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Groups, Geometry, and Dynamics

Algorithmic aspects of branched coverings I/V. Van Kampen's theorem for bisets

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Abstract. We develop a general theory of *bisets*: sets with two commuting group actions. They naturally encode topological correspondences.

Just as van Kampen's theorem decomposes into a graph of groups the fundamental group of a space given with a cover, we prove analogously that the biset of a correspondence decomposes into a *graph of bisets*: a graph with bisets at its vertices, given with some natural maps. The *fundamental biset* of the graph of bisets recovers the original biset.

We apply these results to decompose the biset of a Thurston map (a branched self-covering of the sphere whose critical points have finite orbits) into a graph of bisets. This graph closely parallels the theory of Hubbard trees.

This is the first part of a series of five articles, whose main goal is to prove algorithmic decidability of combinatorial equivalence of Thurston maps.

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Contents

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1. Introduction

This is the first of a series of five articles, and develops the theory of decompositions of bisets. For an overview of the series, see [1].

1.1. Bisets. Bisets are algebraic objects used to describe continuous maps up to isotopy. Classically, a map $f\colon(Y,\dagger)\to(X,*)$ between pointed spaces induces a homomorphism between the fundamental groups $f_*\colon\pi_1(Y,\dagger)\to\pi_1(X,*)$. If, however, f does not preserve any natural base points, it is much more convenient to consider a weaker kind of relation between $\pi_1(Y,\dagger)$ and $\pi_1(X,*)$: this is precisely a *biset*, namely a set $B(f)$ with two commuting group actions, of $\pi_1(Y,\dagger)$ on the left and of $\pi_1(X,*)$ on the right. Bisets may naturally be multiplied, and the product of bisets corresponds to composition of the maps.

Another advantage of bisets is that only bijective maps may be inverted, while every biset B has a *contragredient* $B^\vee$, obtained by dualizing the actions. Thus it is almost as easy to handle correspondences $Y\overset{i}{\leftarrow}Z\overset{f}{\rightarrow}X$ as genuine maps $Y\to X$, see §4: the biset of the correspondence is $B(i)^\vee\otimes B(f)$.

Van Kampen's theorem is a cornerstone in algebraic topology: given a space X covered by (say) open subsets (U_α), the theorem expresses the fundamental group of X in terms of the fundamental groups of the pieces U_α and of their intersections. This is best expressed in terms of a *graph of groups*, namely a simplicial graph with groups attached to its vertices and edges. The *fundamental group* of a graph of groups is algebraically constructed in terms of the graph data, and recovers the fundamental group of X.

The main construction in this article is a *graph of bisets* expressing the decomposition of a continuous map, or more generally a topological correspondence, between spaces given with compatible covers, see Definition 3.8. We single out the subclass of *fibrant* graphs of bisets. Graphs of bisets can be multiplied, and the product of fibrant graphs of bisets is again fibrant.

We construct in Definition 3.13 the fundamental biset of a graph of bisets, and show in Corollary 3.20 how it can be conveniently computed for a fibrant graph of bisets.

Our central result, Theorem 4.8, is an analogue of van Kampen's theorem in the language of bisets. It expresses the biset of a correspondence as the fundamental biset of a graph of bisets constructed from restrictions of the correspondence to elements of the cover.

1.2. Applications. We then apply these results, in §5, to complex dynamics. Following Thurston, we consider branched coverings of the sphere, namely self-maps $f\colon S^2 \circlearrowleft$ that are locally modeled on $z\mapsto z^d$ in complex charts. The subset of S^2 at which the local model is z^d with $d>1$ is called the *critical set* of f, and the *post-critical set* $P(f)$ is the strict forward orbit of the critical set.

A *Thurston map* is a branched covering of the sphere for which $P(f)$ is finite. Such maps f are studied as correspondences

$$S^2 \setminus P(f) \hookleftarrow S^2 \setminus f^{-1}(P(f)) \xrightarrow{f} S^2 \setminus P(f).$$

The simplest example of all is the map $f(z) = z^d$ itself, with $P(f) = \{0, \infty\}$. The map f is a covering on $\widehat{\mathbb{C}} \setminus \{0, \infty\}$, and $\pi_1(\widehat{\mathbb{C}} \setminus \{0, \infty\}, 1) = \mathbb{Z}$. In this case, we can give the biset of f quite explicitly: it is $B(f) = \mathbb{Z}$, with right and left actions given by $m \cdot b \cdot n = dm + b + n$. We call such bisets *regular cyclic bisets*, and we define a *cyclic biset* as a transitive biset over cyclic groups.

The next simplest maps are rational maps $f(z)$ with finite post-critical set, such as $f(z) = z^2 - 1$. In particular, complex polynomials have been intensively studied via their *Hubbard tree*, see [7, 8, 16]. This is a dynamically-defined f-invariant tree containing $P(f)$ and embedded in $\mathbb{C}$. We apply the van Kampen Theorem 4.8 to the Hubbard tree and obtain in this manner a decomposition of the biset of f as a graph of cyclic bisets, see Theorem 5.5.

One of the advantages in working with branched coverings rather than rational maps is that surgery operations are possible. For example, given a Thurston map f with a fixed point z mapping locally to itself by degree $d > 1$, and given a (topological) polynomial g of degree d, a small neighbourhood of f may be removed to be replaced by a sphere, punctured at ∞, on which g acts. One calls the resulting map the *tuning* of f by g.

This operation has a transparent interpretation in terms of graphs of bisets: it amounts to replacing a cyclic biset, at a vertex of the graph of bisets of f, with the biset $B(g)$; see Theorem 5.8.

In case f itself also has degree d, one calls the resulting map h the *mating* of f and g. The sphere on which h acts is naturally covered by the punctured spheres on which f, respectively g act, and the "equator" on which they overlap. Thus h is naturally expressed by a graph of bisets with two vertices corresponding to f and g respectively, and an edge corresponding to the equator; see Theorem 5.9.

Laminations allow Julia sets of polynomials to be obtained out of the Julia set of z^d, namely a circle, by pinching. Similarly, van Kampen's Theorem 4.8 applied to the lamination of f decomposes the biset $B(f)$ into a graph of bisets made of trivial bisets and one regular cyclic biset (the biset of z^d). We compute it explicitly for the map $f(z) = z^2 - 1$, and describe in this manner, in §5.6, the mating of $z^2 - 1$ with an arbitrary quadratic polynomial.

1.3. Notation. We introduce some convenient notations, which we will follow throughout the series of articles. For a set S, we write $S\wr$ for its permutation group. This has the mnemonic advantage that $\#(S\wr) = (\#S)!$. We always write $\mathbb{1}$ for the identity map from a set to itself. The restriction of a map $f\colon Y \to X$ to a subset $Z \subset Y$ is written $f{\restriction}_Z$. Self-maps are written $f\colon X \circlearrowleft$ in preference to $f\colon X \to X$.

Paths are continuous maps $\gamma\colon[0,1]\to X$; the path starts at $\gamma(0)$ and ends at $\gamma(1)$. We write $\gamma\#\delta$ for concatenation of paths, following first γ and then δ; this is defined only when $\gamma(1)=\delta(0)$. We write γ^{-1} for the inverse of a path.

We write $\approx$ for isotopy of paths, maps etc, $\sim$ for conjugacy or combinatorial equivalence, and $\cong$ for isomorphism of algebraic objects. Similarly h^ϕ denotes the image of h under ϕ. Similarly, $g\circ f$ is the composition of maps, first f then g, and fg is the composition of homomorphisms, first f then g.

A graphs of groups is a graph $\mathfrak{X}$ with groups associated with $\mathfrak{X}$'s vertices and edges. We *always* write G_x for the group associated with $x\in\mathfrak{X}$; thus if $\mathfrak{Y}$ is another graph of groups, we will write G_y for the group associated with $y\in\mathfrak{Y}$, and no relationship should be assumed between G_x and G_y. Similarly, in a graph of bisets $\mathfrak{B}$ there are bisets B_z associated with $z\in\mathfrak{B}$, and different graphs of bisets will all have their bisets written in this manner.

2. Bisets

We show, in this section, how topological data can be conveniently converted to group theory. We shall extend, along the way, the classical dictionary between topology and group theory.

Consider a continuous map $f\colon Y\to X$ between path connected topological spaces, and basepoints $\dagger\in Y$ and $*\in X$. Denote by $H=\pi_1(Y,\dagger)$ and $G=\pi_1(X,*)$ their fundamental groups. If $f(\dagger)=*$, then f induces a homomorphism $f_*\colon H\to G$; however, no such natural map exists if f does not preserve the basepoints.

A solution would be to express f in the fundamental groupoid $\pi_1(X)$, whose objects are X and whose morphisms from x to y consist of all paths from x to y in X up to homotopy rel their endpoints. However, for computational purposes, a much more practical solution exists: one expresses f as an *H-G-biset*.

Definition 2.1 (bisets). Let H,G be two groups. An *H-G-biset* is a set B equipped with a left H-action and a right G-action that commute; namely, a set B and maps $H\times B\to B$ and $B\times G\to B$, both written $\cdot$, such that

$$h\cdot h'\cdot(b\cdot gg')=hh'\cdot(b\cdot gg')=(hh'\cdot b)\cdot gg'=(hh'\cdot b)\cdot g\cdot g',$$

so that no parentheses are needed to write any product of h's, b, and g's. We will also omit the $\cdot$, and write the actions as multiplication.

A *G-biset* is a G-G-biset.

An H-G-set B is *left-free* if, qua left H-set, it is isomorphic to $H\times S$ for a set S (where, implicitly, the left action of H is by multiplication on the first coördinate); it is *right-free* if, qua right G-set, B is isomorphic to $T\times G$. It is *left-principal*, respectively *right-principal*, if furthermore S, respectively T may be chosen a singleton, so that the respective action is simply transitive; more generally, it is left, respectively right *free of rank* r if $\#S=r$, respectively $\#T=r$.

Bisets should be thought of as generalizations of homomorphisms. Indeed, if $\phi\colon H \to G$ is a group homomorphism, one associates with it the H-G-biset B_ϕ, which, qua right G-set, is plainly G; the left H-action is by

$$h \cdot b = h^\phi b.$$

All bisets of the form B_ϕ are right-principal; and they are left-free if and only if ϕ is injective. Recall that h^ϕ denotes the image of h under ϕ.

Let us return to our continuous map $f\colon Y \to X$, but drop the assumption $f(\dagger) = *$. The H-G-biset of f is defined as homotopy classes of paths rel their endpoints:

$$B(f) = B(f, \dagger, *) = \{\gamma\colon [0,1] \to X \mid \gamma(0) = f(\dagger),\ \gamma(1) = *\}/{\sim}. \qquad (1)$$

For paths $\gamma, \delta\colon [0,1] \to X$ with $\gamma(1) = \delta(0)$, we denote by $\gamma\#\delta$ their *concatenation*, defined by

$$(\gamma\#\delta)(t) = \begin{cases} \gamma(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \delta(2t-1) & \text{if } \frac{1}{2} \le t \le 1; \end{cases}$$

and, for a path $\gamma\colon [0,1] \to X$, its *reverse* γ^{-1} is defined by

$$\gamma^{-1}(t) = \gamma(1-t).$$

The left action of H on $B(f)$ is, for a loop λ in Y based at $\dagger$,

$$[\lambda] \cdot [\gamma] = [(f \circ \lambda)\#\gamma],$$

and the right action of G is, for a loop μ in X based at $*$,

$$[\gamma] \cdot [\mu] = [\gamma\#\mu].$$

It is then clear that $B(f)$ is a right-principal biset. If $f(\dagger) = *$, then $B(f)$ is naturally isomorphic to B_{f_*}.

As we will see in §4, bisets actually encode topological correspondences, as generalizations of continuous maps.

2.1. Morphisms. We consider three different kinds of maps between bisets.

Definition 2.2 (biset morphisms). Let ${}_H B_G$ and ${}_H B'_G$ be two H-G-bisets. A *morphism* between them is a map $\beta\colon B \to B'$ such that

$$h b^\beta g = (hbg)^\beta \qquad \text{for all } h \in H, b \in B, g \in G.$$

The definitions of endomorphism, isomorphism, automorphism, monomorphism, and epimorphism are standard.

Let now ${}_H B_G$ and ${}_{H'} B'_{G'}$ be two bisets. An *intertwiner* is a triple (ψ, β, ϕ) of maps, with $\psi\colon H \to H'$ and $\phi\colon G \to G'$ group homomorphisms, and $\beta\colon B \to B'$ a map, such that

$$h^\psi b^\beta g^\phi = (hbg)^\beta \quad \text{for all } h \in H, b \in B, g \in G.$$

The map β itself is called a (ψ, ϕ)-intertwiner; therefore a morphism is the same thing as a $(\mathbb{1}, \mathbb{1})$-intertwiner. An intertwiner is injective if all ψ, β, ϕ are injective, and similarly for surjective etc. A *congruence* is an invertible intertwiner.

Consider finally a G-biset ${}_G B_G$ and a G'-biset ${}_{G'} B'_{G'}$. Apart from morphisms and intertwiners, a third (intermediate) notion relates them: a *semiconjugacy* is a pair (ϕ, β) of maps, with $\phi\colon G \to G'$ and $\beta\colon B \to B'$, such that

$$h^\phi b^\beta g^\phi = (hbg)^\beta \quad \text{for all } g, h \in G, b \in B.$$

In other words, β is a (ϕ, ϕ)-intertwiner. Note that we do not require, as is sometimes customary, that β be surjective. A *conjugacy* is an invertible semiconjugacy, i.e. a semiconjugacy in which both ϕ and β are invertible.

In summary,

$$\begin{array}{ccccc}
\begin{array}{c}\text{intertwiners}\\(\psi,\beta,\phi)\end{array} & \supset & \begin{array}{c}\text{semiconjugacies}\\ \psi=\phi\end{array} & \supset & \begin{array}{c}\text{morphisms}\\ \psi=\phi=\mathbb{1}\end{array}\\
\cup & & \cup & & \cup\\
\begin{array}{c}\text{congruences}\\(\psi,\beta,\phi)\end{array} & \supset & \begin{array}{c}\text{conjugacies}\\ \psi=\phi\end{array} & \supset & \begin{array}{c}\text{isomorphisms}\\ \psi=\phi=\mathbb{1}.\end{array}
\end{array}$$

Note now the following important, if easy, fact: the isomorphism class of a biset B_ϕ remembers precisely the homomorphism ϕ up to inner automorphisms. More precisely,

Lemma 2.3. *Let $\phi, \psi\colon H \to G$ be two homomorphisms. Then the bisets B_ϕ and B_ψ are congruent if and only if there exists an automorphism η of G such that $\psi = \phi\eta$; and they are isomorphic if and only if η may be chosen to be inner.*

Proof. We only prove the second assertion ("isomorphic bisets if and only if η is inner").

Assume first $\psi = \phi\eta$, and let η be conjugation by g. Then an isomorphism between the bisets B_ϕ and B_ψ is given by $\beta\colon b \mapsto g^{-1}b$. Indeed,

$$(h \cdot b)^\beta = (h^\phi b)^\beta = g^{-1}h^\phi b = h^{\phi\eta}g^{-1}b = h^\psi b^\beta = h \cdot b^\beta.$$

Conversely, if B_ϕ and B_ψ are isomorphic, let β be such an isomorphism, and set $g = (1^\beta)^{-1}$. Because β commutes with the right G-action, which is principal, the map β must have the form $b^\beta = 1^\beta b = g^{-1}b$, and the same computation as above shows that $\psi = \phi\eta$, with η conjugation by g. □

2.2. Products. The *product* of the H-G-biset B with the G-F-biset C is the H-F-biset $B \otimes_G C$, defined as

$$B \otimes_G C = B \times C \big/ \{(b, g \cdot c) = (b \cdot g, c) \text{ for all } b \in B, g \in G, c \in C\};$$

it is naturally an H-F-biset for the left H-action on B and the right F-action on C. We have the easy

Lemma 2.4. *Let $\phi: H \to G$ and $\psi: G \to F$ be group homomorphisms. Then*

$$B_{\phi\psi} \cong B_\phi \otimes_G B_\psi.$$

Recall that we stick to the usual topological convention that $g \circ f$ is first f, then g; but we use the algebraic order on composition of homomorphisms and bisets, so that fg means 'first f, then g' for composable algebraic objects f and g. In other words, we may write $g \circ f = fg$.

The *contragredient* of the H-G-biset B is the G-H-biset $B^\vee$, which is B as a set (but with elements written $b^\vee$), and actions

$$g \cdot b^\vee \cdot h = (h^{-1} \cdot b \cdot g^{-1})^\vee.$$

If $\phi: H \to G$ is invertible, then we have $(B_\phi)^\vee = B_{\phi^{-1}}$. In all cases, we have a canonical isomorphism $(B \otimes C)^\vee = C^\vee \otimes B^\vee$.

Remark 2.5. Answering a question of A. Epstein, we may define $\operatorname{Hom}(B, C) = B^\vee \otimes C$, and then note that '$\otimes$' is the adjoint to this internal Hom functor, namely $\operatorname{Hom}(C \otimes B, D) = \operatorname{Hom}(B, \operatorname{Hom}(C, D))$ is natural.

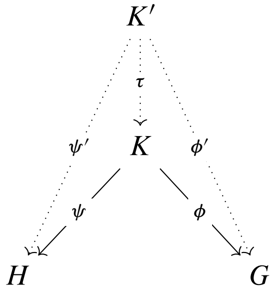
Products allow us to write intertwiners in terms of morphisms:

Lemma 2.6. *Let $\psi: H \to H'$ and $\phi: G \to G'$ be homomorphisms, let B be an H-G-biset, and let B' be an H'-G'-biset. Then there is an equivalence between intertwiners $(\psi, \beta, \phi): B \to B'$ and morphisms $\gamma: B_\psi^\vee \otimes B \otimes B_\phi \to B'$, in the following sense: there is a natural map $\theta: B \to B_\psi^\vee \otimes B \otimes B_\phi$ defined by $b \mapsto 1^\vee \otimes b \otimes 1$, and the intertwiner (ψ, β, ϕ) factors as $(\psi, \beta, \phi) = (\psi, \theta, \phi)\gamma$.*

In particular, the bisets ${}_G B_G$ and ${}_{G'} B'_{G'}$ are conjugate if and only if there exists an isomorphism $\phi: G \to G'$ with $B' \cong B_\phi^\vee \otimes B \otimes B_\phi = B_{\phi^{-1}} \otimes B \otimes B_\phi$.

2.3. Transitive bisets. We call an H-G-biset *transitive* if it consists of a single $H \times G$-orbit. If desired, transitive bisets may be viewed as quotients of right-principal bisets (namely, bisets coming from homomorphisms), as follows:

Lemma 2.7. *Let B be a transitive H-G-biset. Then there exist a group K and homomorphisms $\phi\colon K \to G$, $\psi\colon K \to H$ such that $B = B_\psi^\vee \otimes_K B_\phi$. Furthermore, there exists a unique minimal such K, in the sense that if (K', ϕ', ψ') also satisfy $B = B_{\psi'}^\vee \otimes_{K'} B_{\phi'}$ then there exists a homomorphism $\tau\colon K' \to K$ with $\tau\phi = \phi'$ and $\tau\psi = \psi'$:*



Proof. Choose a basepoint $b \in B$, and define

$$K = \{(h, g) \in H \times G \mid bg = hb\}.$$

The homomorphisms ψ, ϕ are given by projection on the first, respectively second coördinate. To construct an isomorphism $\beta\colon B_\psi^\vee \otimes B_\phi \to B$, set $(h^\vee \otimes g)^\beta := h^{-1}bg$; note that this is well-defined because $h^\vee \otimes g = (h')^\vee \otimes g'$ if and only if there exists $(h'', g'') \in K$ with $g' = g''g$ and $h' = h''h$; and then $h^{-1}bg = (h')^{-1}bg'$.

To prove the unicity of K, consider a group K' with two homomorphisms $\phi'\colon K' \to G$ and $\psi'\colon K' \to H$, and an isomorphism $\beta\colon B_{\psi'}^\vee \otimes_{K'} B_{\phi'} \to B$. Define then the map $\tau\colon K' \to K$ as follows. Write $\beta^{-1}(b) = (b_1, b_2)$. For $k' \in K'$, define $h \in H, g \in G$ by

$$\beta(b_1, k'b_2) = hb = bg \text{ and then } \tau(k') = (h, g). \qquad \square$$

Note that the product of two bisets, when expressed as groups K, L with homomorphisms as in the lemma, is nothing but the fibre product of the corresponding groups K, L.

If furthermore B is left-free, then the construction can be made even more explicit: the biset $B_\psi^\vee$ is a subbiset of B, and B_ϕ is left- and right-free. We summarize this in the

Proposition 2.8. *Let B be a left-free transitive H-G-biset. Choose $b \in B$. Define*

- *$G_b := \{g \in G \mid bg \in Hb\}$, the right stabilizer of b;*
- *$D := {}_{G_b}G_G$, the natural G_b-G biset;*
- *$C^\vee := Hb$, an H-G_b subbiset of ${}_HB_{G_b}$.*

Then C is a right-principal biset and $B \cong C^\vee \otimes_{G_b} D$.

2.4. Combinatorial equivalence of bisets. Let ${}_G B_G$ be a left-free biset. For every $n \geq 0$ we have a right G-action on $1 \otimes_G B_G^{\otimes n}$, which gives a well defined action on the set

$$T := \bigsqcup_{n \geq 0} 1 \otimes_G B_G^{\otimes n}.$$

The G-set T naturally has the structure of a rooted tree, by putting an edge from $1 \otimes b_1 \otimes \cdots \otimes b_n$ to $1 \otimes b_1 \otimes \cdots \otimes b_n \otimes b_{n+1}$ for all $b_i \in B$. It is called the *Fock tree* of B in [14]. The action of G on T is self-similar in an appropriate labeling of T, see [13, Chapter 2].

Set $G' := G/\ker(\text{action})$; thus G' is the maximal quotient of G such that the induced action on T is faithful. As in [15] we say that ${}_G B_G$ is *combinatorially equivalent* to

$${}_{G'} B'_{G'} := G' \otimes_G B \otimes_G G'.$$

(The motivation is dynamical: if ${}_G B_G$ is contracting, then so is ${}_{G'} B'_{G'}$ and, moreover, the limit dynamical systems associated with ${}_G B_G$ and ${}_{G'} B'_{G'}$ are topologically conjugate, see e.g. [13, Corollary 3.6.7].)

We say that two left-free ${}_G B_G$ and ${}_H C_H$ are *combinatorially equivalent* if their quotients ${}_{G'} B'_{G'}$ and ${}_{H'} C'_{H'}$ are conjugate.

It is easy to check that any surjective semi-conjugacy $(\phi, \beta)\colon {}_G B_G \to {}_H C_H$ between left-free bisets descends to a semi-conjugacy $(\phi', \beta')\colon {}_{G'} B'_{G'} \to {}_{H'} C'_{H'}$. We say that a surjective semi-conjugacy $(\phi, \beta)\colon {}_G B_G \to {}_H C_H$ between left-free bisets *respects combinatorics* if $(\phi', \beta')\colon {}_{G'} B'_{G'} \to {}_{H'} C'_{H'}$ is a conjugacy (so that ${}_G B_G$ and ${}_H C_H$ are combinatorially equivalent).

2.5. Biset presentations. Let B be a left-free H-G-biset. Choose a *basis* of B, namely a subset S of B such that B, qua left H-set, is isomorphic to $H \times S$. In other words, S contains precisely one element from each H-orbit of B.

The structure of the biset is then determined by the right action in that description. For $g \in G$ and $s \in S$, there are unique $(h, t) \in H \times S$ such that $sg = ht$. The choice of basis therefore leads to a map $S \times G \to H \times S$, or, which is the same, a map $\Phi\colon G \to (H \times S)^S$.

Associativity of the biset operations yields, as is easy to see, that Φ is a group homomorphism $G \to H \wr S\!\!\downarrow$. This last group, the *wreath product* of H with the symmetric group $S\!\!\downarrow$ on S, is by definition the semidirect product of H^S with $S\!\!\downarrow$, in which $S\!\!\downarrow$ acts by permutations of the coördinates in H^S. We call Φ a *wreath map* of the biset B.

Wreath products are best thought of in terms of *decorated permutations*: one writes permutations of S in the standard arrow diagram notation, but adds a label belonging to H on each arrow in the permutation. Permutations are multiplied by concatenating arrow diagrams and multiplying the labels.

A *presentation* of B is a choice of generating sets Γ, Δ for G, H respectively, and for each $g \in \Gamma$ an expression of the form

$$g = \langle\!\langle h_1, \dots, h_d \rangle\!\rangle (i_1, \dots),$$

describing $\Phi(g)$; the h_i are words in Δ, and $(i_1, \dots)$ is a permutation of $S \cong \{1, \dots, d\}$ in disjoint cycle format.

Lemma 2.9. *If $\Phi, \Psi \colon G \to H \wr d{\downarrow}$ are two wreath maps of the same biset ${}_H B_G$, then there exists $w \in H \wr d{\downarrow}$ such that $\Psi = \Phi \cdot (h \mapsto h^w)$.*

Proof. Let S, T be the bases of B in which Φ, Ψ are computed, and identify S, T with $\{1, \dots, d\}$ by writing $S = \{s_1, \dots, s_d\}$ and $T = \{t_1, \dots, t_d\}$. In B, write $s_i = w_i \cdot t_{i^\pi}$ for $i = 1, \dots, d$, defining thus $w = \langle\!\langle w_1, \dots, w_d \rangle\!\rangle \pi \in H \wr d{\downarrow}$.

Consider an arbitrary $g \in G$, and write

$$\Phi(g) = \langle\!\langle g_1, \dots, g_d \rangle\!\rangle \sigma, \quad \Psi(g) = \langle\!\langle h_1, \dots, h_d \rangle\!\rangle \tau.$$

We therefore have $s_i g = g_i s_{i^\sigma}$ and $t_i g = h_i t_{i^\tau}$ for all i. Now

$$h_{i^\pi} t_{i^{\pi\tau}} = t_{i^\pi} g = w_i^{-1} s_i g = w_i^{-1} g_i s_{i^\sigma} = w_i^{-1} g_i w_{i^{\sigma\pi}} t_{i^{\sigma\pi}},$$

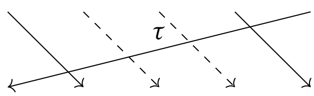
so $\Phi(g)^w = \Psi(g)$. □

It may help to introduce an example here. Consider the map $f(z) = z^d$ from the cylinder $X = \mathbb{C} \setminus \{0\}$ to itself. Choose $* = \dagger = 1$ as basepoints, and identify $\pi_1(X, *)$ with $\mathbb{Z}$ by choosing as generator the loop $\tau(t) = \exp(2i\pi t)$.

Since it is right-(free and transitive), the biset $B(f)$ is in bijection with the set of homotopy classes of loops at 1, and so may be naturally identified with $\mathbb{Z}$. The biset structure is given by $n \cdot b \cdot m = dn + b + m$ for $n, m \in \mathbb{Z} \cong \pi_1(X, *)$ and $b \in \mathbb{Z} \cong B(f)$. Thus $B(f)$ is left-free of rank d, and a basis is a complete set of congruence representatives modulo d, for example $\{0, 1, \dots, d-1\}$. In that basis, the wreath map reads

$$\Phi(\tau) = \langle\!\langle 1, \dots, 1, \tau \rangle\!\rangle (1, 2, \dots, d),$$

or in diagram notation

$\tau =$  .

Since it is left-free of rank d, the biset $B(f)$ is also in bijection with the homotopy classes of paths from $*$ to an f-preimage of $*$, namely to a d-th root of unity. A basis of $B(f)$ may be chosen as $\{\exp(2i\pi tk/d) \mid t \in [0,1]\}_{k=0,\dots,d-1}$, and in this basis one recovers the wreath map of τ given above.

2.6. Conjugacy classes in groups. We will, in later articles, consider conjugacy classes so as to represent unbased loops in topological spaces. There are various applications; for example, a fundamental construction by Goldman [9] defines a Lie bracket structure on the vector space spanned by conjugacy classes in $\pi_1(X, *)$ for a closed surface X. We consider, here, the general case of groups and bisets.

Let $\mathcal{C}(G)$ denote the set of conjugacy classes of the group G, and consider a left-free H-G-biset B. Choose a basis S of B, whence a wreath map $\Phi\colon G \to H \wr S\!\downarrow$. Consider $g^G \in \mathcal{C}(G)$, write $\Phi(g) = \langle\!\langle h_1, \dots, h_d \rangle\!\rangle \pi$, and let $S_1, \dots, S_\ell$ be the orbits of π on S. For each $j = 1, \dots, \ell$, let k_j be the product of the h_i's along the orbit S_j; namely, if $S_j = \{s_1, \dots, s_{d_j}\}$ with $s_i^\pi = s_{i+1}$, indices being computed modulo d_j, then $k_j = h_{s_1} h_{s_2} \cdots h_{s_{d_j}}$. The multiset $\{(d_j, k_j^H) \mid i = j, \dots, \ell\}$ consisting of degrees and conjugacy classes in H is called the *lift* of g^G.

Lemma 2.10. *The lift of g^G is independent of all the choices made: of g in its conjugacy class, of the basis S, and of the cyclic ordering of the orbit S_j.*

Proof. Different choices of bases give conjugate wreath maps, by Lemma 2.9. □

Let $\mathbb{Q}\mathcal{C}(G)$ denote the vector space spanned by the conjugacy classes $\mathcal{C}(G)$, and consider again a left-free H-G-biset B. Then the lift operation gives rise to a linear map $B^*\colon \mathbb{Q}\mathcal{C}(G) \to \mathbb{Q}\mathcal{C}(H)$, defined on the basis by

$$B^*(g^G) := \sum_{(d_i, h_i^H) \in \text{lift of } g^G} \frac{k_j^H}{d_j}, \tag{2}$$

called the *Thurston endomorphism* of B.

3. Graphs of bisets

We define here the fundamental notion in this article: viewing groups as fundamental groups of spaces, bisets describe continuous maps between spaces. Graphs of groups describe decompositions of spaces, and graphs of bisets describe continuous maps that are compatible with the decompositions.

3.1. Graphs of groups. We begin by precising the notion of graph we will use; it is close to Serre's [17], but allows more maps between graphs. In particular, we allow graph morphisms that send edges to vertices.

Definition 3.1 (graphs). A *graph* $\mathfrak{X}$ is a set $\mathfrak{X} = V \sqcup E$, consisting of two subsets called *vertices* and *edges* respectively, equipped with two self-maps $x \mapsto x^-$ and $x \mapsto \bar{x}$, and subject to axioms

$$\text{for all } x \in \mathfrak{X}, \quad \bar{\bar{x}} = x, \quad x^- \in V, \quad \text{and} \quad x = x^- \iff x = \bar{x} \iff x \in V. \tag{3}$$

The object $\bar{x}$ is called the *reverse* of x. Setting $x^+ = (\bar{x})^-$, the vertices x^-, x^+ are respectively the *origin* and *terminus* of x.

We write $V(\mathfrak{X})$ and $E(\mathfrak{X})$ for the sets of vertices and edges respectively in a graph $\mathfrak{X}$. A *path* is a sequence $(e_1, \dots, e_n)$ of edges with $e_i^+ = e_{i+1}^-$ for all $i = 1, \dots, n-1$. A graph is *connected* if there exists a path joining any two objects. A *circuit* is a sequence $(e_1, \dots, e_n)$ of edges with $e_i^+ = e_{i+1}^-$ for all i, indices taken modulo n. A *tree* is a graph with no non-trivial circuits; that is in every circuit $(e_1, \dots, e_n)$ one has $e_{i+1} = \overline{e_i}$ for some i.

A *graph morphism* is a map $\theta\colon \mathfrak{Y} \to \mathfrak{X}$ satisfying $\theta(\bar{y}) = \overline{\theta(y)}$ and $\theta(y^-) = \theta(y)^-$ for all $y \in \mathfrak{Y}$. Note that θ maps the vertices of $\mathfrak{Y}$ to those of $\mathfrak{X}$. It is *simplicial* if furthermore θ maps the edges of $\mathfrak{Y}$ to those of $\mathfrak{X}$.

A *graph of groups* is a connected graph $\mathfrak{X} = V \sqcup E$, with a group G_x associated with every $x \in \mathfrak{X}$, and homomorphisms $(\cdot)^-\colon G_x \to G_{x^-}$ and $\overline{(\cdot)}\colon G_x \to G_{\bar{x}}$ for each $x \in \mathfrak{X}$, satisfying the same axioms as (3), namely the composition $G_x \to G_{\bar{x}} \to G_{\bar{\bar{x}}} = G_x$ is the identity for every $x \in \mathfrak{X}$, and if $x \in V$ then the homomorphisms $G_x \to G_{x^-}$ and $G_x \to G_{\bar{x}}$ are the identity. For $g \in G_e$ we write $g^+ = (\bar{g})^-$. The graph of groups is still denoted $\mathfrak{X}$.

Let $\mathfrak{X} = V \sqcup E$ be a graph of groups. For $v, w \in V$, consider the set

$$\begin{aligned}\Pi_{v,w} = \{(g_0, x_1, g_1, \dots, x_n, g_n) \mid x_i \in \mathfrak{X}, x_1^- = v, x_i^+ = x_{i+1}^-, x_n^+ = w,\\ g_i \in G_{x_i^+} = G_{x_{i+1}^-}\},\end{aligned} \tag{4}$$

of (group-decorated) paths from v to w. As a special case, if $v = w$ then we allow $n = 0$ and $g_0 \in G_v = G_w$. Say that two paths are equivalent, written $\sim$, if they differ by a finite sequence of elementary local transformations of the form $(gh) \leftrightarrow (g, x, 1, \bar{x}, h)$ for some $x \in \mathfrak{X}$ and $g, h \in G_{x^-}$, or of the form $(gh^-, x, k) \leftrightarrow (g, x, h^+k)$ for some $x \in \mathfrak{X}$ and $g \in G_{x^-}, h \in G_x, k \in G_{x^+}$, or of the form $(g, x, h) \leftrightarrow (gh)$ for some $x \in V$ and $g, h \in G_x$.

The product of two paths $(g_0, x_1, \dots, g_m)$ and $(h_0, y_1, \dots, h_n)$ is defined if $x_m^+ = y_1^-$, and equals $(g_0, x_1, \dots, g_m h_0, y_1, \dots, h_n)$; this product $\Pi_{u,v} \times \Pi_{v,w} \to \Pi_{u,w}$ is compatible with the equivalence relation. The *fundamental groupoid* $\pi_1(\mathfrak{X})$ of $\mathfrak{X}$ is a groupoid with object set V, and with morphisms between v and w the set $\pi_1(\mathfrak{X}, v, w) = \Pi_{v,w}/{\sim}$ of equivalence classes of paths from v to w.

In particular, for $v \in V$, the *fundamental group* $\pi_1(\mathfrak{X}, v)$ is $\Pi_{v,v}/{\sim}$. If $\mathfrak{X}$ is connected, then $\pi_1(\mathfrak{X}, v)$ is up to isomorphism independent of the choice of $v \in V$.

In fact, the elements x_i in a path may be assumed to all belong to E, and if $x_{i+1} = \overline{x_i}$ then the element g_{i+1} may be assumed not to belong to $(G_{x_{i+1}})^-$. Such paths are called *reduced*, and $\Pi_{v,w}/\sim$ may be identified with the set of reduced paths from v to w.

In a more algebraic language, the fundamental groupoid $\pi_1(\mathfrak{X})$ is the universal groupoid with object set V, and whose morphism set is generated by $\mathfrak{X} \sqcup \bigsqcup_{v \in V} G_v$; the source and target of $x \in \mathfrak{X}$ are x^- and x^+ respectively; the source and target of $g \in G_v$ are v; the relations are those of the G_v as well as $v = 1 \in G_v$ for all $v \in V$, and $x\bar{x} = 1 \in G_{x^-}$ and $g^- x = x g^+$ for all $x \in \mathfrak{X}$ and $g \in G_x$. The path $(g_0, x_1, \dots, g_n)$ is identified with $g_0 x_1 \dots g_n$. The following property follows easily from the definitions:

Lemma 3.2 (e.g. Serre [17, §5.2]). *Let $\mathfrak{X}$ be a graph of groups. If all morphisms $G_x \to G_{x^-}$ are injective, then all natural maps $G_x \to \pi_1(\mathfrak{X}, *)$ are injective.*

3.2. Decompositions and van Kampen's theorem. Graphs of groups, and their fundamental group, generalize decompositions of spaces and their fundamental group. Here is a useful example of graphs of groups:

Example 3.3. If X be a path connected surface with at least one puncture, consider a graph $\mathfrak{X}$ drawn on X which contains precisely one puncture in each face. Then X deformation retracts to $\mathfrak{X}$, and the groups $\pi_1(X, *)$ and $\pi_1(\mathfrak{X}, *)$ are isomorphic. Here we consider $\mathfrak{X}$ as a graph of groups in which all groups G_x are trivial.

Definition 3.4 (1-dimensional covers). Consider a path connected space X, covered by a collection of path connected subspaces $(X_v)_{v \in V}$. It is a 1*-dimensional open cover* of X if for all $u, v, w \in V$

(1) all path connected components of $X_u \cap X_v$ are of the form X_t for some $t \in V$;
(2) if $X_u \subseteq X_v \subseteq X_w$ then $u = v$ or $v = w$.

We order V by writing $u < v$ if $X_u \subsetneqq X_v$.

A 1*-dimensional cover* of X is a cover $(X_v)_{v \in V}$ of X satisfying (1) and (2) and such that for every family of open neighbourhoods $\mathfrak{U}_v \supseteq X_v$ of the X_v there exists a family $\widetilde{X}_v \subseteq \mathfrak{U}_v$ of path connected open subneighbourhoods such that the natural map $\pi_1(X_v) \to \pi_1(\widetilde{X}_v)$ is an isomorphism for all $v \in V$ and the $(\widetilde{X}_v)_{v \in V}$ form a 1-dimensional open cover as above, with same combinatorics as $(X_v)_{v \in V}$.

Traditionally, the sets X_v in a cover are assumed open so that every curve $\gamma \subset X$ is a concatenation of finitely many curves γ_i where each γ_i lies entirely in some $X_{v(i)}$.

For our dynamical applications, we may also wish to consider closed subsets X_v of X: typical dynamically-invariant subsets (such as the Julia set) are closed but not open. We may use these more general covers in place of open covers, thanks to the following fact:

Lemma 3.5. *Let $f\colon Y \to X$ be a continuous map, and suppose that f extends to $\tilde{f}\colon \widetilde{Y} \to \widetilde{X}$ with $X \subset \widetilde{X}$ and $Y \subset \widetilde{Y}$. Assume that all spaces are path connected and that $X \hookrightarrow \widetilde{X}$ and $Y \hookrightarrow \widetilde{Y}$ induce isomorphisms on fundamental groups. Then the bisets of $B(f)$ and $B(\widetilde{f})$ are isomorphic.*

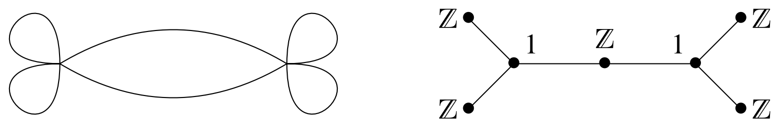
Proof. Suppose that $\dagger \in Y$ and $* \in X$ are the basepoints; we can assume that $f(\dagger) = *$. We need to show that $f_* = \tilde{f}_*$. Suppose that $f_* \neq \tilde{f}_*$. Then there is a $\gamma \in \pi_1(Y, \dagger)$ such that $f_*(\gamma) \neq 1$ but $\tilde{f}_*(\gamma) = 1$. This means that $f(\gamma)$ is not trivial in $\pi_1(X, *)$ but is trivial in $\pi_1(\widetilde{X}, *)$; this contradicts $\pi_1(X, *) \simeq \pi_1(\widetilde{X}, *)$. □

Definition 3.6 (graphs of groups from covers). Consider a path connected space X with a 1-dimensional cover $(X_v)_{v \in V}$. It has an associated graph of groups $\mathfrak{X}$, defined as follows. The vertex set of $\mathfrak{X}$ is V. For every pair $u < v$ there are edges e and $\bar{e}$ connecting $u = e^- = \bar{e}^+$ and $v = e^+ = \bar{e}^-$, and we let E be the set of these edges. Set $\mathfrak{X} = V \sqcup E$.

Choose basepoints $*_v \in X_v$ for all $v \in V$, and set $G_v := \pi_1(X_v, *_v)$. For every edge e with $e^- < e^+$ choose a path $\gamma_e\colon [0,1] \to X_{e^+}$ from $*_{e^-}$ to $*_{e^+}$, set $G_e := G_{e^-}$ and define $G_e \to G_{e^-} := \mathbb{1}$ and $G_e \to G_{e^+}$ by $\gamma \mapsto \gamma_e^{-1}\#\gamma\#\gamma_e$. Finally for edges e with $e^- > e^+$ set $G_e := G_{\bar{e}}$ and $\gamma_{\bar{e}} = \gamma_e^{-1}$ and define morphisms $G_e \to G_{e^-}$ and $G_e \to G_{e^+}$ as $G_{\bar{e}} \to G_{\bar{e}^+}$ and $G_{\bar{e}} \to G_{\bar{e}^-}$ respectively.

The graph of groups $\mathfrak{X}$ depends on the choice of basepoints $*_v$ and paths γ_e, but mildly: we will show in Lemma 4.9 that the congruence class of $\mathfrak{X}$ is independent of the choice of $*_v$ and γ_e.

Here is a simple example: on the left, the subspaces X_v are the simple loops and the two triple intersection points; on the right, the corresponding graph of groups, with trivial or infinite cyclic groups.



Theorem 3.7 (van Kampen's theorem). *Let X be a path connected space with 1-dimensional cover (X_v) and a choice of basepoints $*_v \in X_v$ as well as paths connecting $*_u$ and $*_v$ if $X_u \subsetneq X_v$. Let $\mathfrak{X}$ be the associated graph of groups.*

Then, for every v, the groups $\pi_1(X, *_v)$ and $\pi_1(\mathfrak{X}, v)$ are isomorphic.

Sketch of proof. The isomorphism $\theta\colon \pi_1(\mathfrak{X}, v) \to \pi_1(X, *_v)$ is defined by

$$\theta(g_0, e_1, g_1, \dots, e_n, g_n) = g_0 \# \gamma_{e_1} \# \dots \gamma_{e_n} \# g_n.$$

We start by assuming that the 1-dimensional cover (X_v) consists of open sets. Then every loop $\gamma\colon [0,1] \to X$ with $\gamma(0) = \gamma(1) = *_v$ is homotopic to a concatenation of the above type, so θ is surjective. By the classical van Kampen argument (see [12, Chapter IV]), the path $\theta(g)$ is homotopic to $\theta(g')$ in X if and only if g and g' are equivalent in $\pi_1(\mathfrak{X}, v)$; thus θ is injective.

If the X_v are not open, then they may be slightly enlarged enlarged to open sets $\widetilde{X}_v$, on which the theorem applies; the graphs of groups associated to the covers (X_v) and $(\widetilde{X}_v)$ are isomorphic. □

In case $\mathfrak{X}$ is a tree, $\pi_1(\mathfrak{X}, *)$ is an iterated free product with amalgamation of the G_v's, amalgamated over the G_e's. It may be constructed explicitly as follows: consider the set of finite sequences over the alphabet $\bigsqcup_{v \in V} G_v$; identify two sequences if they differ on subsequences respectively of the form $\{(1), (\cdot)\}$; or $\{(g,h), (gh)\}$ for g, h in the same G_v; or $\{(g), (g)\}$ where the first, respectively second, 'g' denote the image of $g \in G_e$ in G_{e^-}, respectively G_{e^+}; and quotient by the equivalence relation generated by these identifications.

Note the following three operations on a graph of groups $\mathfrak{X}$, which does not change its fundamental group.

(1) Split an edge. Choose an edge $e \in E$. Add a new vertex v to V, and replace $e, \bar{e}$ by new edges $e_0, \overline{e_0}, e_1, \overline{e_1}$ with $e_0^+ = e_1^- = v$ and $e_0^- = e^-$ and $e_1^+ = e^+$. Define the new groups by $G_v = G_{e_0} = G_{e_1} = G_e$, with the obvious maps between them;

(2) Add an edge. Choose a vertex $v \in V$, and a subgroup $H \subseteq G_v$. Add a new vertex w to V, and add new edges $e, \bar{e}$ with $e^- = v$ and $e^+ = w$. Define the new groups by $G_e = G_v = H$, with the obvious maps between them;

(3) Barycentric subdivision. Construct a new graph $\mathfrak{X}' = V' \sqcup E'$ with $V' = \mathfrak{X}/\{x = \bar{x}\}$ and $E' = E \times \{+,-\}$; for $e \in E$ and $\varepsilon \in \{\pm 1\}$, set $(e,\varepsilon)^\varepsilon = e^\varepsilon$ and $(e,\varepsilon)^{-\varepsilon} = [e]$ and $\overline{(e,\varepsilon)} = (\bar{e}, -\varepsilon)$. Define finally a graph of groups structure $(G'_{[x]})$ on $\mathfrak{X}'$ by setting $G'_{[x]}$ to be G_x for all $[x] \in V'$ (recalling that G_x and $G_{\bar{x}}$ are isomorphic), and $G'_{(e,\varepsilon)} = G_e$, with the obvious maps between these groups.

Recall that graph morphisms may send edges to vertices, but not vertices to (midpoints of) edges. If such a map between the topological realizations of the graphs is needed, it may be expressed as a map between their barycentric subdivisions.

If $\mathfrak{X}$ be a graph of groups, let $\mathfrak{X}'$ denote the underlying graph of $\mathfrak{X}$, but with trivial groups. Then, on the one hand, $\pi_1(\mathfrak{X}')$ is the usual fundamental groupoid of $\mathfrak{X}'$, and $\pi_1(\mathfrak{X}', v)$ is a free group; on the other hand, there exists a natural morphism of groupoids $\pi_1(\mathfrak{X}) \to \pi_1(\mathfrak{X}')$.

If $\mathfrak{X}, \mathfrak{Y}$ be graphs of groups, $\theta\colon \mathfrak{Y} \to \mathfrak{X}$ be a graph morphism, and θ_y be, for every $y \in \mathfrak{Y}$, a homomorphism $G_y \to G_{\theta(y)}$ such that $\theta_y(g)^- = \theta_{y^-}(g^-)$ and $\overline{\theta_y(g)} = \theta_{\bar{y}}(\bar{g})$ for all $g \in G_y$, then there is an induced morphism of groupoids $\pi_1(\mathfrak{Y}) \to \pi_1(\mathfrak{X})$. We do not give a name to these kinds of maps, because they are too restrictive; see the discussion in the introduction of [5]. We will give later, in §3.9, more examples of graphs of groups; we now describe in more detail the precise notion of morphisms between graphs of groups that we will use.

3.3. Graphs of bisets. Just as bisets generalize appropriately to our setting homomorphisms between groups, so do "graphs of bisets" also generalize the notion of homomorphisms between graphs of groups. In fact, our notion also extends the definition of morphisms between graphs of groups given in [5], in that a "right-principal graph of bisets" is the same thing as a morphism between graphs of groups in the sense of Bass.

Definition 3.8 (graph of bisets). Let $\mathfrak{X}, \mathfrak{Y}$ be two graphs of groups. A *graph of bisets* ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ between them is the following data:

- a graph $\mathfrak{B}$, not necessarily connected;
- graph morphisms $\lambda\colon \mathfrak{B} \to \mathfrak{Y}$ and $\rho\colon \mathfrak{B} \to \mathfrak{X}$;
- for every $z \in \mathfrak{B}$, a $G_{\lambda(z)}$-$G_{\rho(z)}$-biset B_z, an intertwiner $(\cdot)^-\colon B_z \to B_{z^-}$ with respect to the homomorphisms $G_{\lambda(z)} \to G_{\lambda(z)^-}$ and $G_{\rho(z)} \to G_{\rho(z)^-}$, and a congruence $\overline{(\cdot)}\colon B_z \to B_{\bar{z}}$ with respect to the isomorphisms $G_{\lambda(z)} \to G_{\overline{\lambda(z)}}$ and $G_{\rho(z)} \to G_{\overline{\rho(z)}}$.

  These homomorphisms must satisfy the same axioms as (3), namely: the composition $B_z \to B_{\bar{z}} \to B_{\bar{\bar{z}}} = B_z$ is the identity for every $z \in \mathfrak{X}$, and if $z \in V$, then the homomorphisms $B_z \to B_{z^-}$ and $B_z \to B_{\bar{z}}$ are the identity. For $b \in B_z$ we write $b^+ = \bar{b}^-$.

We call $\mathfrak{B}$ a $\mathfrak{Y}$-$\mathfrak{X}$-*biset*.

Example 3.9. A graph morphism $\theta\colon \mathfrak{Y} \to \mathfrak{X}$, given with homomorphisms $\theta_y\colon G_y \to G_{\theta(y)}$ for all $y \in \mathfrak{Y}$ such that $(\cdot)^- \circ \theta_y = \theta_{y^-} \circ (\cdot)^-$, naturally gives rise to a $\mathfrak{Y}$-$\mathfrak{X}$-biset $\mathfrak{B}_\theta$. Its underlying graph is $\mathfrak{B} = \mathfrak{Y}$; the maps are $\lambda = \mathbb{1}$ and $\rho = \theta$. For every $z \in \mathfrak{B}$, the biset B_z is $G_{\theta(z)}$ with its natural right $G_{\theta(z)}$-action, and its left G_z-action is $g \cdot b = \theta_z(g)b$.

Definition 3.10 (product of graphs of bisets). Let $\mathfrak{B}$ be a $\mathfrak{Y}$-$\mathfrak{X}$-biset, and let $\mathfrak{C}$ be a $\mathfrak{X}$-$\mathfrak{W}$-biset. The *product* of $\mathfrak{B}$ and $\mathfrak{C}$ is the following graph of bisets ${}_{\mathfrak{Y}}\mathfrak{D}_{\mathfrak{W}} = \mathfrak{B} \otimes_{\mathfrak{X}} \mathfrak{C}$: its underlying graph is the fibre product

$$\mathfrak{D} = \{(b, c) \in \mathfrak{B} \times \mathfrak{C} \mid \rho(b) = \lambda(c)\},$$

with $(b, c)^- = (b^-, c^-)$ and $\overline{(b, c)} = (\bar{b}, \bar{c})$. The map $\lambda\colon \mathfrak{D} \to \mathfrak{Y}$ is $\lambda(b, c) = \lambda(b)$, and the map $\rho\colon \mathfrak{D} \to \mathfrak{W}$ is $\rho(b, c) = \rho(c)$. The biset $B_{(b,c)}$ is $B_b \otimes_{G_{\rho(b)}} B_c$.

Properties of the product will be investigated in §3.7. For now we content ourselves with some illustrative examples.

Lemma 3.11 (see Example 3.9). *Let* $\theta\colon \mathfrak{Y} \to \mathfrak{X}$ *and* $\kappa\colon \mathfrak{X} \to \mathfrak{W}$ *be graph morphisms, given with homomorphisms* $\theta_y\colon G_y \to G_{\theta(y)}$ *and* $\kappa_x\colon G_x \to G_{\kappa(x)}$. *Then their bisets satisfy* $\mathfrak{B}_\theta \otimes_{\mathfrak{X}} \mathfrak{B}_\kappa = \mathfrak{B}_{\theta\kappa}$.

Proof. We have natural identifications $G_y \otimes_{G_x} G_x = G_y$ between the bisets of $\mathfrak{B}_\theta \otimes_{\mathfrak{X}} \mathfrak{B}_\kappa$ and those of $\mathfrak{B}_{\theta\kappa}$. □

The contragredient $\mathfrak{B}^\vee$ of the graph of bisets $\mathfrak{B}$ is defined by exchanging ρ and λ, and taking the contragredients of all bisets B_z. If $\theta\colon \mathfrak{Y} \to \mathfrak{X}$ is bijective and all θ_y are isomorphisms, then the contragredient of $\mathfrak{B}_\theta$ is $\mathfrak{B}_{\theta^{-1}}$.

The identity biset for products is the following biset ${}_{\mathfrak{X}}\mathfrak{I}_{\mathfrak{X}}$: its underlying graph is $\mathfrak{X}$, with maps $\lambda = \rho = \mathbb{1}$, and bisets $B_z = G_z$.

Example 3.12 (see Lemma 2.7; a partial converse to Example 3.9). Let ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ be a graph of bisets such that the graph $\mathfrak{B}$ is connected and B_z is transitive for every $z \in \mathfrak{B}$. Suppose that for every $z \in \mathfrak{B}$ we are given an element $b_z \in B_z$ such that $\overline{b_z} = b_{\bar z}$ and $(b_z)^- = b_{z^-}$ for all $z \in \mathfrak{B}$. Then there exist a graph of groups $\mathfrak{Z}$ and morphisms of graphs of groups $\lambda\colon \mathfrak{Z} \to \mathfrak{Y}$ and $\rho\colon \mathfrak{Z} \to \mathfrak{X}$ as in Example 3.9 such that $\mathfrak{B}$ is isomorphic to $\mathfrak{B}_\lambda^\vee \otimes \mathfrak{B}_\rho$.

Note, however, that it is not always possible to make coherent choices of basepoints $b_z \in B_z$. For example, it may happen that $(B_e)^- \cap (B_f)^- = \emptyset$ for two edges e, f with $e^- = f^-$.

Here is a dynamical situation in which this happens. Let f be a rational map whose Julia set is a Sierpiński carpet, and assume that its Fatou set has two invariant components $\mathcal{U}, \mathcal{V}$, perforce with disjoint closures, on which f acts as $z \mapsto z^m, z^n$ respectively. Remove these components, and attach on each of them a sphere with a self-map that has an invariant Fatou component mapped to itself by the same degree. To these data correspond a graph of groups $\mathfrak{X}$ with three vertices (the main sphere and the two grafted ones), and a graph of bisets ${}_{\mathfrak{X}}\mathfrak{B}_{\mathfrak{X}}$ describing the self-map. The basepoint on the main sphere cannot be chosen to coincide with the basepoints of the grafted spheres, because the Fatou components of the Sierpiński map have disjoint closures.

3.4. The fundamental biset

Definition 3.13 (fundamental biset of graph of bisets). Let $\mathfrak{B}$ be a $\mathfrak{Y}$-$\mathfrak{X}$-biset; choose $* \in \mathfrak{X}$ and $\dagger \in \mathfrak{Y}$. Write $G = \pi_1(\mathfrak{X}, *)$ and $H = \pi_1(\mathfrak{Y}, \dagger)$. The *fundamental biset* of $\mathfrak{B}$ is an H-G-biset $B = \pi_1(\mathfrak{B}, \dagger, *)$, constructed as follows:

$$B = \frac{\bigsqcup_{z \in V(\mathfrak{B})} \pi_1(\mathfrak{Y}, \dagger, \lambda(z)) \otimes_{G_{\lambda(z)}} B_z \otimes_{G_{\rho(z)}} \pi_1(\mathfrak{X}, \rho(z), *)}{\left\{ q b^- p = q\lambda(z) b^+ \overline{\rho(z)} p \quad \text{for all } \begin{array}{c} q \in \pi_1(\mathfrak{Y}, \dagger, \lambda(z)^-), b \in B_z, \\ p \in \pi_1(\mathfrak{X}, \rho(z)^-, *), z \in E(\mathfrak{B}) \end{array} \right\}}. \quad (5)$$

In other words, elements of B are sequences $(h_0, y_1, h_1, \dots, y_n, b, x_1, \dots, g_{m-1}, x_n, g_n)$ subject to the equivalence relations used previously to define $\pi_1(\mathfrak{X})$, as well as $(y_n, hb^- g, x_1) \leftrightarrow (y_n, h, \lambda(z), b^+, \overline{\rho(z)}, g, x_1)$ for all $z \in \mathfrak{B}$, $b \in B_z$, $h \in G_{\lambda(z)^-}$, $g \in G_{\rho(z)^-}$.

In particular, if $\mathfrak{Y} = \mathfrak{X}$ are graphs with one vertex and no edges and associated groups H, G respectively, the definition simplifies a little: then B is the quotient of $\bigsqcup_{z \in V(\mathfrak{B})} B_z$ by the relation identifying b^- with b^+, for all edges $z \in \mathfrak{B}$ and all $b \in B_z$, with respective images $b^\pm$ in $B_{z^\pm}$.

If $f: Y \to X$ be a continuous map and the spaces X, Y admit decompositions giving a splitting of their fundamental group as a graph of groups, as in Theorem 3.7, and if f is compatible with the decompositions of X and Y, then $B(f)$ will decompose into a graph of bisets, as we will see in Theorem 4.8.

Let $\mathfrak{X}$ be a graph of groups. Choose $* \in \mathfrak{X}$, and consider the graph $\{*\}$ with a single vertex and no edge. We treat it as a graph of groups, with group $\pi_1(\mathfrak{X}, *)$ attached to $*$, and denote it by $\mathfrak{X}_0$. Consider now the following biset ${}_{\mathfrak{X}}(\mathfrak{X}, *)_{\mathfrak{X}_0}$: its underlying graph is $\mathfrak{X}$; the maps are $\lambda = \mathbb{1}$ and $\rho(z) = *$ for all $z \in \mathfrak{X}$; and the bisets are $B_z = \pi_1(\mathfrak{X}, z^-, *)$; for vertices, this is just $\pi_1(\mathfrak{X}, z, *)$, while for edges this is a left G_z-set via the embedding $G_z \to G_{z^-}$. The embedding $B_z \to B_{z^-}$ is the identity, while the map $B_z \to B_{\bar z}$ is $b \mapsto \bar z b$.

Lemma 3.14. *Write $G = \pi_1(\mathfrak{X}, *)$. Then we have an isomorphism of G-G-bisets*

$$\pi_1((\mathfrak{X}, *), *, *) \cong {}_G G_G.$$

The bisets $(\mathfrak{X}, *)$ and $(\mathfrak{Y}, \dagger)$ enable us, using (5), to write $\pi_1(\mathfrak{B}, \dagger, *)$ more simply. We will prove Lemma 3.15 in §3.7:

Lemma 3.15. *Let ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ be a graph of bisets, and choose $* \in \mathfrak{X}, \dagger \in \mathfrak{Y}$. Then $\pi_1(\mathfrak{B}, \dagger, *) = \pi_1((\mathfrak{Y}, \dagger)^\vee \otimes \mathfrak{B} \otimes (\mathfrak{X}, *), \dagger, *)$.*

Note, in particular, the expression $\pi_1((\mathfrak{X}, \dagger)^\vee \otimes_{\mathfrak{X}} (\mathfrak{X}, *)) = \pi_1(\mathfrak{X}, \dagger, *)$ of the set of paths from $\dagger$ to $*$ in $\mathfrak{X}$ in terms of graphs of bisets.

3.5. Fibrant and covering bisets.

We now define a class of graphs of bisets for which the fundamental biset admits a convenient description.

Definition 3.16 (fibrant and covering graph of bisets). Let $\mathfrak{B}$ be a $\mathfrak{Y}$-$\mathfrak{X}$-graph of bisets. Then $\mathfrak{B}$ is a *left-fibrant* graph of bisets if $\rho: \mathfrak{B} \to \mathfrak{X}$ is a simplicial graph-map, and for every vertex $v \in \mathfrak{B}$ and every edge $f \in \mathfrak{X}$ with $f^- = \rho(v)$ the map

$$\bigsqcup_{\substack{e \in \rho^{-1}(f)\\ e^- = v}} G_{\lambda(v)} \otimes_{G_{\lambda(e)}} B_e, \longrightarrow B_v, \quad g \otimes b \longmapsto g b^-, \tag{6}$$

is a $G_{\lambda(v)}$-G_f biset isomorphism, for the action of G_f on B_v given via the map $(\cdot)^-: G_f \to G_{\rho(v)}$.

If in addition B_z is a left-free biset for every object $z \in \mathfrak{X}$, then $\mathfrak{B}$ is a *covering*, or *left-free*, graph of bisets.

If furthermore B_z is left-principal for every $z \in \mathfrak{B}$ and $\rho\colon \mathfrak{B} \to \mathfrak{Y}$ is a graph isomorphism, then $\mathfrak{B}$ is *left-principal*. Right-fibrant, -free, and -principal graphs of bisets are defined similarly. For example, the biset associated with a morphism of groups as in Example 3.9 is right-principal.

We think about (6) as a unique lifting property: we may always rewrite $pbq \in \pi_1(\mathfrak{B}, \dagger, *)$ as $p'b' \in \pi_1(\mathfrak{B}, \dagger, *)$ in an essentially unique way, as we will see in Corollary 3.20. We will show in Lemma 3.28 that for left-principal graphs of bisets (6) follows automatically.

For every object $y \in \mathfrak{Y}$ let us denote by $\widehat{G}_y$ the image of G_y in the fundamental groupoid $\pi_1(\mathfrak{Y})$; we write $g \mapsto \hat{g}$ the associated natural quotient map $G_y \to \widehat{G}_y$. We define $\widehat{G}_x$ and $g \to \hat{g}$ similarly for $x \in \mathfrak{X}$. Finally, for $z \in \mathfrak{B}$ we set

$$\widehat{B}_z := \widehat{G}_{\lambda(z)} \otimes_{G_{\lambda(z)}} B_z \otimes_{G_{\rho(z)}} \widehat{G}_{\rho(z)}$$

and denote by $b \mapsto \hat{b}$ the natural quotient map $B_z \to \widehat{B}_z$.

Let $\widehat{\mathfrak{B}}$ be the graph of bisets obtained from $\mathfrak{B}$ by replacing all G_x, G_y, B_z by $\widehat{G}_x, \widehat{G}_y, \widehat{B}_z$ respectively. It is immediate that $\pi_1(\mathfrak{B}, \dagger, *)$ and $\pi_1(\widehat{\mathfrak{B}}, \dagger, *)$ are naturally isomorphic via $ybx \mapsto \hat{y}\hat{b}\hat{x}$.

Theorem 3.17. *Suppose that $\mathfrak{B}$ is a left-fibrant biset. Then*

(1) *$\widehat{\mathfrak{B}}$ is left-fibrant;*

(2) *$\widehat{B}_z \cong \widehat{G}_{\lambda(z)} \otimes_{G_{\lambda(z)}} B_z$ via the natural map $1 \otimes b \otimes 1 \leftarrow 1 \otimes b$;*

(3) *$(\cdot)^-\colon \widehat{B}_e \to \widehat{B}_{e^-}$ are monomorphisms for all $e \in \mathfrak{B}$; and*

(4) *if $\mathfrak{B}$ is left-free, respectively left-principal, then so is $\widehat{\mathfrak{B}}$.*

Proof. For $z \in \mathfrak{B}$ let us write $B'_z := \widehat{G}_{\lambda(z)} \otimes B_z$, viewed as a right $G_{\rho(z)}$-set. For $x \in \mathfrak{X}$ let us write

$$C_x := \bigsqcup_{z \in \rho^{-1}(x)} B'_z,$$

viewed as a right G_x-set. In particular, if $f \in \mathfrak{X}$ is an edge, then (6) gives a bijection

$$\bigsqcup_{e \in \rho^{-1}(f)} (\widehat{G}_{\lambda(e^-)}/\widehat{G}_{\lambda(e)}) \times B'_e \longrightarrow C_{f^-} \tag{7}$$

of right G_f-sets. Finally, for $x \in \mathfrak{X}$ we denote by $\operatorname{Aut}(C_x)$ the set of *pure automorphisms* of C_x:

$$\operatorname{Aut}(C_x) := \{\phi = (\phi_z)_{z \in \rho^{-1}(x)} \mid \phi_z \in (B'_z)^{\downarrow}, \\ (gb)^{\phi_z} = g\, b^{\phi_z} \text{ for all } b \in B'_z, g \in \widehat{G}_{\lambda(z)}\}.$$

The right action of G_x on C_x induces a homomorphism

$$\theta_x \colon G_x \longrightarrow \operatorname{Aut}(C_x).$$

Lemma 3.18. *Suppose $\mathfrak{B}$ is a left-fibrant biset. Then for every edge $f \in \mathfrak{X}$ there is a natural embedding*

$$\begin{aligned} (\cdot)^- \colon \operatorname{Aut}(C_f) &\hookrightarrow \operatorname{Aut}(C_{f^-}), \\ (\phi_e)_{e \in \rho^{-1}(f)} &\longmapsto [(g,b) \in (\widehat{G}_{\lambda(e^-)}/\widehat{G}_{\lambda(e)}) \times B'_e \longmapsto (g, b^{\phi_e})], \end{aligned} \tag{8}$$

under the identification of C_{f^-} given by (7). Moreover, $(\cdot)^-$ embeds $\theta_f(G_f)$ into $\theta_{f^-}(G_{f^-})$.

Proof. The contents of (7) that we use is that every G_f-set occurring in C_f occurs as a summand of C_{f^-}, with multiplicity ≥ 1. Therefore $(\cdot)^-$ is injective. Since the map $(\cdot)^- \colon G_f \to G_{f^-}$ is equivariant with respect to the actions on C_f and C_{f^-}, we get $\theta_{f^-}(G_{f^-}) \cong (\theta_f(G_f))^- \leq \theta_{f^-}(G_{f^-})$. △

Let $\mathfrak{X}'$ be the graph of groups obtained from $\mathfrak{X}$ by replacing each G_x with $\theta_x(G_x)$, with maps $(\cdot)^-$ given by (8). By Lemma 3.2, every $\theta_x(G_x)$ embeds into the fundamental groupoid $\pi_1(\mathfrak{X}')$. We also have an epimorphism $\theta \colon \pi_1(\mathfrak{X}) \to \pi_1(\mathfrak{X}')$ induced by the epimorphisms $\theta_x \colon G_x \to \theta_x(G_x)$. Therefore, $\theta_x \colon G_x \to \theta_x(G_x)$ descends to a map $\hat{\theta}_x \colon \widehat{G}_x \to \theta_x(G_x)$. The action of $\widehat{G}_{\rho(z)}$ on $\widehat{B}_z$ therefore lifts to B'_z, and this completes the proof of Claim (2).

Multiplying (6) on the left by $\widehat{G}_{\lambda(v)}$ gives then Claim (1) as well as Claim (3).

Finally, if B_z is left-free then so is $\widehat{G}_{\lambda(z)} \otimes_{G_{\lambda(z)}} B_z = B'_z \cong \widehat{B}_z$, and similarly if B_z is left-principal then so is $\widehat{B}_z$; this finishes the proof of Theorem 3.17. □

3.6. Canonical form for fibrant bisets. We now derive a canonical expression for element in a left-fibrant biset. Let us set

$$\begin{aligned} S &:= \bigsqcup_{z \in V(\mathfrak{B})} \pi_1(\mathfrak{Y}, \dagger, \lambda(z)) \otimes_{\widehat{G}_{\lambda(z)}} \widehat{B}_z \otimes_{\widehat{G}_{\rho(z)}} \pi_1(\mathfrak{X}, \rho(z), *) \\ &\cong \bigsqcup_{z \in V(\mathfrak{B})} \pi_1(\mathfrak{Y}, \dagger, \lambda(z)) \otimes_{G_{\lambda(z)}} B_z \otimes_{G_{\rho(z)}} \pi_1(\mathfrak{X}, \rho(z), *). \end{aligned}$$

Then $\pi_1(\mathfrak{B}, \dagger, *) \cong S/_\sim$, where $\sim$ is the equivalence relation from (5). For any path $p \in \pi_1(\mathfrak{X}, x, *)$ starting at some vertex $x \in \mathfrak{X}$, we consider the following subset of S:

$$S(p) := \bigcup_{z \in \rho^{-1}(x)} \pi_1(\mathfrak{Y}, \dagger, \lambda(z)) \otimes_{\widehat{G}_{\lambda(z)}} \widehat{B}_z \otimes_{\widehat{G}_{\rho(z)}} p$$

viewed as a left $\pi_1(\mathfrak{Y}, \dagger)$-set.

Proposition 3.19. *Suppose $\mathfrak{B}$ is a left-fibrant graph of bisets. Then for every $p_1, p_2 \in \pi_1(\mathfrak{X}, -, *)$ the relation $\sim$ on $S(p_1) \times S(p_2)$ is, in fact, a $\pi_1(\mathfrak{Y}, \dagger)$-set isomorphism between $S(p_1)$ and $S(p_2)$.*

Proof. If $p \in \pi_1(\mathfrak{X}, x, *)$ then, clearly, $S(p) = S(gp)$ for every $g \in G_x$. We now show that for every edge $f \in \mathfrak{X}$ with $f^+ = x$ the equivalence $qb^- p = q\lambda(z)b^+\overline{\rho(z)}p$ in (5) induces an isomorphism between $S(fp)$ and $S(p)$. Since $\widehat{B}_e^+$ is isomorphic to $\widehat{B}_e^-$ by Theorem 3.17(3) we have the required isomorphism:

$$\begin{aligned}
S(fp) &= \bigsqcup_{z\in\rho^{-1}(f^-)} \pi_1(\widehat{\mathfrak{Y}}, \dagger, \lambda(z)) \otimes_{\widehat{G}_{\lambda(z)}} \widehat{B}_z \otimes fp \\
&= \bigsqcup_{z\in\rho^{-1}(f^-)} \pi_1(\widehat{\mathfrak{Y}}, \dagger, \lambda(z)) \otimes_{\widehat{G}_{\lambda(z)}} \Big(\bigsqcup_{\substack{e\in\rho^{-1}(f)\\ e^-=z}} \widehat{G}_{\lambda(e^-)} \otimes \widehat{B}_e^- \Big) \otimes fp \quad \text{using (6)} \\
&= \bigsqcup_{e\in\rho^{-1}(f)} \pi_1(\widehat{\mathfrak{Y}}, \dagger, \lambda(e^-)) \otimes_{\widehat{G}_{\lambda(e^-)}} \widehat{B}_e^- \otimes fp \\
&\cong \bigsqcup_{e\in\rho^{-1}(f)} \pi_1(\widehat{\mathfrak{Y}}, \dagger, \lambda(e^-)) \otimes (\lambda(e)\widehat{B}_e^+ \overline{\rho(e)}) \otimes fp \quad \text{using } qb^-p \sim q\lambda(z)b^+\overline{\rho(z)}p \\
&= \bigsqcup_{e\in\rho^{-1}(f)} \pi_1(\widehat{\mathfrak{Y}}, \dagger, \lambda(e^+)) \otimes \widehat{B}_e^+ \otimes p = S(p).
\end{aligned}$$

□

Corollary 3.20 (canonical form of $\pi_1(\mathfrak{B}, \dagger, *)$). *Let ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ be a left-fibrant graph of bisets. Then its fundamental graph of bisets has the following description*

$$\pi_1(\mathfrak{B}, \dagger, *) = \bigsqcup_{z\in\rho^{-1}(*)} \pi_1(\mathfrak{Y}, \dagger, \lambda(z)) \otimes_{G_{\lambda(z)}} B_z, \tag{9}$$

with right action given by lifting of paths in $\pi_1(\mathfrak{X}, *)$.

Proof. Follows immediately from Proposition 3.19, since the right-hand side of (9) is $S(1)$. □

Since $\pi_1(\mathfrak{B}, \dagger, *) \cong \pi_1(\widehat{\mathfrak{B}}, \dagger, *)$, by Theorem 3.17 we may rewrite (9) as

$$\pi_1(\mathfrak{B}, \dagger, *) \cong \bigsqcup_{z\in\rho^{-1}(*)} \pi_1(\widehat{\mathfrak{Y}}, \dagger, \lambda(z)) \otimes_{\widehat{G}_{\lambda(z)}} \widehat{B}_z. \tag{10}$$

Corollary 3.21. *Let ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ be a left-free graph of bisets. Then $\pi_1(\mathfrak{B}, \dagger, *)$ is a left-free biset of degree equal to that of $\bigsqcup_{z\in\rho^{-1}(*)} B_z$.*

In particular, the number of left orbits of $\bigsqcup_{z\in\rho^{-1}(*)} \widehat{B}_z$ is independent on $* \in \mathfrak{B}$.

Proof. Follows immediately from (10) and Theorem 3.17 Claim 4. □

Lemma 3.22. *Suppose* $\mathfrak{B}$ *is a left-fibrant graph of bisets. Then* $\pi_1(\mathfrak{B}, \dagger, *)$ *is left-free if and only if* $\widehat{\mathfrak{B}}$ *is a left-free graph of bisets.*

Proof. It follows from (10) that $\pi_1(\mathfrak{B}, \dagger, *)$ is left-free if and only if every $\widetilde{B}_z$ is left-free. □

The following corollary is an adaptation of Lemma 3.2 to the context of bisets:

Corollary 3.23. *Let* $\mathfrak{B}$ *be a left-fibrant graph of bisets. If for* $y \in \mathfrak{Y}$ *the maps* $(\cdot)^{-}\colon G_y \to G_{y^-}$ *are injective, then the natural maps*

$$B_z \longrightarrow \pi_1(\mathfrak{B}, \lambda(z), \rho(z)), \quad b \longmapsto 1 \otimes b \otimes 1,$$

are also injective for all vertices $z \in \mathfrak{B}$.

Proof. By Lemma 3.2 we have $G_y \cong \widehat{G}_y$ for all $y \in \mathfrak{Y}$. Therefore, by Theorem 3.17(2) we have $B_z \cong \widehat{B}_z$ for all $z \in \mathfrak{B}$. In particular, all $B_z \to \pi_1(\mathfrak{B}, \lambda(z), \rho(z))$ are injections by Corollary 3.20. □

Consider a graph of bisets ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$. As for graphs of groups, ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ has a barycentric subdivision ${}_{\mathfrak{Y}'}\mathfrak{B}'_{\mathfrak{X}'}$: all graphs of $\mathfrak{Y}', \mathfrak{B}', \mathfrak{X}'$ are the barycentric subdivisions of those of $\mathfrak{Y}, \mathfrak{B}, \mathfrak{X}$ respectively, the vertex groups and bisets G_y, G_x, B_z with $y \in \mathfrak{Y}', x \in \mathfrak{X}', z \in \mathfrak{B}'$ are the groups and bisets of the associated objects in ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$, the edge groups and bisets in ${}_{\mathfrak{Y}'}\mathfrak{B}'_{\mathfrak{X}'}$ represent the group morphisms and the biset intertwiners of ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$.

Lemma 3.24. *Let* ${}_{\mathfrak{Y}'}\mathfrak{B}'_{\mathfrak{X}'}$ *be the barycentric subdivision of a graph of bisets* ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$. *Then for* $\dagger \in \mathfrak{Y}$ *and* $* \in \mathfrak{X}$ *we have a natural isomorphism* $\pi_1(\mathfrak{B}, \dagger, *) \cong \pi_1(\mathfrak{B}', \dagger, *)$.

If ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ *is left-fibrant (left-covering, etc.), then so is* ${}_{\mathfrak{Y}'}\mathfrak{B}'_{\mathfrak{X}'}$.

Proof. Write $\mathfrak{B}'_{\dagger,*} := (\mathfrak{Y}', \dagger)^{\vee} \otimes \mathfrak{B}' \otimes (\mathfrak{X}', *)$ and $\mathfrak{B}_{\dagger,*} := (\mathfrak{Y}, \dagger)^{\vee} \otimes \mathfrak{B} \otimes (\mathfrak{X}, *)$. By Lemma 3.15, the fundamental bisets of $\mathfrak{B}'_{\dagger,*}$ and $\mathfrak{B}_{\dagger,*}$ are isomorphic to the fundamental bisets of $\mathfrak{B}'$ and $\mathfrak{B}$ respectively. It is also easy to see that $\mathfrak{B}'_{\dagger,*}$ is (isomorphic to) to the barycentric subdivision of $\mathfrak{B}_{\dagger,*}$ because $\pi_1(\mathfrak{Y}', \dagger) \cong \pi_1(\mathfrak{Y}, \dagger)$ and $\pi_1(\mathfrak{X}', *) \cong \pi_1(\mathfrak{X}, *)$. This reduces the problem to the case when $\mathfrak{Y}$ and $\mathfrak{X}$ are graphs with one vertex. In this case we have a simple description of the fundamental biset B of $\mathfrak{B}$, and similarly of $\mathfrak{B}'$, see §3.4: B is the quotient of $\bigsqcup_{z \in V(\mathfrak{B})} B_z$ by the relation identifying b^- with b^+, for all edges $z \in \mathfrak{B}$ and all $b \in B_z$, with respective images $b^{\pm}$ in $B_{z^{\pm}}$. The first claim of the lemma easily follows.

The second part of the lemma is straightforward: the unique lifting property (6) is respected by passing to the barycentric subdivision and (6) holds for the new vertices of $\mathfrak{B}'$ which are the former edges of $\mathfrak{B}$. □

3.7. Properties of products

Lemma 3.25. *For every graphs of bisets ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ and ${}_{\mathfrak{X}}\mathfrak{C}_{\mathfrak{W}}$ there exists a biset morphism*

$$\pi_1(\mathfrak{B} \otimes_{\mathfrak{X}} \mathfrak{C}, \dagger, \ddagger) \longrightarrow \pi_1(\mathfrak{B}, \dagger, *) \otimes_{\pi_1(\mathfrak{X},*)} \pi_1(\mathfrak{C}, *, \ddagger) \tag{11}$$

defined as follows: consider $(v, w) \in V(\mathfrak{B} \otimes_{\mathfrak{X}} \mathfrak{C})$ and $b \otimes c \in B_{(v,w)}$. Consider paths $q \in \pi_1(\mathfrak{Y}, \dagger, \lambda(v))$ and $r \in \pi_1(\mathfrak{W}, \rho(w), \ddagger)$. Then a typical element of $\pi_1(\mathfrak{B} \otimes_{\mathfrak{X}} \mathfrak{C}, \dagger, \ddagger)$ is of the form $q(b \otimes c)r$, and its image under (11) is defined to be $(qbp) \otimes (p^{-1}cr)$ for any choice of path $p \in \pi_1(\mathfrak{X}, \rho(v), *) = \pi_1(\mathfrak{X}, \lambda(w), *)$.

Proof. To show that (11) is well-defined, we must see that it is independent of the choice of $q(b \otimes c)r$ in its $\sim$-equivalence class and of the choice of p. Clearly the product over $\mathfrak{X}$ is independent of the choice of p. Changing $b \otimes c$ into $bg \otimes g^{-1}c$ is the same as changing p into gp. For an edge (e, f) in $\mathfrak{B} \otimes_{\mathfrak{X}} \mathfrak{C}$, changing $q(b \otimes c)^{-}r$ into $q\lambda(e)(b \otimes c)^{+}\overline{\rho(f)}r$ is the same as changing p into $\lambda(e)p$. It is immediate that (11) is a biset morphism. □

In general, the morphism in Lemma 3.25 need not be an isomorphism. For instance, $\mathfrak{B} \otimes_{\mathfrak{X}} \mathfrak{C}$ is the empty graph of bisets if the images of the graphs $\mathfrak{B}$ and $\mathfrak{C}$ do not intersect in $\mathfrak{X}$. However,

Lemma 3.26. *Let $\mathfrak{B}$ be a left-fibrant $\mathfrak{Y}$-$\mathfrak{X}$-biset, and let $\mathfrak{C}$ be a $\mathfrak{X}$-$\mathfrak{W}$-biset. Then (11) induces an isomorphism*

$$\pi_1(\mathfrak{B}, \dagger, *) \otimes_{\pi_1(\mathfrak{X},*)} \pi_1(\mathfrak{C}, *, \ddagger) \cong \pi_1(\mathfrak{B} \otimes_{\mathfrak{X}} \mathfrak{C}, \dagger, \ddagger).$$

Proof. By definition, every element in $\pi_1(\mathfrak{B}, \dagger, *) \otimes_{\pi_1(\mathfrak{X},*)} \pi_1(\mathfrak{C}, *, \ddagger)$ is of the form $qbpcr$ for paths q, p, r in $\mathfrak{Y}, \mathfrak{X}, \mathfrak{W}$ respectively. By Corollary 3.20 the subexpression qbp can be rewritten in a unique way as $q'b'$. This defines a biset morphism

$$\pi_1(\mathfrak{B}, \dagger, *) \otimes_{\pi_1(\mathfrak{X},*)} \pi_1(\mathfrak{C}, *, \ddagger) \longrightarrow \pi_1(\mathfrak{B} \otimes_{\mathfrak{X}} \mathfrak{C}, \dagger, \ddagger)$$

by $qbpcr \mapsto q'(b' \otimes c)r$, which is clearly the inverse of (11). □

Lemma 3.27. *Let ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ and ${}_{\mathfrak{X}}\mathfrak{C}_{\mathfrak{W}}$ be left-fibrant (respectively left-free, left-principal) bisets. Then $\mathfrak{B} \otimes_{\mathfrak{X}} \mathfrak{C}$ is a left-fibrant (respectively left-free, left-principal) graph of bisets.*

Proof. Observe first that $\rho \colon \mathfrak{B} \otimes_{\mathfrak{X}} \mathfrak{C} \to \mathfrak{W}$ is simplicial. Indeed, if $\rho(v, w)$ is a vertex for $(v, w) \in \mathfrak{B} \otimes_{\mathfrak{X}} \mathfrak{C}$, then w is a vertex of $\mathfrak{C}$ because $\rho \colon \mathfrak{C} \to \mathfrak{W}$ is simplicial, and $\lambda(w)$ is a vertex of $\mathfrak{X}$ because morphisms send vertices to vertices, so $v \in \rho^{-1}(\lambda(w))$ is a vertex of $\mathfrak{B}$ because $\rho \colon \mathfrak{B} \to \mathfrak{X}$ is simplicial. Similarly, if $\rho(v, w)$ is an edge, then so is $w \in \mathfrak{C}$ because $\rho \colon \mathfrak{C} \to \mathfrak{W}$ is simplicial; thus (v, w) is an edge in $\mathfrak{B} \otimes_{\mathfrak{X}} \mathfrak{C}$.

Consider now a vertex $(v,w) \in \mathfrak{B} \otimes_{\mathfrak{X}} \mathfrak{C}$ and an edge $f \in \mathfrak{W}$ with $f^- = \rho(v,w)$. Let us verify (6). Note first that if $(\ell,e) \in \rho^{-1}(f)$ with $(\ell,e)^- = (v,w)$, then $\ell = v$ if and only if $\lambda(\ell) = \lambda(v)$, because $\rho\colon \mathfrak{B} \to \mathfrak{X}$ is simplicial. We have

$$\begin{aligned}
B_{(v,w)} &= B_v \otimes B_w \\
&\cong B_v \otimes \bigsqcup_{\substack{e \in \rho^{-1}(f)\\ e^- = w}} G_{\lambda(w)} \otimes_{G_{\lambda(e)}} B_e \quad \text{using (6) for } \mathfrak{B}_w \\
&= \bigsqcup_{\substack{e \in \rho^{-1}(f)\\ e^- = w}} (B_v \otimes G_{\lambda(w)}) \otimes_{G_{\lambda(e)}} B_e \\
&= \bigsqcup_{\substack{e \in \rho^{-1}(f)\\ e^- = w}} B_v \otimes_{G_{\lambda(e)}} B_e \quad \text{because } \rho(v) = \lambda(w) \\
&= \Big(\bigsqcup_{\substack{e \in \rho^{-1}(f)\\ e^- = w\\ \lambda(e) = \rho(v)}} B_v \otimes_{G_{\lambda(e)}} B_e \Big) \sqcup \Big(\bigsqcup_{\substack{e \in \rho^{-1}(f)\\ e^- = w\\ \lambda(e) \neq \rho(v)}} B_v \otimes_{G_{\lambda(e)}} B_e \Big) \\
&\cong \Big(\bigsqcup_{\substack{(v,e) \in \rho^{-1}(f)\\ (v,e)^- = (v,w)}} B_{(v,e)} \Big) \sqcup \Big(\bigsqcup_{\substack{e \in \rho^{-1}(f)\\ e^- = w\\ \lambda(e) \neq \rho(v)}} \Big(\bigsqcup_{\substack{\ell \in \rho^{-1}(\lambda(e))\\ \ell^- = v}} G_{\lambda(v)} \otimes_{G_{\lambda(\ell)}} B_\ell \Big) \otimes_{G_{\lambda(e)}} B_e \Big) \\
&\cong \Big(\bigsqcup_{\substack{(v,e) \in \rho^{-1}(f)\\ (v,e)^- = (v,w)}} G_{\lambda(v)} \otimes B_{(v,e)} \Big) \sqcup \Big(\bigsqcup_{\substack{(\ell,e) \in \rho^{-1}(f)\\ (\ell,e)^- = (v,w)\\ \ell \neq v}} G_{\lambda(v)} \otimes_{G_{\lambda(\ell)}} B_{(\ell,e)} \Big) \\
&\cong \bigsqcup_{\substack{(\ell,e) \in \rho^{-1}(f)\\ (\ell,e)^- = (v,w)}} G_{\lambda(v)} \otimes_{G_{\lambda(\ell)}} B_{(\ell,e)} \\
&= \bigsqcup_{\substack{(\ell,e) \in \rho^{-1}(f)\\ (\ell,e)^- = (v,w)}} G_{\lambda(v,w)} \otimes_{G_{\lambda(\ell,e)}} B_{(\ell,e)}.
\end{aligned}$$

This shows that $\mathfrak{B} \otimes_{\mathfrak{X}} \mathfrak{C}$ is a left-fibrant graph of bisets. If, moreover, ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ and ${}_{\mathfrak{X}}\mathfrak{C}_{\mathfrak{W}}$ are left-free (respectively left-principal), then all $B_{(v,w)}$ are left-free (respectively left-principal); thus $\mathfrak{B} \otimes_{\mathfrak{X}} \mathfrak{C}$ is a left-free (respectively left-principal) graph of bisets. □

We are now ready to prove Lemma 3.15:

Proof of Lemma 3.15. Both $(\mathfrak{B}, \dagger)$ and $(\mathfrak{X}, *)$ are left-principal graphs of bisets. The proof now follows from Lemmas 3.26 and 3.27. □

3.8. Principal and biprincipal graphs of bisets. Let us now simplify the part of Definition 3.16 concerning principal graphs of bisets.

Lemma 3.28. *A graph of bisets ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ is left-principal if and only if $\rho\colon \mathfrak{B} \to \mathfrak{X}$ is a graph isomorphism and B_z are left-principal for all $z \in \mathfrak{B}$.*

Proof. We verify that the conditions stated in Lemma 3.28 imply (6).

Since $\rho\colon \mathfrak{B} \to \mathfrak{X}$ is a graph isomorphism, (6) takes the following form: for every $z \in \mathfrak{B}$ the map

$$G_{\lambda(e^-)} \otimes_{G_{\lambda(e)}} B_e \longrightarrow B_{e^-}, \quad g \otimes b \longmapsto gb^-, \tag{12}$$

is a $G_{\lambda(e^-)}$-$G_{\rho(e)}$ biset isomorphism (for the action of $G_{\rho(e)}$ on B_{e^-} given via $(\cdot)^-\colon G_{\rho(e)} \to G_{\rho(e)^-}$). Let us verify (12).

Consider an edge $e \in \mathfrak{B}$. We claim that ${}_{(G_{\lambda(e)})^-}(B_e)^-{}_{G_{\rho(e)}}$ is isomorphic to $(G_{\lambda(e)})^- \otimes_{G_{\lambda(e)}} (B_e)_{G_{\rho(e)}}$. Indeed, ${}_{(G_{\lambda(e)})^-}(B_e)^-{}_{G_{\rho(e)}}$ is left-free as a sub-biset of a left-free biset B_{e^-}. Furthermore, ${}_{(G_{\lambda(e)})^-}(B_e)^-{}_{G_{\rho(e)}}$ has a single orbit under the action of $(G_{\lambda(e)})^-$ because this property holds for B_e. Therefore, ${}_{(G_{\lambda(e)})^-}(B_e)^-{}_{G_{\rho(e)}}$ is left-principal. As a set it is isomorphic to $(G_{\lambda(e)})^-$; this proves ${}_{(G_{\lambda(e)})^-}(B_e)^-{}_{G_{\rho(e)}} \cong (G_{\lambda(e)})^- \otimes_{G_{\lambda(e)}} (B_e)_{G_{\rho(e)}}$. Since B_{e^-} is left-principal we obtain a natural isomorphism of $G_{\lambda(e^-)}$-$G_{\rho(e)}$-bisets

$$B_{e^-} = G_{\lambda(e^-)}(B_e)^- \cong G_{\lambda(e^-)} \otimes_{G_{\lambda(e)}} B_e$$

which is (12). ☐

An example of left-principal graph of bisets is ${}_{\mathfrak{X}}(\mathfrak{X}, *)_{\mathfrak{X}_0}$ defined in §3.4.

A biset ${}_H B_G$ is *biprincipal* if it is left- and right-principal. Clearly, if ${}_H B_G$ is biprincipal, then the groups G and H are isomorphic. Similarly, a graph of bisets ${}_{\mathfrak{Y}}\mathfrak{I}_{\mathfrak{X}}$ is *biprincipal* if it is left- and right-principal. By Lemma 3.28 a graph of bisets ${}_{\mathfrak{Y}}\mathfrak{I}_{\mathfrak{X}}$ is biprincipal if and only if

(1) $\lambda\colon \mathfrak{I} \to \mathfrak{Y}$ and $\rho\colon \mathfrak{I} \to \mathfrak{X}$ are graph isomorphisms; and

(2) B_z are biprincipal for all objects $z \in \mathfrak{I}$.

Definition 3.29 (congruence of graphs of groups and bisets). Two graphs of groups $\mathfrak{Y}$, $\mathfrak{X}$ are called *congruent* if there is a biprincipal graph of bisets ${}_{\mathfrak{Y}}\mathfrak{I}_{\mathfrak{X}}$.

Isomorphism of graphs of bisets is meant in the strongest possible sense: isomorphism of the underlying graphs, and isomorphisms of the respective bisets.

Two graphs of bisets ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ and ${}_{\mathfrak{Y}'}\mathfrak{C}_{\mathfrak{X}'}$ are *congruent* if there are biprincipal graph of bisets ${}_{\mathfrak{Y}}\mathfrak{I}_{\mathfrak{Y}'}$ and ${}_{\mathfrak{X}}\mathfrak{L}_{\mathfrak{X}'}$ such that ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ is isomorphic to $\mathfrak{I} \otimes_{\mathfrak{Y}'} \mathfrak{C} \otimes_{\mathfrak{X}'} \mathfrak{L}^\vee$.

Finally, two graphs of bisets ${}_{\mathfrak{X}}\mathfrak{B}_{\mathfrak{X}}$ and ${}_{\mathfrak{X}'}\mathfrak{C}_{\mathfrak{X}'}$ are *conjugate* if ${}_{\mathfrak{X}}\mathfrak{B}_{\mathfrak{X}}$ is isomorphic to $\mathfrak{I} \otimes_{\mathfrak{X}'} \mathfrak{C} \otimes_{\mathfrak{X}'} \mathfrak{I}^\vee$ for a biprincipal graph of bisets ${}_{\mathfrak{X}}\mathfrak{I}_{\mathfrak{X}'}$.

Let ${}_H B_G$ be a biprincipal biset and let H' be a subgroup of H. For every $b \in B$ we consider the subgroup $(H')^b$ of G defined by the equality

$$H'b = b(H')^b \quad \text{in } B.$$

It is easy to see that the conjugacy class of $(H')^b$ in G does not depend on b.

Lemma 3.30. *Let $\mathfrak{Y}$, $\mathfrak{X}$ be graphs of groups and let $h\colon \mathfrak{Y} \to \mathfrak{X}$ be an isomorphism of the underlying graphs. Suppose also that a G_y-$G_{h(y)}$ biprincipal biset B_y is given for every object $y \in \mathfrak{Y}$. Set $\mathfrak{I}$ to be $\mathfrak{Y}$ as a graph. Then the data $\{B_z\}_{z\in\mathfrak{I}}$ extends (via appropriate embeddings of edge bisets into vertex bisets) to a biprincipal graph of bisets ${}_{\mathfrak{Y}}\mathfrak{I}_{\mathfrak{X}}$ if and only if for every edge $e \in \mathfrak{I}$ the following holds. For any, or equivalently for every, $b \in B_e$ the groups $(G_e^-)^b$ and $G_{h(e)}^-$ are conjugate as subgroups of $G_{h(e^-)}$.*

Proof. Follows immediately from the definition: if $(G_e^-)^b$ and $G_{h(e)}^-$ are conjugate, say $((G_e^-)^b)^g = G_{h(e)}^-$, then we may define $(\cdot)^-\colon B_e \to B_{e^-}$ by mapping B_e into $G_e^-(bg)G_{h(e)}^-$. The converse is obvious. □

3.9. Examples. We will give more examples and applications in §5. Nevertheless, we give here an example that illustrates the main features of Definition 3.13. First let us introduce some conventions simplifying descriptions of the examples.

Let the graph $\mathfrak{X} = V \sqcup E$ be a graph as in Definition 3.1. By an *undirected graph* we mean $\mathfrak{X}_0 := \mathfrak{X}/\{z = \bar{z}\}$ with vertex set $V_0 = V$ and *geometric* edge set $E_0 := E/\{e = \bar{e}\}$. The undirected graph $\mathfrak{X}_0$ is endowed with an adjacency relation. A map $\theta\colon \mathfrak{X}_0 \to \mathfrak{Y}_0$ is a *weak morphism* if θ sends adjacent objects into adjacent or equal objects. If $\mathfrak{X}'$ and $\mathfrak{Y}'$ are the barycentric subdivisions of $\mathfrak{X}$ and $\mathfrak{Y}$ as in §3.2, then $\theta\colon \mathfrak{X}_0 \to \mathfrak{Y}_0$ uniquely determines a graph morphism $\theta\colon \mathfrak{X}' \to \mathfrak{Y}'$.

Convention. We will often describe a graph of bisets as ${}_{\mathfrak{Y}_0}\mathfrak{B}_0{}_{\mathfrak{X}_0}$ with

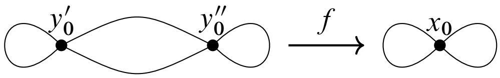
- undirected graphs $\mathfrak{Y}_0, \mathfrak{B}_0, \mathfrak{X}_0$;
- weak graph morphisms $\lambda\colon \mathfrak{B}_0 \to \mathfrak{Y}_0$ and $\rho\colon \mathfrak{B}_0 \to \mathfrak{X}_0$;
- groups G_y, G_x and $G_{\lambda(z)}$-$G_{\rho(z)}$ bisets B_z for all $y \in \mathfrak{Y}_0$, $x \in \mathfrak{X}_0$ and $z \in \mathfrak{B}_0$ respectively;
- intertwiners from B_e to B_v and B_w for all edges $e \in \mathfrak{B}_0$ adjacent to vertices $v, w \in \mathfrak{B}_0$; and, similarly, group morphisms from B_e to B_v and B_w for all edges $e \in \mathfrak{Y}_0 \sqcup \mathfrak{X}_0$ adjacent to vertices v, w.

By passing to a barycentric subdivision of ${}_{\mathfrak{Y}_0}\mathfrak{B}_0{}_{\mathfrak{X}_0}$ we obtain a graph of bisets ${}_{\mathfrak{Y}'}\mathfrak{B}'{}_{\mathfrak{X}'}$ satisfying Definition 3.8. Following Lemma 3.24 we set

$$\pi_1({}_{\mathfrak{Y}_0}\mathfrak{B}_0{}_{\mathfrak{X}_0}, \dagger, *) := \pi_1({}_{\mathfrak{Y}'}\mathfrak{B}'{}_{\mathfrak{X}'}, \dagger, *).$$

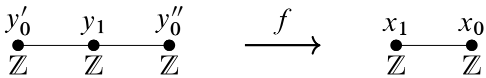
Example 3.31. Consider the biset corresponding to the "Basilica map" $f(z) = z^2 - 1$, from $(\mathbb{C}, \{-1, 0, 1\})$ to $(\mathbb{C}, \{0, -1\})$.

On the one hand, choose a basepoint $x_0 \cong -0.6$ in $\mathbb{C} \setminus \{0, -1\}$, and choose a graph $\mathfrak{X}$ with one vertex x_0 to which $\mathbb{C} \setminus \{0, -1\}$ deformation retracts; let $\mathfrak{Y}$ be the full preimage of $\mathfrak{X}$ under f:



There, the graphs of groups $\mathfrak{X}$ and $\mathfrak{Y}$ have trivial groups, and the biset $\mathfrak{B}$ corresponding to f is $\mathfrak{Y}$ as a graph, with $\lambda = 1$ and $\rho = f$.

This biset may be encoded differently, by considering rather an orbispace structure on $\mathbb{C}$ with non-trivial groups at $0, \pm 1$; the complex plane deformation retracts to the intervals $[-1, 0]$ and $[-1, 1]$ respectively:



The graphs $\mathfrak{X}$ and $\mathfrak{Y}$ have the group $\mathbb{Z}$ attached to each vertex, and the edge groups are all trivial. The biset $\mathfrak{B}$ has $\mathfrak{Y}$ as underlying graph; the map λ is the retraction to the segment $[y_0', y_1] = [x_1, x_0]$ and ρ folds $\mathfrak{Y}$ around its middle point. The bisets $B_{y_0'}$ and $B_{y_0''}$ are ${}_{\mathbb{Z}}\mathbb{Z}_{\mathbb{Z}}$, while $B_{y_1} = {}_{2\mathbb{Z}}\mathbb{Z}_{\mathbb{Z}}$ is free of rank 2 on the left. The edge bisets embed in B_{y_1} as two elements of opposite parity.

Note that the underlying graph of $\mathfrak{B}$ is the "Hubbard tree" of f; that will be explained in more details in §5.2.

Proposition 3.32. *Let $\phi\colon H \to G$ be a group homomorphism, and let $\mathfrak{X}$ be a graph of groups with fundamental group G. Then there exists a graph of groups decomposition of H and a graph of bisets with fundamental biset B_ϕ. Furthermore, there exists a unique minimal such decomposition for H in the sense that other decompositions are refinements of it.*

Proof. Let G act on the tree $\widetilde{\mathfrak{X}}$, the universal cover of $\mathfrak{X}$ (also known as the *Bass-Serre tree* of G, see [17]), with stabilizers the groups G_x of $\mathfrak{X}$. Then H acts on $\widetilde{\mathfrak{X}}$ via ϕ. We define the graph of groups $\mathfrak{Y}$ as $\widetilde{\mathfrak{X}}//H$, namely as the quotient of $\widetilde{\mathfrak{X}}$ by the action of H, remembering the stabilizers in H of vertices and edges in the quotient.

We next define a graph of bisets $\mathfrak{B}$ as follows. Its underlying graph is $\widetilde{\mathfrak{X}}/H$. The graph morphism $\lambda\colon \mathfrak{B} \to \mathfrak{Y}$ is the identity, and the graph morphism $\rho\colon \mathfrak{B} \to \mathfrak{X}$ is the natural quotient map $xH \mapsto xG$. The biset at the vertex or edge xH is $G_{\rho(xH)}$, on which $G_{\rho(xH)}$ acts naturally by left multiplication, while H_{xH} acts on the right via ϕ. □

Remark 3.33. In a manner similar to Lemma 2.7, every graph of bisets ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ may be factored as $\mathfrak{B} = \mathfrak{A}^\vee \otimes_{\mathfrak{K}} \mathfrak{C}$ for some graph of groups $\mathfrak{K}$ and some right-principal bisets ${}_{\mathfrak{K}}\mathfrak{A}_{\mathfrak{Y}}$ and ${}_{\mathfrak{K}}\mathfrak{C}_{\mathfrak{X}}$. Furthermore, there is an essentially unique minimal such $\mathfrak{K}$.

Lemma 3.34. *Let ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ and ${}_{\mathfrak{Y}}\mathfrak{C}_{\mathfrak{X}}$ be graphs of bisets over the same graphs, and let $\theta\colon \mathfrak{B} \to \mathfrak{C}$ be a "semiconjugacy of graphs of bisets", namely a graph morphism $\theta\colon \mathfrak{B} \to \mathfrak{C}$ and compatible surjective biset morphisms $\theta_z\colon B_z \to C_{\theta(z)}$. Then θ may be factored as $\theta = \kappa \circ \iota$ through semiconjugacies $\iota\colon \mathfrak{B} \to \mathfrak{C}'$ and $\kappa\colon \mathfrak{C}' \to \mathfrak{C}$, in such a manner that ι induces an isomorphism between $\pi_1(\mathfrak{B})$ and $\pi_1(\mathfrak{C}')$, and the underlying graph of $\mathfrak{C}'$ is the same as that of $\mathfrak{C}$.*

4. Correspondences

Bisets are well adapted to encode more general objects than continuous maps, *topological correspondences*.

Definition 4.1 (correspondences). Let X, Y be topological spaces. A *topological correspondence from Y to X* is a triple (Z, f, i) consisting of a topological space Z and continuous maps $f\colon Z \to X$ and $i\colon Z \to Y$. It is often abbreviated (f, i), and written $Y \leftarrow Z \to X$.

In the special case that f is a covering, (Z, f, i) is called a *covering correspondence*. If $X = Y$, then (Z, f, i) is called a *self-correspondence*. If $X = Y$ and f is a covering, then (Z, f, i) is called a *covering pair*, and is also written $f, i\colon Z \rightrightarrows X$.

Ultimately, we will be interested in dynamical systems that are partially defined topological coverings $f\colon X \dashrightarrow X$, for path connected topological spaces X; the dashed arrow means that the map is defined on an open subset $Y \subseteq X$; these can naturally be viewed as correspondences (Y, f, i) with i the inclusion.

4.1. The biset of a correspondence.

Let $Y \leftarrow Z \to X$ be a correspondence, and assume that X and Y are path connected. Assume first that Z is path connected, and fix a basepoint $\ddagger \in Z$. By (1) we have a $\pi_1(Z,\ddagger)$-$\pi_1(X,*)$-biset $B(f)$ and a $\pi_1(Z,\ddagger)$-$\pi_1(Y,\dagger)$-biset $B(i)$; we *define* the biset $B(f,i)$ to be $B(i)^\vee \otimes_{\pi_1(Z,\ddagger)} B(f)$; this amounts to first "lifting" from Y to Z, and then projecting Z back to X. If Z is not path connected, then we *define* $B(f,i)$ to be the disjoint union of the bisets $B(i)^\vee \otimes_{\pi_1(Z,p)} B(f)$ with one p per path connected component of Z.

The following alternate construction of $B(f,i)$ avoids an unnecessary reference to the basepoint and fundamental group of Z:

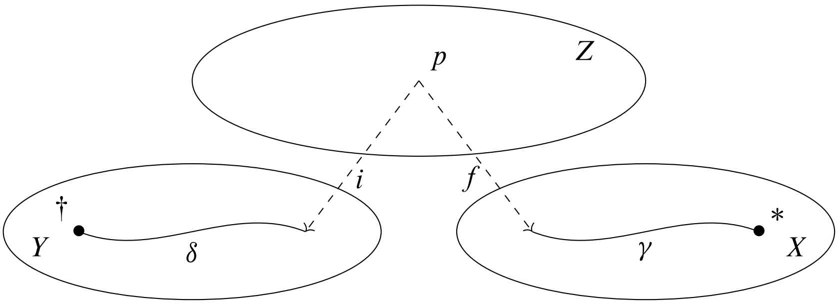
Lemma 4.2. *The biset $B(f,i)$ may be constructed directly as follows, from the correspondence*:

$$B(f,i) = \left\{ \left(\begin{array}{c} \delta\colon [0,1] \to Y \\ p \in Z \\ \gamma\colon [0,1] \to X \end{array} \right) \middle| \begin{array}{ll} \delta(0) = \dagger, & \delta(1) = i(p) \\ \gamma(0) = f(p), & \gamma(1) = * \end{array} \right\} \Big/ \sim \tag{13}$$

in which the equivalence relation $\sim$ allows p to move in Z as long as δ, γ move continuously with it[1].

The left action is by pre-composition of δ by loops in Y at $\dagger$, and the right action is by post-composition of γ by loops in X at $*$:

$$[\alpha] \cdot ([\delta], p, [\gamma]) \cdot [\varepsilon] = ([\alpha\#\delta], p, [\gamma\#\varepsilon]).$$



Proof. By the definition of products of bisets, $B(f,i) = \bigsqcup_z \{(\delta^\vee, \gamma)\}/\sim$, with $\delta\colon [0,1] \to Y$ ending at $\dagger$ and $\gamma\colon [0,1] \to X$ ending at $*$ and $\delta(0) = i(z)$ and $\gamma(0) = f(z)$, and with one z per path connected component of Z. By the homotopy relation, we may also allow z to move freely, arriving at the formulation of the lemma in which no condition on z is imposed. □

Recall that a continuous map $f\colon Y \to X$ is a *fibration*, or a *fibrant map* if it has the homotopy lifting property with respect to arbitrary spaces: for every space Z, every homotopy $g\colon Z \times [0,1] \to X$ and every $h\colon Z \times \{0\} \to Y$ with $f \circ h = g$ on $Z \times \{0\}$ there exists an extension of h to $Z \times [0,1]$ with $f \circ h = g$. Clearly, the composition of fibrations is a fibration. If X is path connected, then in particular all fibres $f^{-1}(x)$ are homotopy equivalent.

Remark 4.3. In fact, we may consider at no cost a larger class of maps: let us say that a map f is a π_1-*fibration* if f has the lifting property with respect to contractible spaces. This is sufficient for our purposes; it implies, for example, that all $f^{-1}(x)$ have isomorphic fundamental groups.

[1] Note the similarity with Definition 3.13

Lemma 4.4. *Assume that $f\colon Z \to X$ is a fibration. Then the biset of f has the following description*:

$$B(f) = \{\beta\colon [0,1] \longrightarrow Z \mid \beta(0) = \dagger,\ f(\beta(1)) = *\}/\sim, \tag{14}$$

with $\beta \sim \beta'$ if and only if there is a path $\varepsilon\colon [0,1] \to f^{-1}(*)$ connecting $\beta(1)$ to $\beta'(1)$ such that $\beta\#\varepsilon$ is homotopic to β'.

Assume that (Z, f, i) is a correspondence with f fibrant. Then in (13) we may assume that γ is constant, and write

$$B(f,i) = \{(\delta\colon [0,1] \longrightarrow Y, p \in Z) \mid \delta(0) = \dagger, \delta(1) = i(p), f(p) = *\}/\sim, \tag{15}$$

with $(\delta, p) \sim (\delta', p')$ if and only if there exists a path $\varepsilon\colon [0,1] \to f^{-1}(*) \subseteq Z$ connecting p to p', such that $\delta\#(i \circ \varepsilon)$ is homotopic to δ'.

Proof. Recall from (1) that $B(f)$ is defined as $\{\gamma\colon [0,1] \to X \mid \gamma(0) = f(\dagger),\ \gamma(1) = *\}/\sim$. Since f is fibrant, every $\gamma \in B(f)$ has a lift, say β, with $\beta(0) = \dagger$ and $f \circ \beta = \gamma$. We need to show that if γ_0 is homotopic to γ_1, say via γ_t, then their lifts β_0 and β_1 have the property that β_0 is homotopic to $\beta_1\#\varepsilon$ for some ε in $f^{-1}(*)$. Consider γ_t as a map $\gamma\colon [0,1] \times [0,1] \to X$, where t is the first variable and the second variable parameterizes γ_t. By construction, $\gamma(t,0) = f(\dagger)$ and $\gamma(t,1) = *$. Since f is fibrant, there is a lift $\epsilon\colon [0,1] \times [0,1] \to Z$ of γ such that $\epsilon(t,0) = \dagger$, $\epsilon f = \gamma$, $\epsilon(0,\tau) = \beta_0(\tau)$, and $\epsilon(1,\tau) = \beta_1(\tau)$. Define $\varepsilon(\tau) := \epsilon(1,\tau)$. Then $\beta_0^{-1}\#\beta_1\#\varepsilon$ is homotopic to $\epsilon\restriction_{\partial([0,1]\times[0,1])}$. Therefore, β_0 is homotopic to $\beta_1\#\varepsilon$.

Let us now prove the second statement. Using the first part, the biset $B(f,i)$ in the sense of (13) is identified with

$$\{(\delta\colon [0,1] \longrightarrow Y, \beta\colon [0,1] \longrightarrow Z) \mid \delta(0) = \dagger, \delta(1) = i(\beta(0)), f(\beta(1)) = *\}/\sim,$$

where $(\delta_0, \beta_0) \sim (\delta_1, \beta_1)$ if there is a path $\varepsilon\colon [0,1] \to f^{-1}(*)$ such that (δ_0, β_0) is homotopic to $(\delta_1, \beta_1\varepsilon)$. Projecting $\beta_1\varepsilon$ to Y yields the required description. □

Example 4.5. Here are some extreme examples of correspondences worth keeping in mind. If $Z = \{\ddagger\}$ is a point, then $B(f,i) = \pi_1(Y,\dagger) \times \pi_1(X,*)$. On the other hand, if $Z = X = Y$ and f and i are the identity, then $B(f,i) \cong \pi_1(Y,\dagger) \cong \pi_1(X,*)$. Finally, one may consider $Z = Y \times X$ with i, f the first and second projections respectively; then $B(f,i)$ consists of a single element.

If f is a covering, then the biset $B(f,i)$ is left-free. Indeed, consider (15) as a description of $B(f,i)$. Suppose $f^{-1}(*) = \{p_i\}_{i\in I}$. For every i choose $\delta_i\colon [0,1] \to Y$ with $\delta_i(0) = \dagger$ and $\delta_i(1) = i(p_i)$. Then $(\delta_i, p_i)_{i\in I}$ forms a basis of $B(f,i)$.

4.2. Van Kampen's theorem for correspondences. We describe in this section the decomposition of the biset of a correspondence into a graph of bisets, using a decomposition of the underlying spaces. We start by the following straightforward lemma.

Lemma 4.6. *Let*

- $f\colon Y \to X$ *be a continuous map*;
- X' *and* Y' *be path connected subsets of* X *and* Y *with* $f(Y') \subseteq X'$;
- $\dagger$, $\dagger'$, $*$, *and* $*'$ *be base points in* Y, Y', X, *and* X'; *and*
- $\gamma_* \subset X$ *and* $\gamma_\dagger \subset Y$ *be curves connecting* $*$ *to* $*'$ *and* $\dagger$ *to* $\dagger'$ *respectively.*

We allow $X' = X$; *in this case we suppose that* γ_* *is the constant path and* $*' = *$.

Then the maps $\delta \to \gamma_\dagger \delta \gamma_\dagger^{-1}$, $\delta \to \gamma_* \delta \gamma_*^{-1}$, *and* $b \to f(\gamma_\dagger) b \gamma_*^{-1}$ *define a biset congruence*

$$_{\pi_1(Y',\dagger')}B(f{\downharpoonleft}_{Y'}, \dagger', *')_{\pi_1(X',*')} \to {}_{\pi_1(Y,\dagger)}B(f, \dagger, *)_{\pi_1(X,*)}.$$

Let $f\colon Z \to X$ be a continuous map between path connected spaces. Suppose that (U_α) and (W_γ) are 1-dimensional covers X and Z respectively, as in Definition 3.4. Choose base points $*_\alpha \in U_\alpha$ and $\dagger_\gamma \in W_\gamma$ as well as connections between $*_{\alpha'}$ and $*_\alpha$ if $U_{\alpha'} \subsetneqq U_\alpha$; and similarly for (W_γ). Consider the associated graphs of groups $\mathfrak{X}$ and $\mathfrak{Z}$ according to Definition 3.6. We have $\pi_1(X, *_\alpha) = \pi_1(\mathfrak{X}, \alpha)$ and $\pi_1(Z, \dagger_\gamma) = \pi_1(\mathfrak{Z}, \gamma)$, by Theorem 3.7. Suppose furthermore that (U_α) and (W_γ) are *compatible* with f in the following sense: for every γ there is $\alpha =: \rho(\gamma)$ with $f(W_\gamma) \subset U_\alpha$.

To the above data there is an associated graph of bisets $_{\mathfrak{Z}}\mathfrak{B}(f)_{\mathfrak{X}}$ defined as follows. As a graph, $\mathfrak{B}$ is $\mathfrak{Z}$, with $\lambda\colon \mathfrak{B} \to \mathfrak{Z}$ the identity and $\rho\colon \mathfrak{B} \to \mathfrak{X}$ given above. For every vertex $\gamma \in \mathfrak{B}$, the biset B_γ is $B(f\colon W_\gamma \to U_{\rho(\gamma)}, \dagger_{\lambda(\gamma)}, *_{\rho(\gamma)})$. For every edge $e \in \mathfrak{B}$ representing the embedding $W_{\gamma'} \subsetneqq W_\gamma$ the biset B_e is $B_{\gamma'}$. If e is oriented so that $e^- = \gamma'$ and $e^+ = \gamma$, the intertwiners $(\cdot)^\pm$ are $(\cdot)^- = \mathbb{1}\colon B_e \to B_{\gamma'}$ and $(\cdot)^+\colon B_e \to B_\gamma$ given by Lemma 4.6.

By construction, $_{\mathfrak{Z}}\mathfrak{B}(f)_{\mathfrak{X}}$ is a right-principal graph of bisets.

Consider now a correspondence (Z, f, i), with $f\colon Z \to X$ and $i\colon Z \to Y$, between path connected spaces X and Y. Suppose (Z_k) are the path connected components of Z, and suppose (U_α), (V_β), and (W_γ) are 1-dimensional covers of X, Y, and Z respectively, *compatible with* f and i: for every γ there are $\lambda(\gamma)$ and $\rho(\gamma)$ such that $f(W_\gamma) \subset U_{\rho(\gamma)}$ and $i(W_\gamma) \subset V_{\lambda(\gamma)}$.

Let (U_α), (V_β), and (W_γ) be 1-dimensional covers of X, Y, and Z compatible with f and i as above. We then have graphs of bisets $_{\mathfrak{Z}_k}\mathfrak{B}(f{\downharpoonleft}_{Z_k})_{\mathfrak{X}}$ and $_{\mathfrak{Z}_k}\mathfrak{B}(i{\downharpoonleft}_{Z_k})_{\mathfrak{Y}}$. We define the graph of bisets of the correspondence (Z, f, i) as

$$_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}} := \bigsqcup_k {}_{\mathfrak{Y}}\mathfrak{B}(i{\downharpoonleft}_{Z_k})^\vee_{\mathfrak{Z}_k} \otimes_{\mathfrak{Z}_k} {}_{\mathfrak{Z}_k}\mathfrak{B}(f{\downharpoonleft}_{Z_k})_{\mathfrak{X}}.$$

The following equivalent description of ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ follows immediately from the definition; see Lemma 4.2.

Lemma 4.7. *Let (Z, f, i) be a topological correspondence from Y to X, for path connected spaces X, Y. Let (U_α), (V_β), and (W_γ) be 1-dimensional covers of X, Y, and Z compatible with f and i as above. Then the graph of bisets ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ of (f, i) with respect to the above data is as follows:*

- *the graphs of groups $\mathfrak{X}$ and $\mathfrak{Y}$ are constructed as in Definition 3.6 using the covers (U_α) and (V_β) of X, Y respectively. Choices of paths ℓ_e, m_e were made for edges e in $\mathfrak{X}$, $\mathfrak{Y}$ respectively;*
- *the underlying graph of $\mathfrak{B}$ is similarly constructed using the cover (W_γ) of Z. Note that neither Z nor the underlying graph of $\mathfrak{B}$ need be connected;*
- *for every vertex $z \in \mathfrak{B}$ the biset ${}_{G_{\lambda(z)}}(B_z)_{G_{\rho(z)}}$ is $B(f\downharpoonleft_{W_z}, i\downharpoonleft_{W_z})$;*
- *for every edge $e \in \mathfrak{B}$ representing the embedding $W_{z'} \subsetneqq W_z$ the biset B_e is $B_{z'}$, and if e is oriented so that $e^- = z'$ then the intertwiners $(\cdot)^\pm$ are the maps $(\cdot)^- = \mathbb{1}\colon B_e \to B_{z'}$ and $(\cdot)^+\colon B_e \to B_z$ given by $(\gamma^{-1}, \delta) \mapsto (m_{\lambda(e)}^{-1}\#\gamma^{-1}, \delta\#\ell_{\rho(e)})$ in the description of B_e as $B(i\downharpoonleft_{W_{e^-}})^\vee \otimes B(f\downharpoonleft_{W_{e^-}})$, see Lemma 4.6.*

Theorem 4.8 (van Kampen's theorem for correspondences). *Let (Z, f, i) be a topological correspondence from a path connected space Y to a path connected space X, and let ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ be the graph of bisets subordinate to compatible 1-dimensional covers of spaces in question.*

Then for every $v \in \mathfrak{Y}$ and $u \in \mathfrak{X}$ we have

$$B(f, i, \dagger_v, *_u) \cong \pi_1(\mathfrak{B}, v, u),$$

where $\dagger_v$ and $*_u$ are basepoints.

Proof. Recall from §3.2 that for all vertices $y \in \mathfrak{Y}$ there are chosen basepoints $\dagger_y \in V_y$ identifying the groups G_y with $\pi_1(V_y, \dagger_y)$ and for all edges $e \in \mathfrak{Y}$ there are curves γ_e connecting $\dagger_{e^-}$ to $\dagger_{e^+}$ satisfying $\gamma_y = \gamma_{\bar{y}}^{-1}$ and describing $(\cdot)^-$-maps. Similarly, for objects in $\mathfrak{X}$ there are basepoints $*_x \in U_x$ and curves $\gamma_e \subset X$.

Every biset element $\bar{b} = (h_0, y_1, \dots, b, \dots x_n, g_n) \in \pi_1(\mathfrak{B}, \dagger, *)$ defines a certain element $\theta(\bar{b}) \in B((Z, f, i), \dagger_\dagger, *_*)$ that is the concatenation $h_0\gamma_{y_1}\#\dots\#b\dots\#\gamma_{x_n}g_n$. We get a biset morphism

$$\theta\colon \pi_1(\mathfrak{B}, \dagger, *) \longrightarrow B((Z, f, i), \dagger_\dagger, *_*).$$

We first assume that all the 1-dimensional covers are open. Then θ is surjective, because every element in $B((Z, f, i), \dagger_\dagger, *_*)$ can be presented as a concatenation $h_0\gamma_{y_1}\#\dots\#b\dots\#\gamma_{x_n}g_n$. To prove that θ is injective, we show that if $\theta(\bar{b}_1)$ is homotopic to $\theta(\bar{b}_2)$, then $\bar{b}_1 = \bar{b}_2$.

By the classical van Kampen argument (see [12, Chapter IV]) a homotopy between $\theta(\bar{b}_1)$ and $\theta(\bar{b}_2)$ can be expressed as a combination of the following operations: (1) homotopies within U_i and V_j; (2) replacement of g_k or h_k by $\gamma_z g_k \gamma_z^{-1}$ or $\gamma_z h_k \gamma_z^{-1}$ for some z, and (3) replacement of b by $\gamma_{\lambda(z)}^{-1} \# b \# \gamma_{\rho(z)}$ for some $z \in \mathfrak{B}$. All the above operations fix the corresponding elements in $\pi_1(\mathfrak{B}, \dagger, *)$.

If the 1-dimensional covers are not open, we first slightly enlarge the covers (U_α) and (V_β) into equivalent 1-dimensional open covers $(\widetilde{U}_\alpha)$ and $(\widetilde{V}_\beta)$, and then enlarge the 1-dimensional cover (W_γ) of Z into a 1-dimensional open cover $(\widetilde{W}_\gamma)$ small enough that $\widetilde{W}_\gamma \subseteq i^{-1}(V_{\lambda(\gamma)}) \cap f^{-1}(U_{\rho(\gamma)})$. We then apply the first part to these compatible open covers, and conclude by Lemma 3.5 that the graphs of bisets are isomorphic. □

Lemma 4.9. *Consider a path connected space X with a 1-dimensional cover $(X_v)_{v \in V}$ and the associated graph of groups $\mathfrak{X}$ as in Definition 3.6.*

Then, up to congruence by a biprincipal biset whose fundamental biset is $\approx B(\mathbb{1})$, the graph of groups $\mathfrak{X}$ is independent of the choices of basepoints $\{*_v\}$ and connecting paths $\{\gamma_e\}$.

Similarly, the graph of bisets associated with a topological correspondence is independent of the choices of basepoints $\{*_v\}$ and connecting paths $\{\gamma_e\}$.

Proof. Let $\mathfrak{X}$, $\mathfrak{X}'$ be two graphs of groups associated with X and with a fixed 1-dimensional cover $(X_v)_{v \in V}$ but with different choices of basepoints and connecting paths. Consider a topological correspondence (X, f, i) such that $f\colon X \circlearrowleft$ and $i\colon X \circlearrowleft$ are the identity maps. We may assume that $\mathfrak{X}'$, resp $\mathfrak{X}$, is the graph of groups associated with the range of f, resp i. Let ${}_{\mathfrak{X}}\mathfrak{B}_{\mathfrak{X}'}$ be the graph of bisets associated with (X, f, i) and the cover $(X_v)_{v \in V}$. Then $\mathfrak{B}$ is a biprincipal graph of bisets because $(f{\downharpoonright}_{X_v}, i{\downharpoonright}_{X_v})$ is biprincipal for every v.

The second claim follows from the first applied to the identity map. □

4.3. Products of correspondences. Correspondences, just as continuous maps, may be composed; the operation is given by fiber products.

Let (Z_1, f_1, i_1) and (Z_2, f_2, i_2) be two correspondences such that $f_1\colon Z_1 \to X$ and $i_2\colon Z_2 \to X$ have the same range X. Their product is the correspondence (Z, f, i) given by

$$Z = \{(z_1, z_2) \in Z_1 \times Z_2 \mid f_1(z_1) = i_2(z_2)\},$$
$$f(z_1, z_2) = f_2(z_2), \quad i(z_1, z_2) = i_1(z_1).$$

We have natural maps $\widetilde{i_2}\colon Z \to Z_1$ and $\widetilde{f_1}\colon Z \to Z_2$ given respectively by

$$\widetilde{i_2}(z_1, z_2) = z_1, \quad \widetilde{f_1}(z_1, z_2) = z_2.$$

It is easy to check that f is a fibration, respectively a covering, if both f_1 and f_2 are fibrations, respectively coverings.

The biset of a product of two correspondences is, in favourable cases, the product of the corresponding bisets:

Lemma 4.10. *Let (Z, f, i) be the product of two correspondences (Z_1, f_1, i_1) and (Z_2, f_2, i_2), with $f_1\colon Z_1 \to X$ and $i_2\colon Z_2 \to X$ and X path connected. Then there is a biset morphism*

$$\begin{aligned} B(f, i) &\longrightarrow B(f_1, i_1) \otimes B(f_2, i_2), \\ (\delta, p, \gamma) &\longmapsto (\delta, \widetilde{i_2}(p), \varepsilon^{-1}) \otimes (\varepsilon, \widetilde{f_1}(p), \gamma), \end{aligned} \tag{16}$$

for any choice of path ε from the basepoint of X to $f_1(\widetilde{i_2}(p)) = i_2(\widetilde{f_1}(p))$.

Example 4.11. In general, the map in the above lemma need not be an isomorphism; for instance, the ranges of i_2 and f_1 need not intersect, in which case $Z = \emptyset$ so $B(f, i) = \emptyset$, while $B(f_1, i_1)$ and $B(f_2, i_2)$ are non-empty so their product is not empty.

For a less artificial example, consider the correspondence shown on Figure 1. Denote by 1 the trivial group; we also view 1 as a set consisting of one element. Then $B(i_1, f_1) = {}_1\mathbb{Z}_{\mathbb{Z}}$, $B(i_2, f_2) = {}_{\mathbb{Z}}1_{\mathbb{Z}}$; thus $B(i_1, f_1) \otimes B(i_2, f_2) = {}_1 1_{\mathbb{Z}}$. On the other hand, $B(\widetilde{i_2} i_1, \widetilde{f_1} f_2) = {}_1\mathbb{Z}_{\mathbb{Z}}$.

Sufficient conditions on the map (16) being an isomorphism are given by Lemma 4.12, in analogy with Lemma 3.26.

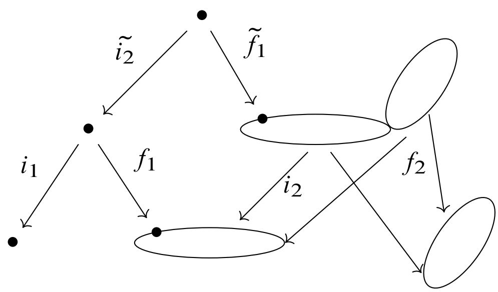


Figure 1. An example with $B(f, i) \not\cong B(f_1, i_1) \otimes B(f_2, i_2)$.

Lemma 4.12. *If in Lemma 4.10 at least one of the maps i_2, f_1 is fibrant, then (16) is an isomorphism of bisets.*

Proof. We construct an inverse to the map (16), with the notations of Lemma 4.10. Suppose without loss of generality that i_2 is fibrant. Consider $b = (\delta_1, p_1, \gamma_1) \otimes (\delta_2, p_2, \gamma_2) \in B(f_1, i_1) \otimes B(f_2, i_2)$. Changing the basepoint in X to $* = f_1(p_1)$ and denoting by $*$ the constant path in X at $*$, we rewrite b as $(\delta_1, p_1, *) \otimes (\gamma_1 \# \delta_2, p_2, \gamma_2)$. Since i_2 is fibrant, the curve $\gamma_1 \# \delta_2$ admits a lift ε to Z_2 with $\varepsilon(1) = p_2$. Write $p_2' = \varepsilon(0)$. We then have $b = (\delta_1, p_1, *) \otimes (*, p_2', (f_2 \circ \varepsilon) \# \gamma_2)$, see Lemma 4.4. Since $f_1(p_1) = * = i_2(p_2')$, there is a unique $p \in Z$ with $\widetilde{i_2}(p) = p_1$ and $\widetilde{f_1}(p) = p_2'$. Then $(\delta_1, p, (f_2 \circ \varepsilon) \# \gamma_2) \in B(f, i)$ is a preimage of b under (16). □

4.4. Fibrant maps and covers. Suppose $f : Z \to X$ is a fibrant map between path connected spaces and (U_α) is a cover of X consisting of path connected sets U_α. The *pullback* $(W_\gamma) := f^*(U_\alpha)$ is the cover of Z consisting of all path connected components of $f^{-1}(U_\alpha)$, for all U_α in the cover. If (U_α) is 1-dimensional, then so is (W_γ). It is also immediate that all $f{\downharpoonleft}_{W_\gamma}$ are fibrant maps.

Proposition 4.13. *Suppose ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ is the graph of bisets of a topological correspondence (Z, f, i) from Y and X subordinate to 1-dimensional covers (U_α), (V_β), and (W_γ) of X, Y, and Z respectively.*

If $f : Z \to X$ is fibrant, respectively a covering, and $(W_\gamma) = f^*(U_\alpha)$, then ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ is a left-fibrant, respectively a left-free, graph of bisets.

The proof of the above proposition is based on the following property.

Lemma 4.14. *Let*

- *$f : W \to U$ be a fibrant map between path connected spaces, with biset $B(f, \dagger, *)$;*
- *$U' \subset U$ be path connected with basepoint $*'$;*
- *$\{W_z'\}_{z \in I}$ be the set of connected components of $f^{-1}(U')$;*
- *$B(f{\downharpoonleft}_{W_z'}, \dagger_z', *')$ be the biset of $f : W_z' \to U'$;*
- *$(\cdot)^- : {}_{\pi_1(W_z', \dagger_z')}B(f{\downharpoonleft}_{W_z'}, \dagger_z', *')_{\pi_1(U', *')} \to {}_{\pi_1(W, \dagger)}B(f, \dagger, *)_{\pi_1(U, *)}$ be intertwiners as in Lemma 4.6.*

Then we have a $\pi_1(W, \dagger)$-$\pi_1(U', *)$ isomorphism

$$\bigsqcup_{z \in I} \pi_1(W, \dagger) \otimes_{\pi_1(W_z', \dagger_z')^-} B(f{\downharpoonleft}_{W_z'}, \dagger_z', *') \longrightarrow B(f{\downharpoonleft}_W, \dagger, *), \quad g \otimes b \longmapsto g b^- \tag{17}$$

with right action of $\pi_1(U', *')$ on $B(f{\downharpoonleft}_W, \dagger, *)$ given via $(\cdot)^- : \pi_1(U', *') \to \pi_1(U, *)$.

Proof. The statement is clearly stable under motion of $*$, so we may assume $* = *'$ and that $(\cdot)^{-}\colon \pi_1(U', *') \to \pi_1(U, *)$ is given by taking each $\gamma \in \pi_1(U', *')$ modulo homotopy in U. By Lemma 4.4

$$B(f, \dagger, *) = \{\beta\colon [0,1] \to W \mid \beta(0) = \dagger, f(\beta(1)) = *\}/\sim,$$

with $\beta \sim \beta'$ if and only if there is a path $\varepsilon\colon [0,1] \to f^{-1}(*)$ connecting $\beta(1)$ to $\beta'(1)$ such that $\beta\#\varepsilon$ is homotopic to β'. Therefore, each $\beta\colon [0,1] \to W$ in $B(f)$ ends at a unique W'_z independent of the choice of β in its homotopy class. It is now easy to see that the $\pi_1(W, \dagger)$-$\pi_1(U', *)$ subbiset of $B(f, \dagger, *)$ consisting of all $b \in B(f, \dagger, *)$ terminating at W'_z is exactly $\pi_1(W, \dagger) \otimes_{\pi_1(W'_z, \dagger'_z)^{-}} B(f{\downharpoonleft}_{W'_z}, \dagger'_z, *')$. This implies (17). □

Proof of Proposition 4.13. Let (Z_k) be the path connected components of Z. By definition, ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}} = \bigsqcup_k {}_{\mathfrak{Y}}\mathfrak{B}(i{\downharpoonleft}_{Z_k})^{\vee}_{\mathfrak{Z}_k} \bigotimes_{\mathfrak{Z}_k} {}_{\mathfrak{Z}_k}\mathfrak{B}(f{\downharpoonleft}_{Z_k})_{\mathfrak{X}}$. Lemma 4.14 implies that every ${}_{\mathfrak{Z}_k}\mathfrak{B}(f{\downharpoonleft}_{Z_k})_{\mathfrak{X}}$ is left-fibrant: equation (17) is exactly (6). Since the product of left-fibrant bisets is left-fibrant (Lemma 3.27), ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$ is a left-fibrant biset. This proves the part of Proposition 4.13 concerning fibrant maps. In the case of covering maps observe that all bisets in $\mathfrak{B}(f{\downharpoonleft}_{Z_k})$ are left-free because $f{\downharpoonleft}_{Z_k}$ are coverings while all bisets in $\mathfrak{B}(i{\downharpoonleft}_{Z_k})^{\vee}$ are left-principal; hence all bisets in $\mathfrak{B}_{\mathfrak{X}} = \bigsqcup_k {}_{\mathfrak{Y}}\mathfrak{B}(i{\downharpoonleft}_{Z_k})^{\vee}_{\mathfrak{Z}_k} \bigotimes_{\mathfrak{Z}_k} {}_{\mathfrak{Z}_k}\mathfrak{B}(f{\downharpoonleft}_{Z_k})$ are left-free. □

4.5. Partial self-coverings. We now turn to a restricted class of covering correspondences $(Z, f\colon Z \to X, i\colon Z \to Y)$, in which the map i is an inclusion, namely a homeomorphism on its image. We then view Z as a subset of Y, and write the correspondence as a *partial covering* $f\colon Y \dashrightarrow X$. Here the dashed arrow means that the map is defined on the image of i.

In this case, the definition of the biset of a correspondence (see §4.1) can be simplified as follows:

Lemma 4.15. *Let $f\colon X \dashrightarrow X$ be a partially defined self-covering. The biset $B(f)$ may then be constructed directly as follows*:

$$B(f) = \{\gamma\colon [0,1] \to X \mid \gamma(0) = * = f(\gamma(1))\}/\sim.$$

The left action is by pre-composition by loops in X at $*$, and the right action is by lifting loops through f:

$$[\alpha] \cdot [\gamma] = [\alpha\#\gamma], \qquad [\gamma] \cdot [\alpha] = [\gamma\#\tilde{\alpha}]$$

for the unique f-lift $\tilde{\alpha}$ of α that starts at $\gamma(1)$.

Proof. Simply remark that, in the notation of Lemma 4.2, the point p is determined as $i^{-1}(\gamma(1))$, so may be removed from the definition. □

Let us consider now a self-correspondence $(Z, f\colon Z \to X, i\colon Z \to X)$. It may arise as follows: $f\colon X \circlearrowleft$ is a branched covering, Z is the subset of X on which f is a genuine covering, and i is the inclusion.

Such correspondences may be iterated; however, their iterates are not defined on the same Z anymore. We set $X^{(1)} := Z$ and, more generally,

$$X^{(n)} = \{(x_1, \dots, x_n) \in Z^n \mid f(x_i) = i(x_{i+1})\}.$$

Define the maps $f^{(n)}(x_1, \dots, x_n) = f(x_n)$ and $i^{(n)}(x_1, \dots, x_n) = i(x_1)$. Then the n-th power, in the sense of §4.3, of (Z, f, i) is $(X^{(n)}, f^{(n)}, i^{(n)})$.

If f is fibrant, then so are all $f^{(n)}$, and naturally (Lemma 4.12), the biset of $(X^{(n)}, f^{(n)}, i^{(n)})$ is none other than $B(Z, f, i)^{\otimes n}$.

If $i\colon X^{(1)} \to X$ is an inclusion, then so are all $i^{(n)}\colon X^{(n)} \to X$. Then (Z, f, i) is identified with a partially defined map $f\colon X \dashrightarrow X$; and all $(X^{(n)}, f^{(n)}, i^{(n)})$ are identified with $f^{(n)}\colon X \dashrightarrow X$. If f is fibrant, then the bisets of $(X^{(n)}, f^{(n)}, i^{(n)})$ and of $f^{(n)}$ are naturally isomorphic.

Iteration may also be interpreted purely in the language of homomorphisms, using Lemma 2.7. Given a G-G-biset B, write $G_0 = G$, and find a group G_1 such that B decomposes as $B_{\psi_1}^\vee \otimes_{G_1} B_{\phi_1}$, for homomorphisms $\phi_1, \psi_1\colon G_1 \to G_0$. Define then iteratively G_{n+1} as the fibre product of G_n with G_n over G_{n-1}:

$$\begin{array}{ccc} G_{n+1} & \overset{\phi_{n+1}}{\dashrightarrow} & G_n \\ {\scriptstyle \psi_{n+1}} \downarrow & & \downarrow {\scriptstyle \psi_n} \\ G_n & \xrightarrow{\phi_n} & G_{n-1}. \end{array}$$

Then $B^{\otimes n} = (B_{\psi_n \cdots \psi_1})^\vee \otimes_{G_n} B_{\phi_n \cdots \phi_1}$.

4.6. Generic maps. We can slightly relax the previous setting, in which $i\colon Z \to Y$ is an inclusion, to "generic" maps in the following sense.

Definition 4.16 (generic maps). A continuous map $f\colon Y \to X$ is *generic* if there exists a continuous map $g\colon X \to Y$ such that $f \circ g$ is isotopic to the identity on X.

Here is a typical example: Consider a topological space X and an injective path $\gamma\colon [0, 1] \to X$, such that a neighbourhood of the image of γ is contractible in X. Then the inclusion $f\colon X \setminus \{\gamma(0), \gamma(1)\} \to X \setminus \{\gamma(0)\}$ is a generic map. Indeed contractibility of the image of γ implies the existence of a homeomorphism $g\colon X \setminus \{\gamma(0)\} \to X \setminus \gamma([0, 1])$.

Lemma 4.17. *Let $f\colon Y \to X$ be a generic map, and let $\dagger \in Y$ be a basepoint. Write $f(\dagger) = *$. Then $f_*\colon \pi_1(Y, \dagger) \to \pi_1(X, *)$ is a split epimorphism.*

Therefore, the biset of f is left-invertible and right-principal; *namely, it is a $\pi_1(Y,\dagger)$-$\pi_1(X,*)$-biset B such that $B^\vee \otimes_{\pi_1(Y,\dagger)} B \cong {}_{\pi_1(X,*)}\pi_1(X,*)_{\pi_1(X,*)}$.*

Proof. Let g be a homotopy left inverse of f. The first statement follows simply from $g_* f_* = \mathbb{1}_X$. The second one follows from the first. □

4.7. Conjugacy classes in bisets. Let ${}_G B_G$ be a biset. Its set of conjugacy classes is

$$\mathcal{C}(B) := {}_G B_G / \{b = gbg^{-1} \mid b \in B, g \in G\}.$$

Consider a self-correspondence $f, i\colon Z \rightrightarrows X$. A homotopy *pseudo-fixed point* [10] is the data (p,γ) such that $p \in Z$ and $\gamma\colon [0,1] \to X$ with $\gamma(0) = f(p)$ and $\gamma(1) = i(p)$. In other words, γ encodes a homotopy difference between $f(p)$ and $i(p)$. If γ is a constant path, then p is a *fixed point of* (Z, f, i). Two homotopy pseudo-fixed points (p,γ) and (q,δ) are *conjugate* if there is a path $\ell\colon [0,1] \to Z$ with $\ell(0) = p$, $\ell(1) = q$ such that $f(\ell)\#\beta\#i(\ell^{-1})$ is homotopic to γ.

The set of fixed points conjugate to a given fixed point $p \in Z$ is also known as the *Nielsen class* of p. The *Nielsen number* $N(f,i)$ is the set of fixed points of (Z, f, i) considered up to conjugacy.

Every $(\delta^{-1}, p, \gamma) \in B(f,i)$, in the notation of Lemma 4.2, naturally defines a homotopy pseudo-fixed point $(p, \gamma\#\delta)$. Conversely, if X is path connected, then for every homotopy pseudo-fixed point (p,γ) we may choose a path $\ell\colon [0,1] \to X$ with $\ell(0) = *$ and $\ell(1) = f(p)$ and construct an element $(\ell^{-1}, p, \gamma\#\ell) \in B(f,i)$ encoding (p,γ).

The following proposition is immediate.

Proposition 4.18. *Two elements $(\delta^{-1}, p, \gamma), (\delta'^{-1}, p', \gamma') \in B(f,i)$ are conjugate as elements of the biset $B(f,i)$ if and only if the homotopy fixed points $(p, \gamma\#\delta), (p', \gamma'\#\delta')$ are conjugate.*

As a corollary, if X is path connected, then the Nielsen number $N(f,i)$ is bounded by the cardinality of $\mathcal{C}(B)$.

In [3, 4] we further investigate homotopy pseudo-periodic orbits of a Thurston map.

5. Dynamical systems

We turn now to applications of the previous sections. They will mainly be to the theory of iterations of branched self-coverings of surfaces. The main objective is an algorithmic understanding of these maps up to isotopy, and will be developed in later articles. Here are some more elementary byproducts.

Consider first a polynomial $p(z) \in \mathbb{C}[z]$. We recall some basic definitions and properties; see [7, 8] for details.

Let us denote by $\widetilde{P}(p) \subset \mathbb{C}$ the forward orbit of p's critical points:

$$\widetilde{P}(p) := \{p^n(z) \mid p'(z) = 0,\, n \geq 0,\, z \in \mathbb{C}\}.$$

The *post-critical set* $P(p) := p(\widetilde{P}(p))$ is the forward orbit of p's critical values. The polynomial p is *post-critically finite* if $P(p)$ is finite. The *Julia set* $J(p)$ of p is the closure of the repelling periodic points of p. Equivalently, $J(p)$ is the boundary of the filled-in Julia set $K(p)$:

$$K(p) := \{z \in \mathbb{C} \mid (p^n(z))_{n\in\mathbb{N}} \text{ is bounded}\}, \quad J_c = \partial K_c.$$

The *Fatou set* $F(p)$ is the complement of $J(p)$.

Let us assume that p is post-critically finite. This assumption is essential to obtain simple combinatorial descriptions of p. Then each bounded connected component of $F(p)$ is a disk, and p acts on this set of disks. Furthermore, the boundary of each disk component is a circle in $J(p)$.

Furthermore, every grand orbit of $p\colon \widetilde{P}(p) \to \widetilde{P}(p)$ contains a unique periodic cycle. If a periodic cycle $C \subset \widetilde{P}(p)$ contains a critical point of p, then C lies entirely in the Fatou set, and moreover each element $c \in C$ lies in a different disk of $F(p)$. On the other hand, if a cycle C contains no critical point of p, then C lies in $J(p)$. If no cycle of $P(p)$ contains a critical point of p, then $F(p)$ has no bounded component and $J(p)$ is a *dendrite*.

Let $\mathfrak{T}$ be the smallest tree in $K(p)$ containing $\widetilde{P}(p)$ and containing $P(p)$ in its vertex set such that $\mathfrak{T}$ intersects $F(p)$ along radial arcs. Since $\widetilde{P}(p)$ is forward invariant we get a self-map $p\colon \mathfrak{T} \circlearrowleft$. For every pair of adjacent edges e_1, e_2 there is an angle $\sphericalangle(e_1, e_2) \in \mathbb{Q}/\mathbb{Z}$ uniquely specified by the following conditions, see [16]:

- $\sphericalangle(e_1, e_2) = 0$ if and only if $e_1 = e_2$;
- if v is a common vertex of e_1 and e_2 and $\deg_v(p)$ is the local degree of p at v, then
$$\sphericalangle(p(e_1), p(e_2)) = \deg_v(p)\sphericalangle(e_1, e_2);$$
- for all edges e_1, e_2, e_3 adjacent to a common vertex we have
$$\sphericalangle(e_1, e_3) = \sphericalangle(e_1, e_2) + \sphericalangle(e_2, e_3).$$

Definition 5.1 (Hubbard trees). The (*angled*) *Hubbard tree* of a complex polynomial p is the data consisting of $p\colon \mathfrak{T} \circlearrowleft$, the values $\deg_p(v) \in \mathbb{N}$ measuring the local degrees of p at vertices $v \in p^{-1}(\mathfrak{T})$, and the angle structure "$\sphericalangle$" on $\mathfrak{T}$ satisfying the above axioms.

In [16] post-critically finite polynomials are classified in terms of their Hubbard trees.

Remark 5.2. If p has degree 2, then the angled structure of $p\colon \mathcal{T} \circlearrowleft$ is uniquely determined by how $\mathcal{T}$ is topologically embedded into the plane. However, in general, the planarity of $\mathcal{T}$ is insufficient to recover p; one needs to endow $p\colon \mathcal{T} \circlearrowleft$ with extra information sufficient to recover the embedding of $p^{-1}(\mathcal{T})$ into the plane.

It may be more convenient to consider a variant, the *Hubbard complex*, which is a special case of topological automaton (see [15, §3]):

Definition 5.3 (Hubbard complexes). Let p be a complex polynomial. Its *Hubbard complex* is the self-correspondence $p, i\colon H^1 \rightrightarrows H^0$ defined as follows:

- H^0 is the smallest 1-dimensional subcomplex of the plane containing $P(p) \cap J(p)$ and all ∂D for $D \subseteq F(p)$ a Fatou component intersecting $P(p)$;
- H^1 is the smallest 1-dimensional subcomplex of the plane containing $p^{-1}(P(p)) \cap J(p)$ and all ∂D for $D \subseteq F(p)$ a Fatou component intersecting $p^{-1}(P(p))$;
- $p\colon H^1 \to H^0$ is the restriction of p to H^1 and is a covering map;
- $i\colon H^1 \to H^0$ retracts H^1 into H^0.

(As for the Hubbard tree, we assume that H^1, H^0 intersects $F(p)$ along radial lines. Points in $H^0 \cap P(p)$ and in $H^1 \cap \widetilde{P}(p)$ are treated as orbifold points.)

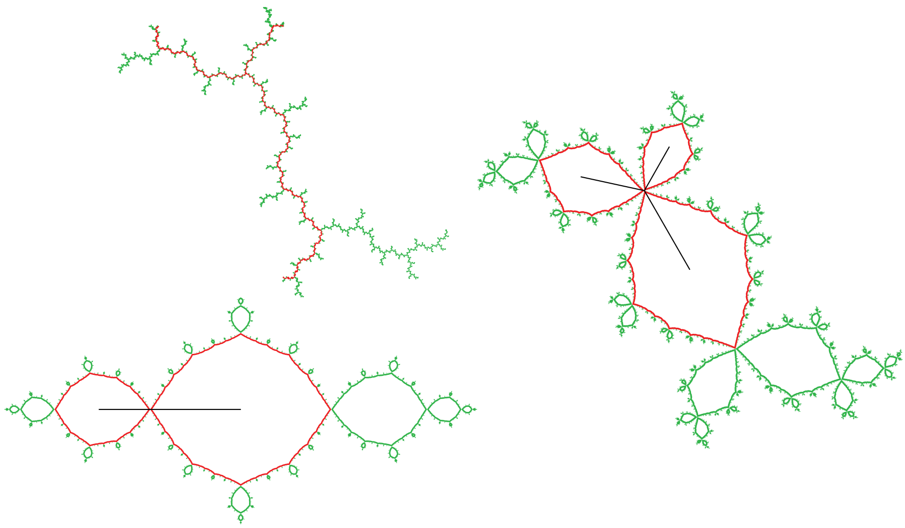


Figure 2. The Julia sets of the maps $z^2 + i$, $z^2 - 1$ (the "Basilica") and $z^2 + c$ for $(c^2 + c)^2 + c = 0$ (the "Rabbit", right). The Hubbard complex is drawn in red, and (if it differs) the Hubbard tree is drawn in black.

5.1. Thurston maps. We may consider the more general situation of a *branched self-covering* of the sphere S^2, namely a map $p\colon S^2 \circlearrowleft$ that is locally modelled, in complex charts, by $z \mapsto z^n$ for some $n \in \mathbb{N}$. Those points $z \in S^2$ at which $n \geq 2$ are *critical points*, and $P(p)$ is the forward orbit of p's critical values. If furthermore there is a point $\infty \in S^2$ with $p^{-1}(\infty) = \{\infty\}$, then p is a topological polynomial. If $P(p)$ is finite, then p is called a *Thurston map*.

Unless p expands a metric on S^2, there is no well-defined notion of Julia set. There is, however, a convenient encoding of p by a biset. One sets $X = Y = S^2 \setminus P(p)$ and $Z = S^2 \setminus p^{-1}(P(p))$, with maps $i\colon Z \to Y$ the inclusion and $p\colon Z \to X$ the restriction of p. Thus $p\colon S^2 \circlearrowleft$ is given by a covering correspondence $p, i\colon Z \rightrightarrows X$.

Fix a basepoint $* \in X$, and write $G = \pi_1(X, *)$. Let $B(p)$ denote the biset of the above correspondence; it is a G-G-biset, left-free of degree $\deg(p)$.

Let p_0, p_1 be Thurston maps. They are called *combinatorially equivalent* if there is an isotopy $(p_t)_{t\in[0,1]}$ from p_0 to p_1 along which $P(p)$ moves continuously (and, in particular, has constant cardinality). The biset of a Thurston map is a complete invariant for combinatorial equivalence:

Theorem 5.4 (Kameyama [11], Nekrashevych [13, Theorem 6.5.2]). *Let p_0, p_1 be Thurston maps. Then p_0, p_1 are Thurston equivalent if and only if the bisets $B(p_0), B(p_1)$ are conjugate by an isomorphism*

$$\pi_1(S^2 \setminus P(p_0), *_0) \longrightarrow \pi_1(S^2 \setminus P(p_1), *_1)$$

induced by a surface homeomorphism.

By definition, p behaves locally as $z \mapsto z^{\deg_z(p)}$ at a point $z \in S^2$; set

$$\operatorname{ord}(v) = \operatorname{l.c.m.}\{\deg_z(p^n) \mid n \geq 0, z \in p^{-n}(v)\}.$$

Clearly, $\operatorname{ord}(z) > 1$ if and only if $z \in P(p)$, and $\operatorname{ord}(v) = \infty$ if and only if v is in a periodic cycle containing a critical point. For example, the degrees at $P(p)$ are all ∞ for the Basilica and the Rabbit maps, and are 2 at $i, -i, i-1$ for $z^2 + i$.

This order function ord defines an *orbispace* structure on S^2: in our simplified context, a topological space with the extra data of non-trivial groups attached at a discrete set of points. We will not go into details of orbispaces, but simply note that, if v is a point with group G_v attached to it, then v has canonical neighbourhoods with fundamental group isomorphic to G_v. In our situation, the group attached to $v \in P(p)$ is cyclic of order $\operatorname{ord}(v)$. If $\operatorname{ord}(v) = \infty$, then the point v may be treated as a puncture rather than as a point with $\mathbb{Z}$ attached to it.

For each $z \in P(p)$, let γ_z denote a small loop around z, and identify γ_z with a representative of a conjugacy class in $\pi_1(S^2 \setminus P(p), *)$. It follows that the fundamental group of the orbispace defined by ord is given as follows:

$$G_p = \pi_1(S^2 \setminus P(p), *)/\langle \gamma_z^{\operatorname{ord}(z)} : z \in P(p)\rangle. \tag{18}$$

For every $z \in S^2$ define $\operatorname{ord}^1(z) := \operatorname{ord}(p(z))/\deg_z(p)$. Then $p\colon (S^2, \operatorname{ord}^1) \to (S^2, \operatorname{ord})$ is a covering between orbispaces while $(S^2, \operatorname{ord}^1) \hookrightarrow (S^2, \operatorname{ord})$ is an orbispace inclusion. This defines the G_p-biset $B(p)$. Note that Theorem 5.4 was stated for bisets over the group $\pi_1(S^2 \setminus P(p))$, not for bisets over the orbispace fundamental group G_p. However, an analogue of Theorem 5.4 is also true in that context; this will be proven in [2].

Much structure in the space of Thurston maps can be obtained by comparing, or deriving, maps from the simplest example $f(z) = z^d$. This map has $J(f) = \{|z| = 1\}$ and $P(f) = \{0, \infty\}$ so that $\pi_1(\mathbb{C} \setminus P(f), *) = \langle t\rangle \cong \mathbb{Z}$. We call the corresponding biset the *regular cyclic biset of degree d*: as a left $\langle t\rangle$-set, it is $\langle t\rangle \times \{1, \dots, d\}$; and the right action on the basis $\{1, \dots, d\}$ is given by

$$1 \cdot t = 2, \quad \dots, \quad (d-1) \cdot t = d, \quad d \cdot t = t \cdot 1.$$

It may also be defined more economically as $B(z^d) = \{t^{j/d} \mid j \in \mathbb{Z}\}$, with left and right actions given by $t^i \cdot t^{j/d} \cdot t^k = t^{i+j/d+k/d}$.

Since topological polynomials of degree d behave as z^d in a neighbourhood of ∞, their bisets contain a copy of the regular degree-d cyclic biset. More generally, if p has a fixed point in a neighbourhood of which it acts as $z \mapsto z^n$, then p contains a regular degree-n cyclic subbiset.

The graph of bisets decompositions that we shall consider essentially attempt to describe bisets of Thurston maps in terms of cyclic bisets.

5.2. (Graphs of) bisets from Hubbard trees. Let us consider the (angled) Hubbard tree $p\colon \mathcal{T} \circlearrowleft$ of a complex polynomial p. We will now apply Van Kampen's theorem to $p\colon \mathcal{T} \circlearrowleft$ to decompose $B(p)$ as a graph of bisets ${}_{\mathfrak{X}}\mathfrak{T}_{\mathfrak{X}}$ as in Example 3.31.

Let $\mathcal{T}^1$ be the preimage of $\mathcal{T}$ under $p\colon \mathbb{C} \circlearrowleft$. We note that $\mathcal{T}^1$ is easily reconstructible from the data $p\colon \mathcal{T} \circlearrowleft$ and $\sphericalangle$ and $\deg_v(p)$ for all $v \in \mathcal{T}$. The angled structure of $\mathcal{T}$ lifts via $p\colon \mathcal{T}^1 \to \mathcal{T}$ to an angled structure on $\mathcal{T}^1$. We denote by $\iota\colon \mathcal{T}^1 \to \mathcal{T}$ the natural retraction of $\mathcal{T}^1$ into its subtree. We write $\lambda(z) = v$ if an object $z \in \mathcal{T}^1$ retracts into a vertex $v \in \mathcal{T}$ and we write $\lambda(z) = e$ if an object $z \in \mathcal{T}^1$ retracts into a subset of an edge $e \in \mathcal{T}$ but $\iota(z)$ is not a vertex of $\mathcal{T}$. This defines a graph morphism $\lambda\colon \mathcal{T}^1 \to \mathcal{T}$ between undirected graphs, see §3.9 such that the images of adjacent objects are adjacent or equal objects.

Let X be the space obtained from $\mathcal{T}$ by blowing up each vertex $v \in \mathcal{T}$ into a closed unit disc D_v with $\partial D_v \cong \mathbb{R}/\mathbb{Z}$ and by blowing down each edge e, say adjacent to vertices v and w, into a point D_e with $\{D_e\} = D_v \cap D_w$ such that for every pair of edges e_1, e_2 adjacent to a common vertex v we have

$$\sphericalangle(e_1, e_2) = D_{e_1} - D_{e_2} \in \mathbb{R}/\mathbb{Z} \cong \partial D_v.$$

For every vertex $v \in \mathcal{T}$ put an orbifold point of order $\operatorname{ord}(v)$ at the center of D_v, so that the fundamental group of D_v is a cyclic group of order $\operatorname{ord}(v)$.

Similarly, let X^1 be the space obtained from $\mathcal{T}^1$ by blowing up each vertex $v \in \mathcal{T}^1$ into a closed disc D_v with orbifold point of order $\operatorname{ord}^1(v)$ at the center of D_v and by blowing down each edge e into a point D_e satisfying same properties as above.

Then $p\colon \mathcal{T}^1 \to \mathcal{T}$ naturally induces a covering map $p\colon X^1 \to X$ specified so that all maps between unit discs are of the form $z \to z^d$. Furthermore, we have a natural retraction $\iota\colon X^1 \to X$ satisfying $\iota(D_z) \subset D_{\lambda(z)}$ for every object $z \in \mathcal{T}^1$. Applying Van Kampen's theorem Theorem 4.8 to the covering pair (X^1, p, ι) subordinate to covers $\{D_z\}_{z\in\mathcal{T}^1}, \{D_z\}_{z\in\mathcal{T}}, p, \lambda\colon \mathcal{T}^1 \to \mathcal{T}$ we get the graph of bisets ${}_{\mathfrak{X}}\mathfrak{T}_{\mathfrak{X}}$.

This graph of bisets can in fact also directly be described out of the Hubbard tree data. We present below an algorithm that computes ${}_{\mathfrak{X}}\mathfrak{T}_{\mathfrak{X}}$.

We say that a vertex $v \in \mathcal{T}^1$ is *essential* if it is the image of a vertex under the embedding $\mathcal{T} \hookrightarrow \mathcal{T}^1$. By definition, every vertex $v \in \mathcal{T}$ has a unique essential preimage under λ. We say a vertex $v \in \mathcal{T}^1$ is *critical* if $\operatorname{ord}(v) > 1$. Observe that if v is a critical but non-essential, then $\lambda(v)$ is an edge.

On the level of graphs, $\mathfrak{X} \overset{\lambda}{\leftarrow} \mathfrak{T} \overset{\rho}{\rightarrow} \mathfrak{Y}$ is the barycentric subdivision of $\mathcal{T} \overset{\lambda}{\leftarrow} \mathcal{T}^1 \overset{p}{\rightarrow} \mathcal{T}$. For convenience let us write $\operatorname{ord}(e) = 1$ for every edge $e \in \mathcal{T}$. For every object $z \in \mathcal{T}$ set

$$G_z := \mathbb{Z}/\operatorname{ord}(z).$$

In particular, G_z is a non-trivial group if and only if z is an essential vertex. This constructs the graph of groups $\mathfrak{X}$. The fundamental group of $\mathfrak{X}$ is isomorphic to a free product of G_v over all essential vertices $v \in \mathcal{T}$. For every $z \in \mathcal{T}^1$ set

$$B_z := \left(\tfrac{1}{\deg_z(p)}\mathbb{Z}\right)/\operatorname{ord}(\lambda(v)) \tag{19}$$

as a set with $G_{\lambda(z)}$-$G_{p(z)}$ actions given by

$$m \cdot b \cdot n = \begin{cases} m + b + \frac{n}{\deg_z(p)} & \text{if } z \text{ is an essential or a critical vertex,} \\ m + b & \text{otherwise.} \end{cases} \tag{20}$$

It remains to specify an intertwiner from B_e into B_v for every edge $e \in \mathcal{T}^1$ adjacent to $v \in \mathcal{T}^1$. Suppose first that v is a non-essential non-critical vertex. Then $\lambda(e) = \lambda(v)$ and we define $B_e \to B_v$ to be the natural bijection coming from (19); this is a well defined intertwiner between bisets because the right actions are trivial.

Let $e \in \mathcal{T}^1$ be an edge adjacent to either an essential or a critical vertex $v \in \mathcal{T}^1$. Then the intertwiner $B_e \to B_v$ is given by

$$b \longmapsto b + \lfloor D_e \rfloor$$

with D_e treated as an element of $\mathbb{R}/\mathbb{Z} \cong \partial D_v$ and $\lfloor D_e \rfloor$ means "round D_e down to an element in B_v".

Theorem 5.5. *The fundamental group of $\mathfrak{X}$ is isomorphic to G_p from (18). The fundamental biset of ${}_{\mathfrak{X}}\mathfrak{T}_{\mathfrak{X}}$ constructed above is isomorphic to $B(p)$. All bisets $B_z, z \in \mathfrak{T}$ are cyclic.*

Proof. By construction, the covering pair $((\mathbb{C}, \mathrm{ord}^1), p, \mathbb{1})$ is homotopic to the covering pair (X^1, p, ι) along a homotopy path of covering pairs. Therefore, by Theorem 3.7 the fundamental group of $\mathfrak{X}$ is isomorphic to G_p; and by Theorem 4.8 the fundamental biset of $\pi_1({}_{\mathfrak{X}}\mathfrak{T}_{\mathfrak{X}})$ is isomorphic to $B(p)$. Let us verify that the above algorithm computes ${}_{\mathfrak{X}}\mathfrak{T}_{\mathfrak{X}}$.

Let us assume that $0 \in \mathbb{R}/\mathbb{Z} \cong \partial D_v$ is the basepoint of $\pi_1(D_v)$ for every vertex $v \in \mathcal{T}$. For every edge e adjacent to v let $[0, D_e] \subset \mathbb{R}/\mathbb{Z} \cong \partial D_v$ be the path connecting the basepoint of D_v to the basepoint of D_e (recall that D_e is a singleton).

For every vertex $v \in \mathcal{T}^1$ there is $a_v \in \mathbb{R}/\mathbb{Z}$ such that the map $p\colon \partial D_v \to \partial D_{p(v)}$ is given by $x \mapsto \deg_v(p)x + a_v$ in the $\mathbb{R}/\mathbb{Z}$-coördinates. Moreover, for every essential vertex $v \in \mathcal{T}^1$ we may choose coördinates $\partial D_v \cong \mathbb{R}/\mathbb{Z}$ such that the map $\iota\colon \partial D_v \to \partial D_{\lambda(v)}$ is the identity in $\mathbb{R}/\mathbb{Z}$-coördinates.

Clearly, G_z is isomorphic to $\mathbb{Z}/\operatorname{ord}(z)$ for every object $z \in \mathcal{T}$.

For every essential vertex $v \in \mathcal{T}^1$ the biset B_v is computed by (15) as

$$B_v = \{b\colon [0,1] \to \mathbb{R}/\mathbb{Z} \cong \partial D_{\lambda(v)} \mid b(0) = 0, \deg_v(p)b(1) + a_v = 0\}/{\sim}$$

with $G_{\lambda(v)}$ actions given by pre-concatenation and by post-concatenation via lifting. Writing

$$\frac{t}{\deg_v(p)} := \left[0, \frac{t - a_v}{\deg_v(p)}\right] \subset \mathbb{R}/\mathbb{Z} \cong \partial D_{\lambda(v)}$$

we see that B_v takes the form given in (19) and (20). The case of non-essential vertices is immediate because the right action is trivial.

If $e \in \mathcal{T}^1$ is an edge adjacent to a non-essential vertex $v \in \mathcal{T}^1$, then the intertwiner $B_e \to B_v$ respects (19). Suppose $e \in \mathcal{T}^1$ is an edge adjacent to an essential vertex $v \in \mathcal{T}^1$. If $\lambda(e)$ is an edge, then $B_e \cong \{D_e\}$ embeds into B_v as $[0, D_e] \subset \mathbb{R}/\mathbb{Z} \cong \partial D_v$ followed by the lift of $[\deg_v(p)D_e + a_v, 0] \subset \partial D_{p(v)}$ under $p\colon \partial D_v \to \partial D_{p(v)}$ starting at $D_e \in \partial D_v$. This gives an element $\lfloor D_e \rfloor \in B_v$ with B_v viewed in the form (19).

If $\lambda(e)$ is a vertex, then $\lambda(e) = v$ and the 0-element of B_e is $[0, D_e] \subset \mathbb{R}/\mathbb{Z} \cong D_v$. Again, this element is mapped to $\lfloor D_e \rfloor$ via the intertwiner $B_e \to B_v$. ☐

5.3. (Graphs of) bisets from Hubbard complexes. Consider now a Hubbard complex $p, i\colon H^1 \rightrightarrows H^0$. As in the case of Hubbard trees §5.2 we blow up each vertex $v \in H^1 \sqcup H^0$ into a closed disc D_v with an orbifold point of order $\operatorname{ord}(v)$ at the center of D_v, and we blow down each edge $e \in H^1 \sqcup H^0$ into a point D_e. Observe that all D_v have finite fundamental groups. Then $p, i\colon H^1 \rightrightarrows H^0$

naturally induces a correspondence $p, i : Y^1 \rightrightarrows Y^0$ normalized so that all maps between unit discs are of the form $z \to z^d$. As in §5.2 we specify a graph morphism $\lambda : H^1 \to H^0$ by $i(D_z) \subset D_{\lambda(z)}$ for every object $z \in H^1$. Applying Van Kampen's Theorem 4.8 to the covering pair (Y^1, p, i) subordinate to covers $\{D_z\}_{z \in H^1}$, $\{D_z\}_{z \in H^0}$ with maps $p, \lambda : H^1 \to H^0$ we get a graph of bisets ${}_{\mathfrak{Y}}\mathfrak{H}_{\mathfrak{Y}}$. We remark that ${}_{\mathfrak{Y}}\mathfrak{H}_{\mathfrak{Y}}$ could be explicitly computed out of $p, i : H^1 \to H^0$ in the same way as ${}_{\mathfrak{X}}\mathfrak{T}_{\mathfrak{X}}$ was computed out of the Hubbard tree in §5.2.

Theorem 5.6. *The fundamental group of $\mathfrak{Y}$ is isomorphic to G_p from (18). The fundamental biset of ${}_{\mathfrak{Y}}\mathfrak{H}_{\mathfrak{Y}}$ is isomorphic to $B(p)$.*

All groups G_y, $y \in \mathfrak{Y}$ and all bisets $B_z, z \in \mathfrak{H}$ are finite.

Proof. The proof is the same as that of Theorem 5.5. The claim about finiteness of G_y and B_z is straightforward. □

5.4. (Graphs of) bisets from subdivision rules. Let us now generalize the setup of §5.2. Suppose that $p : S^2 \circlearrowleft$ is a *topological* Thurston map. By a *subdivision rule* or a "puzzle partition" we mean graphs $G^0 \subset G^1 \subset S^2$ such that

- $G^1 = p^{-1}(G^0)$;
- there is a retraction $\iota : G^1 \to G^0$, with $\iota\restriction_{G_0} = \mathbb{1}$ such that G^1 is homotopic in S^2 to $\iota(G^1)$ rel the post-critical set;
- each connected component $S^2 \setminus G^1$ contains at most one post-critical point.

The last condition guarantees that $p, \iota : G^1 \rightrightarrows G^0$ captures all combinatorial information of $p : S^2 \circlearrowleft$.

Assume, furthermore, that ι can be chosen in such a way that it maps vertices and edges of G^1 into vertices and edges of G^1. We may construct a graph of bisets ${}_{\mathfrak{Z}}\mathfrak{G}_{\mathfrak{Z}}$ associated with $p, \iota : G^1 \rightrightarrows G^0$. As in §5.2 we blow up each vertex $v \in G^1 \sqcup G^0$ into a closed disc D_v with orbifold point of order $\operatorname{ord}(v)$ at the center of D_v and we blow down each edge $e \in G^1 \sqcup G^0$ into a point D_e. Then $p, \iota : G^1 \rightrightarrows G^0$ naturally descends to a correspondence $p, \iota : Z^1 \rightrightarrows Z^0$ normalized so that all maps between unit discs are of the form $z \mapsto z^d$. As in §5.2 we specify a graph morphism $\lambda : G^1 \to G^0$ by $\iota(D_z) \subset D_{\lambda(z)}$ for every object $z \in G^1$. Applying Van Kampen's Theorem 4.8 to the covering pair (Z^1, p, i) subordinate to covers $\{D_z\}_{z \in G^1}, \{D_z\}_{z \in G^0}$ with $p, \lambda : G^1 \to G^0$ we get the graph of bisets ${}_{\mathfrak{Z}}\mathfrak{G}_{\mathfrak{Z}}$.

Theorem 5.7. *There is a natural epimorphism $\phi : \pi_1(\mathfrak{Z}, *) \to G_p$ and there is a natural surjective map $\beta : \pi_1({}_{\mathfrak{Z}}\mathfrak{G}_{\mathfrak{Z}}, *) \to B(p)$ such that*

$$(\phi, \beta) : \pi_1({}_{\mathfrak{Z}}\mathfrak{G}_{\mathfrak{Z}}) \longrightarrow {}_{G_p}B(p)_{G_p}$$

is a semi-conjugacy respecting combinatorics (see §2.4).

Proof. The space Z^0 has a natural embedding into S^2, unique up to homotopy rel the post-critical set, such that Z^0 separates post-critical points. Therefore, there is a natural epimorphism $\phi\colon \pi_1(\mathfrak{G}) \to G_p$.

The embedding of $p, \iota\colon Z^1 \rightrightarrows Z^0$ into S^2 defines a semi-conjugacy

$$(\phi, \beta)\colon B(Z^1, p, \iota) \longrightarrow B(p).$$

Observe that if a loop $\gamma \in \pi_1(Z^0)$ is trivial in G_p, then all lifts of γ via $p\colon Z^1 \to Z^0$ are loops. Therefore, γ has trivial monodromy action, so (ϕ, β) respects combinatorics. □

5.5. Tuning and mating. The *tuning* operation takes as input a topological polynomial p, a periodic cycle $z_0, z_1, \dots, z_n = z_0$ for p, and n topological polynomials $q_0, \dots, q_{n-1}$ with $\deg(q_i) = \deg_{z_i}(p)$ for all $i = 0, \dots, n-1$; and produces a topological polynomial of degree $\deg(p)$.

For each i, we give ourselves a set P_i containing the critical values of q_{i-1} and such that $q_i(P_i) \subset P_{i+1}$. In particular, P_i contains the post-critical set of $q_{i-1} \circ \cdots \circ q_{i+1} \circ q_i$.

For simplicity, let us assume that the cycle $z_0, z_1, \dots, z_n = z_0$ has no iterated critical preimages outside of the cycle. This condition could be lifted, but the construction would require slight modifications.

Up to isotopy, we may assume that around each z_i there is a small closed topological disc F_i such that p restricts to a covering map $p\colon F_i \setminus \{z_i\} \to F_{i+1} \setminus \{z_{i+1}\}$ and such that there are homeomorphisms (known as "Böttcher coördinates") $\psi_i\colon \operatorname{int}(F_i) \to \mathbb{C}$ so that $\psi_{i+1} \circ p \circ \psi_i^{-1}$ is the map $z \mapsto z^{\deg_{z_i}(p)}$. Also, up to isotopy, we may assume that $q_i(z) = z^{\deg(q_i)} + o(z^{\deg(q_i)})$ so that the extension of q_i to the circle at infinity coincides with the extension of $z \mapsto z^{\deg_{z_i}(p)}$. We consider the following *tuning* map $t(p, \{z_i\}, \{q_i\})$:

$$t(z) = \begin{cases} \psi_{i+1}^{-1}(q_i(\psi_i(z))) & \text{if } z \in \operatorname{int}(F_i) \text{ for some } i, \\ p(z) & \text{otherwise.} \end{cases}$$

Note that if $q_i = z^{\deg_{z_i}(p)}$ for all i then t is isotopic to p.

Theorem 5.8. *Suppose that $t = t(p, \{z_i\}, \{q_i\})$ is the tuning of a complex post-critically finite polynomial p with polynomials $q_i\colon \mathbb{C} \setminus P_i \dashrightarrow \mathbb{C} \setminus P_{i+1}$ as above. Suppose also $\operatorname{ord}(z_i) = \infty$ for all i. Let ${}_{\mathfrak{X}}\mathfrak{T}_{\mathfrak{X}}$ denote the graph of bisets constructed out of the Hubbard tree of p, see Theorem 5.5. Set $G_i = \pi_1(\mathbb{C} \setminus P_i, *_i)$ for a basepoint $*_i \gg 0$, and let $\phi_i\colon G_{z_i} \to G_i$ be the monomorphism defined by identifying $\operatorname{int}(F_i)$ with $\mathbb{C}$ via ψ_i, so that the circle ∂F_i corresponds to the circle at infinity in $\mathbb{C} \setminus P_i$. Let $\beta_i\colon B_{z_i} \to B(q_i)$ be the (ϕ_i, ϕ_{i+1})-intertwiner defined by viewing $p\colon \partial F_i \to \partial F_{i+1}$ as a map at infinity of $q_i\colon \mathbb{C} \setminus P_i \dashrightarrow \mathbb{C} \setminus P_{i+1}$.*

For all i, replace the cyclic group G_{z_i} at $z_i \in \mathfrak{X}$ by $G_i := \pi_1(\mathbb{C} \setminus P_i)$, and replace the regular cyclic biset B_{z_i} at $z_i \in \mathfrak{B}$ by the G_i-G_{i+1}-biset $B(q_i)$. Modify accordingly all non-essential bisets: replace all B_z with $z \in \lambda^{-1}(z_i) \setminus \{z_i\}$ by $G_i \otimes_{G_{z_i}} B_z$ and declare for all $z \in p^{-1}(z_{i+1}) \setminus \{z_i\}$ the right G_{i+1}-action on B_z to be trivial. Intertwiners of edge bisets into $B(q_i)$ are given through the intertwiner $\beta_i : B_{z_i} \to B(q_i)$. This defines a new graph of bisets $\mathfrak{B}$ with same underlying graph as $\mathfrak{T}$.

Then $B(t)$ and $\pi_1(\mathfrak{B})$ are isomorphic.

Proof. Let $p, \iota : X^1 \rightrightarrows X$ be the correspondence from §5.2 associated with the Hubbard tree of p. Let $t, \iota : \widetilde{X}^1 \rightrightarrows \widetilde{X}$ be the correspondence obtained by replacing each $p : \operatorname{int}(D_{z_i}) \to \operatorname{int}(D_{z_{i+1}})$ with $q_i : \mathbb{C} \setminus P_i \dashrightarrow \mathbb{C} \setminus P_{i+1}$ appropriately glued with $\partial D_i, \partial D_{i+1}$. Since t is isotopic to $t, \iota : \widetilde{X}^1 \rightrightarrows \widetilde{X}$, the biset of t is isomorphic to the biset of a covering pair $(\widetilde{X}^1, t, \iota)$ which by Van Kampen's Theorem 4.8 is $\pi_1(\mathfrak{B})$. □

This operation may be performed with p a Thurston map, not necessarily a topological polynomial. As a special case, a topological polynomial has a critical fixed point ∞. The *formal mating* $p \sqcup q$ of two topological polynomials p, q of same degree is the tuning $t(p, \{\infty\}, q)$. In other words, $p \sqcup q$ is a topological map on a two dimensional sphere where p acts on the southern hemisphere while q acts on the northern hemisphere.

Since the vertex ∞ does not belong to the Hubbard tree of p, Theorem 5.8 cannot apply. We do have a simple description of the mating in terms of graphs of bisets, though:

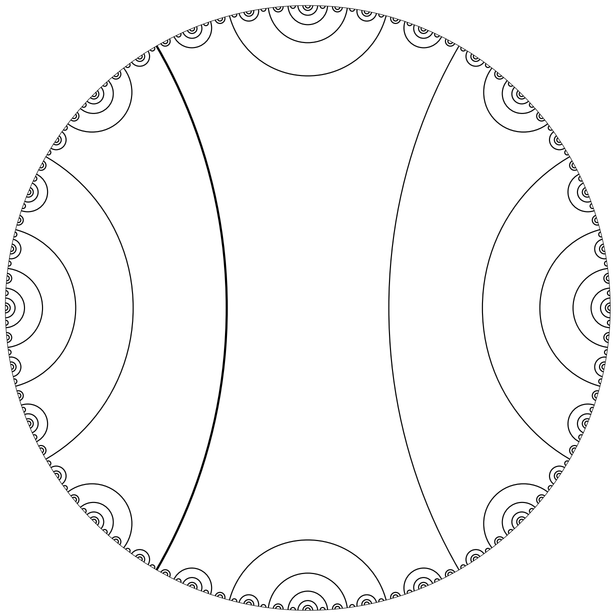
Theorem 5.9. *Let p, q be two topological polynomials of same degree d. Write $G_p = \pi_1(\mathbb{C} \setminus P(p), *)$ and $G_q = \pi_1(\mathbb{C} \setminus P(q), \dagger)$ with $*, \dagger$ close to ∞. Consider the graph of groups $\mathfrak{X}$ with one edge e and two vertices p, q. The groups at p, q are G_p, G_q respectively; the group at e is $\mathbb{Z}$, included in G_p and G_q as the loop around infinity. Consider the graph of bisets $\mathfrak{B}$ with one edge and two vertices. The graph maps λ, ρ are the identity, the biset B_e on the edge is regular cyclic of degree d, and the bisets at the vertices are $B(p), B(q)$ respectively. The intertwiners $B_e \hookrightarrow B(p)$ and $B_{\bar{e}} \hookrightarrow B(q)$ are defined by viewing $z \mapsto z^d : S^1 \circlearrowleft$ as a map on the circle at infinity of $p : \mathbb{C} \setminus P(p) \dashrightarrow \mathbb{C} \setminus P(p)$ and $q : \mathbb{C} \setminus P(q) \dashrightarrow \mathbb{C} \setminus P(q)$.*

Then $\pi_1(\mathfrak{B})$ is isomorphic to $B(p \sqcup q)$.

Proof. Applying Theorem 4.8 to $p \sqcup q$ subordinate to the cover consisting of the closed southern hemisphere, the closed northern hemisphere and the equator, we obtain a graph of bisets isomorphic to the barycentric subdivision of $\mathfrak{B}$. □

5.6. Laminations. Consider a complex post-critically finite polynomial p of degree d. The Fatou component around ∞ admits a Böttcher parameterization $\phi\colon F_\infty \to \{|z| < 1\}$ such that $\phi(p(z)) = \phi(z)^d$. Since $J(p) = \partial F_\infty$, every element of $J(p)$ may be described as $\lim_{r\to 1}\phi^{-1}(re^{i\theta})$ for some (non-unique) angle θ; this gives a surjective map $c_p\colon\{|z| = 1\} \to J(p)$ encoding the Julia set. In fact, the biset of p may be read from the base-d expansion of the *kernel* of c_p, namely the equivalence relation $\Xi_p = \{(z_1, z_2) \mid c_p(z_1) = c_p(z_2)\}$.

The equivalence relation may be presented by a *lamination* of the disk $\{|z| \leq 1\}$: the disjoint collection of subsets of the closed disk, called *leaves*, that are convex hulls (in the hyperbolic metric, say) of equivalence classes of Ξ_p.



The Julia set $J(p)$ is the quotient of the circle by the relation Ξ_p, so the filled-in Julia set is the quotient of the disk obtained by contracting all leaves of the lamination to points. The dynamics on the circle is the "multiply the angle by d" map. The van Kampen theorem, Theorem 3.7, may therefore be applied to the covering consisting of the boundary circle and the leaves. In fact, if p is post-critically finite it suffices to consider a finite collection of leaves. We consider, as a simple illustration, the Basilica map $z^2 - 1$ obtained from the circle by pinching the above lamination.

The space under consideration consists of a circle C and an arc A connecting the points $x = \exp(2i\pi/3)$ and $y = \exp(4i\pi/3)$, see Figure 3 right. We fix basepoints $* = 1$ on C and $\dagger = x$ on A.

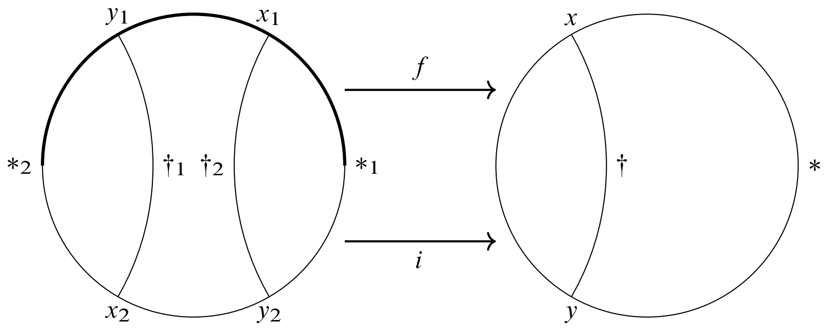
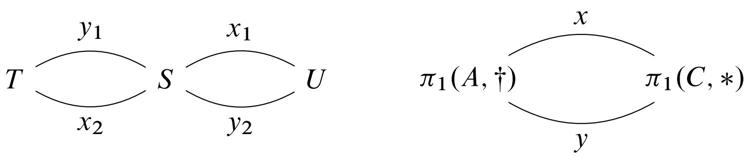


Figure 3. The correspondence generating the lamination of $z^2 - 1$. Note that the two preimages of x are $x_1, x_2 = \pm\sqrt{x}$, and similarly for y.

Let us denote by S, T, U the bisets of $f, i\colon C \rightrightarrows C$, $f, i\colon A \rightrightarrows A$, and $f\colon -A \to A$, $i\colon -A \to C$ respectively. We thus have the following graph of bisets $\mathfrak{B}$:



in which $S = {}_{2\mathbb{Z}}\mathbb{Z}_{\mathbb{Z}}$ is the regular cyclic biset encoding the doubling map on C and the other bisets T, U are trivial. The maps ρ, λ are given by

$$\rho(T) = \rho(U) = \lambda(T) = \pi_1(A, \dagger), \qquad \rho(S) = \lambda(S) = \lambda(U) = \pi_1(C, *).$$

The structure of $\mathfrak{B}$ will have been completely given when we fix bases of S, T, U and describe the edge bisets x_1, y_1, x_2, y_2 as maps from T or U into S. Let us write $H = \pi_1(C, *) = \langle t \rangle$ and $1 = \pi_1(A, \dagger)$. Then

$${}_H S = H \times \{1, 2\}, \quad {}_1 T = 1 \times \{3\}, \quad {}_H U = H \times \{4\}.$$

Recall that elements of a biset $B(f, i, *'', *')$ are written, in their most general form (13), as (α, p, β) for paths α, β with $\alpha(0) = *'', \alpha(1) = i(p), \beta(0) = f(p), \beta(1) = *'$. We fix paths p, x, ℓ, y starting at $* = 1$, turning counterclockwise on C, and ending respectively at $\exp(2i\pi/6), x, *_2, y$. We also write ϵ for any constant path. In this notation, the bases $1, 2, 3, 4$ are respectively

$$1 = (\epsilon, *_1, \epsilon), \quad 2 = (\ell, *_2, \epsilon), \quad 3 = (\epsilon, \dagger_1, \epsilon), \quad 4 = (p, \dagger_2, \epsilon).$$

We may now view y_1, x_2 as maps $T \to S$ and x_1, y_2 as maps $U \to S$. They are given by

$$y_1(3) = (x, \dagger_1, y^{-1}) = 1, \qquad x_1(4) = (p, \dagger_2, x^{-1}) = 1,$$
$$x_2(3) = (y, \dagger_1, x^{-1}) = 2, \qquad y_2(4) = (p, \dagger_2, y^{-1}) = t^{-1}2.$$

We are finally ready to compute the fundamental biset B of $\mathfrak{B}$. It is a G-G-biset for the group $G = H * \langle yx^{-1}\rangle = \langle t, yx^{-1}\rangle$. We may keep the basis $\{1,2\}$ of S, so ${}_G B = G \times \{1,2\}$, and we have, just as in S,

$$1 \cdot t = 2, \quad 2 \cdot t = t \cdot 1.$$

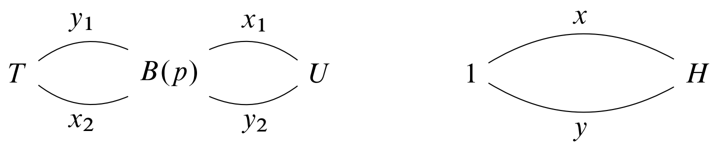
We now compute

$$1 \cdot yx^{-1} = y_1(3)yx^{-1} = x3x^{-1} = xy^{-1}x_2(3) = xy^{-1} \cdot 2,$$
$$2 \cdot yx^{-1} = ty_2(4)yx^{-1} = t4x^{-1} = tx_1(4) = t \cdot 1.$$

In this manner, we formally recover the presentation of the Basilica biset from the graph of bisets $\mathfrak{B}$, using the relations $y_1(3)y = x3$ etc. appearing in Definition 3.13.

There is another sort of mating, obtained directly from the Julia sets. The *geometric mating* of two polynomials p, q of same degree is the map obtained from $z \mapsto z^{\deg(p)} : (\mathbb{C} \cup \{\infty\}) \circlearrowleft$ through the quotient of $\{|z| \le 1\}$ by the lamination of p and through the quotient of $\{|z| \ge 1\}$ by the lamination of q embedded into the outer disc via the map $z \mapsto \frac{1}{z}$. In case both p and q are post-critically finite quadratic polynomials that are not in the conjugate limbs of the Mandelbrot set, the formal and geometric matings of p and q are isotopic. See [6] for a discussion on the various types of mating.

Theorem 5.10. *Let p be a post-critically finite quadratic polynomial that lies outside of the Basilica limb. Write $H = \pi_1(\mathbb{C} \setminus P(p), *)$. Then the biset of $(z^2 - 1) \sqcup p$ is the fundamental biset of the following graph of bisets*:



with the same maps ρ, λ and inclusions as above.

Proof. The map $(z^2 - 1) \sqcup p$ is homotopic to the map shown on Figure 3 with $\{|z| \ge 1\}$ replaced by the filled-in Julia set of p. The statement becomes a corollary of Theorem 4.8. □

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