rangle$ is a proper subgroup of $\langle \gamma_i, e_j \rangle$ because the order of e_j is finite. Then, we see from Lemma 2.1 again that $X_i \cap \mathrm{Fix}(e_j)$ is properly contained in X_i by replacing γ_i with some γ_i^k if necessary. This contradicts the fact that every point in $X_i \cap V$ is fixed by e_j.

The same proof can be applied if we restrict all the limit sets to an open subset U. Thus, the general statement is also valid. □

Remark. The assumption of Lemma 8.1 that Γ is countable can be removed. This will be seen in Section 12.

Lemma 8.2. *Let Γ be a subgroup of* $\mathrm{Mod}(R)$. *If $p \in \Lambda(\Gamma) - \Lambda_0(\Gamma)$ has a neighborhood U such that $U \cap \Lambda(\Gamma) \subset \Lambda_\infty(\Gamma)$, then p belongs to $E(\Gamma)$.*

Proof. By Proposition 4.3, we may assume that the neighborhood U of p is equivariant under Γ, i.e., $\gamma(U) = U$ for every γ in the isotropy subgroup $\Gamma_0 = \mathrm{Stab}_\Gamma(p)$ and $\gamma(U) \cap U = \emptyset$ for every $\gamma \in \Gamma - \Gamma_0$. Then, $U \cap \Lambda(\Gamma) = U \cap \Lambda(\Gamma_0)$ and $U \cap \Lambda_\infty(\Gamma) = U \cap \Lambda_\infty(\Gamma_0)$. On the other hand, the assumption implies that $U \cap \Lambda(\Gamma) = U \cap \Lambda_\infty(\Gamma)$. Hence, $U \cap \Lambda(\Gamma_0) = U \cap \Lambda_\infty(\Gamma_0)$. Since Γ_0 is countable, it follows from Lemma 8.1 that $U \cap \Lambda_\infty(\Gamma_0) = U \cap \Lambda^2_\infty(\Gamma_0)$. Here, again by the equivariance of U under Γ, we conclude that $U \cap \Lambda_\infty(\Gamma) = U \cap \Lambda^2_\infty(\Gamma)$; thus, p belongs to $E(\Gamma)$ by definition. □

Now, we can formulate the density of generic limit points in the following form. This is the best possible assertion if we assume the existence of exceptional limit points.

Theorem 8.3. *For a subgroup Γ of* $\mathrm{Mod}(R)$, *the set of generic limit points $\Lambda_0(\Gamma)$ is dense in $\Lambda(\Gamma) - E(\Gamma)$.*

Proof. Take a limit point $p \in \Lambda(\Gamma) - E(\Gamma) - \Lambda_0(\Gamma)$. If there exists a neighborhood U of p such that $U \cap \Lambda(\Gamma) \subset \Lambda_\infty(\Gamma)$, then p belongs to $E(\Gamma)$ by Lemma 8.2. This is a contradiction; thus, there is no such neighborhood. This means that there is a sequence of points in $\Lambda(\Gamma) - \Lambda_\infty(\Gamma) \subset \Lambda_0(\Gamma)$ that converges to p. □

In Proposition 4.3, we have seen certain equivalent conditions for the action of an isometry group to be weakly discontinuous in a general setting on metric spaces. Here, we add a specific condition obtained by Theorem 8.3 in the case of a Teichmüller space.

Corollary 8.4. *Let* Γ *be a subgroup of* $\mathrm{Mod}(R)$. *Then, the following conditions are equivalent*:

(1) Γ *acts weakly discontinuously on* $T(R)$;

(2) $\Lambda_0(\Gamma) = \emptyset$;

(3) $\Lambda(\Gamma) = E(\Gamma)$.

In particular, condition (3) implies that $\Lambda(\Gamma) = \Lambda_\infty(\Gamma)$. We will consider the converse implication later on in Section 12.

We will show that the isotropy subgroup of an exceptional limit point contains a subgroup that has the same algebraic property as the isotropy subgroup of an isolated limit point. Recall that the existence of such a group has been stated in Section 7.

Theorem 8.5. *For an exceptional limit point* $p \in E(\Gamma)$ *of a subgroup* $\Gamma \subset \mathrm{Mod}(R)$, *the isotropy subgroup* $\mathrm{Stab}_\Gamma(p)$ *contains a finitely generated infinite group* Γ_0 *whose proper subgroups are all finite.*

Proof. Let $\{\Gamma_n\}_{n=1}^\infty$ be the family of all infinite subgroups of $\mathrm{Stab}_\Gamma(p)$. We will show that the union $\bigcup_{n=1}^\infty \mathrm{Fix}(\Gamma_n)$ of all fixed point loci of the subgroups Γ_n is a closed set. Let us suppose that the opposite is true. Then, there exists a sequence $\{p_m\}_{m=1}^\infty$ in

$$\overline{\bigcup_{n=1}^\infty \mathrm{Fix}(\Gamma_n)} - \bigcup_{n=1}^\infty \mathrm{Fix}(\Gamma_n)$$

such that $p_m \to p$ as $m \to \infty$ and $p_m \in \Lambda_\infty^2(\Gamma)$. Set $H_m = \mathrm{Stab}_\Gamma(p_m)$ for each m, which is an infinite group. If $p \in \mathrm{Fix}(H_m)$, then $H_m \subset \mathrm{Stab}_\Gamma(p)$; however, this contradicts $p_m \notin \bigcup_{n=1}^\infty \mathrm{Fix}(\Gamma_n)$. Hence, we have $p \notin \mathrm{Fix}(H_m)$. This implies that there is some $\gamma_m \in H_m$ for each m such that $\gamma_m(p) \neq p$. On the other hand, $\gamma_m(p_m) = p_m$ and $p_m \to p$ yield $\gamma_m(p) \to p$ as $m \to \infty$. However, since $p \notin \Lambda_0(\Gamma)$, this is impossible. Thus, we have shown that $\bigcup_{n=1}^\infty \mathrm{Fix}(\Gamma_n)$ is closed.

We apply the Baire category theorem to the complete metric space $\bigcup_{n=1}^\infty \mathrm{Fix}(\Gamma_n)$ with the restriction of the Teichmüller distance, where each $\mathrm{Fix}(\Gamma_n)$ is a closed subset. Then, there is some Γ_0 in the family $\{\Gamma_n\}_{n=1}^\infty$ such that $\mathrm{Fix}(\Gamma_0)$ has an interior point in $\bigcup_{n=1}^\infty \mathrm{Fix}(\Gamma_n)$. In particular, this implies that there is no $\mathrm{Fix}(\Gamma_n)$ that contains $\mathrm{Fix}(\Gamma_0)$ properly. Thus, by possibly replacing Γ_0 with a subgroup of finite index, we see from Lemma 2.1 that Γ_0 has no proper infinite subgroup in it. This property also forces Γ_0 to be finitely generated. □

Corollary 8.6. *If a subgroup* Γ *of* $\mathrm{Mod}(R)$ *does not contain a finitely generated infinite group* Γ_0 *whose proper subgroups are all finite, then* $E(\Gamma) = \emptyset$.

If the group structure does not allow $\Gamma \subset \mathrm{Mod}(R)$ to have such a subgroup Γ_0, e.g., if Γ is abelian, then $E(\Gamma) = \emptyset$.

Finally, we give a necessary condition for $\Gamma \subset \mathrm{Mod}(R)$ to act weakly discontinuously on $T(R)$, which is equivalent to the condition $\Lambda(\Gamma) = E(\Gamma)$, in terms of the algebraic properties of the isotropy subgroups.

Proposition 8.7. *If a subgroup* Γ *of* $\mathrm{Mod}(R)$ *acts on* $T(R)$ *weakly discontinuously, then for every* $p \in T(R)$*, the isotropy group* $\Gamma_0 = \mathrm{Stab}_\Gamma(p)$ *has no infinite descending sequence of proper subgroups. In particular, every element of* Γ_0 *is of finite order and every subgroup of* Γ_0 *is finitely generated.*

Proof. If Γ acts weakly discontinuously, then the orbit $\Gamma(q)$ is a discrete set for every $q \in T(R)$. Then, by Theorem 6.4, Γ_0 cannot contain an infinite descending sequence $\{\Gamma_n\}_{n=1}^{\infty}$ of proper subgroups as in its statement. $\square$

9. Partial discreteness of the length spectrum

For an analytically finite Riemann surface R, it is well known that the length spectrum $\mathrm{LS}(p)$ is discrete (in a stronger sense, the multiplicity of each point spectrum is at most finite) for every $p \in T(R)$, from which the discontinuity of the action of $\mathrm{Mod}(R)$ on $T(R)$ follows. Although $\mathrm{LS}(p)$ is not necessarily discrete for a Riemann surface R in general, the distribution of $\mathrm{LS}(p)$ gives certain information on the action of $\mathrm{Mod}(R)$ locally at $p \in T(R)$. Recall that $\mathrm{LS}(p)$ is defined as the closure of the set $\{\log \ell_p(c)\}_{c \in \mathcal{S}(R)}$ of all point spectra.

Definition. An accumulation point of $\mathrm{LS}(p)$ is called an *essential spectrum* and the closed subset of all essential spectra is denoted by $\mathrm{LS}_{\mathrm{ess}}(p)$. We assume that a point of infinite multiplicity is an essential spectrum. A point in the complement $\mathrm{LS}(p) - \mathrm{LS}_{\mathrm{ess}}(p)$ is called a *point spectrum*. Let $r_x(p)$ denote the Euclidean distance from $x \in \mathbb{R}$ to the closed set $\mathrm{LS}_{\mathrm{ess}}(p)$. For each $c \in \mathcal{S}(R)$, in particular, we define $r_c(p)$ to be $r_x(p)$ where $x = \log \ell_p(c)$. Let $r(p)$ be the supremum of $r_x(p)$ taken over all the spectra:

$$r(p) = \sup\{r_x(p) \mid x \in \mathrm{LS}(p)\} = \sup\{r_c(p) \mid c \in \mathcal{S}(R)\}.$$

This represents the gap of $\mathrm{LS}_{\mathrm{ess}}(p)$ relative to $\mathrm{LS}(p)$.

Proposition 9.1. *The function $r(p)$ is invariant under* Mod(R) *and continuous on $T(R)$. More precisely, if $r(p) < \infty$ for every $p \in T(R)$, then it satisfies*

$$|r(p) - r(q)| \leq 2d_{\mathrm{LS}}(p,q) \leq 2d_T(p,q).$$

If $r(p) = \infty$ for some $p \in T(R)$, then $r(p) = \infty$ for every $p \in T(R)$.

Proof. The invariance under Mod(R) is obvious. The second inequality is due to Proposition 3.1. We will prove the first inequality below.

Suppose that $r(p) < \infty$ for every $p \in T(R)$. For an arbitrary $\varepsilon > 0$, there exists $c \in \mathcal{S}(R)$ such that $r_c(p) > r(p) - \varepsilon$. Then, there is no point of $\mathrm{LS}_{\mathrm{ess}}(p)$ within distance $r(p) - \varepsilon$ from $\log \ell_p(c)$, where at most finitely many point spectra exist. Thus, we see that there is no point of $\mathrm{LS}_{\mathrm{ess}}(q)$ within distance $r(p) - 2d_{\mathrm{LS}}(p,q) - \varepsilon$ from $\log \ell_q(c)$. This implies that $r_c(q) \geq r(p) - 2d_{\mathrm{LS}}(p,q) - \varepsilon$; hence, $r(q) \geq r(p) - 2d_{\mathrm{LS}}(p,q) - \varepsilon$ for any $q \in T(R)$. Since ε is arbitrary, we have $r(p) - r(q) \leq 2d_{\mathrm{LS}}(p,q)$. By exchanging the roles of p and q, we obtain the desired inequality.

If $r(p) = \infty$ for some $p \in T(R)$, then for an arbitrary $M > 0$, there exists $c \in \mathcal{S}(R)$ such that $r_c(p) > M$. By an argument similar to the one presented above, we have $r(q) \geq M - 2d_{\mathrm{LS}}(p,q)$ for any $q \in T(R)$. This implies that $r(q) = \infty$. □

If $\mathrm{LS}(p)$ is discrete, i.e., $\mathrm{LS}_{\mathrm{ess}}(p) = \emptyset$ or equivalently $r_c(p) = \infty$ for all $c \in \mathcal{S}(R)$, then $r(p) = \infty$. Conversely, we do not know whether $r(p) = \infty$ implies that $\mathrm{LS}_{\mathrm{ess}}(p) = \emptyset$ or not. As the above proof indicates, these conditions are independent of $p \in T(R)$; if a condition is satisfied for some p, then it is satisfied for all p. By contrast, there exists a case in which $\mathrm{LS}(p)$ is totally indiscrete, i.e., $\mathrm{LS}_{\mathrm{ess}}(p) = \mathrm{LS}(p)$. This is equivalent to the condition that $r(p) = 0$. For example, if p is a stabilized limit point of Mod(R), then $r(p) = 0$.

Here, we consider a situation where $\mathrm{LS}(p)$ is partially discrete in the sense that $\mathrm{LS}_{\mathrm{ess}}(p) \neq \mathrm{LS}(p)$, or equivalently $r(p) > 0$. We remark that the conditions $r(p) > 0$ and $r_c(p) > 0$ that appear below include the cases $r(p) = \infty$ and $r_c(p) = \infty$, respectively.

Theorem 9.2. *If $p \in T(R)$ satisfies $r(p) > 0$, then p belongs to the region of stability $\Phi(\Gamma)$ for $\Gamma =$ Mod(R). In addition, if R satisfies the bounded geometry condition, then p belongs to the region of discontinuity $\Omega(\Gamma)$.*

Proof. Since $r(p) > 0$, there exists $c \in \mathcal{S}(R)$ such that $r_c(p) > 0$. In the case where $r_c(p) = \infty$, we assume that $r_c(p)$ takes an arbitrary positive constant. Then, the length spectra belonging to an open interval $I(\log \ell_p(c), r_c(p)/2) \subset \mathbb{R}$ with center $\log \ell_p(c)$ and radius $r_c(p)/2$ is finite, and we denote the corresponding elements in $\mathcal{S}(R)$ by $\{c_1, \ldots, c_k\}$ including c. Let $U(p, r_c(p)/2) \subset T(R)$ be an open ball with center $p = [f]$ and radius $r_c(p)/2$. If an orbit point $\gamma(p)$ is in $U(p, r_c(p)/2)$ for $\gamma = [g]_* \in \Gamma$, then Proposition 3.1 implies that the quasiconformal automorphism g^{-1} of R must send c to one of $\{c_1,\dots,c_k\}$. Hence, $f\circ g^{-1}\circ f^{-1}$ for all such g are quasiconformal automorphisms of $R_p=f(R)$ with bounded maximal dilatation and satisfy the stationary condition. As in the proof of Theorem 5.1, it follows that the orbit $\Gamma(p)$ restricted to $U(p,r_c(p)/2)$ is closed and the isotropy subgroup $\mathrm{Stab}_\Gamma(p)$ is finite. Since the same properties are satisfied for each point in the orbit $\Gamma(p)$, we conclude that $p\in\Phi(\Gamma)$. The latter statement then directly follows from Corollary 5.4. □

We can refine this conclusion quantitatively by the value $r(p)$.

Corollary 9.3. *If $p\in T(R)$ satisfies $r(p)>0$, then $U(p,r(p)/2)$ is contained in $\Phi(\Gamma)$ for $\Gamma=\mathrm{Mod}(R)$. In addition, if R satisfies the bounded geometry condition, then $U(p,r(p)/2)\subset\Omega(\Gamma)$. When $r(p)=\infty$, the above conclusions are expressed as $\Phi(\Gamma)=T(R)$ and $\Omega(\Gamma)=T(R)$, respectively.*

Proof. Suppose that $r(p)<\infty$. Every $q\in U(p,r(p)/2)$ satisfies $d_T(p,q)=r(p)/2-\epsilon/2$ for some $\epsilon>0$. By the definition of $r(p)$, there exists some $c\in\mathcal{S}(R)$ such that $r_c(p)>r(p)-\epsilon$. Since $|\log\ell_p(c')-\log\ell_q(c')|\le r(p)/2-\epsilon/2$ for every $c'\in\mathcal{S}(R)$ by Proposition 3.1, we see that $r_c(q)>0$; hence, $r(q)>0$. Then, from Theorem 9.2, we conclude that $q\in\Phi(\Gamma)$. The additional statement is due to Corollary 5.4. If $r(p)=\infty$, then $r(q)=\infty$ for every $q\in T(R)$ by Proposition 9.1. Hence, $\Phi(\Gamma)=T(R)$ by Theorem 9.2. □

Based on a primary version of the above arguments, it was proved in [9] that $\Omega(\Gamma)$ is not empty for $\Gamma=\mathrm{Mod}(R)$ and for a Riemann surface R satisfying the bounded geometry condition. An application of the property $\Omega(\Gamma)\neq\emptyset$ to the infinite-dimensional Teichmüller theory can be found in [12]. In the next two sections, we will employ the above results to show more detailed properties of $\Phi(\Gamma)$.

We define the bottom of the spectra as

$$\lambda_0(p)=\inf\{x\in\mathbb{R}\mid x\in\mathrm{LS}(p)\}=\inf\{\log\ell_p(c)\mid c\in\mathcal{S}(R)\}$$

and the bottom of the essential spectra as

$$\lambda_{\mathrm{ess}}(p)=\inf\{x\in\mathbb{R}\mid x\in\mathrm{LS}_{\mathrm{ess}}(p)\}.$$

Obviously, $\lambda_0(p)\le\lambda_{\mathrm{ess}}(p)$, and if $\lambda_0(p)<\lambda_{\mathrm{ess}}(p)$, then the partial discreteness condition $r(p)\ge\lambda_{\mathrm{ess}}(p)-\lambda_0(p)>0$ follows. Furthermore, they are continuous functions on $T(R)$ invariant under $\mathrm{Mod}(R)$ satisfying

$$|\lambda_0(p)-\lambda_0(q)|,\ |\lambda_{\mathrm{ess}}(p)-\lambda_{\mathrm{ess}}(q)|\le d_{\mathrm{LS}}(p,q)\le d_T(p,q)$$

if they are finite over $T(R)$.

We investigate the variation of the bottom of the (essential) spectra under a quasiconformal deformation. First, we consider the case that the bottom is $-\infty$ or

$+\infty$. Note that the condition $\lambda_0(p) = -\infty$ implies that the hyperbolic Riemann surface R_p does not satisfy the lower boundedness condition for the injectivity radii. Since any K-quasiconformal homeomorphism can move $\log \ell_p(c)$ by at most $\log K$ for each $c \in \mathcal{S}(R)$, the conditions $\lambda_0(p) = -\infty$ and $\lambda_{\text{ess}}(p) = -\infty$ are consistent throughout $p \in T(R)$. On the other hand, the condition $\lambda_{\text{ess}}(p) = +\infty$ implies that $\text{LS}_{\text{ess}}(p) = \emptyset$ and this is equivalent to the condition that $\text{LS}(p)$ is discrete. For the same reason as that stated above, we see that this condition is also consistent throughout $p \in T(R)$. In other words, we see the following.

Proposition 9.4. *If $\lambda_{\text{ess}}(p) = \pm\infty$ for some $p \in T(R)$, then $\lambda_{\text{ess}}(p) = \pm\infty$ for every $p \in T(R)$. Hence, $\lambda_{\text{ess}}(p) = \pm\infty$ is a property of the Teichmüller space $T(R)$.*

Next, we consider the general case. By virtue of Corollary 3.4, we have the invariance of the bottom of the essential spectra under a quasiconformal deformation whose support is on a compact subset.

Theorem 9.5. *If there exists a quasiconformal homeomorphism $f: R_p \to R_q$ between Riemann surfaces corresponding to p and q in $T(R)$ such that f is conformal off a compact subset $E \subset R_p$, then $\lambda_{\text{ess}}(p) = \lambda_{\text{ess}}(q)$.*

Proof. We have only to consider the case that $\lambda_{\text{ess}}(p) \neq \pm\infty$, and prove the claim that, for every $\epsilon > 0$, the number of $c \in \mathcal{S}(R)$ satisfying $\log \ell_q(c) \leq \lambda_{\text{ess}}(p) - \epsilon$ is finite. Then, we have $\lambda_{\text{ess}}(p) \leq \lambda_{\text{ess}}(q)$. By exchanging the roles of p and q considering f^{-1}, we conclude that $\lambda_{\text{ess}}(p) = \lambda_{\text{ess}}(q)$.

Let $K \geq 1$ be the maximal dilatation of the quasiconformal homeomorphism f. If $\log \ell_p(c) \geq \lambda_{\text{ess}}(p) + \log K$, then $\log \ell_q(c) \geq \lambda_{\text{ess}}(p)$ by Proposition 3.1. Hence, we have only to consider such $c \in \mathcal{S}(R)$ that satisfy $\log \ell_p(c) < \lambda_{\text{ess}}(p) + \log K$. On the other hand, Corollary 3.4 asserts that, if $c \in \mathcal{S}(R)$ satisfies $\log \ell_p(c) \geq \lambda_{\text{ess}}(p) - \epsilon/2$ and $\log \ell_q(c) \leq \lambda_{\text{ess}}(p) - \epsilon$, then

$$\log \alpha = \log \left\{ K + (1 - K) \frac{2}{\pi} \arctan(\sinh d_h(c, E)) \right\} \geq \frac{\epsilon}{2}.$$

This implies that the distances $d_h(c, E)$ are bounded above for such c. In addition, their lengths are bounded above by $K \exp(\lambda_{\text{ess}}(p))$. Hence, such c are finitely many. Since the number of $c \in \mathcal{S}(R)$ satisfying the condition $\log \ell_p(c) < \lambda_{\text{ess}}(p) - \epsilon/2$ is also finite, we obtain the above claim. □

Remark. By arguments similar to those presented above, we can extend Theorem 9.5 to the claim that $\mathrm{LS}_{\mathrm{ess}}(p) = \mathrm{LS}_{\mathrm{ess}}(q)$ is satisfied under any quasiconformal homeomorphism $f: R_p \to R_q$ with the dilatation on a compact support. Moreover, this is also true when f is an asymptotically conformal homeomorphism. The proof can be given by applying Lemma 3.7 in [11], which is a generalization of Corollary 3.4 in the present paper.

We conclude this section by presenting another continuous map on the Teichmüller space $T(R)$ invariant under $\mathrm{Mod}(R)$, which is given by using the length spectrum. Let $C(\mathbb{R})$ be the family of all closed subsets in $\mathbb{R}$ equipped with the Hausdorff distance H. We define the map $\eta: T(R) \to C(\mathbb{R})$ by $p \mapsto \mathrm{LS}(p)$. By Proposition 3.1, it is easy to see that η satisfies $H(\mathrm{LS}(p), \mathrm{LS}(q)) \le d_T(p,q)$; in particular, η is Lipschitz continuous.

10. Density of the region of stability

In this section, we prove that the region of stability $\Phi(\Gamma)$ for $\Gamma = \mathrm{Mod}(R)$ is dense in $T(R)$. Actually, we show the density of points $q \in T(R)$ satisfying the partial discreteness condition $r(q) > 0$ for the length spectrum. Then, by Theorem 9.2, we have the required result.

Theorem 10.1. *In every neighborhood U_p of every $p \in T(R)$, there exists q such that $r(q) > 0$.*

Corollary 10.2. *The region of stability $\Phi(\Gamma)$ for $\Gamma = \mathrm{Mod}(R)$ is dense in $T(R)$. In addition, if R satisfies the bounded geometry condition, then the region of discontinuity $\Omega(\Gamma)$ is dense in $T(R)$.*

The proof of Theorem 10.1 is divided into two cases according to the bottom of the spectra: Lemma 10.3 deals with the case $\lambda_{\mathrm{ess}}(p) > -\infty$ for $p \in T(R)$ and Lemma 10.4 deals with the case $\lambda_{\mathrm{ess}}(p) = -\infty$ included in the case $\lambda_0(p) = -\infty$ where R does not satisfy the lower boundedness condition. Recall Proposition 9.4, which states that these conditions are regarded as assumptions on the Teichmüller space $T(R)$.

Lemma 10.3. *Suppose that $\lambda_{\mathrm{ess}}(p) > -\infty$ for some $p \in T(R)$. Then, for every $\epsilon > 0$, there exist $q \in U(p,\epsilon)$ and $c \in \mathcal{S}(R)$ such that*

$$r_c(q) \ge \lambda_{\mathrm{ess}}(q) - \log \ell_q(c) > 0.$$

Proof. If $\lambda_{\mathrm{ess}}(p) = \infty$, then the statement is clearly satisfied. Hence, we may assume that $\lambda_{\mathrm{ess}}(p) < \infty$. Set an angle

$$\psi = 2 \arctan \frac{1}{\sinh(\exp \lambda_{\mathrm{ess}}(p))}.$$

Choose an amount $\phi \in (0, \pi)$ such that $\log(1 + \phi/\psi) < \epsilon$. Then, take $c \in \mathcal{S}(R)$ satisfying

$$\log \ell_p(c) - \lambda_{\mathrm{ess}}(p) < \log \left(1 + \frac{\phi}{\pi}\right) < \log 2.$$

Since $\ell_p(c)/2 < \exp \lambda_{\mathrm{ess}}(p)$, the collar lemma implies that there is a collar of the angle ψ for the corresponding simple closed geodesic $f(c)$ on $R_p = f(R)$.

Consider the canonical quasiconformal homeomorphism $\chi_{f(c),\phi}$ of R_p induced by the ϕ-grafting with respect to $f(c)$ and set $q = [\chi_{f(c),\phi} \circ f]$. Then, $d_T(p,q) \leq \log\{(\psi + \phi)/\psi\} < \epsilon$. By Lemma 3.5, the geodesic length $\ell_q(c)$ satisfies

$$\log \ell_q(c) \leq \log \ell_p(c) - \log \frac{\pi + \phi}{\pi} < \lambda_{\mathrm{ess}}(p),$$

where $\lambda_{\mathrm{ess}}(p) = \lambda_{\mathrm{ess}}(q)$ by Theorem 9.5. This implies that $r_c(q) \geq \lambda_{\mathrm{ess}}(q) - \log \ell_q(c) > 0$. □

Lemma 10.4. *Suppose that $\lambda_0(p) = -\infty$ for some $p \in T(R)$. Then, for every $\epsilon > 0$, there exist $q \in U(p,\epsilon)$ and $c \in \mathcal{S}(R)$ such that $r_c(q) > 0$.*

Proof. Take an element $c \in \mathcal{S}(R)$ of sufficiently small $\ell_p(c)$ satisfying

$$\psi := 2 \arctan \frac{1}{\sinh(2\ell_p(c))} \geq \frac{11}{12}\pi.$$

Choose an amount $\phi \in (0,\pi)$ such that $3\rho := \log\left(1 + \frac{\phi}{\psi}\right) < \epsilon$. Then, a simple calculation gives

$$2\rho < \log\left(1 + \frac{\phi}{\pi}\right) < \log 2.$$

Consider an open interval $I(x,\rho) \subset \mathbb{R}$ with center $x := \log \ell_p(c)$ and radius ρ. Let $\{c_i\}_{i=1}^{\infty}$ be a family of all elements in $\mathcal{S}(R)$ except c such that $\log \ell_p(c_i)$ belongs to $I(x,\rho)$. Since $\rho < \log 2/2 < \log 4$, it follows that $\ell_p(c_i)/2 < 2\ell_p(c)$, from which the collar lemma ensures that there is a collar of the angle $\psi > \pi/2$ for each simple closed geodesic $f(c_i)$ on $R_p = f(R)$. In particular, $\sinh(\ell_p(c_i)/2) < 1$. On the other hand, the width ω_i of the canonical collar of $f(c_i)$ satisfies

$$\sinh \omega_i = \frac{1}{\sinh(\ell_p(c_i)/2)} > 1.$$

This implies that $\omega_j > \operatorname{arcsinh} 1 > \ell_p(c_i)/2$ for any i and j. If $f(c_i)$ intersects a distinct $f(c_j)$, it must take at least length $2\omega_j$ to pass the canonical collar of $f(c_j)$. Hence, this inequality guarantees that the simple closed geodesics $\{c_i\}$ are mutually disjoint.

For each $i \in \mathbb{N}$, we perform a grafting by an amount ϕ with respect to $f(c_i)$. This is obtained on the canonical collar $A^*(f(c_i))$ as a quasiconformal homeomorphism of the maximal dilatation not greater than $(\psi+\phi)/\psi = \exp(3\rho)$. Let χ be the quasiconformal homeomorphism of R_p induced by all these graftings with respect to $\{f(c_i)\}$ and set $q = [\chi \circ f]$. Then, χ is $\exp(3\rho)$-quasiconformal; hence, $d_T(p, q) < \epsilon$.

By Lemma 3.5 and the subsequent remark, the geodesic length $\ell_q(c_i)$ satisfies

$$\log \ell_q(c_i) \le \log \ell_p(c_i) - \log \frac{\pi + \phi}{\pi} < \log \ell_p(c_i) - 2\rho.$$

Hence, $\log \ell_q(c_i) \notin I(x, \rho)$ for every i.

Next, we consider all $c' \in \mathcal{S}(R)$ with $\log \ell_p(c') \in I(x, 4\rho) - I(x, \rho)$ or with $c' = c$. Any other $c'' \in \mathcal{S}(R)$ with $\log \ell_p(c'') \notin I(x, 4\rho)$ does not satisfy $\log \ell_q(c'') \in I(x, \rho)$ because χ is $\exp(3\rho)$-quasiconformal; hence, $|\log \ell_p(c'') - \log \ell_q(c'')| \le 3\rho$ by Proposition 3.1. Since $4\rho < \log 4$, we still have $\ell_p(c')/2 < 2\ell_p(c)$, which implies that each $f(c')$ has a collar of the angle ψ disjoint from all $A^*(f(c_i))$. Since χ is conformal on $A^*(f(c'))$, Theorem 3.3 yields

$$\begin{aligned} |\log \ell_p(c') - \log \ell_q(c')| &\le \log \left\{1 + \frac{(\exp(3\rho) - 1)(\pi - \psi)}{\pi}\right\} \\ &< (\exp(3\rho) - 1)\frac{\pi - \psi}{\pi} \\ &< 6\rho \cdot \frac{1}{12} = \frac{\rho}{2}. \end{aligned}$$

This implies that $\log \ell_q(c)$ belongs to $I(x, \rho/2)$ but $\log \ell_q(c')$ does not belong to $I(x, \rho/2)$ for any other $c' \in \mathcal{S}(R)$. Thus, we have $r_c(q) > 0$. □

Consider the projection $\pi\colon T(R) \to M(R)$ and the moduli space of the stable points $M_\Phi(R) = \pi(\Phi(\Gamma))$. Corollary 10.2 implies that $M_\Phi(R)$ is dense in $M(R)$. Moreover, since $\Phi(\Gamma)$ is open by Theorem 5.2, the complement $M(R) - M_\Phi(R)$ is closed; hence, it is nowhere dense. On the other hand, the moduli space $M(R)$ can fail the first separability axiom as Corollary 6.5 shows and the closure of a point set can be a larger set in this case. However, we see that this closure cannot be so large.

Proposition 10.5. *The closure of a point set $\{\sigma\}$ for any $\sigma \in M(R)$ has no interior.*

Proof. If $\sigma \in M_\Phi(R)$, then the closure of $\{\sigma\}$ coincides with itself. If $\sigma \in M(R) - M_\Phi(R)$, then the closure of $\{\sigma\}$ is contained in the closed set $M(R) - M_\Phi(R)$, which is nowhere dense. □

In Section 13, we will extend this result to any countable subset of $M(R)$.

11. Connectivity of the region of stability

In this section, we will prove that the region of stability $\Phi(\Gamma)$ for $\Gamma = \mathrm{Mod}(R)$ is connected. The method for showing this property is similar to the method presented in the previous section. Specifically, we again utilize the partial discreteness of the length spectrum. In fact, we prove a stronger result than the global connectivity of $\Phi(\Gamma)$ as follows.

Theorem 11.1. *For every $p \in T(R)$ and every $r > 0$, there exists a positive number $C > 0$ depending continuously on p and r that satisfies the following property: any distinct points q_1 and q_2 in $U(p,r) \cap \Phi(\Gamma)$ for $\Gamma = \mathrm{Mod}(R)$ can be connected by a path in $\Phi(\Gamma)$ whose length is less than $C d_T(q_1, q_2)$.*

Corollary 11.2. *The region of stability $\Phi(\Gamma)$ for $\Gamma = \mathrm{Mod}(R)$ is connected. In addition, if R satisfies the bounded geometry condition, then the region of discontinuity $\Omega(\Gamma)$ is connected.*

Proof of Theorem 11.1. We divide the proof into two cases as in the proof of Theorem 10.1:

(a) $\lambda_{\mathrm{ess}}(p) > -\infty$;

(b) $\lambda_0(p) = -\infty$.

We may also assume that $\lambda_{\mathrm{ess}}(p) < \infty$. Fix $\epsilon > 0$ such that $U(q_1, \epsilon) \subset U(p,r) \cap \Phi(\Gamma)$ and $U(q_2, \epsilon) \subset U(p,r) \cap \Phi(\Gamma)$.

Case (a). By Lemma 10.3, for each $i = 1, 2$, there exist $q_i' = [f_i] \in U(q_i, \epsilon)$ and $c_i \in \mathcal{S}(R)$ such that $\lambda_{\mathrm{ess}}(q_i') - \log \ell_{q_i'}(c_i) > 0$. Moreover, we can choose c_1 and c_2 such that the hyperbolic distance $d_h(c_1, c_2)$ is sufficiently large. Take a K-quasiconformal homeomorphism f of $R_{q_1'} = f_1(R)$ onto $R_{q_2'} = f_2(R)$ such that $0 < \log K < d_T(q_1, q_2) + 2\epsilon$.

Set $\phi := (K^2 - 1)\pi$ and consider a one-parameter family of the canonical quasiconformal homeomorphisms $\chi_{f_1(c_1), t\phi}$ of $R_{q_1'} = f_1(R)$ induced by the $(t\phi)$-grafting with respect to $f_1(c_1)$ for $0 \le t \le 1$. This defines a path $\{\alpha(t)\}_{0 \le t \le 1}$ in $T(R)$ by $\alpha(t) = [\chi_{f_1(c_1), t\phi} \circ f_1]$. By Lemma 3.5, the geodesic length of c_1 satisfies

$$\ell_{\alpha(t)}(c_1) \le \frac{\pi}{\pi + t\phi} \ell_{\alpha(0)}(c_1) \le \ell_{q_1'}(c_1).$$

Since $\lambda_{\mathrm{ess}}(\alpha(t)) = \lambda_{\mathrm{ess}}(\alpha(0)) = \lambda_{\mathrm{ess}}(q_1')$ for every t by Theorem 9.5, we see that

$$r_{c_1}(\alpha(t)) \ge \lambda_{\mathrm{ess}}(\alpha(t)) - \log \ell_{\alpha(t)}(c_1) > 0,$$

which implies that the path $\alpha(t)$ is contained in $\Phi(\Gamma)$. Moreover, for $t = 1$, we have

$$\lambda_{\mathrm{ess}}(\alpha(1)) - \log \ell_{\alpha(1)}(c_1) > \log \frac{\pi + \phi}{\pi} = 2 \log K.$$

Let ψ be the angle of the canonical collar $A^*(f_1(c_1))$. Since each $\chi_{f_1(c_1),t\phi}$ is obtained by linearly stretching $A^*(f_1(c_1))$, we can estimate the distance between any two points on the path α and hence the length of α (alternatively, we may consider a Beltrami disk $\mathbb{D} \to T(R)$ defined by the complex dilatation of $\chi_{f_1(c_1),\phi}$ and obtain this estimate as explained below). It is bounded above by $\log K'$ for

$$K' = \frac{\psi + \phi}{\psi} = 1 + \frac{(K^2-1)\pi}{\psi},$$

which is the maximal dilatation of $\chi = \chi_{f_1(c_1),\phi}$. Here, the angle ψ of $A^*(f_1(c_1))$ is estimated by

$$\psi = 2\arctan\frac{1}{\sinh(\ell_{q_1'}(c_1)/2)} > 2\arctan\frac{1}{\sinh\{\exp(\lambda_{\mathrm{ess}}(p)+r)/2\}}.$$

Next, we consider a deformation of $R_{q_2'} = f_2(R)$. Define a Beltrami coefficient μ on $R_{q_2'}$ by $\mu = 0$ on $R_{q_2'} - f(A^*(f_1(c_1)))$ and $\mu = \mu_{\chi\circ f^{-1}}$ on $f(A^*(f_1(c_1)))$, where $\mu_{\chi\circ f^{-1}}$ denotes the complex dilatation of $\chi \circ f^{-1}$. Then, take a one-parameter family of quasiconformal deformations h_t of $R_{q_2'}$ for $0 \le t \le 1$, where h_t is the quasiconformal homeomorphism with the complex dilatation $t\mu$. This defines a path $\{\beta(t)\}_{0\le t\le 1}$ in $T(R)$ by $\beta(t) = [h_t \circ f_2]$. Under this deformation, Corollary 3.4 ensures that the geodesic length $\ell_{\beta(t)}(c_2)$ does not change significantly and hence satisfies the condition $\lambda_{\mathrm{ess}}(q_2') - \log \ell_{\beta(t)}(c_2) > 0$, since we have chosen the hyperbolic distance $d_h(c_1, c_2)$ to be sufficiently large. Again, by $\lambda_{\mathrm{ess}}(\beta(t)) = \lambda_{\mathrm{ess}}(\beta(0)) = \lambda_{\mathrm{ess}}(q_2')$, we have

$$r_{c_2}(\beta(t)) \ge \lambda_{\mathrm{ess}}(\beta(t)) - \log \ell_{\beta(t)}(c_2) > 0,$$

which implies that the path $\beta(t)$ is contained in $\Phi(\Gamma)$. Moreover, the length of β is bounded above by $\log K(\chi \circ f^{-1}) \le \log K' + \log K$. Indeed, we consider a holomorphic map (Beltrami disk) from the unit disk $\mathbb{D}$ into $T(R)$ by assigning $z \in \mathbb{D}$ to a quasiconformal deformation of $R_{q_2'}$ with a Beltrami coefficient $z\mu/\|\mu\|_\infty$. Then, the path $\beta(t)$ is the image of the interval $[0, \|\mu\|_\infty] \subset \mathbb{D}$, and the contraction of the Kobayashi distance, which coincides with the Teichmüller distance, gives the claim.

Finally, we connect $\alpha(1)$ and $\beta(1)$ by a path in $\Phi(\Gamma)$. We define a Beltrami coefficient μ' on $R_{\alpha(1)} = \chi(R_{q_1'})$ by $\mu' = 0$ on $\chi(A^*(f_1(c_1)))$ and $\mu' = \mu_{f\circ\chi^{-1}}$ on $\chi(R_{q_1'} - A^*(f_1(c_1)))$. Then, take a one-parameter family of quasiconformal deformations h_t' of $R_{\alpha(1)}$ for $0 \le t \le 1$, where h_t' is the quasiconformal homeomorphism with the complex dilatation $t\mu'$. This defines a path $\{\eta(t)\}_{0\le t\le 1}$ in $T(R)$ by $\eta(t) = [h_t' \circ \chi \circ f_1]$ with $\eta(0) = \alpha(1)$ and $\eta(1) = \beta(1)$. We remark that the maximal dilatation of h_t' is bounded above by K because χ^{-1} is conformal outside $\chi(A^*(f_1(c_1)))$. Hence,

$$\begin{aligned} r_{c_1}(\eta(t)) &\geq \lambda_{\mathrm{ess}}(\eta(t)) - \log \ell_{\eta(t)}(c_1) \\ &\geq (\lambda_{\mathrm{ess}}(\eta(0)) - \log K) - (\log \ell_{\eta(0)}(c_1) + \log K) \\ &= \lambda_{\mathrm{ess}}(\alpha(1)) - \log \ell_{\alpha(1)}(c_1) - 2\log K > 0 \end{aligned}$$

for every t. This implies that the path $\eta(t)$ is contained in $\Phi(\Gamma)$. For the same reason as before, the length of η is bounded above by $\log K$.

Therefore, q_1' and q_2' are connected by the composition of the paths $\alpha \cdot \eta \cdot \beta^{-1}$ in $\Phi(\Gamma)$ whose length is bounded above by

$$\begin{aligned} 2\log K + 2\log K' &= 2\log K + 2\log\Big(1 + \frac{(K^2-1)\pi}{\psi}\Big) \\ &= 2\Big(1 + \frac{2\pi}{\psi}\Big)(\log K - \varepsilon(K,\psi)), \end{aligned}$$

where $\varepsilon(K,\psi)$ is some positive function of $K > 1$ and $0 < \psi < 2\pi$. Hence, q_1 and q_2 can be connected by a path in $\Phi(\Gamma)$ whose length is 2ϵ greater than that of the above path. Recall that $\log K < d_T(q_1,q_2) + 2\epsilon$. Since $\epsilon > 0$ can be taken to be arbitrarily small, we set $\epsilon = \varepsilon(K,\psi)/3$. Then, we conclude that the length of this path in $\Phi(\Gamma)$ connecting q_1 and q_2 is less than $C d_T(q_1,q_2)$, where $C = 2(1 + 2\pi/\psi)$ depends only on p and r.

Case (b). By Lemma 10.4, for each $i = 1,2$, there exist $q_i' = [f_i] \in U(q_i,\epsilon)$ and $c_i \in \mathcal{S}(R)$ such that $r_{c_i}(q_i') > 0$. We also require that there is another $c_0 \in \mathcal{S}(R)$ such that $r_{c_0}(q_1') > 0$ and that the geodesic length of c_0 is sufficiently small. This is possible by arguments similar to Lemma 10.4 for finding q_1' that makes the two spectra isolated simultaneously. Take a K-quasiconformal homeomorphism f of $R_{q_1'} = f_1(R)$ onto $R_{q_2'} = f_2(R)$ such that $\log K < d_T(q_1,q_2) + 2\epsilon$.

We define a one-parameter family of quasiconformal deformations χ_t of $R_{q_1'}$ for $0 \leq t \leq 1$ as follows. Consider all $c' \in \mathcal{S}(R)$ except c_0 such that $\log \ell_{q_1'}(c')$ belongs to an open interval $I = I(\log \ell_{q_1'}(c_0), \frac{9}{4}\log K)$. Since c_0 can be taken to be arbitrarily short, we may assume that all c' are mutually disjoint and that c_1 and c_2 are not among such c'. Set $\phi := (K^{9/2} - 1)\pi$ and perform grafting by an amount $t\phi$ with respect to each $f_1(c')$. In each canonical collar $A^*(f_1(c'))$, we take a smaller collar $A^{**}(f_1(c'))$ with a uniform angle $\psi = \psi(K)$ such that the distance to the boundary $\partial A^*(f_1(c'))$ is sufficiently large and ψ is sufficiently close to π. The collar lemma makes this possible by choosing c_0 to be arbitrarily short. Then, χ_t is defined to be a quasiconformal homeomorphism obtained through linear stretching of all $A^{**}(f_1(c'))$ by $t\phi$. This determines a path $\{\alpha(t)\}_{0\leq t\leq 1}$ in $T(R)$ by $\alpha(t) = [\chi_t \circ f_1]$.

Since the support of χ_t is taken to be far from any simple closed geodesic disjoint from all c', Corollary 3.4 implies that this grafting process does not influence the value r_{c_1} significantly. Hence, the condition $r_{c_1}(\alpha(t)) > 0$ is maintained throughout; thus, the path $\alpha(t)$ is contained in $\Phi(\Gamma)$. By Lemma 3.5, we have

$$\log \ell_{\alpha(0)}(c') - \log \ell_{\alpha(1)}(c') \geq \log \frac{\pi + \phi}{\pi} = \frac{9}{2} \log K.$$

This implies that $\log \ell_{\alpha(1)}(c') \notin I$ for all c'. Owing to the slight influence on the other geodesic lengths, again we see that the difference between $\log \ell_{\alpha(1)}(c_0)$ and any other $\log \ell_{\alpha(1)}(c)$ ($c \neq c_0$) is greater than $\frac{17}{8} \log K$, which implies that $r_{c_0}(\alpha(1)) > 2 \log K$. The length of α is bounded above by $\log K'$ for the maximal dilatation $K' = (\psi + \phi)/\psi$ of $\chi = \chi_1$. Since $\psi = \psi(K)$ is arbitrarily close to π and ϕ is chosen to be $(K^{9/2} - 1)\pi$, we can represent $K' = K^5/\exp(\varepsilon(K))$ by using a positive function $\varepsilon(K)$ of $K > 1$.

The deformation of $R_{q_2'}$ is defined similarly to case (a). Set a Beltrami coefficient μ on $R_{q_2'}$ by $\mu = 0$ on $R_{q_2'} - f(\bigcup A^{**}(f_1(c')))$ and $\mu = \mu_{\chi \circ f^{-1}}$ on $f(\bigcup A^{**}(f_1(c')))$, where the union is taken over all c' for which grafting has been performed. Then, take a one-parameter family of quasiconformal deformations h_t of $R_{q_2'}$ with the complex dilatation $t\mu$ for $0 \leq t \leq 1$. This defines a path $\{\beta(t)\}_{0 \leq t \leq 1}$ in $T(R)$ by $\beta(t) = [h_t \circ f_2]$. Corollary 3.4 again states that h_t does not change the value of r_{c_2} significantly; hence, $r_{c_2}(\beta(t)) > 0$ for every t, which implies that the path $\beta(t)$ is contained in $\Phi(\Gamma)$. The length of β is bounded above by $\log K(\chi \circ f^{-1}) \leq \log K' + \log K$.

We connect $\alpha(1)$ and $\beta(1)$ by a path in $\Phi(\Gamma)$. As before, a Beltrami coefficient μ' on $R_{\alpha(1)} = \chi(R_{q_1'})$ is defined by $\mu' = 0$ on $\chi(\bigcup A^{**}(f_1(c')))$ and $\mu' = \mu_{f \circ \chi^{-1}}$ on $\chi(R_{q_1'} - \bigcup A^{**}(f_1(c')))$. Then, take a one-parameter family of quasiconformal deformations h_t' of $R_{\alpha(1)}$ with the complex dilatation $t\mu'$ for $0 \leq t \leq 1$. This defines a path $\{\eta(t)\}_{0 \leq t \leq 1}$ in $T(R)$ by $\eta(t) = [h_t' \circ \chi \circ f_1]$ with $\eta(0) = \alpha(1)$ and $\eta(1) = \beta(1)$. Since $r_{c_0}(\eta(0)) = r_{c_0}(\alpha(1)) > 2 \log K$ and the maximal dilatation of h_t' is bounded above by K, we see that $r_{c_0}(\eta(t)) > 0$ for every t; hence, the path $\eta(t)$ is contained in $\Phi(\Gamma)$. The length of η is bounded above by $\log K$.

Therefore, q_1' and q_2' are connected by the composition of the paths $\alpha \cdot \eta \cdot \beta^{-1}$ in $\Phi(\Gamma)$ whose length is bounded above by

$$2 \log K + 2 \log K' = 2 \log K + 2 \log \left(\frac{K^5}{\exp(\varepsilon(K))} \right) = 12 \log K - 2\varepsilon(K).$$

Hence, q_1 and q_2 can be connected by a path in $\Phi(\Gamma)$ whose length is 2ϵ greater than that of the above path. Since $\log K < d_T(q_1, q_2) + 2\epsilon$ and $\epsilon > 0$ can be taken to be arbitrarily small, we set $\epsilon = \varepsilon(K)/13$. Then, we conclude that the length of this path in $\Phi(\Gamma)$ connecting q_1 and q_2 is less than $12 d_T(q_1, q_2)$. □

Remark. The positive number $C > 0$ in Theorem 11.1 has been taken locally uniformly in case (a) but globally uniformly in case (b). We expect that there should be a globally uniform constant C for every Teichmüller space.

12. Stabilized limit points are not dense

We will prove that the set of stabilized limit points is not dense in the limit set. This is in contrast to the nature of familiar dynamics, such as Kleinian groups and iterations of rational maps. Strictly speaking, exceptional cases might exist where the above statement is not true, e.g., the case in which $\Lambda(\Gamma)$ coincides with the exceptional limit set $E(\Gamma)$. Hence, a certain restriction on the limit set is necessary to justify the claim.

Definition. For a subgroup $\Gamma \subset \mathrm{Mod}(R)$, a limit point $p \in \Lambda(\Gamma)$ belongs to the *practically exceptional limit set* denoted by $[E](\Gamma)$ if $p \notin \overline{\Lambda_0(\Gamma)}$ and if there exists a neighborhood U of p in $T(R)$ such that $U \cap \Lambda(\Gamma) \subset \Lambda^2_\infty(\Gamma)$.

By definition, $E(\Gamma) \subset [E](\Gamma)$ is obvious. We expect these sets to be coincident, but do not pursue this problem herein. Hence, we employ the practically exceptional limit set $[E](\Gamma)$ instead of $E(\Gamma)$ for our arguments and formulate our statement as follows.

Theorem 12.1. *For a subgroup Γ of $\mathrm{Mod}(R)$, if $\Lambda(\Gamma) - \overline{[E](\Gamma)}$ is not empty, then the stabilized limit set $\Lambda_\infty(\Gamma)$ is nowhere dense in $\Lambda(\Gamma) - \overline{[E](\Gamma)}$.*

First, we consider the subset $\Lambda^1_\infty(\Gamma)$ of $\Lambda_\infty(\Gamma)$ and prove that $\Lambda^1_\infty(\Gamma)$ is nowhere dense in the entire limit set $\Lambda(\Gamma)$. This is a crucial step in the proof of Theorem 12.1.

Theorem 12.2. *Let p_0 be a point in $\Lambda^1_\infty(\Gamma)$ for a subgroup Γ of $\mathrm{Mod}(R)$. Then, in every neighborhood U of p_0, there exists a generic limit point $q \in \Lambda_0(\Gamma)$ that does not belong to the closure $\overline{\Lambda_\infty(\Gamma)}$ of the stabilized limit set. In particular, $\Lambda^1_\infty(\Gamma)$ is nowhere dense in $\Lambda(\Gamma)$.*

In particular, this result implies that the limit set $\Lambda(\Gamma)$ contains a strictly smaller Γ-invariant closed subset $\overline{\Lambda^1_\infty(\Gamma)}$ whenever Γ contains an elliptic element of infinite order. Hence, in this case, the orbit of any limit point of Γ is not dense in $\Lambda(\Gamma)$. Moreover, Theorem 12.2 extends Lemma 8.1, where we imposed countability on the subgroup Γ.

Corollary 12.3. *For an arbitrary subgroup Γ of $\mathrm{Mod}(R)$ and for every open subset U of $T(R)$, if $U \cap \Lambda(\Gamma) = U \cap \Lambda_\infty(\Gamma)$, then they coincide with $U \cap \Lambda^2_\infty(\Gamma)$.*

In Corollary 8.4, we had given conditions equivalent to weak discontinuity. Theorem 12.1 further yields the relationship of these conditions with the fullness of the stabilized limit set as in the following corollary. We remark that $[E](\Gamma) = E(\Gamma)$ is satisfied under the assumption that $\Lambda(\Gamma) = \Lambda_\infty(\Gamma)$ because this assumption is equivalent to the condition $\Lambda(\Gamma) = \Lambda^2_\infty(\Gamma)$ by Corollary 12.3.

Corollary 12.4. *The condition* $\Lambda(\Gamma) = E(\Gamma)$ *equivalent to the weak discontinuity of* $\Gamma \subset \mathrm{Mod}(R)$ *satisfies the following implication*:

$$\Lambda(\Gamma) = E(\Gamma) \implies \Lambda(\Gamma) = \Lambda_\infty(\Gamma) \implies \Lambda(\Gamma) = \overline{E(\Gamma)}.$$

We expect the first implication above to be strict, but we do not expect the second one to be strict.

Proof of Theorem 12.2. Without loss of generality, we may assume $p_0 \in \Lambda^1_\infty(\Gamma)$ to be the origin o of the Teichmüller space $T(R)$. There is a conformal automorphism $g \in \mathrm{Conf}(R) \subset \mathrm{MCG}(R)$ of infinite order such that $[g]_* \in \Gamma$.

We will find a simple closed geodesic c on R in the following manner. If the set of lengths of all simple closed geodesics on R modulo multiplicity by $\langle g \rangle$ has an isolated point, then we choose c corresponding to this point. Otherwise, (a) if the lengths of simple closed geodesics on R are bounded from below, then we choose c whose geodesic length is sufficiently close to the infimum; (b) if R has an arbitrarily short simple closed geodesic, then we choose a sufficiently short c. Note that, in case (b), $\{g^n(c)\}_{n \in \mathbb{Z}}$ are mutually disjoint for any sufficiently short c.

We observe the images of the simple closed geodesic c under $\langle g \rangle$. Since $\langle g \rangle$ acts properly discontinuously on R, there is a positive integer t such that the images $\{g^{tn}(c)\}_{n \in \mathbb{Z}}$ are mutually disjoint. Then, by replacing g with g^t, we have a quotient Riemann surface $\widehat{R} = R/\langle g \rangle$ on which c projects injectively. In addition, by choosing a larger t, we may assume that the distance between c and $g(c)$ is sufficiently large. We can also avoid the case in which $\dim T(\widehat{R}) = 0$ by this replacement.

Choose an arbitrary neighborhood U of $p_0 = o \in T(R)$. This defines a neighborhood $\widehat{U}$ of the origin $\hat{o}$ in $T(\widehat{R})$ such that $\widehat{U}$ is embedded in U by the inclusion $T(\widehat{R}) \hookrightarrow T(R)$. Recall that, for the elliptic modular transformation $\gamma = [g]_* \in \mathrm{Mod}(R)$, the Teichmüller space $T(\widehat{R})$ is identified with the fixed point locus $\mathrm{Fix}(\gamma)$ in $T(R)$. The mapping class $[g]$ has a conformal representative on the Riemann surface R_p corresponding to any $p \in \mathrm{Fix}(\gamma)$.

We give a deformation of $\widehat{R}$ within $\widehat{U}$ to find a point $p \in U \cap \mathrm{Fix}(\gamma)$ in $T(R)$ having a suitable property. Let $\hat{c}$ be the simple closed geodesic on $\widehat{R}$, which is the injective image of c under the projection $R \to \widehat{R}$. In case (a), by arguments similar to those given in the proof of Lemma 10.3, we have $p \in U \cap \mathrm{Fix}(\gamma)$ such that $\log \ell_p(c)$ is minimal and isolated in $\mathrm{LS}(p)$ by decreasing the length of $\hat{c}$. In case (b), we use the arguments for Lemma 10.4 to make $\log \ell_p(c)$ isolated in

$\mathrm{LS}(p)$ by sweeping out all nearby lengths of simple closed geodesics. In both these cases, we have the isolated point $\log \ell_p(c)$ in $\mathrm{LS}(p)$ modulo multiplicity by $\langle g \rangle$.

We apply the following lemma to this situation.

Lemma 12.5. *Let* $[g] \in \mathrm{MCG}_p(R)$ *be a conformal mapping class at* $p \in T(R)$*, and assume that the length spectrum* $\mathrm{LS}(p)$ *has an isolated point of infinite multiplicity owing to mutually disjoint simple closed geodesics* $c_n = g^{-n}(c)$ *for all* $n \in \mathbb{Z}$*. Then, there exists a neighborhood* V *of* p *such that every conformal mapping class* $[h] \in \mathrm{MCG}_{p'}(R)$ *of infinite order at* $p' \in V$ *acts on the family* $\{c_n\}_{n \in \mathbb{Z}}$ *as a translation, i.e., there is some* $k \in \mathbb{Z} - \{0\}$ *such that* $h(c_n) \sim c_{n+k}$ *for every* $n \in \mathbb{Z}$*.*

Proof. By contrast, suppose that there is no such neighborhood of p. Then, there are $p' \in T(R)$ arbitrarily close to p and $[h] \in \mathrm{MCG}_{p'}(R)$ of infinite order such that $[h]$ does not act on $\{c_n\}_{n \in \mathbb{Z}}$ as a translation. Since p and p' are close, $[h]$ is realized on R_p as a quasiconformal automorphism with sufficiently small dilatation. Then, $[h]$ must give a permutation on the family $\{c_n\}$ because $\{\log \ell_p(c_n)\}$ are isolated in $\mathrm{LS}(p)$ and Proposition 3.1 makes it impossible for such a quasiconformal automorphism to send c_n to a simple closed curve different from $\{c_n\}_{n \in \mathbb{Z}}$ by jumping the spectral gap. On the other hand, $[h]$ is not a translation by assumption, nor is it an involution of the form $h(c_n) \sim c_{-n+k}$ for some $k \in \mathbb{Z}$. Indeed, if so, $h^2(c_n) \sim c_n$ and the conformal mapping class $[h^2]$ would keep each simple closed geodesic c_n invariant. This is possible only if $[h]$ is of finite order, which violates the assumption.

If $[h]$ give a permutation on $\{c_n\}$ but it is neither a translation nor an involution, then there must be consecutive pairs c_n and c_{n+1} such that their images $h(c_n)$ and $h(c_{n+1})$ are not consecutive. Then, we see that the distance between the geodesic realizations of c_n and c_{n+1} on R_p is strictly smaller than the distance between the geodesic realizations of $h(c_n)$ and $h(c_{n+1})$ on R_p. To see this claim, we consider the lifts of $\{c_n\}$ to the universal cover $\mathbb{D}$ of R_p and their intersection or shortest connection with the axis corresponding to $[g]$. If the lifts of $\{c_n\}$ intersect the axis, the claim is clear; otherwise, hyperbolic trigonometry on right-angled hexagons yields the claim. However, this situation is impossible for the quasiconformal automorphism realizing $[h]$ with sufficiently small dilatation, which can be seen from Proposition 3.2. Thus, we have reached a contradiction and the proof is complete. □

The proof of Theorem 12.2 also requires the following fact, which has been used in [13] to find a generic limit point of an infinite elliptic cyclic subgroup $\Gamma \subset \mathrm{Mod}(R)$ that does not lie on $\overline{\Lambda_\infty(\Gamma)}$ for a particular Riemann surface R.

Proposition 12.6. *Consider the Banach space ℓ^∞ of all bilateral infinite sequences of real numbers with supremum norm, i.e.,*

$$\ell^\infty = \{\xi = (\xi_n)_{n\in\mathbb{Z}} \mid \|\xi\|_\infty = \sup|\xi_n| < \infty\}.$$

Let $\sigma\colon \ell^\infty \to \ell^\infty$ be a shift operator defined by $(\xi_n) \mapsto (\xi_{n+1})$. Then, there exists an element $\xi = (\xi_n) \in \ell^\infty$ with $0 \le \xi_n \le 1$ for all $n \in \mathbb{Z}$ and with $\xi_0 = 0$ that satisfies the following properties:

(1) *there exists a subsequence $\{k(j)\}_{j\in\mathbb{N}} \subset \mathbb{Z}$ such that $\|\sigma^{k(j)}(\xi) - \xi\|_\infty \to 0$ as $j \to \infty$;*

(2) *for every $k \in \mathbb{Z} - \{0\}$, there exists an integer $m \in \mathbb{Z}$ such that $\xi_{mk} \ge 1/2$.*

Proof of Theorem 12.2 continued. For the given neighborhood U of $p_0 = o \in T(R)$ and the selected point $p = [f] \in U$, we choose a neighborhood V of p as in Lemma 12.5, satisfying $V \subset U$. We will find a point q in V that belongs to $\Lambda_0(\Gamma)$ but not to $\overline{\Lambda_\infty(\Gamma)}$.

Take the canonical collar $A^*(f(c))$ of the simple closed geodesic $f(c)$ on $R_p = f(R)$ whose angle is $\psi = 2\arctan(\sinh\omega)$ for $\sinh\omega = 1/\sinh(\ell_p(c)/2)$. We do this for each $c_n = g^{-n}(c)$ $(n \in \mathbb{Z})$ and have the canonical collars $A^*(f(c_n))$, which are mutually disjoint by the collar lemma. For the element $\xi = (\xi_n)_{n\in\mathbb{Z}} \in \ell^\infty$ as in Proposition 12.6 and for a positive constant $\theta > 0$, let χ be a quasiconformal homeomorphism of R_p defined by the $(\xi_n\theta)$-grafting with respect to c_n for all $n \in \mathbb{Z}$. We remark that χ is conformal off the union of the canonical collars $\bigcup_{n\in\mathbb{Z}} A^*(f(c_n))$. Set $q = [\chi \circ f]$. By choosing θ to be sufficiently small, we may assume that the maximal dilatation of χ is sufficiently small for q to stay within V. We will show that q satisfies the required properties.

We choose a subsequence $\{k(j)\}_{j=1}^\infty$ as in property (1) of Proposition 12.6 and consider the sequence $\{\gamma^{k(j)}(q)\}$ for $\gamma = [g]_* \in \mathrm{Mod}(R)$. For each $j \in \mathbb{N}$, there exists a quasiconformal homeomorphism between the Riemann surfaces corresponding to q and $\gamma^{k(j)}(q)$, which is obtained by linearly stretching the annuli $\chi(A^*(f(c_n)))$ for all n. It maps each annulus conformally equivalent to $A^{\psi+\xi_n\theta}$ onto an annulus $A^{\psi+\xi_{n+k(j)}\theta}$ with the maximal dilatation

$$\max\left\{\frac{\psi+\xi_{n+k(j)}\theta}{\psi+\xi_n\theta}, \frac{\psi+\xi_n\theta}{\psi+\xi_{n+k(j)}\theta}\right\}.$$

Then, the global maximal dilatation is bounded above by

$$\frac{\psi+\theta\|\sigma^{k(j)}(\xi)-\xi\|_\infty}{\psi}.$$

Hence, we have $d_T(\gamma^{k(j)}(q), q) \to 0$ as $j \to \infty$. This implies that $q \in \Lambda(\Gamma)$.

Here, we will estimate the lengths $\ell_q(c_n)$ from above for all $n \in \mathbb{Z} - \{0\}$ and from below for $n = 0$. Recall that $\chi\colon R_p \to R_q$ is given by the $(\xi_n\theta)$-grafting on each $A^*(f(c_n))$. Then, we see from Lemma 3.5 and the subsequent remark that

$$\ell_q(c_n) \leq \frac{\pi}{\pi + \xi_n \theta} \ell_p(c_0)$$

for $n \in \mathbb{Z} - \{0\}$. On the other hand, since $\chi \colon R_p \to R_q$ is conformal outside $\bigcup_{n \neq 0} A^*(f(c_n))$ and its maximal dilatation is bounded above by $(\psi + \theta)/\psi$, Corollary 3.4 yields $\ell_q(c_0) \geq \ell_p(c_0)/\alpha$, where

$$\alpha = \frac{\psi + \theta}{\psi} + \Big(1 - \frac{\psi + \theta}{\psi}\Big) \frac{2}{\pi} \arctan(\sinh \omega)$$

and ω is the distance from the geodesic realization of $f(c_0)$ to $\bigcup_{n \neq 0} A^*(f(c_n))$, which was assumed to be sufficiently large (where we chose the integer t), say,

$$\arctan(\sinh \omega) \geq \frac{\pi}{2} - \frac{\psi}{8}.$$

Then, the previous inequality becomes

$$\ell_q(c_0) \geq \frac{\pi}{\pi + \theta/4} \ell_p(c_0).$$

First, we will prove that q is not in the closure of $\Lambda^1_\infty(\Gamma)$. Suppose that there exists $p' \in \Lambda^1_\infty(\Gamma)$ in the neighborhood V, i.e., there exists some conformal mapping class $[h] \in \mathrm{MCG}_{p'}(R)$ of infinite order for $p' \in V$. Then, by Lemma 12.5, $[h]$ acts on $\{c_n\}$ by $h(c_n) = c_{n+k}$ for some $k \in \mathbb{Z} - \{0\}$. By property (2) of Proposition 12.6, we know that there is some m such that $\xi_{mk} \geq 1/2$. We compare the lengths $\ell_q(c_{mk})$ with $\ell_q(c_0)$. By the estimates obtained in the previous paragraph, we have

$$\frac{\ell_q(c_0)}{\ell_q(c_{mk})} \geq \frac{\pi + \theta/2}{\pi + \theta/4} =: C > 1.$$

By Proposition 3.1, this means that the maximal dilatation of any quasiconformal realization of the mapping class $[h^{mk}]$ on R_q is not less than C. In other words, $d_T(q, [h^{mk}]_*(q)) \geq \log C$. On the other hand, $[h^{mk}]_*(p') = p'$ since $[h]$ has a conformal realization on $R_{p'}$. Hence,

$$\begin{aligned}
2 d_T(q, p') &= d_T(q, p') + d_T([h^{mk}]_*(q), [h^{mk}]_*(p')) \\
&= d_T(q, p') + d_T(p', [h^{mk}]_*(q)) \\
&\geq d_T(q, [h^{mk}]_*(q)) \\
&\geq \log C,
\end{aligned}$$

from which we have $d_T(q, p') \geq \log C/2$. This implies that no limit point $p' \in \Lambda^1_\infty(\Gamma)$ can enter within the distance $\log C/2$ of q; hence, $q \notin \overline{\Lambda^1_\infty(\Gamma)}$.

Finally, we will show that q is not in the closure of $\Lambda^2_\infty(\Gamma)$ either. Then, we have $q \notin \overline{\Lambda_\infty(\Gamma)}$, which completes the proof. Suppose that there exists $p' \in \Lambda^2_\infty(\Gamma)$ in the neighborhood V. Then, $R_{p'}$ has infinitely many conformal automorphisms of finite order. By arguments similar to but easier than those of Lemma 12.5, we see that each mapping class of this conformal automorphism should keep every c_n invariant since a conformal mapping class of finite order cannot give a translation on the infinite family $\{c_n\}_{n\in\mathbb{Z}}$. However, it is impossible for the infinite group of conformal automorphisms to keep the same simple closed geodesic invariant because it acts on $R_{p'}$ properly discontinuously. □

Proof of Theorem 12.1. We prove that, in every neighborhood of $p_0 \in \Lambda_\infty(\Gamma) - \overline{[E](\Gamma)}$, there exists a limit point $q \in \Lambda_0(\Gamma)$ that does not belong to $\overline{\Lambda_\infty(\Gamma)}$. If $p_0 \in \Lambda^1_\infty(\Gamma)$, then Theorem 12.2 verifies this claim. Actually, this theorem obviously asserts a slightly stronger claim: if $p_0 \in \overline{\Lambda^1_\infty(\Gamma)}$, then we have a limit point $q \notin \overline{\Lambda_\infty(\Gamma)}$. Hence, we have only to consider the case where $p_0 \in \Lambda^2_\infty(\Gamma) - \overline{[E](\Gamma)} - \overline{\Lambda^1_\infty(\Gamma)}$. This condition implies that there exists a neighborhood W of p_0 that intersects neither $[E](\Gamma)$ nor $\Lambda^1_\infty(\Gamma)$.

By contrast, suppose that there exists a neighborhood U of $p_0 \in \Lambda^2_\infty(\Gamma)$ such that every limit point of Γ in U belongs to $\overline{\Lambda_\infty(\Gamma)}$. We may assume that U is contained in W, which intersects neither $[E](\Gamma)$ nor $\Lambda^1_\infty(\Gamma)$. By the definition of $[E](\Gamma)$, we see that every limit point of Γ in U belongs to both $\overline{\Lambda^2_\infty(\Gamma)}$ and $\Lambda_0(\Gamma)$. However, the following lemma shows that this is impossible. □

Lemma 12.7. *Let p_0 belong to $\Lambda^2_\infty(\Gamma)$ for a subgroup $\Gamma \subset \mathrm{Mod}(R)$. Assume that there exists a neighborhood U of p_0 in $T(R)$ such that $\Lambda(\Gamma) \cap U \subset \Lambda_0(\Gamma)$. Then, there exists a limit point $q \in U$ that does not belong to $\overline{\Lambda_\infty(\Gamma)}$.*

For the proof of this lemma, we prepare two claims, both of which are technical (see the remark after the proof of Proposition 12.9).

Proposition 12.8. *Let R be a planar Riemann surface and let G be an infinite subgroup of $\mathrm{Conf}(R)$, all of whose elements are of finite order. Assume that the orbifold $\widehat{R} = R/G$ is topologically infinite. Then, there exists a simple closed geodesic c on R such that $R - c$ consists of two topologically infinite subsurfaces of R.*

Proof. We choose a topologically finite geodesic subsurface $\widehat{S}$ of the hyperbolic orbifold $\widehat{R}$ with a geodesic boundary component $\hat{c}$ such that the connected component $\widehat{R}'$ of $\widehat{R} - \widehat{S}$ having $\hat{c}$ as a boundary component is topologically infinite. Note that any connected component of the inverse image $\kappa^{-1}(\hat{c})$ under the covering projection $\kappa\colon R \to \widehat{R}$ is also a simple closed geodesic since every element of G is of finite order. Furthermore, we may assume that a connected component

S of $\kappa^{-1}(\widehat{S}) \subset R$ has multiple boundary components of $\kappa^{-1}(\hat{c})$. Indeed, by taking a sufficiently large $\widehat{S}$, we can make the stabilizer of S in G non-cyclic, which implies that $\kappa^{-1}(\hat{c}) \cap S$ consists of multiple connected components.

Let c be a connected component of $\kappa^{-1}(\hat{c}) \cap S$. Since R is planar, $R - c$ consists of two connected components. Let R' be the component of $R - c$ disjoint from S. Since $\widehat{R}'$ is topologically infinite, so is R'. The other component of $R - c$ is also topologically infinite since it contains another component of $\kappa^{-1}(\hat{c}) \cap S$ and hence another component of $\kappa^{-1}(\widehat{R}')$. □

Let $\mathcal{S}^{\#}(R)$ be a subset of $\mathcal{S}(R)$ consisting of free homotopy classes of simple closed geodesics c on R that does not divide R or that divides R into two topologically infinite subsurfaces. Then, we define the restricted length spectrum

$$\mathrm{LS}^{\#}(p) = \mathrm{Cl}\{\log \ell(c) \mid c \in \mathcal{S}^{\#}(R)\}$$

for $p \in T(R)$ as well as $\lambda_0^{\#}(p) = \min \mathrm{LS}^{\#}(p)$. For $p = o$, we may use $\mathrm{LS}^{\#}(R)$ and $\lambda_0^{\#}(R)$ instead. If R is not planar, then $\mathrm{LS}^{\#}(R) \neq \emptyset$. However, even if R is planar, we can also assume that $\mathrm{LS}^{\#}(R) \neq \emptyset$ under the circumstances of Proposition 12.8.

Proposition 12.9. *Let R be a Riemann surface with $\mathrm{LS}^{\#}(R) \neq \emptyset$. For a subgroup G of $\mathrm{Conf}(R)$, let $\kappa\colon R \to \widehat{R}$ be the projection onto the orbifold $\widehat{R} = R/G$. Then, there exists some constant $\delta > 0$ such that if $c \in \mathcal{S}^{\#}(R)$ satisfies $\ell(c) < \exp(\lambda_0^{\#}(R)) + \delta$, then the image $\kappa(c)$ is a simple closed geodesic on $\widehat{R}$ (possibly, $\kappa|_c$ is not injective), including the case that $\kappa(c)$ is a geodesic segment connecting two cone points of order 2.*

Proof. We consider any sequence $\{c_n\}_{n=1}^{\infty} \subset \mathcal{S}^{\#}(R)$ such that $\ell(c_n)$ converges to $\exp(\lambda_0^{\#}(R))$ as $n \to \infty$. Suppose that $\hat{c}_n = \kappa(c_n)$ is a closed geodesic but not simple on $\widehat{R}$. This implies that there exists some $g_n \in G$ such that $g_n(c_n)$ and c_n intersect transversely. Moreover, the angle of the intersection is uniformly bounded away from 0. Then, we can find a simple closed curve $c_n' \in \mathcal{S}^{\#}(R)$ composed of some portions of $g_n(c_n)$ and c_n whose geodesic length $\ell(c_n')$ is less than $\ell(c_n)$ by a positive constant uniformly bounded away from 0. It follows that $\ell(c_n') < \exp(\lambda_0^{\#}(R))$ for a sufficiently large n, but this is a contradiction. Thus, we can find a desired constant $\delta > 0$ as in the statement. □

Remark. In general, for a non-simple closed geodesic $\hat{c}_n$ on $\widehat{R}$, we cannot always find a simple closed curve c_n' as above that has a geodesic representative in its free homotopy class. This happens when any simple closed curve contained in $\hat{c}_n$ that is liftable to a simple closed curve on R surrounds a puncture. Simple closed geodesics are restricted to $\mathcal{S}^{\#}(R)$ in order to avoid this situation.

Proof of Lemma 12.7. Without loss of generality, we may assume that p_0 is the origin $o \in T(R)$. Set $\Gamma_0 = \mathrm{Stab}_\Gamma(o)$ and consider $\widehat{R} = R/G_0$ for $G_0 \subset \mathrm{Conf}(R)$ corresponding to Γ_0. If $\widehat{R}$ is topologically finite, then the set of the lengths of all closed geodesics that are not necessarily simple is discrete. Hence, $\mathrm{LS}(R)$ is discrete modulo multiplicity by G_0. In this case, we apply the following arguments for $p = o$.

If $\widehat{R}$ is topologically infinite, Propositions 12.8 and 12.9 assert that for a simple closed geodesic $c \in \mathcal{S}^\#(R)$ such that $\log \ell(c)$ is sufficiently close to $\lambda_0^\#(R)$, the image $\kappa(c)$ is a simple closed geodesic on $\widehat{R}$. For a given neighborhood U of $o \in T(R)$, we consider the corresponding neighborhood $\widehat{U}$ of the origin $\hat{o} \in T(\widehat{R})$ under the inclusion $T(\widehat{R}) \hookrightarrow T(R)$. Then, by lifting a quasiconformal deformation of $\widehat{R}$ to a quasiconformal homeomorphism f of R, we have $p = [f] \in U$ such that $\log \ell_p(c)$ for some simple closed geodesic c is isolated in $\mathrm{LS}^\#(p)$ modulo multiplicity by $G_0 \subset \mathrm{MCG}(R)$. This is similar to the argument in the former part of the proof of Theorem 12.2.

Since $p \in \Lambda(\Gamma) \cap U$, it belongs to $\Lambda_0(\Gamma)$ by assumption. Then, there exists a sequence $\gamma_n = [g_n]_* \in \Gamma$ such that $\gamma_n(p) = p_n \neq p$ converges to p as $n \to \infty$. Let c be the simple closed geodesic on R such that the geodesic lengths $\ell_p(h(c))$ for all $h \in G_0$ are the same but isolated from each other. Furthermore, there exist a neighborhood $U' \subset U$ of p and a constant $\varepsilon > 0$ such that the set $\{\log \ell_{p'}(h(c))\}_{h \in G_0}$ for every $p' \in U'$ is in the interval $I(\log \ell_p(c), \varepsilon)$ that includes no other spectrum of $\mathrm{LS}^\#(p')$. Hence, for any sufficiently large n with $p_n \in U'$, the mapping class $[g_n]$ keeps the set $\{h(c)\}_{h \in G_0}$ invariant. In particular, there exists some $h_n \in G_0$ such that $g_n \circ h_n(c) \sim c$. Moreover, for $\gamma_n' = [g_n \circ h_n]_*$, we have $\gamma_n'(p) = p_n$, which converges to p as $n \to \infty$. Thus, we see that a sequence of some representatives of the mapping classes $[g_n \circ h_n]$ converge locally uniformly to a conformal mapping class in $\mathrm{MCG}_p(R)$ that fixes c.

Next, we consider the ϕ-grafting $\chi_{f(c),\phi}$ with respect to $f(c)$ on R_p and set $q = [\chi_{f(c),\phi} \circ f]$. We choose the amount ϕ to be sufficiently small such that $q \in U'$. Then, we see that $q \in \Lambda(\Gamma)$. Actually, $\gamma_n'(q) \to q$ as $n \to \infty$. Indeed, by the above argument, there are quasiconformal automorphisms $\tilde{g}_n$ of R_p homotopic to $f \circ g_n \circ h_n \circ f^{-1}$ that converge to a conformal automorphism $\tilde{g} \in \mathrm{Conf}(R_p)$ locally uniformly with the maximal dilatation $K(\tilde{g}_n)$ tending to 1 as $n \to \infty$, where $\tilde{g}$ fixes the simple closed geodesic in the homotopy class of $f(c)$. We may assume that each $\tilde{g}_n$ is identical to $\tilde{g}$ on the canonical collar $A^*(f(c))$. Then, the quasiconformal automorphisms $\chi_{f(c),\phi} \circ \tilde{g}_n \circ \chi_{f(c),\phi}^{-1}$ of R_q are conformal on the extended collar $A^*(f(c), \phi)$ and conformally conjugate to $\tilde{g}_n$ outside $A^*(f(c), \phi)$. This implies that $K(\chi_{f(c),\phi} \circ \tilde{g}_n^{-1} \circ \chi_{f(c),\phi}^{-1}) \to 1$, i.e., $\gamma_n'(q) \to q$.

Finally, we will show that $q \notin \overline{\Lambda_\infty(\Gamma)}$. By Corollary 3.4 and Lemma 3.5, there exist some constant $\delta > 0$ and a finite subset J of G_0 such that $\log \ell_q(h(c)) - \log \ell_q(c) > \delta$ for every $h \in G_0 - J$. Then, there is a neighborhood $U'' \subset U'$ of q such that $\ell_{q'}(h(c)) \neq \ell_{q'}(c)$ for every $h \in G_0 - J$ and for every $q' \in U''$.

By contrast, suppose that $q \in \overline{\Lambda_\infty(\Gamma)}$. Then, there exists some $q' \in U''$ that belongs to $\Lambda_\infty(\Gamma)$. Specifically, $\mathrm{Stab}_\Gamma(q')$ and the corresponding subgroup $G' \subset \mathrm{MCG}_{q'}(R)$ are infinite. By the definition of U', we see that G' keeps the set $\{h(c)\}_{h \in G_0}$ invariant. Hence, there exists some $h \in G_0 - J$ and $g' \in G'$ such that $g'(c) \sim h(c)$. In particular, $\ell_{q'}(h(c)) = \ell_{q'}(g'(c)) = \ell_{q'}(c)$. However, this contradicts the condition that $\ell_{q'}(h(c)) \neq \ell_{q'}(c)$. □

Before concluding this section, we will consider a certain problem related to the above arguments. Epstein [8] proved that the set $O(\Gamma)$ of all points $p \in T(R)$ where $\mathrm{Stab}_\Gamma(p)$ is trivial for $\Gamma \subset \mathrm{Mod}(R)$ is residual in $T(R)$, which means that it is the complement of a countable union of nowhere dense subsets. In particular, $O(\Gamma)$ is dense in $T(R)$. Let $F(\Gamma)$ denote the complement of $O(\Gamma)$, which is the union of all fixed point loci for elliptic elements of Γ. However, since the number of elliptic elements in Γ can be uncountable, these loci are not suitable for showing that $O(\Gamma)$ is residual. Instead, another locus is defined in [8] by

$$V_{(c,c')} = \{p \in T(R) \mid \ell_p(c) = \ell_p(c')\}$$

for any pair of distinct elements c and c' in $\mathcal{S}(R)$, which is nowhere dense in $T(R)$. Here, the set $I = \{(c, c')\}$ of all these pairs is countable. Then, $F(\Gamma)$ is contained in $\bigcup_{(c,c') \in I} V_{(c,c')}$; hence, it is a countable union of nowhere dense subsets.

We will further prove that $O(\Gamma)$ contains an open dense subset in $T(R)$, which is equivalent to saying that $F(\Gamma)$ is nowhere dense. Actually, Theorem 12.2 asserts that $\Lambda^1_\infty(\Gamma)$, which is the set of all points $p \in T(R)$ where $\mathrm{Stab}_\Gamma(p)$ contains an element of infinite order, is nowhere dense in $\Lambda(\Gamma)$ and hence in $T(R)$. Similar arguments are applicable to the fixed points of elliptic elements of finite order, and we can conclude the following. We include a more direct and easy proof for completeness.

Theorem 12.10. *For a subgroup $\Gamma \subset \mathrm{Mod}(R)$, the set $F(\Gamma)$ of all fixed points of elliptic elements of Γ is nowhere dense in $T(R)$.*

Proof. It suffices to prove the statement for $\Gamma = \mathrm{Mod}(R)$. We will show that, in every open subset U of $T(R)$, there exists $q \in U$ that does not belong to the closure $\overline{F(\Gamma)}$. By the proofs of Lemmas 10.3 and 10.4, we can find $c \in \mathcal{S}(R)$ and $q \in U$ such that $\log \ell_q(c)$ is isolated in the length spectrum $\mathrm{LS}(q)$. Moreover, by the above-mentioned result that $O(\Gamma)$ is dense in $T(R)$, we may assume that q is in $O(\Gamma)$.

By contrast, suppose that $q \in \overline{F(\Gamma)}$. Then, there is a sequence of points $p_n \in F(\Gamma)$ that converges to q. Let $\gamma_n = [g_n]_*$ be an elliptic element of Γ that fixes p_n. Since the mapping class $[g_n]$ is realized by a conformal automorphism of the Riemann surface R_{p_n}, it is realized by a quasiconformal automorphism of R_q, which we denote by $\tilde{g}_n$. The maximal dilatation of $\tilde{g}_n$ converges to 1; hence, by Proposition 3.1, $[g_n]$ must keep c invariant for all sufficiently large n.

In particular, this forces the order of $[g_n]$ to be finite and uniformly bounded. Then, a subsequence of quasiconformal automorphisms $\tilde{g}_n$ converges to a conformal automorphism $\tilde{g}$ uniformly on each compact subset of R_q. Since the order of $[g_n]$ is uniformly bounded, we see that $\tilde{g}$ is not the identity. This implies that $q \notin O(\Gamma)$, which is a contradiction. □

We remark that, if $F(\Gamma)$ is known to be a closed set, then the statement of Theorem 12.10 follows immediately from the fact that the complement $O(\Gamma)$ is dense in $T(R)$. However, we cannot expect that this will always be true without any restriction. Recall that, in Lemma 7.6, we have imposed an assumption that the union of the fixed point loci is closed in order to prove the statement; however, if this condition were always true, we would have a solution for the existence of an isolated limit point.

13. The moduli space is not separable

In this section, we will prove that the topological moduli space $M(R)$ of a topologically infinite Riemann surface R is not separable, and hence, neither is the geometric moduli space $M_*(R)$. This is an immediate consequence of the following stronger assertion.

Theorem 13.1. *Let $M(R)$ be the topological moduli space of a topologically infinite Riemann surface R. Then, a countable subset Σ is nowhere dense in $M(R)$, i.e., the closure $\overline{\Sigma}$ has no interior point.*

Corollary 13.2. *The geometric moduli space $M_*(R)$ of a topologically infinite Riemann surface R does not satisfy the second countability axiom.*

Proof. By Theorem 5.2 and Corollary 10.2, we see that the moduli space of the stable points $M_\Phi(R)$ is open and dense in $M(R)$. Since $M(R)$ is not separable by Theorem 13.1, neither is $M_\Phi(R)$. On the other hand, $M_*(R)$ also contains an open dense subset that is homeomorphic to $M_\Phi(R)$. Hence, $M_*(R)$ is not separable either. For the metric space $M_*(R)$, this is equivalent to saying that $M_*(R)$ does not satisfy the second countability axiom. □

Take an arbitrary simple closed geodesic $c_0 \in \mathcal{S}(R)$ and consider the relative Teichmüller space $T^{c_0}(R) = T(R)/\operatorname{Mod}_{c_0}(R)$ with respect to c_0, which has been defined in Section 5. We prove Theorem 13.1 by lifting the countable set Σ to $T^{c_0}(R)$.

Theorem 13.3. *For a topologically infinite Riemann surface R, every countable set in $T^{c_0}(R)$ is nowhere dense. In particular, $T^{c_0}(R)$ is not separable, which is equivalent to saying that the metric space $T^{c_0}(R)$ does not satisfy the second countability axiom.*

Note that Theorem 13.3 is evident when R satisfies the bounded geometry condition because, by Theorems 5.1 and 5.3, $\mathrm{Mod}_{c_0}(R)$ acts discontinuously on the non-separable space $T(R)$ in this case.

Proof of Theorem 13.1. For every countable set $\Sigma \subset M(R)$, the inverse image $\pi_{c_0}^{-1}(\Sigma)$ under the projection

$$\pi_{c_0}\colon T^{c_0}(R) = T(R)/\operatorname{Mod}_{c_0}(R) \longrightarrow M(R) = T(R)/\operatorname{Mod}(R)$$

is a countable set. This is because $\mathrm{Mod}_{c_0}(R)$ is of countable index in $\mathrm{Mod}(R)$ by Theorem 5.1. Then, $\pi_{c_0}^{-1}(\Sigma)$ is nowhere dense by Theorem 13.3. Since π_{c_0} is continuous and open, Σ is also nowhere dense in $M(R)$. □

In the remainder of this section, we will prove Theorem 13.3 by constructing a continuous surjective map from a certain subset in any open set of $T^{c_0}(R)$ onto a non-separable space. This function is defined by the hyperbolic lengths of an appropriate choice of infinitely many simple closed geodesics on R.

Definition. The *multiple length spectrum* $\widetilde{\mathrm{LS}}(R, c_0)$ for a hyperbolic Riemann surface R with respect to $c_0 \in \mathcal{S}(R)$ is a set of pairs $(\log \ell(c), \log \rho(c)) \in \mathbb{R}^2$ for all $c \in \mathcal{S}(R)$ with $c \cap c_0 = \emptyset$ respecting multiplicity, where $\rho(c)$ is the hyperbolic distance between the simple closed geodesics c and c_0 on R. We set

$$S_{c_0}(R) = \{c \in S(R) \mid c \cap c_0 = \emptyset\}.$$

For each $p = [f]$ in the Teichmüller space $T(R)$, let $\rho_p(c)$ be the hyperbolic distance between the simple closed geodesics $f(c)$ and $f(c_0)$ on $f(R)$. Then, the multiple length function $L_\bullet(c)\colon T(R) \to \mathbb{R}^2$ for $c \in \mathcal{S}_{c_0}(R)$ is defined by $L_p(c) = (\log \ell_p(c), \log \rho_p(c))$, which is well defined for the Teichmüller class p. The multiple length spectrum at $p \in T(R)$ is defined by

$$\widetilde{\mathrm{LS}}(p, c_0) = \{L_p(c) \in \mathbb{R}^2 \mid c \in \mathcal{S}_{c_0}(R)\}.$$

Proposition 13.4. (1) *The multiple length spectrum* $\widetilde{\mathrm{LS}}(R, c_0)$ *is always discrete in* $\mathbb{R}^2$ *with at most finite multiplicity.* (2) *If* p *and* q *in* $T(R)$ *are equivalent under* $\mathrm{Mod}_{c_0}(R)$*, then* $\widetilde{\mathrm{LS}}(p, c_0) = \widetilde{\mathrm{LS}}(q, c_0)$*. In other words, the multiple length spectrum with respect to* c_0 *is an invariant for an element of the relative Teichmüller space* $T^{c_0}(R)$.

Proof. The first assertion follows from the fact that the number of simple closed geodesics of bounded lengths intersecting a compact subset of R is finite. The second assertion is obvious. □

We provide the supremum norm $\|\cdot\|_\infty$ for $\mathbb{R}^2$. Recall that the constant $b = b(K)$ in Proposition 3.2 depends on $K = e^{d_T(p,q)}$ for a quasiconformal homeomorphism $f\colon R_p \to R_q$ and satisfies $b(e^{d_T(p,q)}) \to 0$ as $d_T(p,q) \to 0$. Then, we have the following estimate on the multiple length function.

Lemma 13.5. *The multiple length function for $c \in \mathcal{S}_{c_0}(R)$ satisfies*

$$\|L_p(c) - L_q(c)\|_\infty \leq d_T(p,q) + \frac{b(e^{d_T(p,q)})}{\omega(p,q)},$$

where ω is a positive continuous symmetric function on $T(R) \times T(R)$ invariant under $\mathrm{Mod}_{c_0}(R) \times \mathrm{Mod}_{c_0}(R)$ *and locally uniformly bounded away from* 0.

Proof. By Proposition 3.1, the first coordinate of $L_\bullet(c)$ satisfies

$$|\log \ell_p(c) - \log \ell_q(c)| \leq d_T(p,q).$$

For the estimate of the second coordinate, we apply Proposition 3.2 to c_0 and $c \in \mathcal{S}_{c_0}(R)$ and obtain

$$\frac{\rho_q(c)}{\rho_p(c)} \leq e^{d_T(p,q)} + \frac{b(e^{d_T(p,q)})}{\rho_p(c)} \leq e^{d_T(p,q)}\left\{1 + \frac{b(e^{d_T(p,q)})}{\rho_p(c)}\right\}.$$

Hence,

$$\log \rho_q(c) - \log \rho_p(c) \leq d_T(p,q) + \frac{b(e^{d_T(p,q)})}{\rho_p(c)}.$$

The other inequality obtained by exchanging the roles of p and q is also valid. Here, $\rho_p(c)$ and $\rho_q(c)$ are not less than the widths $\omega(p)$ and $\omega(q)$ of the canonical collars for the geodesic realization of c_0 on R_p and R_q, respectively. Note that, since every element of $\mathrm{MCG}_{c_0}(R)$ preserves c_0, these values are invariant under $\mathrm{Mod}_{c_0}(R)$. By setting $\omega(p,q) = \min\{\omega(p), \omega(q)\}$, we have

$$|\log \rho_p(c) - \log \rho_q(c)| \leq d_T(p,q) + \frac{b(e^{d_T(p,q)})}{\omega(p,q)}.$$

Therefore, the required estimate immediately follows from these inequalities. □

For any infinite discrete subsets P and Q of $\mathbb{R}^2$ counting multiplicity, the Hausdorff distance between P and Q is given by

$$H(P,Q) = \inf_j \sup\{\, \|j(z) - z\|_\infty \mid z \in P,\ j\colon P \to Q\},$$

where the infimum is taken over all bijections $j\colon P \to Q$. For any points $\hat{p}$ and $\hat{q}$ in $T^{c_0}(R)$, the pseudo-distance on $T^{c_0}(R)$ is defined by

$$d_H(\hat{p},\hat{q}) = H(\widetilde{\mathrm{LS}}(p,c_0), \widetilde{\mathrm{LS}}(q,c_0)),$$

where p and q are any points of $T(R)$ that are mapped to $\hat{p}$ and $\hat{q}$, respectively, by the projection $\pi\colon T(R) \to T^{c_0}(R)$.

As a consequence of Lemma 13.5, we see that the pseudo-distance d_H is continuous with respect to the quotient Teichmüller distance $\hat{d}$ on $T^{c_0}(R)$. More precisely, we have the following.

Corollary 13.6. *There exists a continuous symmetric function* $\hat{\beta} \geq 0$ *on* $T^{c_0}(R) \times T^{c_0}(R)$ *such that* $d_H(\hat{p}, \hat{q}) \leq \hat{\beta}(\hat{p}, \hat{q})$ *for any points* $\hat{p}$ *and* $\hat{q}$ *in* $T^{c_0}(R)$ *and* $\hat{\beta}(\hat{p}, \hat{q}) = 0$ *precisely when* $\hat{p} = \hat{q}$.

Proof. For $\hat{p}$ and $\hat{q}$ in $T^{c_0}(R)$, define

$$\hat{\beta}(\hat{p}, \hat{q}) = \inf_{p,q} \left\{ d_T(p,q) + \frac{b(e^{d_T(p,q)})}{\omega(p,q)} \right\} = \hat{d}(\hat{p}, \hat{q}) + \frac{b(e^{\hat{d}(\hat{p},\hat{q})})}{\hat{\omega}(\hat{p}, \hat{q})},$$

where the infimum is taken over all p and q satisfying $\pi(p) = \hat{p}$ and $\pi(q) = \hat{q}$ for the projection $\pi\colon T(R) \to T^{c_0}(R)$. Here, $\hat{\omega}(\hat{p}, \hat{q}) > 0$ is well defined from the function $\omega(p,q)$ in Lemma 13.5 owing to its invariance under $\mathrm{Mod}_{c_0}(R) \times \mathrm{Mod}_{c_0}(R)$. Then, $\hat{\beta}$ satisfies the required properties. □

Let $U(r)$ be an open ball of radius $r > 0$ in $T(R)$ centered at the origin o. We fix the radius $r > 0$ to be sufficiently small such that

$$r + \frac{b(e^r)}{\inf_{p \in U(r)} \omega(p,o)} < \frac{\log 2}{2} - \epsilon_0$$

for the constant b and the function ω in Lemma 13.5 and for some constant $\epsilon_0 > 0$.

Next, we will choose a sequence of simple closed geodesics on R whose lengths can parameterize a slice in the relative Teichmüller space $T^{c_0}(R)$. An X-piece $X(c)$ with a core geodesic $c \in \mathcal{S}_{c_0}(R)$ is a union of two pairs of pants that have a geodesic boundary c in common but no other intersection. Every X-piece has four geodesic boundary components. In a topologically infinite Riemann surface R, we take a sequence of X-pieces $\{X(c_i)\}_{i=1}^{\infty}$ satisfying the following properties:

(1) $X(c_i)$ are mutually disjoint and disjoint from c_0;

(2) $X(c_i)$ escape to infinity in R, i.e., for any compact subsurface S with boundary in R, the number of $X(c_i)$ that intersect S is finite.

Actually, we can always take such a sequence of X-pieces in any topologically infinite Riemann surface R. Indeed, we have only to find a topologically infinite geodesic subsurface R_0 including c_0 such that the complement $R - R_0$ has infinitely many (topologically finite or infinite) connected components R_i containing $X(c_i)$ for $i \in \mathbb{N}$. In this situation, it is clear that $d_h(R_i, c_0) \to \infty$ as $i \to \infty$, which implies that $X(c_i)$ escape to infinity in R.

For each i, let c_i' denote the closest geodesic boundary component of $X(c_i)$ to c_0. Further, set

$$M_i := \max\{d_h(x, c_0) \mid x \in X(c_i)\} \; (> \rho(c_i')).$$

Since $\rho(c_i') \to \infty$ as $i \to \infty$, we may assume that $\rho(c_j') \geq 2M_i + \ell(c_i)$ for any $i < j$ by passing to a subsequence if necessary.

We give a specific deformation of the hyperbolic structure on R restricted to $\{X(c_i)\}_{i=1}^{\infty}$. For an infinite sequence of real numbers $(\xi_1, \xi_2, \ldots) \in \mathbb{R}^{\infty}$, we consider a locally quasiconformal homeomorphism f of R onto another Riemann surface R' satisfying the following properties:

(1) f is isometric (conformal) outside $X(c_i)$ for all i and no twist is given along each geodesic boundary component of $X(c_i)$;

(2) the image $f(c_i)$ itself is a simple closed geodesic on R' and its length satisfies $\log \ell(f(c_i)) = \xi_i$ for each i;

(3) f has a constant directional derivative on each c_i with respect to the geodesic length parameter and no twist is given along c_i.

Let $\tilde{\ell}_i$ be the maximum of the lengths of the four geodesic boundary components of $X(c_i)$ and c_i. Then, as has been proved by Bishop [4], the maximal dilatation of the above quasiconformal homeomorphism f restricted to each $X(c_i)$ can be estimated as

$$K(f|_{X(c_i)}) \leq 1 + C|\log \ell(f(c_i)) - \log \ell(c_i)|$$

if $|\log \ell(f(c_i)) - \log \ell(c_i)| \leq 2$, where $C = C(\tilde{\ell}_i) > 0$ is a constant depending only on $\tilde{\ell}_i$. Hence, for a given dilatation constant $K = e^r$, there exists an open interval $I_i \subset \mathbb{R}$ centered at $\log \ell(c_i)$ for each i such that, if $\xi_i \in I_i$ for all i, then the above map f satisfying $\log \ell(f(c_i)) = \xi_i$ is globally K-quasiconformal.

Thus, we have a function

$$\phi\colon \prod_{i=1}^{\infty} I_i \ (\subset \mathbb{R}^{\infty}) \longrightarrow U(r) \ (\subset T(R))$$

sending $(\xi_1, \xi_2, \ldots)$ to the Teichmüller class $[f]$ of f defined as above. This function ϕ is clearly injective. Moreover, it is real-analytic as a function of a finite number of variables with the other coordinates fixed. Furthermore, for every $c \in \mathcal{S}(R)$, take the length function $\ell_\bullet(c)\colon T(R) \to \mathbb{R}$ defined by $\ell_p(c)$ for $p \in T(R)$ and consider the composition with ϕ. Then, $\ell_\bullet(c) \circ \phi\colon \prod_{i=1}^{\infty} I_i \to \mathbb{R}$ is also real-analytic as a function of a finite number of variables.

For every $c \in \mathcal{S}_{c_0}(R)$, let $E(c)$ be the ϵ_0-neighborhood of the range of the multiple length spectrum $\{L_p(c) \mid p \in U(r)\}$ in $(\mathbb{R}^2, \|\cdot\|_\infty)$. Then, by Lemma 13.5 and the definition of r, the radius of $E(c)$ at $L_o(c)$ has a upper bound less than $(\log 2)/2$ independent of c. Since $\widetilde{\mathrm{LS}}(R, c_0) = \widetilde{\mathrm{LS}}(o, c_0)$ is discrete by Proposition 13.4, for each i, there exist only finitely many $c \in \mathcal{S}_{c_0}(R)$ such that $E(c) \cap E(c_i) \neq \emptyset$.

Proposition 13.7. *For distinct integers $i \neq j$, if a simple closed geodesic $c \in \mathcal{S}_{c_0}(R)$ intersects $X(c_j)$, then $E(c) \cap E(c_i) = \emptyset$.*

Proof. If $c \cap X(c_j) \neq \emptyset$, then

$$\rho(c_j') - \ell(c)/2 \leq \rho(c) \leq M_j$$

is satisfied. By contrast, suppose that $E(c) \cap E(c_i) \neq \emptyset$. Then, since the radii of $E(c)$ and $E(c_i)$ are smaller than $(\log 2)/2$, we have $\|L_o(c) - L_o(c_i)\|_\infty < \log 2$. This implies that

$$\frac{1}{2}\ell(c_i) < \ell(c) < 2\ell(c_i); \quad \frac{1}{2}\rho(c_i) < \rho(c) < 2\rho(c_i).$$

Hence, we have the following two estimates:

$$\rho(c_j') \leq \rho(c) + \ell(c)/2 < 2\rho(c_i) + \ell(c_i);$$

$$\rho(c_i') \leq \rho(c_i) < 2\rho(c) \leq 2M_j.$$

However, when $i < j$, the first inequality violates the condition $\rho(c_j') \geq 2M_i + \ell(c_i)$ for the distribution of the X-pieces since $\rho(c_i) \leq M_i$. When $i > j$, the second inequality also violates the same condition after exchanging the roles of i and j. □

For each $i \in \mathbb{N}$, the composition $\log \ell_\bullet(c_i) \circ \phi\colon \prod_{i=1}^\infty I_i \to \mathbb{R}$ is nothing but the i-th coordinate function $(\xi_1, \xi_2, \ldots) \to \xi_i$. In other words, this is the identity restricted to the i-th coordinate. For $c \in \mathcal{S}_{c_0}(R)$ with $E(c) \cap E(c_i) \neq \emptyset$, we also consider the composition $\log \ell_\bullet(c) \circ \phi\colon \prod_{i=1}^\infty I_i \to \mathbb{R}$. By Proposition 13.7, such a simple closed geodesic c does not intersect $X(c_j)$ for $j \neq i$. Hence, this function also depends only on the i-th coordinate ξ_i; thus, a real-analytic function $h_{c,i}\colon I_i \to \mathbb{R}$ is induced.

Proposition 13.8. *For each c $(\neq c_i) \in \mathcal{S}_{c_0}(R)$ with the property $E(c) \cap E(c_i) \neq \emptyset$, the set of points ξ satisfying $h_{c,i}(\xi) = \xi$ is discrete in $I_i \subset \mathbb{R}$.*

Proof. If a simple closed geodesic c does not intersect $X(c_i)$, then $h_{c,i}(\xi)$ is constant and the claim is obvious. Suppose that c $(\neq c_i)$ intersects $X(c_i)$. By elementary hyperbolic geometry, we see that $h_{c,i}(\xi)$ is not the identity. Hence, the set of points ξ satisfying $h_{c,i}(\xi) = \xi$ should be discrete by the theorem of identity. □

Finally, by choosing an open interval J_i in I_i to be sufficiently small, we have an appropriate parameter space for a subset of $T^{c_0}(R)$.

Lemma 13.9. *There exists an open interval $J_i \subset I_i$ for each $i \in \mathbb{N}$ that satisfies the following properties*:

(1) *the composition of $\phi\colon \prod_{i=1}^{\infty} J_i \to U(r)$ and $\pi\colon T(R) \to T^{c_0}(R)$ is injective*;

(2) *by setting $W = \phi(\prod_{i=1}^{\infty} J_i)$, the inverse function $\Xi\colon \pi(W) \to \prod_{i=1}^{\infty} J_i$ of $\pi \circ \phi$ is continuous with respect to the uniform topology defined by the supremum norm $\|\Xi\|_\infty = \sup_i \xi_i$ for $\Xi = (\xi_1, \xi_2, \dots)$.*

Proof. For each $i \in \mathbb{N}$, the number of $c \in \mathcal{S}_{c_0}(R)$ satisfying $E(c) \cap E(c_i) \neq \emptyset$ is finite. Hence, by Proposition 13.8, there exist a constant $\epsilon_i > 0$ and an open interval $J_i \subset I_i$ such that $|h_{c,i}(\xi) - \xi| \geq 2\epsilon_i$ for every $\xi \in J_i$ and for every $c \neq c_i$ with $E(c) \cap E(c_i) \neq \emptyset$. Furthermore, by taking the interval J_i to be sufficiently small for each $i \in \mathbb{N}$, we can make it satisfy $|J_i| \leq \min\{2\epsilon_0, \epsilon_i\}$.

For any points $\Xi = (\xi_1, \xi_2, \dots)$ and $\Xi' = (\xi'_1, \xi'_2, \dots)$ in $\prod_{i=1}^{\infty} J_i$, consider the images $p = \phi(\Xi)$ and $p' = \phi(\Xi')$ in $W \subset U(r)$. Then, we see that

$$\|L_p(c_i) - L_{p'}(c_i)\|_\infty \geq |\log \ell_p(c_i) - \log \ell_{p'}(c_i)| = |\xi_i - \xi'_i|$$

for each i. We will show that $L_p(c_i)$ is away from $\widetilde{\mathrm{LS}}(p', c_0)$ (or $L_{p'}(c_i)$ is away from $\widetilde{\mathrm{LS}}(p, c_0)$).

By the definition of the neighborhood E, every $c \in \mathcal{S}_{c_0}(R)$ with $E(c) \cap E(c_i) = \emptyset$ satisfies $\|L_q(c_i) - L_{q'}(c)\|_\infty \geq 2\epsilon_0$ for any $q, q' \in U(r)$. Hence,

$$\|L_p(c_i) - L_{p'}(c)\|_\infty \geq 2\epsilon_0 \geq |J_i| \geq |\xi_i - \xi'_i|.$$

On the other hand, every $c\ (\neq c_i) \in \mathcal{S}_{c_0}(R)$ with $E(c) \cap E(c_i) \neq \emptyset$ satisfies

$$|\log \ell_{p'}(c) - \log \ell_{p'}(c_i)| = |h_{c,i}(\xi'_i) - \xi'_i| \geq 2\epsilon_i$$

for p' in W. Hence, in this case, we have

$$\begin{aligned}\|L_p(c_i) - L_{p'}(c)\|_\infty &\geq |\log \ell_p(c_i) - \log \ell_{p'}(c)| \\ &\geq |\log \ell_{p'}(c) - \log \ell_{p'}(c_i)| - |\log \ell_p(c_i) - \log \ell_{p'}(c_i)| \\ &\geq 2\epsilon_i - |\xi_i - \xi'_i| \geq |\xi_i - \xi'_i|.\end{aligned}$$

From these estimates, we see that the distance from $L_p(c_i)$ to $\widetilde{\mathrm{LS}}(p', c_0)$ is not less than $|\xi_i - \xi'_i|$ for each $i \in \mathbb{N}$. Thus, the Hausdorff distance $H(\widetilde{\mathrm{LS}}(p, c_0), \widetilde{\mathrm{LS}}(p', c_0))$ is bounded from below by $|\xi_i - \xi'_i|$. By taking the supremum over all i, we have

$$d_H(\hat{p}, \hat{p}') = H(\widetilde{\mathrm{LS}}(p, c_0), \widetilde{\mathrm{LS}}(p', c_0)) \geq \|\Xi - \Xi'\|_\infty$$

for $\hat{p} = \pi(p)$ and $\hat{p}' = \pi(p')$ in $\pi(W) \subset T^{c_0}(R)$. This implies that the function $\pi \circ \phi$ is injective on $\prod_{i=1}^{\infty} J_i$ and the projection π is injective on W.

Consider the inverse function $\Xi = (\pi \circ \phi)^{-1}$ on $\pi(W) \subset T^{c_0}(R)$. The above estimate implies that $\|\Xi(\hat{p}) - \Xi(\hat{p}')\|_\infty \to 0$ as $d_H(\hat{p}, \hat{p}') \to 0$. Since d_H is continuous with respect to $\hat{d}$ by Corollary 13.6, this is also true when $\hat{d}(\hat{p}, \hat{p}') \to 0$, i.e., Ξ is continuous on $\pi(W)$. □

Using this continuous and bijective map $\Xi\colon \pi(W) \to \prod_{i=1}^\infty J_i$, we now complete the proof of Theorem 13.3. Note that it is easy to see that $\prod_{i=1}^\infty J_i$ is not separable in the uniform topology.

Proof of Theorem 13.3. Suppose that the closure of a countable subset $\overline{\{\hat{p}_n\}}_{n\in\mathbb{N}}$ of $T^{c_0}(R)$ contains an open subset V. Without loss of generality, we may assume that $\pi(o) \in V$. By replacing the radius r with a smaller one, we may assume that $\pi(U(r)) \subset V$. Hence, the set $\pi(W)$ of Lemma 13.9 is contained in V.

For each $n \in \mathbb{N}$, we take a point $\hat{p}'_n \in \pi(W)$ such that $\hat{p}'_n = \hat{p}_n$ if $\hat{p}_n \in \pi(W)$ and

$$\hat{d}(\hat{p}_n, \hat{p}'_n) < \inf_{\hat{p}\in\pi(W)} \hat{d}(\hat{p}_n, \hat{p}) + \frac{1}{n}$$

otherwise. Then, we see that $\pi(W)$ is contained in the closure of the countable set $\overline{\{\hat{p}'_n\}}_{n\in\mathbb{N}}$. Indeed, if not, there is a point $\hat{q} \in \pi(W)$ such that an open ball $V(\hat{q}, 3\epsilon)$ with center $\hat{q}$ and radius 3ϵ for some $\epsilon > 0$ contains no $\hat{p}'_n$. On the other hand, there is some $\hat{p}_n$ in $V(\hat{q}, \epsilon)$ with $1/n < \epsilon$ because $\hat{q}$ belongs to $\overline{\{\hat{p}_n\}} - \{\hat{p}_n\}$. However, this contradicts the way of taking $\hat{p}'_n$ for this $\hat{p}_n$.

Consider a countable subset $\{\Xi(\hat{p}'_n)\}_{n\in\mathbb{N}} \subset \prod_{i=1}^\infty J_i$. Since $\{\hat{p}'_n\}$ is dense in $\pi(W)$ and $\Xi\colon \pi(W) \to \prod_{i=1}^\infty J_i$ is continuous and surjective with respect to the uniform topology by Lemma 13.9, $\{\Xi(\hat{p}'_n)\}$ is dense in $\prod_{i=1}^\infty J_i$. However, this contradicts the fact that $\prod_{i=1}^\infty J_i$ is not separable and thus completes the proof of Theorem 13.3. □

14. The moduli space of the stable points

In this section, we consider the metric completion of the moduli space of the stable points $M_\Phi(R) = \Phi(\mathrm{Mod}(R))/\mathrm{Mod}(R)$. Here, the completion respects the inner distance d_M^i on $M_\Phi(R)$ induced from the pseudo-distance d_M on the topological moduli space $M(R) = T(R)/\mathrm{Mod}(R)$. In other words, the distance $d_M^i(\sigma, \tau)$ between σ and τ in $M_\Phi(R)$ is given by the infimum of the lengths of all paths in $M_\Phi(R)$ measured by d_M that connect σ and τ.

The restriction of d_M to $M_\Phi(R)$ becomes a distance and it clearly satisfies the inequality $d_M \leq d_M^i$. On the other hand, Theorem 11.1 yields a converse estimate as in the following theorem. It turns out that the completions by d_M and d_M^i are homeomorphic. We have seen that the stable points are generic in $T(R)$ in the sense that $\Phi(\mathrm{Mod}(R))$ is an open, dense, and connected set. The fact that d_M and d_M^i are comparable on $M_\Phi(R)$ also reflects a stronger genericity of the stable points.

Theorem 14.1. *For every bounded subset V in $M(R)$, there exists a constant C depending on V such that $d^i_M(\tau_1, \tau_2) \leq C d_M(\tau_1, \tau_2)$ for any τ_1 and τ_2 in $V \cap M_\Phi(R)$.*

Proof. Let r_0 be the diameter of V. Choose $p \in T(R)$ such that $\pi(p)$ belongs to V under the projection $\pi: T(R) \to M(R)$. Then, for an arbitrary $\epsilon > 0$, an open ball $U(p, r_0 + \epsilon)$ with center p and radius $r_0 + \epsilon$ in $T(R)$ covers V by the projection π. Hence, there exists $q_1 \in U(p, r_0 + \epsilon) \cap \Phi(\Gamma)$ for $\Gamma = \mathrm{Mod}(R)$ such that $\pi(q_1) = \tau_1$. Then, since $d_M(\tau_1, \tau_2) \leq r_0$, there exists $q_2 \in \Phi(\Gamma)$ such that $\pi(q_2) = \tau_2$ and $d_T(q_1, q_2) < d_M(\tau_1, \tau_2) + \epsilon$ $(\leq r_0 + \epsilon)$.

By Theorem 11.1, there exists a constant C depending on p and $r = r_0 + \epsilon$ such that q_1 and q_2 can be connected by a path in $\Phi(\Gamma)$ whose length is less than $C d_T(q_1, q_2)$. Then, the projection of this path on $M_\Phi(R)$ connects τ_1 and τ_2, and its length is less than $C\{d_M(\tau_1, \tau_2) + \epsilon\}$. Since ϵ can be arbitrarily small, this implies that $d^i_M(\tau_1, \tau_2) \leq C d_M(\tau_1, \tau_2)$. The constant C depends only on the subset V because p and r are determined by V. □

Corollary 14.2. *The metric completions $\overline{M_\Phi(R)}^{d^i_M}$ and $\overline{M_\Phi(R)}^{d_M}$ of $M_\Phi(R)$ with respect to d^i_M and d_M, respectively, are homeomorphic.*

Remark. From the proof of Theorem 11.1, we see that, if R does not satisfy the lower boundedness condition, then we can choose a uniform constant C in Theorem 14.1. Hence, in this case, there is a bi-Lipschitz homeomorphism between $\overline{M_\Phi(R)}^{d^i_M}$ and $\overline{M_\Phi(R)}^{d_M}$. We expect that this is always the case.

Let $M_*(R) = T(R) /\!/ \mathrm{Mod}(R)$ be the geometric moduli space with the projection $\bar{\pi}: M(R) \to M_*(R)$. As will be seen in the next theorem, the restriction of $\bar{\pi}$ to $M_\Phi(R)$ extends continuously to the completion $\overline{M_\Phi(R)}^{d_M}$, which defines an isometry onto $M_*(R)$.

Theorem 14.3. *There exists a bijective isometry*

$$\iota: \overline{M_\Phi(R)}^{d_M} \longrightarrow M_*(R)$$

that extends $\bar{\pi}|_{M_\Phi(R)}: M_\Phi(R) \to M_*(R)$.

Proof. Let $\tilde{\sigma}$ be an element of $\overline{M_\Phi(R)}^{d_M}$ that is represented by a Cauchy sequence $\{\sigma_n\}_{n=1}^\infty$ in $M_\Phi(R)$. It converges to a point $\sigma \in M(R)$. Choose another representative $\{\sigma'_n\}_{n=1}^\infty$ of $\tilde{\sigma}$. Then, it also converges to another point $\sigma' \in M(R)$. Since $d_M(\sigma_n, \sigma'_n) \to 0$ as $n \to \infty$, we see that $d_M(\sigma, \sigma') = 0$. This implies that $\bar{\pi}(\sigma) = \bar{\pi}(\sigma')$ under the projection $\bar{\pi}: M(R) \to M_*(R)$. Denoting this element by s, we have a well-defined continuous map $\iota: \overline{M_\Phi(R)}^{d_M} \to M_*(R)$ by the correspondence $\tilde{\sigma} \mapsto s$.

The surjectivity of ι is seen from the fact that $M_\Phi(R)$ is dense in $M(R)$, which immediately follows from Corollary 10.2. The injectivity of ι is easily seen. Indeed, for any distinct elements $\tilde{\sigma}$ and $\tilde{\sigma}'$ of $\overline{M_\Phi(R)}^{d_M}$, the Cauchy sequences $\{\sigma_n\}_{n=1}^\infty$ and $\{\sigma_n'\}_{n=1}^\infty$ representing $\tilde{\sigma}$ and $\tilde{\sigma}'$, respectively, satisfy $d_M(\sigma_n, \sigma_n') \not\to 0$ $(n \to \infty)$. Thus, $d_M(\sigma, \sigma') \neq 0$ for their limits σ and σ' in $M(R)$. This implies that $\bar{\pi}(\sigma) \neq \bar{\pi}(\sigma')$ in $M_*(R)$.

It is clear that the restriction of ι to $M_\Phi(R)$ is nothing but $\bar{\pi}|_{M_\Phi(R)}$. Since $\bar{\pi}|_{M_\Phi(R)}$ is isometric, the extension ι is also isometric by the definition of the distance on the metric completion $\overline{M_\Phi(R)}^{d_M}$. □

Corollary 14.4. *If R satisfies the bounded geometry condition, then the geometric moduli space $M_*(R)$ is isometric to the completion $\overline{M_\Omega(R)}^{d_M}$ of the complex Banach orbifold $M_\Omega(R)$.*

By a general theory, it is known that the complete metric space $M_*(R)$ is isometric to the locus of zeros of some holomorphic map between complex Banach spaces, and in particular, it has the structure of a Banach analytic space (see Pestov [29]).

Finally, we conclude this paper by raising a question on more concrete characterizations of an element of $M_*(R)$.

Definition. A *geometric invariant* of the moduli is a $\mathrm{Mod}(R)$-invariant continuous map $\eta\colon T(R) \to Y$ to a metric space Y.

For example, let $Y = C(\mathbb{R})$ be the family of all closed subsets in $\mathbb{R}$ equipped with the Hausdorff distance. Then, the map $T(R) \to C(\mathbb{R})$ defined by $p \mapsto \mathrm{LS}(p)$ satisfies the above conditions. In other words, the length spectrum is a geometric invariant of the moduli.

The following proposition asserts that the geometric moduli space $M_*(R)$ is the universal space for the geometric invariants.

Proposition 14.5. *For every geometric invariant $\eta\colon T(R) \to Y$, there exists a continuous map $\tilde{\eta}\colon M_*(R) \to Y$ satisfying $\eta = \tilde{\eta} \circ \pi_{M_*}$, where $\pi_{M_*}\colon T(R) \to M_*(R) = T(R)/\!/\,\mathrm{Mod}(R)$ is the projection by the closure equivalence.*

Proof. For every $s \in M_*(R)$, take any $p \in T(R)$ such that $\pi_{M_*}(p) = s$ and define $\tilde{\eta}(s)$ to be $\eta(p)$. This is well defined. Indeed, if we take another $q \in T(R)$ such that $\pi_{M_*}(q) = s$, then q is in the closure of the orbit of p under $\mathrm{Mod}(R)$. Since η is invariant under $\mathrm{Mod}(R)$, this implies that $\eta(q)$ is in the closure of the point set $\{\eta(p)\}$. Since Y is a metric space, this implies that $\eta(q) = \eta(p)$. Once $\tilde{\eta}$ is defined in this manner, the condition $\eta = \tilde{\eta} \circ \pi_{M_*}$ is clearly satisfied. □

We propose the problem of finding a better geometric invariant of the moduli, which will give an interpretation for an element of our moduli space $M_*(R)$.

References

[1] S. Adjan and I. Lysënok, On groups all of whose proper subgroups are finite cyclic. *Izv. Akad. Nauk SSSR Ser. Mat.* **55** (1991), no. 5, 933–990. In Russian. English translation, *Math. USSR-Izv.* **39** (1992), no. 2, 905–957. Zbl 0771.20015 MR 1149884

[2] A. Basmajian, Quasiconformal mappings and geodesics in the hyperbolic plane. In I. Kra and B. Maskit (eds.), *In the tradition of Ahlfors and Bers.* (Stony Brook, N.Y., 1998.) Contemporary Mathematics, 256. American Mathematical Society, Providence, R.I., 2000, 1–4. Zbl 0971.30024 MR 1759665

[3] L. Bers, An extremal problem for quasiconformal mappings and a theorem by Thurston. *Acta Math.* **141** (1978), no. 1-2, 73–98. Zbl 0389.30018 MR 0477161

[4] C. Bishop, Quasiconformal mappings of Y-pieces. *Rev. Mat. Iberoamericana* **18** (2002), no. 3, 627–652. Zbl 1064.30045 MR 1954866

[5] P. Buser *Geometry and spectra of compact Riemann surfaces.* Progress in Mathematics, 106. Birkhäuser Boston, Boston, MA, 1992. Zbl 0770.53001 MR 1183224

[6] C. Earle, F. Gardiner, and N. Lakic, Teichmüller spaces with asymptotic conformal equivalence. IHES preprint, 1995.

[7] C. Earle and C. McMullen, Quasiconformal isotopies. In D. Drasin, C. J. Earle, F. W. Gehring, I. Kra, and A. Marden (eds.), Holomorphic functions and moduli. I. Mathematical Sciences Research Institute Publications, 10. Springer-Verlag, New York, 1988, 143–154. Zbl 0659.30018 MR 0955816

[8] A. Epstein, Effectiveness of Teichmuller modular groups. In I. Kra and B. Maskit (eds.), *In the tradition of Ahlfors and Bers.* (Stony Brook, N.Y., 1998.) Contemporary Mathematics, 256. American Mathematical Society, Providence, R.I., 2000, 69–74. Zbl 0964.30026 MR 1759670

[9] E. Fujikawa, Limit sets and regions of discontinuity of Teichmüller modular groups. *Proc. Amer. Math. Soc.* **132** (2004), no. 1, 117–126. Zbl 1091.30012 MR 2021254

[10] E. Fujikawa, Modular groups acting on infinite dimensional Teichmüller spaces. In W. Abikoff and A. Haas (eds.), *In the tradition of Ahlfors and Bers.* III. (Storrs, CT, 2001.) Contemporary Mathematics, 355. American Mathematical Society, Providence, R.I., 2004, 239–253. Zbl 1072.30032 MR 2145066

[11] E. Fujikawa, The action of geometric automorphisms of asymptotic Teichmüller spaces. *Michigan Math. J.* **54** (2006), no. 2, 269–282. Zbl 1115.30051 MR 2252759

[12] E. Fujikawa, Another approach to the automorphism theorem for Teichmüller spaces. In D. Canary, J. Gilman, J. Heinonen, and H. Masur (eds.), *In the tradition of Ahlfors-Bers.* IV. (Ann Arbor, MI, 2005.) Contemporary Mathematics, 432. American Mathematical Society, Providence, R.I., 2007, 39–44. Zbl 1189.30085 MR 2342805

[13] E. Fujikawa and K. Matsuzaki, Recurrent and periodic points for isometries of L^∞ spaces. *Indiana Univ. Math. J.* **55** (2006), no. 3, 975–997. Zbl 1097.37005 MR 2244594

[14] E. Fujikawa and K. Matsuzaki, Stable quasiconformal mapping class groups and asymptotic Teichmüller spaces. *Amer. J. Math.* **133** (2011), no. 3, 637–675. Zbl 1238.30033 MR 2808328

[15] E. Fujikawa, H. Shiga, and M. Taniguchi, On the action of the mapping class group for Riemann surfaces of infinite type. *J. Math. Soc. Japan* **56** (2004), no. 4, 1069–1086. Zbl 1064.30047 MR 2091417

[16] F. Gardiner and N. Lakic, *Quasiconformal Teichmüller theory.* Mathematical Surveys and Monographs, 76. American Mathematical Society, Providence, R.I., 2000. Zbl 0949.30002 MR 1730906

[17] S. Kerckhoff, The Nielsen realization problem. *Ann. of Math.* (2) **117** (1983), no. 2, 235–265. Zbl 0528.57008 MR 0690845

[18] O. Lehto, *Univalent functions and Teichmüller spaces,* Graduate Texts in Mathematics, 109. Springer-Verlag, New York, 1987. Zbl 0606.30001 MR 0867407

[19] V. Markovic Biholomorphic maps between Teichmüller spaces. *Duke Math. J.* **120** (2003), no. 2, 405–431. Zbl 1056.30045 MR 2019982

[20] V. Markovic, Quasisymmetric groups. *J. Amer. Math. Soc.* **19** (2006), no. 3, 673–715. Zbl 1096.20042 MR 2220103

[21] K. Matsuzaki, Dynamics of Teichmüller modular groups and general topology of moduli spaces: Announcement. *RIMS Kokyuroku* **1387** (2004), 81–94. http://www.kurims.kyoto-u.ac.jp/~kyodo/kokyuroku/contents/pdf/1387-11.pdf

[22] K. Matsuzaki, Inclusion relations between the Bers embeddings of Teichmüller spaces. *Israel J. Math.* **140** (2004), 113–123. Zbl 1056.30046 MR 2054840

[23] K. Matsuzaki, A classification of the modular transformations of infinite dimensional Teichmüller spaces. In D. Canary, J. Gilman, J. Heinonen, and H. Masur (eds.), *In the tradition of Ahlfors-Bers.* IV. (Ann Arbor, MI, 2005.) Contemporary Mathematics, 432. American Mathematical Society, Providence, R.I., 2007, 167–178. Zbl 1170.30020 MR 2342814

[24] K. Matsuzaki, Quasiconformal mapping class groups having common fixed points on the asymptotic Teichmüller spaces. *J. Anal. Math.* **102** (2007), 1–28. Zbl 1134.30036 MR 2346552

[25] K. Matsuzaki, *Infinite dimensional Teichmüller spaces and modular groups.* In A. Papadopoulos (ed.), *Handbook of Teichmüller theory.* Vol. IV. IRMA Lectures in Mathematics and Theoretical Physics, 19. European Mathematical Society (EMS), Zürich, 2014, 681–716. Zbl 1311.30023 MR 3289713

[26] C. T. McMullen, Complex earthquakes and Teichmüller theory. *J. Amer. Math. Soc.* **11** (1998), no. 2, 283–320. Zbl 0890.30031 MR 1478844

[27] P. Novikov and S. Adjan, Infinite periodic groups I, II, III. *Izv. Akad. Nauk SSSR Ser. Mat.* **32** (1968), 212–244, 251–524, 709–731. In Russian. English translation, *Math. USSR. Izv.* **2** (1968), 209–236, 241–479, 665–685. Zbl 0194.03301 MR 0240178 (I) MR 0240179 (II) MR 0240180 (III)

[28] A. Ol′shanskii, Groups of bounded period with subgroups of prime order. *Algebra i Logika* **21** (1982), no. 5, 553–618. In Russian. English translation, *Algebra and Logic* **21** (1982), no. 5, 369–418. Zbl 0524.20024 MR 0721048

[29] V. Pestov, Douady's conjecture on Banach analytic spaces. *C. R. Acad. Sci. Paris Sér.* I *Math.* **319** (1994), no. 10, 1043–1048. Zbl 0819.32012 MR 1305674

[30] T. Sorvali, The boundary mapping induced by an isomorphism of covering groups. *Ann. Acad. Sci. Fenn. Ser. A* I **526** (1972), 31 pp. MR 0328066

[31] A. Vasil′ev, *Moduli of families of curves for conformal and quasiconformal mappings.* Lecture Notes in Mathematics, 1788. Springer-Verlag, Berlin, 2002. Zbl 0999.30001 MR 1929066

[32] S. Wolpert, The length spectra as moduli for compact Riemann surfaces. *Ann. of Math.* (2) **109** (1979), no. 2, 323–351. Zbl 0441.30055 MR 0528966

Received October 2, 2013

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